Published for SISSA by Springer
Received: *December 21, 2010*
Accepted: *January 18, 2011*
Published: *February 14, 2011*

A direct proof of AGT conjecture at $\beta = 1$

A. Mironov,[a,b] **A. Morozov,**[b,c] **and Sh. Shakirov**[b,d]

[a] *Theory Department, Lebedev Physics Institute, Leninsky 53, Moscow, Russia*
[b] *ITEP, Bol. Cheremushkinskaya 25, Moscow, Russia*
[c] *Laboratoire de Mathematiques et Physique Theorique, CNRS-UMR 6083, Universite Francois Rabelais de Tours, France*
[d] *MIPT, Dolgoprudny, Russia*

E-mail: mironov@lpi.ru, morozov@itep.ru, shakirov@itep.ru

Abstract: The AGT conjecture claims an equivalence of conformal blocks in 2d CFT and sums of Nekrasov functions (instantonic sums in 4d SUSY gauge theory). The conformal blocks can be presented as Dotsenko-Fateev β-ensembles, hence, the AGT conjecture implies the equality between Dotsenko-Fateev β-ensembles and the Nekrasov functions. In this paper, we prove it in a particular case of $\beta = 1$ (which corresponds to $c = 1$ at the conformal side and to $\epsilon_1 + \epsilon_2 = 0$ at the gauge theory side) in a very direct way. The central role is played by representation of the Nekrasov functions through correlators of characters (Schur polynomials) in the Selberg matrix models. We mostly concentrate on the case of SU(2) with 4 fundamentals, the extension to other cases being straightforward. The most obscure part is extending to an arbitrary β: for $\beta \neq 1$, the Selberg integrals that we use do not reproduce single Nekrasov functions, but only sums of them.

Keywords: Matrix Models, Supersymmetric gauge theory, Conformal and W Symmetry

ArXiv ePrint: 1012.3137

doi:10.1007/JHEP02(2011)067

Contents

1 Introduction

One of the most recent instructive discoveries in string theory, the *AGT conjecture* [1] (see [2–81] for later progress) states an equivalence between conformal blocks in two-dimensional conformal field theory (with W_N symmetry) on one side [82, 83], and the LMNS instanton partition functions [84–87] in four-dimensional supersymmetric (SU(N)) gauge theory on the other. This relation is important for several reasons. Basically, it provides a very explicit (and rigorously formulated) realization of the string theory idea for a similarity between 4d supersymmetric and 2d conformal field theories, much more concrete than the standard AdS/CFT duality. Serving as a bridge between two different fields of research, the AGT relation stimulates progress in the both of them (say, activates the once abandoned studies of conformal blocks of W_N-algebras). It also provides [79] an advanced version [55–58] of the well-known correspondence [88] between 4d effective low-energy actions (Seiberg-Witten prepotentials [118, 119]) and integrable systems (often formulated in terms of 2d bosons and fermions).

Remarkably, apart from the two initial branches of physics connected by the AGT relation, there is still another, third field of research, which gets naturally involved: the theory of matrix models [59, 65–71, 73]. This was, of course, expected from the very beginning that matrix models belong to the same level of complexity as Seiberg-Witten prepotentials, their partition functions are long known to provide solutions to classical integrable hierarchies [120–130], etc (see [131–138] for an exact correspondence between matrix models and Seiberg-Witten theory). Nowadays these expectations turned into a very clearly formulated statement: that matrix models provide explicit integral representations for the conformal blocks. To be more precise, these are integral representations of conformal blocks $B(q)$ in the non-trivially interacting 2d CFT in terms of correlators with screening charge insertions in the free field 2d CFT *a-la* Dotsenko and Fateev [139–142]. A representative example is the four-point spherical conformal block (related to SU(2) Nekrasov function with four fundamental matter hypermultiplets) [69–71]:

