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On some new Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals

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Abstract
In this paper, three fundamental and important Riemann-Liouville fractional integral identities including a twice differentiable mapping are established. Secondly, some interesting Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals for m-convexity and (s,m)-convexity functions, respectively, by virtue of the established integral identities are presented.
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1 Introduction

In 1881, Hermite found the famous Hermite-Hadamard inequality (see Mitrinović and Lacković [1])

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2},$$

where $f : I \subset R \to R$ is a convex function on the interval I of real numbers and $a,b \in I$ with $a < b$. For the contribution on the recent results which generalized, improved, and extended this classical Hermite-Hadamard inequality *via* convex functions, we refer the reader to [2–16] and references therein.

In addition to the classical convex functions, Toader [17], Hudzik and Maligranda [18] and Pinheiro [19] extended the concepts of classical convex functions to the concepts of m-convex function and (s,m)-convex function.

Definition 1.1 The function $f : [0,b^*] \to R$ is said to be m-convex, where $m \in [0,1]$ and $b^* > 0$, if for every $x,y \in [0,b^*]$ and $t \in [0,1]$, we have

$$f\big(tx + m(1-t)y\big) \leq tf(x) + m(1-t)f(y).$$

Definition 1.2 The function $f : [0,b^*] \to R$ is said to be (s,m)-convex, where $(s,m) \in [0,1]^2$ and $b^* > 0$, if for every $x,y \in [0,b^*]$ and $t \in [0,1]$, we have

$$f\big(tx + m(1-t)y\big) \leq t^s f(x) + m\big(1-t^s\big)f(y).$$

Recently, Ödemir *et al.* [5, 6] applied the following two important integral identities, including second-order derivatives, to establish some interesting Hermite-Hadamard type inequalities for m-convexity and (s,m)-convexity functions, respectively.

Lemma 1.3 *Let $f:[a,b]\to R$ be a twice differentiable mapping on (a,b) with $a<b$. If $f''\in L[a,b]$, then*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{(b-a)^2}{2}\int_0^1 t(1-t)f''\bigl(ta+(1-t)b\bigr)\,dt.$$

Lemma 1.4 *Let $f:[a,b]\to R$ be a twice differentiable mapping on (a,b) with $a<b$ and $m\in(0,1]$. If $f''\in L[a,b]$, then*

$$\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(t)\,dt=\frac{(mb-a)^2}{2}\int_0^1 t(1-t)f''\bigl(ta+m(1-t)b\bigr)\,dt.$$

For more recent interesting integral inequalities results for m-convexity and (s,m)-convexity functions, one can see [20–25].

On the other hand, fractional integrals and derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. For more recent development on fractional calculus, one can see the monographs [26–33] and the references therein.

Very recently, Sarikaya *et al.* [34] extended Lemma 1.3 and Lemma 1.4 to the case of Riemann-Liouville fractional integrals.

Lemma 1.5 (see Lemma 2, [34]) *Let $f:[a,b]\to R$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\bigl[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\bigr]\\&\quad=\frac{b-a}{2}\int_0^1\bigl[(1-t)^\alpha-t^\alpha\bigr]f'\bigl(ta+(1-t)b\bigr)\,dt,\end{aligned}$$

where the symbols ${}_{RL}J_{a^+}^{\alpha}f$ and ${}_{RL}J_{b^-}^{\alpha}f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha\in R^+$ that are defined by

$$\bigl({}_{RL}J_{a^+}^{\alpha}f\bigr)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)\,dt\quad(0\le a<x\le b),$$

and

$$\bigl({}_{RL}J_{b^-}^{\alpha}f\bigr)(x)=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)\,dt\quad(0\le a\le x<b),$$

respectively. Here $\Gamma(\cdot)$ is the gamma function.

Thereafter, Wang *et al.* [35, 36] extended Lemma 1.5 to the case of including a twice differentiable function involving Riemann-Liouville fractional integrals and the case of including a first differentiable function involving Hadamard fractional integrals, respectively.

Lemma 1.6 (see Lemma 2.1, [35]) *Let $f : [a,b] \to R$ be a twice differentiable mapping on (a,b) with $a < b$. If $f'' \in L[a,b]$, then the following equality for fractional integrals holds*:

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[{}_{RL}J_{a^+}^{\alpha}f(b) + {}_{RL}J_{b^-}^{\alpha}f(a)\big]\\&\quad = \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} f''\big(ta+(1-t)b\big)\,dt.\end{aligned} \tag{1}$$

Lemma 1.7 (see Lemma 3.1, [36]) *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then*

$$\begin{aligned}&\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b-\ln a)^{\alpha}}\big[{}_{H}J_{a^+}^{\alpha}f(b) +{}_{H}J_{b^-}^{\alpha}f(a)\big]\\&\quad = \frac{\ln b-\ln a}{2}\int_0^1 \big[(1-t)^{\alpha}-t^{\alpha}\big]e^{t\ln a+(1-t)\ln b} f'\big(e^{t\ln a+(1-t)\ln b}\big)\,dt,\end{aligned}$$

where the symbols ${}_{H}J_{a^+}^{\alpha}f$ and ${}_{H}J_{b^-}^{\alpha}f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \in R^+$ that are defined by

$$\big({}_{H}J_{a^+}^{\alpha}f\big)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \left(\ln\frac{x}{t}\right)^{\alpha-1} f(t)\frac{dt}{t} \quad (0 \le a < x \le b),$$

and

$$\big({}_{H}J_{b^-}^{\alpha}f\big)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \left(\ln\frac{t}{x}\right)^{\alpha-1} f(t)\frac{dt}{t} \quad (0 \le a \le x < b),$$

respectively.

Furthermore, Zhu *et al.* [37] established another important Riemann-Liouville fractional integral identity for a differentiable mapping.

Lemma 1.8 (see Lemma 2.1, [37]) *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$, then*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\big[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\big] - f\left(\frac{a+b}{2}\right)\\&\quad = \frac{b-a}{2}\left[\int_0^1 \zeta f'\big(ta+(1-t)b\big)\,dt - \int_0^1 \big[(1-t)^{\alpha}-t^{\alpha}\big]f'\big(ta+(1-t)b\big)\,dt\right],\end{aligned}$$

where

$$\zeta = \begin{cases} 1, & 0 \le t < \frac{1}{2}, \\ -1, & \frac{1}{2} \le t < 1. \end{cases}$$

Motivated by [5, 6, 34, 35, 37], we offer the following two basic questions:

(i) Can we extend Lemma 1.8 to some possible cases of including a twice differentiable mapping? If we can, we must give the concrete form.

(ii) Can we give other more interesting Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals for m-convex or (s,m)-convex functions by virtue of our established integral identities?

The first aim of this paper is to establish three fundamental and important Riemann-Liouville fractional integral identities including a twice differentiable function (see Lemmas 2.1, 2.3 and 2.3). Next, we present some interesting Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals for m-convexity and (s,m)-convexity functions, respectively, by virtue of our established integral identities.

2 Some interesting Riemann-Liouville fractional integral identities

We firstly extend Lemma 1.8 to the following integral identity including a twice differentiable mapping.

Lemma 2.1 *Let $f:[a,b]\to R$ be a twice differentiable mapping on (a,b) with $a<b$. If $f''\in L[a,b]$, then*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\bigl[{}_{RL}J^\alpha_{a^+}f(b)+{}_{RL}J^\alpha_{b^-}f(a)\bigr]-f\left(\frac{a+b}{2}\right)\\&\quad=\frac{(b-a)^2}{2}\int_0^1 m(t)f''\bigl(ta+(1-t)b\bigr)\,dt,\end{aligned}\tag{2}$$

where

$$m(t)=\begin{cases}t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t\in[0,\frac{1}{2}),\\ 1-t-\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t\in[\frac{1}{2},1).\end{cases}$$

Proof Denote

$$J=\int_0^{\frac{1}{2}} tf''\bigl(ta+(1-t)b\bigr)\,dt+\int_{\frac{1}{2}}^1(1-t)f''\bigl(ta+(1-t)b\bigr)\,dt:=J_1+J_2.\tag{3}$$

Integrating by parts, we have

$$\begin{aligned}J_1&=\int_0^{\frac{1}{2}} tf''\bigl(ta+(1-t)b\bigr)\,dt\\&=\frac{1}{a-b}tf'\bigl(ta+(1-t)b\bigr)\Big|_0^{\frac{1}{2}}-\frac{1}{a-b}\int_0^{\frac{1}{2}}f'\bigl(ta+(1-t)b\bigr)\,dt\\&=\frac{1}{2(a-b)}f'\left(\frac{a+b}{2}\right)-\frac{1}{(a-b)^2}\left[f\left(\frac{a+b}{2}\right)-f(b)\right],\end{aligned}\tag{4}$$

and

$$\begin{aligned}J_2&=\int_{\frac{1}{2}}^1(1-t)f''\bigl(ta+(1-t)b\bigr)\,dt\\&=\frac{1}{a-b}(1-t)f'\bigl(ta+(1-t)b\bigr)\Big|_{\frac{1}{2}}^1+\frac{1}{a-b}\int_{\frac{1}{2}}^1f'\bigl(ta+(1-t)b\bigr)\,dt\\&=-\frac{1}{2(a-b)}f'\left(\frac{a+b}{2}\right)+\frac{1}{(a-b)^2}\left[f(a)-f\left(\frac{a+b}{2}\right)\right].\end{aligned}\tag{5}$$

Submitting (4) and (5) to (3), it follows that

$$J = \frac{f(a)+f(b)}{(b-a)^2} - \frac{2f(\frac{a+b}{2})}{(b-a)^2}. \tag{6}$$

Thus, by multiplying both sides of (6) by $\frac{(b-a)^2}{2}$, we have

$$\begin{aligned}\frac{(b-a)^2}{2}&\int_0^1 m(t)f''\bigl(ta+(1-t)b\bigr)\,dt\\ &= \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\\ &\quad - \frac{(b-a)^2}{2}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} f''\bigl(ta+(1-t)b\bigr)\,dt.\end{aligned} \tag{7}$$

On the other hand, by (1) we obtain

$$\begin{aligned}\frac{(b-a)^2}{2}&\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} f''\bigl(ta+(1-t)b\bigr)\,dt\\ &= \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\bigl[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\bigr].\end{aligned} \tag{8}$$

Combing (7) and (8), we obtain the conclusion (2). This completes the proof. □

Next, we establish the following results.

Lemma 2.2 *Let $f:[a,b]\to R$ be a twice differentiable mapping on (a,b) with $a<b$. If $f''\in L[a,b]$, $r>0$, then*

$$\begin{aligned}&\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\bigl[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\bigr]\\ &\quad = (b-a)^2\int_0^1 k(t)f''\bigl(ta+(1-t)b\bigr)\,dt,\end{aligned} \tag{9}$$

where

$$k(t) = \begin{cases} \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{t}{r+1}, & t\in[0,\frac{1}{2}),\\ \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} - \frac{1-t}{r+1}, & t\in[\frac{1}{2},1).\end{cases} \tag{10}$$

Proof By multiplying both sides of (1) by $\frac{1}{r(r+1)}$, we have

$$\begin{aligned}&\frac{f(a)+f(b)}{r(r+1)} - \frac{\Gamma(\alpha+1)}{r(r+1)(b-a)^\alpha}\bigl[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\bigr]\\ &\quad = \frac{(b-a)^2}{r(r+1)}\int_0^1 \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1} f''\bigl(ta+(1-t)b\bigr)\,dt.\end{aligned} \tag{11}$$

By multiplying both sides of (2) by $-\frac{1}{r+1}$, we have

$$\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{(r+1)(b-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\right]$$
$$=-\frac{(b-a)^2}{r+1}\int_0^1 m(t)f''\big(ta+m(1-t)b\big)\,dt. \tag{12}$$

Hence, (11) and (12) yield

$$\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\right]$$
$$=\frac{(b-a)^2}{r(r+1)}\int_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}f''\big(ta+(1-t)b\big)\,dt$$
$$-\frac{(b-a)^2}{r+1}\int_0^1 m(t)f''\big(ta+m(1-t)b\big)\,dt$$
$$=(b-a)^2\int_0^1 k(t)f''\big(ta+(1-t)b\big)\,dt,$$

where $k(t)$ is defined in (10). This completes the proof. □

Lemma 2.3 *Let $f:[a,b]\to R$ be a twice differentiable mapping on (a,b) with $a<mb\le b$. If $f''\in L^1[a,b]$, $r>0$, then*

$$\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(mb)+{}_{RL}J_{mb^-}^{\alpha}f(a)\right]$$
$$=(mb-a)^2\int_0^1 k(t)f''\big(ta+m(1-t)b\big)\,dt,$$

where $k(t)$ is defined in (10).

Proof This is just Lemma 2.2 on the interval $[a,mb]\subset[a,b]$. □

3 Hermite-Hadamard type inequalities for *m*-convex functions

We start by stating the first theorem containing a Hermite-Hadamard type inequality.

Theorem 3.1 *Let $f:[0,b^*]\to R$ be a twice differentiable mapping with $b^*>0$. If $|f''|^q$ is measurable and m-convex on $[a,\frac{b}{m}]$ for some fixed $q\ge 1$, $0\le a<b$ and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, $r>0$, then*

$$\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\right]\right|$$
$$\le(b-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)\left(\frac{|f''(a)|^q+m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}}.$$

Proof Case 1: We suppose that $q = 1$. From Lemma 2.2, we have

$$\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(b)+{}_{RL}J^{\alpha}_{b^-}f(a)\right]\right|$$
$$\leq (b-a)^2\int_0^1\left|k(t)\right|\left|f''\left(ta+(1-t)b\right)\right|dt \tag{13}$$

due to $(1-t)^{\alpha+1}+t^{\alpha+1}\leq 1$ for any $t\in[0,1]$. Since $|f''|$ is m-convex on $[a,\frac{b}{m}]$, we know that for any $t\in[0,1]$,

$$\left|f''\left(ta+(1-t)b\right)\right|\leq t\left|f''(a)\right|+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|.$$

Therefore,

$$\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(b)+{}_{RL}J^{\alpha}_{b^-}f(a)\right]\right|$$
$$\leq (b-a)^2\int_0^{\frac{1}{2}}\left|k_1(t)\right|\left(t\left|f''(a)\right|+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right)dt$$
$$+(b-a)^2\int_{\frac{1}{2}}^1\left|k_2(t)\right|\left(t\left|f''(a)\right|+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right)dt,$$

where

$$k_1(t)=\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{t}{r+1},\qquad k_2(t)=\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}-\frac{1-t}{r+1}.$$

Denote

$$I_1=\int_0^{\frac{1}{2}}\left|k_1(t)\right|\left(t\left|f''(a)\right|+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right)dt,$$
$$I_2=\int_{\frac{1}{2}}^1\left|k_2(t)\right|\left(t\left|f''(a)\right|+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right)dt.$$

Integrating the above equalities by parts, respectively we have

$$I_1\leq\int_0^{\frac{1}{2}}\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}+\frac{t}{r+1}\right]\left(t\left|f''(a)\right|+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right)dt$$
$$=\frac{|f''(a)|}{r(\alpha+1)}\int_0^{\frac{1}{2}}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]t\,dt+\frac{|f''(a)|}{r+1}\int_0^{\frac{1}{2}}t^2\,dt$$
$$+\frac{m|f''(\frac{b}{m})|}{r(\alpha+1)}\int_0^{\frac{1}{2}}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right](1-t)\,dt+\frac{m|f''(\frac{b}{m})|}{r+1}\int_0^{\frac{1}{2}}t(1-t)\,dt$$
$$=\frac{|f''(a)|}{r(\alpha+1)}\left[\frac{1}{8}-\frac{1}{(\alpha+2)(\alpha+3)}+\frac{1}{(\alpha+2)(\alpha+3)2^{\alpha+2}}\right]+\frac{1}{24}\frac{|f''(a)|}{r+1}$$
$$+\frac{m|f''(\frac{b}{m})|}{r(\alpha+1)}\left[\frac{3}{8}-\frac{1}{\alpha+3}-\frac{1}{(\alpha+2)(\alpha+3)2^{\alpha+2}}\right]+\frac{1}{12}\frac{m|f''(\frac{b}{m})|}{r+1}, \tag{14}$$