$$\begin{gathered}
\text{4-point conformal block} = \mathcal{B}(q) = \text{integrated free-field correlator} = \\
= \left\langle\left\langle :e^{\tilde\alpha_1\phi(0)}::e^{\tilde\alpha_2\phi(q)}::e^{\tilde\alpha_3\phi(1)}::e^{\tilde\alpha_4\phi(\infty)}: \left(\int_0^q :e^{b\phi(z)}:\, dz\right)^{N_1} \left(\int_0^1 :e^{b\phi(z)}:\, dz\right)^{N_2} \right\rangle\right\rangle = \\
= q^{\alpha_1\alpha_2/2\beta}\,(1-q)^{\alpha_2\alpha_3/2\beta} \prod_{i=1}^{N_1}\int_0^q dz_i \prod_{i=N_1+1}^{N_1+N_2}\int_0^1 dz_i \prod_{i<j}(z_j-z_i)^{2\beta}\prod_i z_i^{\alpha_1}(z_i-q)^{\alpha_2}(z_i-1)^{\alpha_3}
\end{gathered} \tag{1.1}$$

where $\tilde\alpha_i = \alpha_i/2b$ and $\beta = b^2$. The second line is the correlator of normally ordered chiral vertex operators, corresponding to the initial four external fields and additional $N_1 + N_2$ screening charges, inserted in positions $z_1, \ldots, z_{N_1+N_2}$ and integrated with peculiar choices of integration contours. Such correlators are free field (Gaussian) averages, straightfor-

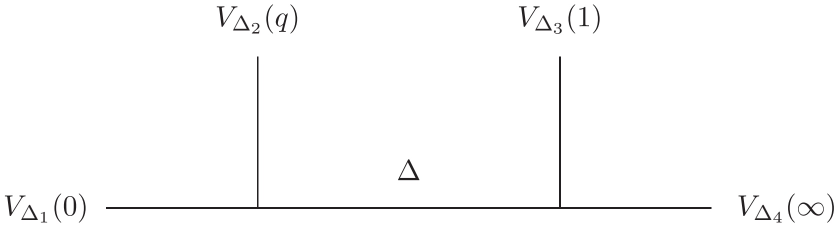


Figure 1. Feynman-like diagram for the 4-point conformal block. The external legs represent primary fields in 2d CFT; the structure of the graph shows the order of contractions in the operator product expansion procedure.

wardly evaluated with help of the Wick theorem

$$\Big\langle\Big\langle : e^{\tilde{\alpha}_1\phi(z_1)} : \ \ldots \ : e^{\tilde{\alpha}_m\phi(z_m)} : \Big\rangle\Big\rangle = \prod_{1\le i<j\le m} (z_j - z_i)^{2\tilde{\alpha}_i\tilde{\alpha}_j} \tag{1.2}$$

and finally put into the form of multiple integral (1.1) similar to matrix model eigenvalue integrals. For a generic $\beta \neq 1$, determined by the value of screening charge b and related to the central charge via $c = 1 - 6(b - 1/b)^2$, the integral is not, strictly speaking, an ordinary matrix model, it is rather a generalization known as β-ensemble [80, 143–147] or "conformal" matrix model [73, 148–150]. The difference, however, is not too drastic: it is well-known that matrix model theory is easily generalizable from $\beta = 1$ to arbitrary values of β, see [80] for a recent summary.

There are many different conformal blocks classified by the three main characteristics: a) conformal diagram, i.e. a graph with external legs, which shows the order of their OPE contraction; b) genus of underlying Riemann surface and c) rank N of the symmetry, which is $N = 2$ for the usual Virasoro conformal blocks, and higher N for conformal blocks of W_N algebras. For all of them, the Dotsenko-Fateev integrals can be straightforwardly written: extra internal dimensions are described by adding screening operators with different integration contours [68–71]; higher genera surfaces are described by substitution of free field Green functions by appropriate theta-functions [59, 75–77]; higher rank symmetries are described by making α's and b's vector-valued [148–150]. Because of this, and also because of their natural simplicity, it is convenient to use the Dotsenko-Fateev integrals to represent the whole variety of conformal blocks in the left hand (conformal) side of the AGT conjecture. This is exactly what we do in the present paper: we use for the conformal blocks the matrix model Dotsenko-Fateev representation [59, 65–71, 73].