and

$$\begin{aligned}
I_2 &\le \int_{\frac{1}{2}}^{1}\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}+\frac{1-t}{r+1}\right]\left(t\left|f''(a)\right|+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|\right)dt \\
&= \frac{|f''(a)|}{r(\alpha+1)}\int_{\frac{1}{2}}^{1}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right]t\,dt+\frac{|f''(a)|}{r+1}\int_{\frac{1}{2}}^{1}(1-t)t\,dt \\
&\quad+\frac{m|f''(\frac{b}{m})|}{r(\alpha+1)}\int_{\frac{1}{2}}^{1}\left[1-(1-t)^{\alpha+1}-t^{\alpha+1}\right](1-t)\,dt+\frac{m|f''(\frac{b}{m})|}{r+1}\int_{\frac{1}{2}}^{1}(1-t)^2\,dt \\
&= \frac{|f''(a)|}{r(\alpha+1)}\left[\frac{3}{8}-\frac{1}{\alpha+3}-\frac{1}{(\alpha+2)(\alpha+3)2^{\alpha+2}}\right]+\frac{1}{12}\frac{|f''(a)|}{r+1} \\
&\quad+\frac{m|f''(\frac{b}{m})|}{r(\alpha+1)}\left[\frac{1}{8}-\frac{1}{(\alpha+2)(\alpha+3)}+\frac{1}{(\alpha+2)(\alpha+3)2^{\alpha+2}}\right]+\frac{1}{24}\frac{m|f''(\frac{b}{m})|}{r+1}.
\end{aligned}\tag{15}$$

Therefore,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\right]\right| \\
&\quad\le \frac{(b-a)^2}{r(\alpha+1)}\left|f''(a)\right|\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)+\frac{1}{8}\frac{(b-a)^2}{r+1}\left|f''(a)\right| \\
&\qquad+\frac{m(b-a)^2}{r(\alpha+1)}\left|f''\left(\frac{b}{m}\right)\right|\left(\frac{1}{2}-\frac{1}{\alpha+2}\right)+\frac{1}{8}\frac{m(b-a)^2}{r+1}\left|f''\left(\frac{b}{m}\right)\right| \\
&\quad=(b-a)^2\left|f''(a)\right|\left(\frac{\alpha}{2r(\alpha+1)(\alpha+2)}+\frac{1}{8(r+1)}\right) \\
&\qquad+m(b-a)^2\left|f''\left(\frac{b}{m}\right)\right|\left(\frac{\alpha}{2r(\alpha+1)(\alpha+2)}+\frac{1}{8(r+1)}\right) \\
&\quad=(b-a)^2\frac{|f''(a)|+m|f''(\frac{b}{m})|}{2}\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right),
\end{aligned}$$

which completes the proof for this case.

Case 2: We suppose that $q>1$. By (9) *via* the power mean inequality for q, it is easy to see

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\right]\right| \\
&\quad\le (b-a)^2\left(\int_0^1\left|k(t)\right|dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1\left|k(t)\right|\left|f''\big(ta+(1-t)b\big)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad=(b-a)^2\left(\int_0^{\frac{1}{2}}\left|k_1(t)\right|dt+\int_{\frac{1}{2}}^{1}\left|k_2(t)\right|dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^{\frac{1}{2}}\left|k_1(t)\right|\left|f''\big(ta+(1-t)b\big)\right|^q dt\right. \\
&\qquad\left.+\int_{\frac{1}{2}}^{1}\left|k_2(t)\right|\left|f''\big(ta+(1-t)b\big)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{16}$$

Clearly,

$$\int_0^{\frac{1}{2}} \big|k_1(t)\big|\,dt \le \frac{\alpha}{2r(\alpha+1)(\alpha+2)} + \frac{1}{8(r+1)}, \tag{17}$$

$$\int_{\frac{1}{2}}^1 \big|k_2(t)\big|\,dt \le \frac{\alpha}{2r(\alpha+1)(\alpha+2)} + \frac{1}{8(r+1)}. \tag{18}$$

Since $|f''|^q$ is m-convex on $[a, \frac{b}{m}]$, we know that for any $t \in [0,1]$,

$$\big|f''\big(ta+(1-t)b\big)\big|^q \le t\big|f''(a)\big|^q + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q. \tag{19}$$

Thus,

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \big|k_1(t)\big|\big|f''\big(ta+(1-t)b\big)\big|^q\,dt \\
&\quad\le \int_0^{\frac{1}{2}} \left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} + \frac{t}{r+1}\right)\left(t\big|f''(a)\big|^q + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q\right)dt \\
&\quad= \frac{|f''(a)|^q}{r(\alpha+1)}\left[\frac{1}{8} - \frac{1}{(\alpha+2)(\alpha+3)} + \frac{1}{(\alpha+2)(\alpha+3)2^{\alpha+2}}\right] + \frac{|f''(a)|^q}{24(r+1)} \\
&\qquad+ \frac{m|f''(\frac{b}{m})|^q}{r(\alpha+1)}\left[\frac{3}{8} - \frac{1}{\alpha+3} - \frac{1}{(\alpha+2)(\alpha+3)2^{\alpha+2}}\right] + \frac{m|f''(\frac{b}{m})|^q}{12(r+1)}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1 \big|k_2(t)\big|\big|f''\big(ta+(1-t)b\big)\big|^q\,dt \\
&\quad\le \frac{|f''(a)|^q}{r(\alpha+1)}\left[\frac{3}{8} - \frac{1}{\alpha+3} - \frac{1}{(\alpha+2)(\alpha+3)2^{\alpha+2}}\right] + \frac{|f''(a)|^q}{12(r+1)} \\
&\qquad+ \frac{m|f''(\frac{b}{m})|^q}{r(\alpha+1)}\left[\frac{1}{8} - \frac{1}{(\alpha+2)(\alpha+3)} + \frac{1}{(\alpha+2)(\alpha+3)2^{\alpha+2}}\right] + \frac{m|f''(\frac{b}{m})|^q}{24(r+1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\right]\right| \\
&\quad\le (b-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)}\right)\left(\frac{|f''(a)|^q + m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}}.
\end{aligned}$$

The proof of this case is completed. □

Remark 3.2 With the same assumptions as in Theorem 3.1, if $|f''(x)| \le M$ on $[a, \frac{b}{m}]$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\right]\right| \\
&\quad\le M(b-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)}\right)\left(\frac{1+m}{2}\right)^{\frac{1}{q}}, \quad q \ge 1.
\end{aligned}$$

Now, we begin by stating the second theorem in this section.

Theorem 3.3 *Let $f : [0, b^*] \to R$ be a twice differentiable mapping with $b^* > 0$. If $|f''|^q$ is measurable and m-convex on $[a, \frac{b}{m}]$ for some fixed $q > 1$, $0 \leq a < b$ and $m \in (0,1]$ with $\frac{b}{m} \leq b^*$, $r > 0$, then*

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a)\right]\right| \leq \frac{(b-a)^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q + m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof From Lemma 2.2 and using the well-known Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a)\right]\right| \leq (b-a)^2\left(\int_0^1 \left|k(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}. \tag{20}$$

On the one hand,

$$\begin{aligned}\int_0^{\frac{1}{2}} \left|k_1(t)\right|^p dt &\leq \int_0^{\frac{1}{2}} \left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}\right)^p dt \\ &= \frac{1}{r^p(\alpha+1)^p}\int_0^{\frac{1}{2}} \left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)^p dt \\ &\leq \frac{1}{r^p(\alpha+1)^p}\int_0^{\frac{1}{2}} \left(1-(1-t)^{p(\alpha+1)}-t^{p(\alpha+1)}\right) dt \\ &= \frac{1}{r^p(\alpha+1)^p}\left(\frac{1}{2}-\frac{1}{p(\alpha+1)+1}\right),\end{aligned} \tag{21}$$

and

$$\int_{\frac{1}{2}}^1 \left|k_2(t)\right|^p dt \leq \frac{1}{r^p(\alpha+1)^p}\left(\frac{1}{2}-\frac{1}{p(\alpha+1)+1}\right), \tag{22}$$

where we use the fact

$$\left(1-(1-t)^{\alpha+1}-t^{\alpha+1}\right)^q \leq 1-(1-t)^{q(\alpha+1)}-t^{q(\alpha+1)}, \tag{23}$$

for any $t \in [0,1]$, which follows from $(A-B)^q \leq A^q - B^q$ for any $A > B \geq 0$ and $q \geq 1$.