On the other (gauge theory) side of the AGT conjecture, there are Nekrasov functions, the ultimate outcome of evaluation of integrals over the instanton moduli spaces in $\mathcal{N} = 2$ SUSY Yang-Mills theories [151, 152]. Since integrals over instanton moduli spaces typically diverge, they need to be regularized, and this, as usual, can be done in many different ways. One of the most popular ways to regularize these integrals [84–87] relies on introduction of the so-called Ω-background and associated deformation parameters ϵ_1, ϵ_2. The integrals over moduli spaces, regularized in this way, were evaluated in [151, 152] and finally represented as series in instanton parameters, with all terms explicit.

JHEP02(2011)067

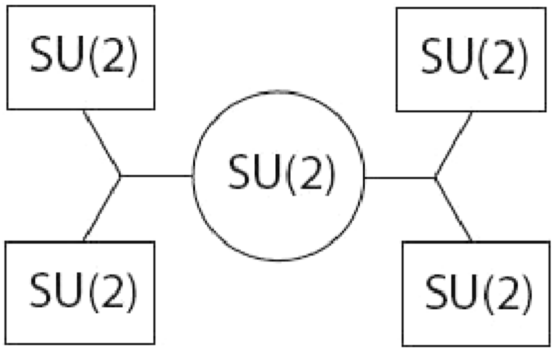


Figure 2. Quiver diagram for the SU(2) Nekrasov functions with four fundamentals. The external boxes represent the matter hypermultiplets, the central circle represents the gauge group, and the structure of the graph shows transformation properties of the matter hypermultiplets under the gauge group action.

There are many different types of Nekrasov functions, classified according to quiver diagrams [1, 153], i.e. graphical representations of the field content of a given theory with detailed indication of gauge groups and transformation properties of the matter multiplets. According to the AGT conjecture [1], each of these types of Nekrasov functions corresponds to a conformal block: the conformal diagram can be simply read off from the quiver diagram, with genus corresponding to the number of loops and with the symmetry (Virasoro or, generally, W_N) fixed by rank of the gauge group (SU(2) or, generally, SU(N)). Such a "dictionary" between 2d and 4d theories extends the one, originally suggested in [88–117], and represents one of the most explicit manifestations of the gauge-string duality over the last decades.

It is natural that, apart from generalizations and possible applications, more and more attention is getting attracted to the questions of understanding and proof of the AGT conjecture. The *understanding* of the otherwise mysterious connection between 2d and 4d theories is generally believed to be based upon existence of a certain unique 6d theory, which is in charge (through compactification) of the AGT relation. However, due to technical complications this direction remains largely philosophical, and has been unable to produce a *proof* yet.

Since the AGT relation is essentially the equality between the Nekrasov functions and Dotsenko-Fateev integrals, a more concrete approach could be to make use of the well-developed methods of matrix models for the proof. Several suggestions have been proposed on how to deal with the Nekrasov functions within the matrix model framework [59, 65–71, 73].

In [80] in order to proof the AGT conjecture in a more concrete way, we suggested to use that the Nekrasov functions are ϵ_1, ϵ_2-deformations of the celebrated Seiberg-Witten prepotentials, and the corresponding Seiberg-Witten theory coincides with the Seiberg-Witten theory of the planar limit of the Dotsenko-Fateev matrix model [59, 65–68]. Then, one may restore the ϵ_1, ϵ_2-deformations of the both Seiberg-Witten theories by the topological recursion [154–164], so that they still would coincide, with the Seiberg-Witten differential in the recursion being given by the exact 1-point resolvent of the matrix model (or, more precisely, of the β-ensemble). Another possibility is to use the Harer-Zagier recursion [80, 154, 165–168]. However, at the moment too little is known about matrix model representation of the Nekrasov functions, thus, this program remains to be accomplished. Development of the Harer-Zagier technique may play an important role here.

In this paper, we suggest to look at Nekrasov functions literally: as explicitly known sums over partitions (Young diagrams, see figure 3). Such series are indeed available for various Nekrasov functions in the literature [151, 152, 169–172]. For example, in the case of SU(2) with four fundamental hypermultiplets (related to the 4-point spherical conformal block (1.1)) the Nekrasov function can be written as a sum

$$Z_{\text{Nek}}(q) \;=\; \sum_{A,B} N_{A,B}\; q^{|A|+|B|} \tag{1.3}$$

over the Young diagrams $A = [A_1 \geq A_2 \geq \dots]$ and $B = [B_1 \geq B_2 \geq \dots]$, with the coefficients $N_{A,B}$ being rational functions of the masses $\mu_1, \mu_2, \mu_3, \mu_4$, of the Coulomb parameter a and of the deformation parameters $\epsilon_{1,2}$. Explicitly, several first coefficients N_{AB} have the form