On the other hand,

$$\begin{aligned}&\int_0^1 \left|f''(ta+(1-t)b\right|^q dt \\ &\quad\leq \int_0^1 \left[\left|f''(a)\right|^q t + m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q\right] dt\end{aligned}$$

$$= \left|f''(a)\right|^q \int_0^1 t\,dt + m\left|f''\left(\frac{b}{m}\right)\right|^q \int_0^1 (1-t)\,dt$$
$$= \frac{|f''(a)|^q + m|f''(\frac{b}{m})|^q}{2}. \tag{24}$$

Finally, submitting (21), (22) and (24) to (20), one can obtain the result immediately. □

Remark 3.4 With the same assumptions as in Theorem 3.3, if $|f''(x)| \leq M$ on $[a, \frac{b}{m}]$, we obtain

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\right]\right|$$
$$\leq \frac{M(b-a)^2}{r(\alpha+1)}\left(1 - \frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1+m}{2}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Another Hermite-Hadamard type inequality for powers in terms of the second derivatives is obtained as follows.

Theorem 3.5 *Let $f : [0, b^*] \to R$ be a twice differentiable mapping with $b^* > 0$. If $|f''|^q$ is measurable and m-convex on $[a, \frac{b}{m}]$ for some fixed $q > 1$, $0 \leq a < b$ and $m \in (0,1]$ with $\frac{b}{m} \leq b^*$, $r > 0$, then*

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\right]\right|$$
$$\leq \frac{(b-a)^2}{r(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(a)|^q + m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}}.$$

Proof From Lemma 2.2 and using the well-known Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\right]\right|$$
$$\leq (b-a)^2\left(\int_0^1 1\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|k(t)f''\bigl(ta+(1-t)b\bigr)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq (b-a)^2\left(\left|f''(a)\right|^q \int_0^1 t\left|k(t)\right|^q dt + m\left|f''\left(\frac{b}{m}\right)\right|^q \int_0^1 (1-t)\left|k(t)\right|^q dt\right)^{\frac{1}{q}}. \tag{25}$$

Calculating by parts, we have

$$\int_0^1 t\left|k(t)\right|^q dt = \int_0^{\frac{1}{2}} t\left|k_1(t)\right|^q dt + \int_{\frac{1}{2}}^1 t\left|k_2(t)\right|^q dt$$
$$\leq \frac{1}{r^q(\alpha+1)^q}\left[\frac{1}{2} - \frac{1}{q(\alpha+1)+1}\right], \tag{26}$$

$$\begin{aligned}\int_0^1 (1-t)\left|k(t)\right|^q dt &= \int_0^{\frac{1}{2}} (1-t)\left|k_1(t)\right|^q dt + \int_{\frac{1}{2}}^1 (1-t)\left|k_2(t)\right|^q dt \\ &\le \frac{1}{r^q(\alpha+1)^q}\left[\frac{1}{2} - \frac{1}{q(\alpha+1)+1}\right]. \end{aligned} \tag{27}$$

Submitting (26) and (27) to (25) *via* (23), one can obtain the result. The proof is completed. □

Remark 3.6 With the same assumptions as in Theorem 3.5, if $|f''(x)| \le M$ on $[a, \frac{b}{m}]$, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a)\right]\right| \\ &\quad\le \frac{M(b-a)^2}{r(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{1+m}{2}\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 3.7 From Theorems 3.1, 3.3 and 3.5, we have

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a)\right]\right| \le \min\{K_1, K_2, K_3\},$$

where

$$\begin{aligned}K_1 &= (b-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)}\right)\left(\frac{|f''(a)|^q + m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}}, \\ K_2 &= \frac{(b-a)^2}{r(\alpha+1)}\left(1 - \frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q + m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}}, \\ K_3 &= \frac{(b-a)^2}{r(\alpha+1)}\left(\frac{q(\alpha+1)-1}{q(\alpha+1)+1}\right)^{\frac{1}{q}}\left(\frac{|f''(a)|^q + m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

From Theorem 3.3 and Theorem 3.5, we use one skill of shrinking about inequality, then we now use another skill of shrinking.

Theorem 3.8 *Let $f : [0, b^*] \to R$ be a twice differentiable mapping with $b^* > 0$. If $|f''|^q$ is measurable and m-convex on $[a, \frac{b}{m}]$ for some fixed $q > 1$, $0 \le a < b$ and $m \in (0,1]$ with $\frac{b}{m} \le b^*$, $r > 0$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a)\right]\right| \\ &\quad\le \left(\frac{2}{p+1}\right)^{\frac{1}{p}}\frac{(b-a)^2}{r+1}\left[\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{p+1} - \left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}} \\ &\qquad\times\left(\frac{|f''(a)|^q + m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof From Lemma 2.2 and using the well-known Hölder inequality, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\right]\right| \\ &\quad\le (b-a)^2\left(\int_0^1 \left|k(t)\right|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f''\bigl(ta+(1-t)b\bigr)\right|^q\,dt\right)^{\frac{1}{q}}. \end{aligned} \tag{28}$$

Note that $(1-t)^{\alpha+1}+t^{\alpha+1}\le 1$ for any $t\in[0,1]$. Calculating by parts, we find

$$\begin{aligned} \int_0^{\frac{1}{2}}\left|k_1(t)\right|^p\,dt &\le \int_0^{\frac{1}{2}}\left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}+\frac{t}{r+1}\right)^p\,dt \\ &\le \int_0^{\frac{1}{2}}\left(\frac{1}{r(\alpha+1)}+\frac{t}{r+1}\right)^p\,dt \\ &= \frac{1}{(r+1)^p}\int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^p\,dt, \\ \int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^p\,dt &= \frac{1}{p+1}\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\frac{1}{p+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}, \end{aligned}$$

and

$$\begin{aligned} \int_{\frac{1}{2}}^1\left|k_2(t)\right|^p\,dt &\le \int_{\frac{1}{2}}^1\left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}+\frac{1-t}{r+1}\right)^p\,dt \\ &\le \int_{\frac{1}{2}}^1\left(\frac{1}{r(\alpha+1)}+\frac{1-t}{r+1}\right)^p\,dt \\ &= \frac{1}{(r+1)^p}\int_{\frac{1}{2}}^1\left(\frac{r+1}{r(\alpha+1)}+1-t\right)^p\,dt, \\ \int_{\frac{1}{2}}^1\left(\frac{r+1}{r(\alpha+1)}+1-t\right)^p\,dt &= \frac{1}{p+1}\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\frac{1}{p+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}. \end{aligned}$$

Thus,

$$\int_0^1\left|k(t)\right|^p\,dt\le\frac{2}{(r+1)^p(p+1)}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]. \tag{29}$$

Moreover,

$$\begin{aligned} \int_0^1\left|f''\bigl(ta+(1-t)b\bigr)\right|^q\,dt &\le \int_0^1\left(\left|f''(a)\right|^q t+m(1-t)\left|f''\left(\frac{b}{m}\right)\right|^q\right)dt \\ &= \left|f''(a)\right|^q\int_0^1 t\,dt+m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1(1-t)\,dt \\ &= \frac{|f''(a)|^q+m|f''(\frac{b}{m})|^q}{2}. \end{aligned} \tag{30}$$

Now submitting (29) and (30) to (28), one can derive the desired result. □

Remark 3.9 With the same assumptions as in Theorem 3.8, if $|f''(x)| \leq M$ on $[a, \frac{b}{m}]$, we obtain

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a)\right]\right|$$
$$\leq \frac{M(b-a)^2}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1} - \left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\left(\frac{1+m}{2}\right)^{\frac{1}{q}},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Another Hermite-Hadamard type inequality for powers in terms of the second derivatives is obtained as follows.

Theorem 3.10 *Let $f:[0,b^*]\to R$ be a twice differentiable mapping with $b^*>0$. If $|f''|^q$ is measurable and m-convex on $[a,\frac{b}{m}]$ for some fixed $q>1$, $0\leq a<b$ and $m\in(0,1]$ with $\frac{b}{m}\leq b^*$, $r>0$, then*

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a)\right]\right|$$
$$\leq \left(\frac{2}{q+1}\right)^{\frac{1}{q}}\frac{(b-a)^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1} - \left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]^{\frac{1}{q}}$$
$$\times\left(\frac{|f''(a)|^q + m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}}.$$

Proof From Lemma 2.2 and using the well-known Hölder inequality, we have

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(b) + {}_{RL}J^\alpha_{b^-}f(a)\right]\right|$$
$$\leq (b-a)^2\left(\int_0^1 1\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|k(t)f''\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\leq (b-a)^2\left(\left|f''(a)\right|^q\int_0^1 t\left|k(t)\right|^q dt + m\left|f''\left(\frac{b}{m}\right)\right|^q\int_0^1(1-t)\left|k(t)\right|^q dt\right)^{\frac{1}{q}}. \tag{31}$$

Calculating by parts, we have

$$\int_0^1 t\left|k(t)\right|^q dt = \int_0^{\frac{1}{2}} t\left|k_1(t)\right|^q dt + \int_{\frac{1}{2}}^1 t\left|k_2(t)\right|^q dt$$

with

$$\int_0^{\frac{1}{2}} t\left|k_1(t)\right|^q dt \leq \int_0^{\frac{1}{2}}\left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}+\frac{t}{r+1}\right)^q t\,dt$$
$$\leq \int_0^{\frac{1}{2}}\left(\frac{1}{r(\alpha+1)}+\frac{t}{r+1}\right)^q t\,dt$$
$$= \frac{1}{(r+1)^p}\int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^q t\,dt,$$

and

$$\int_{\frac{1}{2}}^{1} t\left|k_2(t)\right|^q dt \le \int_{\frac{1}{2}}^{1} \left(\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)} + \frac{1-t}{r+1}\right)^q t\,dt \le \int_{\frac{1}{2}}^{1} \left(\frac{1}{r(\alpha+1)} + \frac{1-t}{r+1}\right)^q t\,dt$$
$$= \frac{1}{(r+1)^q} \int_{\frac{1}{2}}^{1} \left(\frac{r+1}{r(\alpha+1)} + 1 - t\right)^q t\,dt,$$

where

$$\int_0^{\frac{1}{2}} \left(\frac{r+1}{r(\alpha+1)} + t\right)^q t\,dt = \frac{1}{2(q+1)}\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{q+1} - \frac{1}{(q+1)(q+2)}\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{q+2} + \frac{1}{(q+1)(q+2)}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+2},$$

and

$$\int_{\frac{1}{2}}^{1} \left(\frac{r+1}{r(\alpha+1)} + 1 - t\right)^q t\,dt$$
$$= -\frac{1}{q+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1} - \frac{1}{(q+1)(q+2)}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+2} + \frac{1}{2(q+1)}\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{q+1} + \frac{1}{(q+1)(q+2)}\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{q+2}.$$

Thus,

$$\int_0^1 t\left|k(t)\right|^q dt = \frac{1}{(q+1)(r+1)^q}\left[\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{q+1} - \left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]. \tag{32}$$

Clearly,

$$\int_0^1 (1-t)\left|k(t)\right|^q dt = \int_0^{\frac{1}{2}} (1-t)\left|k_1(t)\right|^q dt + \int_{\frac{1}{2}}^{1} (1-t)\left|k_2(t)\right|^q dt$$
$$\le \frac{1}{(q+1)(r+1)^q}\left[\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{q+1} - \left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]. \tag{33}$$

Now, submitting (32) and (33) to (31), one can obtain the result. The proof is completed. □

Remark 3.11 With the same assumptions as in Theorem 3.10, if $|f''(x)| \le M$ on $[a, \frac{b}{m}]$, we obtain

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J_{a^+}^{\alpha} f(b) + {}_{RL}J_{b^-}^{\alpha} f(a)\right]\right|$$
$$\le \left(\frac{2}{q+1}\right)^{\frac{1}{q}} \frac{M(b-a)^2}{r+1}\left[\left(\frac{1}{2} + \frac{r+1}{r(\alpha+1)}\right)^{q+1} - \left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]^{\frac{1}{q}}\left(\frac{1+m}{2}\right)^{\frac{1}{q}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 3.12 From Theorems 3.8 and 3.10, we have

$$\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(b)+{}_{RL}J^{\alpha}_{b^-}f(a)\right]\right|\leq\min\{N_1,N_2\},$$

where

$$N_1=\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\frac{(b-a)^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}$$
$$\times\left(\frac{|f''(a)|^q+m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}},$$
$$N_2=\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\frac{(b-a)^2}{r+1}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right]^{\frac{1}{q}}$$
$$\times\left(\frac{|f''(a)|^q+m|f''(\frac{b}{m})|^q}{2}\right)^{\frac{1}{q}}.$$

4 Hermite-Hadamard type inequalities for (s,m)-convex functions

Theorem 4.1 *Let $f:[a,b]\to R$ be a twice differentiable mapping with $a<b$. If $|f''|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q\geq 1$ and $(s,m)\in(0,1]^2$, $r>0$, then*

$$\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(mb)+{}_{RL}J^{\alpha}_{mb^-}f(a)\right]\right|$$
$$\leq(mb-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left|f''(a)\right|^q I+m\left|f''(b)\right|^q\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\right]^{\frac{1}{q}},\tag{34}$$

where

$$I=\frac{1}{r(s+1)(s+\alpha+2)}-\frac{1}{r(\alpha+1)}B(s+1,\alpha+2)$$
$$+\frac{1}{(r+1)(s+1)(s+2)}\left(1-\left(\frac{1}{2}\right)^{s+1}\right).\tag{35}$$

Proof Case 1: We suppose that $q=1$. From Lemma 2.3, we have

$$\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J^{\alpha}_{a^+}f(b)+{}_{RL}J^{\alpha}_{b^-}f(a)\right]\right|$$
$$\leq(mb-a)^2\int_0^1\left|k(t)f''\big(ta+m(1-t)b\big)\right|dt.\tag{36}$$

Since $|f''|$ is (s,m)-convex on $[a,b]$, we know that for any $t\in[0,1]$,

$$\left|f''\big(ta+m(1-t)b\big)\right|\leq t^s\left|f''(a)\right|+m\big(1-t^s\big)\left|f''(b)\right|.$$

Therefore (36) turns to

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\right]\right|\\&\quad\le(mb-a)^2(I_1+I_2),\end{aligned}$$

where

$$\begin{aligned}I_1&=\int_0^{\frac{1}{2}}\left|k_1(t)\right|\left(t^s\left|f''(a)\right|+m\left(1-t^s\right)\left|f''(b)\right|\right)dt,\\I_2&=\int_{\frac{1}{2}}^1\left|k_2(t)\right|\left(t^s\left|f''(a)\right|+m\left(1-t^s\right)\left|f''(b)\right|\right)dt.\end{aligned}$$

Calculating by parts, we have

$$\begin{aligned}I_1&\le\int_0^{\frac{1}{2}}\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}+\frac{t}{r+1}\right]\left(t^s\left|f''(a)\right|+m\left(1-t^s\right)\left|f''(b)\right|\right)dt\\&\le\frac{|f''(a)|}{r(\alpha+1)}\left[\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}-\frac{1}{s+\alpha+2}\left(\frac{1}{2}\right)^{s+\alpha+2}-\int_0^{\frac{1}{2}}t^s(1-t)^{\alpha+1}\,dt\right]\\&\quad+\frac{|f''(a)|}{r+1}\frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}+\frac{m|f''(b)|}{r(\alpha+1)}\left[\frac{1}{2}-\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}-\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\right.\\&\quad\left.+\frac{1}{\alpha+s+2}\left(\frac{1}{2}\right)^{\alpha+s+2}+\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}-\frac{1}{\alpha+2}+\int_0^{\frac{1}{2}}(1-t)^{\alpha+1}t^s\,dt\right]\\&\quad+\frac{m|f''(b)|}{r+1}\left[\frac{1}{8}-\frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}\right]\end{aligned}$$