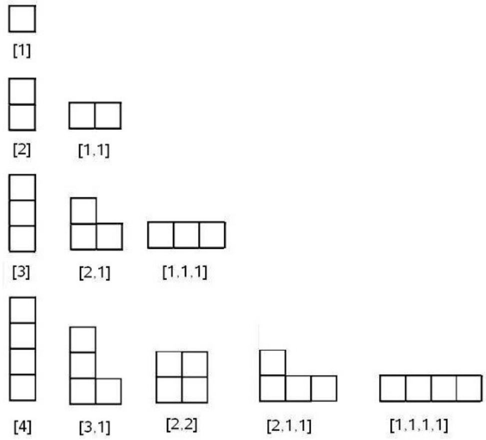


Figure 3. Several first Young diagrams.

$$N_{[1][]} = -\frac{1}{\epsilon_1\epsilon_2}\cdot\frac{\prod_{r=1}^4(a+\mu_r)}{2a(2a+\epsilon_1+\epsilon_2)}, \qquad N_{[][1]} = -\frac{1}{\epsilon_1\epsilon_2}\cdot\frac{\prod_{r=1}^4(a-\mu_r)}{2a(2a-\epsilon_1-\epsilon_2)} \tag{1.4}$$

$$N_{[1][1]} = \frac{1}{\epsilon_1^2\epsilon_2^2}\cdot\frac{\prod_{r=1}^4(a+\mu_r)(a-\mu_r)}{(4a^2-\epsilon_1^2)(4a^2-\epsilon_2^2)} \tag{1.5}$$

$$N_{[2][]} = \frac{1}{2!\,\epsilon_1\epsilon_2^2(\epsilon_1-\epsilon_2)}\cdot\frac{\prod_{r=1}^4(a+\mu_r)(a+\mu_r+\epsilon_2)}{2a(2a+\epsilon_2)(2a+\epsilon_1+\epsilon_2)(2a+\epsilon_1+2\epsilon_2)}, \tag{1.6}$$

$$N_{[][2]} = \frac{1}{2!\,\epsilon_1\epsilon_2^2(\epsilon_1-\epsilon_2)}\cdot\frac{\prod_{r=1}^4(a-\mu_r)(a-\mu_r-\epsilon_2)}{2a(2a-\epsilon_2)(2a-\epsilon_1-\epsilon_2)(2a-\epsilon_1-2\epsilon_2)}, \tag{1.7}$$

$$N_{[11][]} = -\frac{1}{2!\,\epsilon_1^2\epsilon_2(\epsilon_1-\epsilon_2)}\cdot\frac{\prod_{r=1}^4(a+\mu_r)(a+\mu_r+\epsilon_1)}{2a(2a+\epsilon_1)(2a+\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)}, \tag{1.8}$$

$$N_{[][11]} = -\frac{1}{2!\,\epsilon_1^2\epsilon_2(\epsilon_1-\epsilon_2)}\cdot\frac{\prod_{r=1}^4(a-\mu_r)(a-\mu_r-\epsilon_1)}{2a(2a-\epsilon_1)(2a-\epsilon_1-\epsilon_2)(2a-2\epsilon_1-\epsilon_2)} \tag{1.9}$$

and so on (omitting the trivial $N_{[][]} = 1$). For an explicit formula for the generic N_{AB}, see (3.2).

This "sum-over-partitions" point of view allows one to make a direct contact with matrix models, where these sums appear after *character expansion* [173–177]: decomposition of the integrand in a proper basis of symmetric polynomials. At $\beta = 1$, the proper basis is realized by the ordinary Schur polynomials, i.e. by the GL(∞) characters, which are labeled by partitions:

$$\chi_1(p) = p_1 \tag{1.10}$$

$$\chi_2(p) = \frac{p_1^2}{2}+\frac{p_2}{2}, \qquad \chi_{11}(p) = \frac{p_1^2}{2}-\frac{p_2}{2} \tag{1.11}$$

$$\chi_3(p) = \frac{p_3}{3}+\frac{p_1p_2}{2}+\frac{p_1^3}{6}, \qquad \chi_{21}(p) = -\frac{p_3}{3}+\frac{p_1^3}{3}, \qquad \chi_{111}(p) = \frac{p_3}{3}-\frac{p_1p_2}{2}+\frac{p_1^3}{6} \tag{1.12}$$