and

$$\begin{aligned}I_2&\le\int_{\frac{1}{2}}^1\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}+\frac{1-t}{r+1}\right]\left(t^s\left|f''(a)\right|+m\left(1-t^s\right)\left|f''(b)\right|\right)dt\\&\le\frac{|f''(a)|}{r(\alpha+1)}\left[\frac{1}{s+1}-\frac{1}{s+\alpha+2}-\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\right.\\&\quad\left.+\frac{1}{s+\alpha+2}\left(\frac{1}{2}\right)^{s+\alpha+2}-\int_{\frac{1}{2}}^1t^s(1-t)^{\alpha+1}\,dt\right]\\&\quad+\frac{|f''(a)|}{r+1}\left[\frac{1}{s+1}-\frac{1}{s+2}-\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}+\frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}\right]\\&\quad+\frac{m|f''(b)|}{r(\alpha+1)}\left[\frac{1}{2}-\frac{1}{\alpha+2}-\frac{1}{s+1}+\frac{1}{\alpha+s+2}+\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}+\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\right.\\&\quad\left.-\frac{1}{\alpha+s+2}\left(\frac{1}{2}\right)^{\alpha+s+2}-\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}+\int_{\frac{1}{2}}^1(1-t)^{\alpha+1}t^s\,dt\right]\\&\quad+\frac{m|f''(b)|}{r+1}\left[\frac{1}{8}-\frac{1}{s+1}+\frac{1}{s+2}+\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}-\frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}\right].\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(mb)+{}_{RL}J_{mb^-}^{\alpha}f(a)\right]\right| \\
&\quad\le (mb-a)^2\left[\frac{|f''(a)|}{r(\alpha+1)}\left(\frac{\alpha+1}{(s+1)(s+\alpha+2)}-\int_0^1 t^s(1-t)^{\alpha+1}\,dt\right)\right] \\
&\qquad+(mb-a)^2\left[\frac{|f''(a)|}{r+1}\left(\frac{1}{(s+1)(s+2)}-\frac{1}{(s+1)(s+2)}\left(\frac{1}{2}\right)^{s+1}\right)\right] \\
&\qquad+(mb-a)^2\left[\frac{m|f''(b)|}{r(\alpha+1)}\left(1-\frac{2}{\alpha+2}-\frac{1}{s+1}+\frac{1}{\alpha+s+2}+\int_0^1(1-t)^{\alpha+1}t^s\,dt\right)\right] \\
&\qquad+(mb-a)^2\left[\frac{m|f''(b)|}{r+1}\left(\frac{1}{4}-\frac{1}{(s+1)(s+2)}+\frac{1}{(s+1)(s+2)}\left(\frac{1}{2}\right)^{s+1}\right)\right] \\
&\quad=(mb-a)^2\left[\left|f''(a)\right|\left(\frac{1}{r(s+1)(s+\alpha+2)}-\frac{1}{r(\alpha+1)}\int_0^1 t^s(1-t)^{\alpha+1}\,dt\right.\right. \\
&\qquad\left.\left.+\frac{1}{(r+1)(s+1)(s+2)}\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\right)\right] \\
&\qquad+(mb-a)^2\left[m\left|f''(b)\right|\left(\frac{1}{r(\alpha+1)}-\frac{2}{r(\alpha+1)(\alpha+2)}-\frac{1}{r(s+1)(\alpha+s+2)}\right.\right. \\
&\qquad+\frac{1}{r(\alpha+1)}\int_0^1 t^s(1-t)^{\alpha+1}\,dt+\frac{1}{4(r+1)} \\
&\qquad\left.\left.-\frac{1}{(r+1)(s+1)(s+2)}\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\right)\right]
\end{aligned}$$

because $\int_0^1 t^s(1-t)^{\alpha+1}\,dt = B(s+1,\alpha+2)$.

Note that (35), one can derive

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(mb)+{}_{RL}J_{mb^-}^{\alpha}f(a)\right]\right| \\
&\quad\le (mb-a)^2\left[\left|f''(a)\right|I+m\left|f''(b)\right|\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\right],
\end{aligned}$$

which completes the proof for this case.

Case 2: We suppose that $q>1$. Using Lemma 2.3 and the power mean inequality for q, we obtain

$$\begin{aligned}
&\int_0^1\left|k(t)f''\big(ta+m(1-t)b\big)\right|dt \\
&\quad\le\left(\int_0^1\left|k(t)\right|dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1\left|k(t)f''\big(ta+m(1-t)b\big)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}\tag{37}$$

Since $|f''|$ is (s,m)-convex on $[a,b]$, we know that for any $t\in[0,1]$,

$$\left|f''\big(ta+(1-t)b\big)\right|^q\le t^s\left|f''(a)\right|^q+m\big(1-t^s\big)\left|f''(b)\right|^q.\tag{38}$$

Hence, from (37) and (38), we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(mb)+{}_{RL}J_{mb^-}^{\alpha}f(a)\right]\right| \\
&\quad\le (mb-a)^2\left(\int_0^1\left|k(t)\right|dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1\left|k(t)f''\bigl(ta+m(1-t)b\bigr)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad= (mb-a)^2\left(\int_0^1\left|k(t)\right|dt\right)^{1-\frac{1}{q}} \\
&\qquad\times\left(\int_0^{\frac{1}{2}}\left|k_1(t)\right|\left|f''\bigl(ta+m(1-t)b\bigr)\right|^q dt+\int_{\frac{1}{2}}^1\left|k_2(t)\right|\left|f''\bigl(ta+m(1-t)b\bigr)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Calculating by parts, we have

$$\int_0^1\left|k(t)\right|dt\le\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}$$

and

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|k_1(t)\right|\left|f''\bigl(ta+m(1-t)b\bigr)\right|^q dt \\
&\quad\le\int_0^{\frac{1}{2}}\left[\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{r(\alpha+1)}+\frac{t}{r+1}\right]\bigl(t^s\left|f''(a)\right|^q+m\bigl(1-t^s\bigr)\left|f''(b)\right|^q\bigr)dt \\
&\quad=\frac{|f''(a)|^q}{r(\alpha+1)}\left[\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}-\frac{1}{s+\alpha+2}\left(\frac{1}{2}\right)^{s+\alpha+2}-\int_0^{\frac{1}{2}}t^s(1-t)^{\alpha+1}dt\right] \\
&\qquad+\frac{|f''(a)|^q}{r+1}\frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}+\frac{m|f''(b)|^q}{r(\alpha+1)}\left[\frac{1}{2}-\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}-\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\right. \\
&\qquad\left.+\frac{1}{\alpha+s+2}\left(\frac{1}{2}\right)^{\alpha+s+2}+\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}-\frac{1}{\alpha+2}+\int_0^{\frac{1}{2}}(1-t)^{\alpha+1}t^s dt\right] \\
&\qquad+\frac{m|f''(b)|^q}{r+1}\left[\frac{1}{8}-\frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}\right],
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1\left|k_2(t)\right|\left|f''\bigl(ta+m(1-t)b\bigr)\right|^q dt\,dt \\
&\quad\le\frac{|f''(a)|^q}{r(\alpha+1)}\left[\frac{1}{s+1}-\frac{1}{s+\alpha+2}-\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\right. \\
&\qquad\left.+\frac{1}{s+\alpha+2}\left(\frac{1}{2}\right)^{s+\alpha+2}-\int_{\frac{1}{2}}^1 t^s(1-t)^{\alpha+1}dt\right] \\
&\qquad+\frac{|f''(a)|^q}{r+1}\left[\frac{1}{s+1}-\frac{1}{s+2}-\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}+\frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}\right] \\
&\qquad+\frac{m|f''(b)|^q}{r(\alpha+1)}\left[\frac{1}{2}-\frac{1}{\alpha+2}-\frac{1}{s+1}+\frac{1}{\alpha+s+2}+\frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2}+\frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1}\right.
\end{aligned}$$

$$- \frac{1}{\alpha+s+2}\left(\frac{1}{2}\right)^{\alpha+s+2} - \frac{1}{\alpha+2}\left(\frac{1}{2}\right)^{\alpha+2} + \int_{\frac{1}{2}}^{1}(1-t)^{\alpha+1}t^{s}\,dt\Bigg]$$

$$+ \frac{m|f''(b)|^{q}}{r+1}\left[\frac{1}{8} - \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+1}\left(\frac{1}{2}\right)^{s+1} - \frac{1}{s+2}\left(\frac{1}{2}\right)^{s+2}\right].$$

Therefore, using the above facts, one can obtain (34), which completes the proof. □

Remark 4.2 In Theorem 4.1, if we choose $s = m = 1$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(b) + {}_{RL}J_{b^-}^{\alpha}f(a)\right]\right| \\
&\leq (b-a)^{2}\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)}\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left|f''(a)\right|^{q}I' + \left|f''(b)\right|^{q}\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)} + \frac{1}{4(r+1)} - I'\right)\right]^{\frac{1}{q}},
\end{aligned}$$

where

$$I' = \frac{1}{2r(\alpha+3)} - \frac{1}{r(\alpha+1)}B(2,\alpha+2) + \frac{1}{8(r+1)}.$$