JHEP02(2011)067

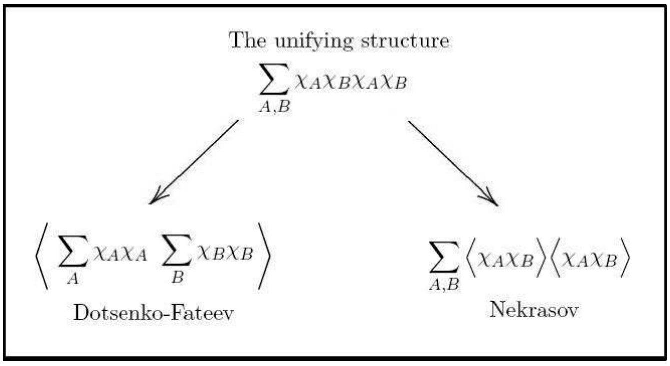


Figure 4. The core idea of the proof. This is a typical duality, when one unifying structure $(\langle\chi_A\chi_B\chi_A\chi_B\rangle)$ decomposes into two different channels $(\chi_A\chi_A)(\chi_B\chi_B)$ and $(\chi_A\chi_B)(\chi_A\chi_B)$.

etc., where $p_k = \sum z_i^k$ are the power sums. For $\beta \neq 1$, the proper deformation of the Schur polynomials is the Jack polynomials (aka β-characters), which depend on p_k and on a single additional parameter β. Further deformations, to the McDonald and, generally, Askey-Wilson polynomials, depend on more additional parameters and are relevant for description of 5d [61, 62] and, perhaps, 6d gauge theories.

To interpret series (1.3) as a character expansion, one needs to express the Nekrasov coefficients $N_{A,B}$ through the Schur polynomials. In this paper, we describe a solution to this problem for $\beta = 1$, which corresponds to the case of $\epsilon_1 + \epsilon_2 = 0$ for the Nekrasov function (the minus in the argument of the Schur function corresponds to the transposed Young diagram, see (A.10)):

$$N_{A,B}\Big(\epsilon_1 = \hbar, \epsilon_2 = -\hbar\Big) = \Big\langle \chi_A\big(-p_k - v_+\big)\ \chi_B\big(p_k\big) \Big\rangle_+ \Big\langle \chi_A\big(p_k\big)\ \chi_B\big(-p_k - v_-\big) \Big\rangle_- \tag{1.13}$$

where the brackets $\langle\rangle_+, \langle\rangle_-$ denote averaging over two Selberg ensembles

$$\Big\langle f \Big\rangle_\pm = \frac{\int_0^1 dz_1 \dots \int_0^1 dz_{N_\pm} \prod_{i<j}(z_i - z_j)^2 \prod_i z_i^{u_\pm}(z_i - 1)^{v_\pm}\ f\big(z_1, \dots, z_{N_\pm}\big)}{\int_0^1 dz_1 \dots \int_0^1 dz_{N_\pm} \prod_{i<j}(z_i - z_j)^2 \prod_i z_i^{u_\pm}(z_i - 1)^{v_\pm}} \tag{1.14}$$

with

$$u_+ = \frac{\mu_2 - \mu_1}{\hbar}, \quad u_- = \frac{\mu_4 - \mu_3}{\hbar} \tag{1.15}$$

$$v_+ = \frac{\mu_1 + \mu_2}{\hbar}, \quad v_- = \frac{\mu_3 + \mu_4}{\hbar} \tag{1.16}$$

$$N_+ = \frac{a - \mu_2}{\hbar}, \quad N_- = \frac{-a - \mu_4}{\hbar} \tag{1.17}$$

After one substitutes (1.13) into (1.3) and takes the sum over A, B, the characters recombine in the way that precisely reproduces the Dotsenko-Fateev integral (1.1), where the above two Selberg ensembles correspond to the two groups of variables z_i with $i \le N_1$

JHEP02(2011)067

and $i > N_1$, respectively ($N_\pm \equiv N_{1,2}$). Thus, the AGT relation for $\beta = 1$ is derived through the character expansion of the Dotsenko-Fateev integral, and can be interpreted as duality (as illustrated in figure 4).