Theorem 4.3 *Let $f : [a,b] \to R$ be a twice differentiable mapping with $a < b$. If $|f''|^{q}$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q > 1$ and $(s,m) \in (0,1]^{2}$, $r > 0$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(mb) + {}_{RL}J_{mb^-}^{\alpha}f(a)\right]\right| \\
&\leq \frac{(mb-a)^{2}}{r(\alpha+1)}\left(1 - \frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{1}{s+1}\left|f''(a)\right|^{q} + \frac{ms}{s+1}\left|f''(b)\right|^{q}\right)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof From Lemma 2.3 and using the well-known Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^{\alpha}}\left[{}_{RL}J_{a^+}^{\alpha}f(mb) + {}_{RL}J_{mb^-}^{\alpha}f(a)\right]\right| \\
&\leq (mb-a)^{2}\int_{0}^{1}\left|k(t)f''\big(ta+m(1-t)b\big)\right|dt \\
&\leq (mb-a)^{2}\left(\int_{0}^{1}\left|k(t)\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f''\big(ta+m(1-t)b\big)\right|^{q}dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Note that

$$\int_{0}^{1}\left|k(t)\right|^{p}dt \leq \frac{1}{r^{p}(\alpha+1)^{p}}\left(1 - \frac{2}{p(\alpha+1)+1}\right),$$

where we use $(1-(1-t)^{\alpha+1}-t^{\alpha+1})^{q} \leq 1-(1-t)^{q(\alpha+1)}-t^{q(\alpha+1)}$ for any $t \in [0,1]$.

Moreover,

$$\int_0^1 \left|f''\left(ta+m(1-t)b\right)\right|^q dt \le \left|f''(a)\right|^q \int_0^1 t^s\,dt + m\left|f''(b)\right|^q \int_0^1 \left(1-t^s\right) dt$$
$$= \frac{1}{s+1}\left|f''(a)\right|^q + \frac{ms}{s+1}\left|f''(b)\right|^q.$$

Therefore,

$$\left|\frac{f(a)+f(mb)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(mb) + {}_{RL}J_{mb^-}^\alpha f(a)\right]\right|$$
$$\le \frac{(mb-a)^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\left|f''(a)\right|^q + \frac{ms}{s+1}\left|f''(b)\right|^q\right)^{\frac{1}{q}},$$

which completes the proof. □

Remark 4.4 In Theorem 4.3, if we choose $s = m = 1$, we obtain

$$\left|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\right]\right|$$
$$\le \frac{(b-a)^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 4.5 *Let $f : [a,b] \to R$ be a twice differentiable mapping $a < b$. If $|f''|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q > 1$ and $(s,m) \in (0,1]^2$, $r > 0$, then*

$$\left|\frac{f(a)+f(mb)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(mb) + {}_{RL}J_{mb^-}^\alpha f(a)\right]\right|$$
$$\le \frac{(mb-a)^2}{r(\alpha+1)}\left[\left|f''(a)\right|^q\left(\frac{1}{s+1} - \frac{1}{q(\alpha+1)+s+1} - B\left(s+1,q(\alpha+1)+1\right)\right)\right.$$
$$+ m\left|f''(b)\right|^q\left(\frac{s}{s+1} - \frac{2}{q(\alpha+1)+1} + \frac{1}{q(\alpha+1)+s+1}\right.$$
$$\left.\left.+ B\left(s+1,q(\alpha+1)+1\right)\right)\right]. \tag{39}$$

Proof From Lemma 2.3 and using the well-known Hölder inequality, we have

$$\left|\frac{f(a)+f(mb)}{r(r+1)} + \frac{2}{r+1}f\left(\frac{a+mb}{2}\right) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[{}_{RL}J_{a^+}^\alpha f(mb) + {}_{RL}J_{mb^-}^\alpha f(a)\right]\right|$$
$$\le (mb-a)^2\left(\int_0^1 1\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|k(t)f''\left(ta+m(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$\le (mb-a)^2\left(\left|f''(a)\right|^q\int_0^1 \left|k(t)\right|^q t^s + m\left|f''(b)\right|^q\int_0^1 \left|k(t)\right|^q\left(1-t^s\right) dt\right)^{\frac{1}{q}}.$$

Calculating by parts, we have

$$\int_0^1 t^s\left|k(t)\right|^q dt \le \frac{1}{r^q(\alpha+1)^q}\left[\frac{1}{s+1} - \frac{1}{q(\alpha+1)+s+1} - \int_0^1 t^s(1-t)^{q(\alpha+1)}\,dt\right]$$

and

$$\begin{aligned}
&\int_0^1 (1-t^s)\left|k(t)\right|^q\,dt \\
&\quad\le \frac{1}{r^q(\alpha+1)^q}\left[1-\frac{1}{s+1}-\frac{2}{q(\alpha+1)+1}+\frac{1}{q(\alpha+1)+s+1}+\int_0^1 t^s(1-t)^{q(\alpha+1)}\,dt\right].
\end{aligned}$$

Therefore, using the above facts *via* $(1-(1-t)^{\alpha+1}-t^{\alpha+1})^q \le 1-(1-t)^{q(\alpha+1)}-t^{q(\alpha+1)}$ for any $t\in[0,1]$, one can derive (39). The proof is completed. □

Remark 4.6 In Theorem 4.5, if we choose $s=m=1$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J_{a^+}^{\alpha}f(b)+{}_{RL}J_{b^-}^{\alpha}f(a)\right]\right| \\
&\quad\le \frac{(b-a)^2}{r(\alpha+1)}\left[\left|f''(a)\right|^q\left(\frac{1}{2}-\frac{1}{q(\alpha+1)+2}-B\bigl(2,q(\alpha+1)+1\bigr)\right)\right. \\
&\qquad\left.+\left|f''(b)\right|^q\left(\frac{1}{2}-\frac{2}{q(\alpha+1)+1}+\frac{1}{q(\alpha+1)+2}+B\bigl(2,q(\alpha+1)+1\bigr)\right)\right].
\end{aligned}$$

Remark 4.7 From Theorems 4.1, 4.3 and 4.5, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[{}_{RL}J_{a^+}^{\alpha}f(mb)+{}_{RL}J_{mb^-}^{\alpha}f(a)\right]\right| \\
&\quad\le \min\{K_1,K_2,K_3\},
\end{aligned}$$

where

$$\begin{aligned}
K_1 &= (mb-a)^2\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left|f''(a)\right|^q I+m\left|f''(b)\right|^q\left(\frac{\alpha}{r(\alpha+1)(\alpha+2)}+\frac{1}{4(r+1)}-I\right)\right]^{\frac{1}{q}}, \\
K_2 &= \frac{(mb-a)^2}{r(\alpha+1)}\left(1-\frac{2}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\left|f''(a)\right|^q+\frac{ms}{s+1}\left|f''(b)\right|^q\right)^{\frac{1}{q}}, \\
K_3 &= \frac{(mb-a)^2}{r(\alpha+1)}\left[\left|f''(a)\right|^q\left(\frac{1}{s+1}-\frac{1}{q(\alpha+1)+s+1}-B\bigl(s+1,q(\alpha+1)+1\bigr)\right)\right. \\
&\quad+m\left|f''(b)\right|^q\left(\frac{s}{s+1}-\frac{2}{q(\alpha+1)+1}+\frac{1}{q(\alpha+1)+s+1}\right. \\
&\quad\left.\left.+B\bigl(s+1,q(\alpha+1)+1\bigr)\right)\right],
\end{aligned}$$

where I is defined in (35).

From Theorem 4.3 and Theorem 4.5, we use one skill of shrinking about inequality, then we now use another skill of shrinking.