It is tempting to generalize identity (1.13) to $\beta \neq 1$, i.e. to $\epsilon_1 + \epsilon_2 \neq 0$. Naively, one just has to substitute the Schur polynomials by the Jack polynomials

$$J_1(p) = p_1 \tag{1.18}$$

$$J_2(p) = \frac{p_2 + \beta p_{11}}{\beta + 1}, \qquad J_{11}(p) = \frac{p_1^2}{2} - \frac{p_2}{2} \tag{1.19}$$

$$J_3(p) = \frac{2p_3 + 3\beta p_1 p_2 + \beta^2 p_1^3}{(\beta+1)(\beta+2)}, \quad J_{21}(p) = \frac{(1-\beta)p_1p_2 - p_3 + \beta p_1^3}{(\beta+1)(\beta+2)}, \quad J_{111}(p) = \frac{p_3}{3} - \frac{p_1p_2}{2} + \frac{p_1^3}{6} \tag{1.20}$$

and change the power β of the Van-der-Monde determinant

$$\prod_{i<j}(z_i - z_j)^2 \quad \mapsto \quad \prod_{i<j}(z_i - z_j)^{2\beta} \tag{1.21}$$

in the definition of the Selberg averages. However, this naive β-deformation fails to reproduce the Nekrasov coefficients $N_{AB}\big(\epsilon_1, \epsilon_2\big)$. In our opinion, the basic reason for the discrepancy is that N_{AB} (if considered as a rational function of a) has a very special structure of poles, which accidentally coincides with that of Selberg integrals at $\beta = 1$, but for generic β is not captured by the Selberg integrals. To reproduce $N_{AB}\big(\epsilon_1, \epsilon_2\big)$ for generic ϵ_1, ϵ_2, some *clever* deformation of the r.h.s. of (1.13) is required. *Clarifying this point would complete the direct proof of the AGT conjecture.*

This paper is organized as follows.

Section 2 is devoted to the simple case of the AGT relation for pure SU(2). Consideration of this simple case helps to elucidate some of the important details of the story. We describe the conformal block (as the Dotsenko-Fateev integral), the Nekrasov functions (as explicit sums over Young diagrams) and state the AGT relation between them. Then, using a pure gauge version of the pair-correlator identity (1.13), we derive the $\epsilon_1 + \epsilon_2 = 0$ Nekrasov function from the $\beta = 1$ Dotsenko-Fateev integral.

Section 3 similarly deals with the AGT relation for SU(2) with four fundamental matter hypermultiplets. We describe, with the help of the pair-correlator identity (1.13), the analytical proof of equality between the $\epsilon_1 + \epsilon_2 = 0$ Nekrasov function and the $\beta = 1$ Dotsenko-Fateev integral, for arbitrary values of masses.

Section 4 is devoted to analysis of the problems, which arise when one attempts to generalize our construction to generic β.

Section 5 is the Conclusion.

The appendix is a list of various known factorizable 1-character (Jack) and 2-character (Jack) averages in the Selberg and BGW matrix model (β-ensemble) theories, for $\beta = 1$ and $\beta \neq 1$, which can also play a role in the future investigations.

JHEP02(2011)067

2 The case of pure SU(2)

2.1 Nekrasov function

The Nekrasov sum over partitions in this case has the form

$$Z^{\rm pure}_{\rm Nek}(\Lambda) \;=\; \sum_{A,B} N^{\rm pure}_{A,B}\;\Lambda^{4|A|+4|B|}, \qquad N^{\rm pure}_{A,B} = \frac{(\epsilon_1\epsilon_2)^{2|A|+2|B|}}{g_{A,A}(0)g_{A,B}(2a)g_{B,A}(-2a)g_{B,B}(0)} \tag{2.1}$$

where $g_{A,B}$ denote the contributions of gauge fields into the Nekrasov function