Theorem 4.8 *Let $f:[a,b]\to R$ be a twice differentiable mapping with $a<b$. If $|f''|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q>1$and $(s,m)\in(0,1]^2$, $r>0$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)\right]\right|\\&\quad\le\frac{(mb-a)^2}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\\&\qquad\times\left(\frac{1}{s+1}\left|f''(a)\right|^q+\frac{ms}{s+1}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof From Lemma 2.3 and using the well-known Hölder inequality, we have

$$\begin{aligned}&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)\right]\right|\\&\quad\le(mb-a)^2\int_0^1\left|k(t)f''\big(ta+(1-t)b\big)\right|dt\\&\quad\le(mb-a)^2\left(\int_0^1\left|k(t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\big(ta+(1-t)b\big)\right|^q dt\right)^{\frac{1}{q}}\\&\quad\le\frac{(mb-a)^2}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\\&\qquad\times\left(\frac{1}{s+1}\left|f''(a)\right|^q+\frac{ms}{s+1}\left|f''(b)\right|^q\right)^{\frac{1}{q}}\end{aligned}$$

because $(1-t)^{\alpha+1}+t^{\alpha+1}\le 1$ for any $t\in[0,1]$, which completes the proof. □

Remark 4.9 With the same assumptions as in Theorem 4.8, if we choose $s=m=1$, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(b)+{}_{RL}J^\alpha_{b^-}f(a)\right]\right|\\&\quad\le\frac{(b-a)^2}{r+1}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{p+1}-\left(\frac{r+1}{r(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\\&\qquad\times\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Another Hermite-Hadamard type inequality for powers in terms of the second derivatives is obtained as follows.

Theorem 4.10 *Let $f:[a,b]\to R$ be a twice differentiable mapping $a<b$. If $|f''|^q$ is measurable and (s,m)-convex on $[a,b]$ for some fixed $q>1$ and $(s,m)\in(0,1]^2$, $r>0$, then the* *following inequality for fractional integrals holds*:

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)\right]\right| \\
&\quad\le\frac{(mb-a)^2}{r+1}\left[\left|f''(a)\right|^qH+m\left|f''(b)\right|^q\left(\frac{2}{q+1}\left(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\right)^{q+1}\right.\right. \\
&\qquad\left.\left.-\frac{2}{q+1}\left(\frac{r+1}{r(\alpha+1)}\right)^{q+1}-H\right)\right],
\end{aligned}\tag{40}$$

where

$$H=\int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^qt^s\,dt+\int_{\frac{1}{2}}^1\left(\frac{r+1}{r(\alpha+1)}+1-t\right)^qt^s\,dt.$$

Proof From Lemma 2.3 and using the well-known Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(mb)}{r(r+1)}+\frac{2}{r+1}f\left(\frac{a+mb}{2}\right)-\frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\left[{}_{RL}J^\alpha_{a^+}f(mb)+{}_{RL}J^\alpha_{mb^-}f(a)\right]\right| \\
&\quad\le(mb-a)^2\int_0^1\left|k(t)f''\left(ta+(1-t)b\right)\right|dt \\
&\quad\le(mb-a)^2\left(\int_0^1 1\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|k(t)f''\left(ta+m(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad\le(mb-a)^2\left(\left|f''(a)\right|^q\int_0^1 t^s\left|k(t)\right|^q dt+m\left|f''(b)\right|^q\int_0^1\left(1-t^s\right)\left|k(t)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Calculating by parts, we have

$$\int_0^1 t^s\left|k(t)\right|^q dt=\int_0^{\frac{1}{2}}t^s\left|k_1(t)\right|^q dt+\int_{\frac{1}{2}}^1 t^s\left|k_2(t)\right|^q dt,$$

where

$$\begin{aligned}
\int_0^{\frac{1}{2}}t^s\left|k_1(t)\right|^q dt&\le\int_0^{\frac{1}{2}}\left(\frac{1}{r(\alpha+1)}+\frac{t}{r+1}\right)^qt^s\,dt \\
&=\frac{1}{(r+1)^q}\int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^qt^s\,dt, \\
\int_{\frac{1}{2}}^1\left|k_2(t)\right|^qt^s\,dt&\le\int_{\frac{1}{2}}^1\left(\frac{1}{r(\alpha+1)}+\frac{1-t}{r+1}\right)^qt^s\,dt \\
&\le\frac{1}{(r+1)^q}\int_{\frac{1}{2}}^1\left(\frac{r+1}{r(\alpha+1)}+1-t\right)^qt^s\,dt.
\end{aligned}$$

Thus,

$$\int_0^1 t^s\left|k(t)\right|^q dt=\frac{1}{(r+1)^q}\left[\int_0^{\frac{1}{2}}\left(\frac{r+1}{r(\alpha+1)}+t\right)^qt^s\,dt+\int_{\frac{1}{2}}^1\left(\frac{r+1}{r(\alpha+1)}+1-t\right)^qt^s\,dt\right].$$

Similarly,

$$\begin{aligned}\int_0^1 (1-t^s)\big|k(t)\big|^q\,dt &= \int_0^{\frac{1}{2}} (1-t^s)\big|k_1(t)\big|^q\,dt + \int_{\frac{1}{2}}^1 (1-t^s)\big|k_2(t)\big|^q\,dt \\ &\leq \frac{1}{(r+1)^q}\Bigg[\frac{2}{q+1}\bigg(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\bigg)^{q+1} - \frac{2}{q+1}\bigg(\frac{r+1}{r(\alpha+1)}\bigg)^{q+1} \\ &\quad - \int_0^{\frac{1}{2}}\bigg(\frac{r+1}{r(\alpha+1)}+t\bigg)^q t^s\,dt - \int_{\frac{1}{2}}^1\bigg(\frac{r+1}{r(\alpha+1)}+1-t\bigg)^q t^s\,dt\Bigg].\end{aligned}$$

Now using the above facts, one can obtain (40). The proof is completed. □

Remark 4.11 With the same assumptions as in Theorem 4.10, if we choose $s = m = 1$, we obtain

$$\begin{aligned}&\bigg|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f\bigg(\frac{a+b}{2}\bigg) - \frac{\Gamma(\alpha+1)}{r(b-a)^\alpha}\big[{}_{RL}J_{a^+}^\alpha f(b) + {}_{RL}J_{b^-}^\alpha f(a)\big]\bigg| \\ &\quad\leq \frac{(b-a)^2}{r+1}\Bigg[\big|f''(a)\big|^q H' + \big|f''(b)\big|^q\bigg(\frac{2}{q+1}\bigg(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\bigg)^{q+1} \\ &\quad\quad - \frac{2}{q+1}\bigg(\frac{r+1}{r(\alpha+1)}\bigg)^{q+1} - H'\bigg)\Bigg],\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, and

$$H' = \frac{1}{q+1}\Bigg[\bigg(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\bigg)^{q+1} - \bigg(\frac{r+1}{r(\alpha+1)}\bigg)^{q+1}\Bigg].$$

Remark 4.12 From Theorems 4.8 and 4.10, we have

$$\begin{aligned}&\bigg|\frac{f(a)+f(mb)}{r(r+1)} + \frac{2}{r+1}f\bigg(\frac{a+mb}{2}\bigg) - \frac{\Gamma(\alpha+1)}{r(mb-a)^\alpha}\big[{}_{RL}J_{a^+}^\alpha f(mb) + {}_{RL}J_{mb^-}^\alpha f(a)\big]\bigg|, \\ &\quad\leq \min\{N_1, N_2\}\end{aligned}$$

where

$$\begin{aligned}N_1 &= \frac{(mb-a)^2}{r+1}\bigg(\frac{2}{p+1}\bigg)^{\frac{1}{p}}\Bigg[\bigg(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\bigg)^{p+1} - \bigg(\frac{r+1}{r(\alpha+1)}\bigg)^{p+1}\Bigg]^{\frac{1}{p}} \\ &\quad\times\bigg(\frac{1}{s+1}\big|f''(a)\big|^q + \frac{ms}{s+1}\big|f''(b)\big|^q\bigg)^{\frac{1}{q}}, \\ N_2 &= \frac{(mb-a)^2}{r+1}\Bigg[\big|f''(a)\big|^q H + m\big|f''(b)\big|^q\bigg(\frac{2}{q+1}\bigg(\frac{1}{2}+\frac{r+1}{r(\alpha+1)}\bigg)^{q+1} \\ &\quad - \frac{2}{q+1}\bigg(\frac{r+1}{r(\alpha+1)}\bigg)^{q+1} - H\bigg)\Bigg],\end{aligned}$$

where H is defined in Theorem 4.10.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between all authors. JRW raised these interesting problems in this research. YZ and JRW proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

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