$$g_{A,B}(x) = \prod_{(i,j)\in A}\Big[x+\epsilon_1 {\rm Arm}_A(i,j)-\epsilon_2 {\rm Leg}_B(i,j)+\epsilon_1\Big]\Big[x+\epsilon_1 {\rm Arm}_A(i,j)-\epsilon_2 {\rm Leg}_B(i,j)-\epsilon_2\Big] \tag{2.2}$$

which has a characteristic form of a product over all the cells of the Young diagram. For the arbitrary Young diagram Y, the symbols ${\rm Arm}_Y(i,j)$ and ${\rm Leg}_Y(i,j)$ denote the arm-length and leg-length of the cell (i,j) in the diagram Y. Algebraically, these lengths are given by the expressions

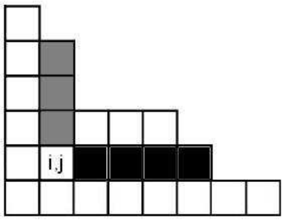


Figure 5. For the cell $(i,j) = (2,2)$, the arm- and leg-length are shown in black and grey, respectively. Note that the cell can lie beyond the diagram.

$${\rm Arm}_Y(i,j) = Y'_j - i, \qquad {\rm Leg}_Y(i,j) = Y_i - j \tag{2.3}$$

where Y' stands for the transposed Young diagram. This algebraic definition is not quite transparent: more enlightening may be the graphical meaning of these quantities, which is shown at figure 5. Several first Nekrasov coefficients for pure SU(2) have the form

$$N_{[1][]} = \frac{-\epsilon_1\epsilon_2}{2a(2a+\epsilon_1+\epsilon_2)}, \qquad N_{[][1]} = \frac{-\epsilon_1\epsilon_2}{2a(2a-\epsilon_1-\epsilon_2)} \tag{2.4}$$

$$N_{[1][1]} = \frac{\epsilon_1^2\epsilon_2^2}{2!(\epsilon_1-\epsilon_2)}\cdot\frac{2(\epsilon_1-\epsilon_2)}{(4a^2-\epsilon_1^2)(4a^2-\epsilon_2^2)} \tag{2.5}$$

$$N_{[2][]} = \frac{\epsilon_1^2\epsilon_2^2}{2!(\epsilon_1-\epsilon_2)}\cdot\frac{\epsilon_1}{2a(2a+\epsilon_2)(2a+\epsilon_1+\epsilon_2)(2a+\epsilon_1+2\epsilon_2)}, \tag{2.6}$$

$$N_{[][2]} = \frac{\epsilon_1^2\epsilon_2^2}{2!(\epsilon_1-\epsilon_2)}\cdot\frac{\epsilon_1}{2a(2a-\epsilon_2)(2a-\epsilon_1-\epsilon_2)(2a-\epsilon_1-2\epsilon_2)}, \tag{2.7}$$

$$N_{[11][]} = \frac{\epsilon_1^2\epsilon_2^2}{2!(\epsilon_1-\epsilon_2)}\cdot\frac{-\epsilon_2}{2a(2a+\epsilon_1)(2a+\epsilon_1+\epsilon_2)(2a+2\epsilon_1+\epsilon_2)}, \tag{2.8}$$

$$N_{[][11]} = -\frac{\epsilon_1^2\epsilon_2^2}{2!(\epsilon_1-\epsilon_2)}\cdot\frac{-\epsilon_2}{2a(2a-\epsilon_1)(2a-\epsilon_1-\epsilon_2)(2a-2\epsilon_1-\epsilon_2)} \tag{2.9}$$

Comparing with eqs. (1.4)–(1.9), one can see that the case of pure SU(2) can be obtained from the more general case of SU(2) with four fundamental matter hypermultiplets by a particular pure gauge limit (PGL) (note that various $\epsilon_{1,2}$-dependent factors emerging in the Nekrasov functions in PGL are completely determined by the way one takes this limit):

$$\mu_1,\mu_2,\mu_3,\mu_4 \to \infty, \qquad q\cdot\frac{\mu_1\mu_2}{\epsilon_1\epsilon_2}\frac{\mu_3\mu_4}{\epsilon_1\epsilon_2} = \Lambda^4 = {\rm fixed} \tag{2.10}$$

As one can see, in this limit the Nekrasov functions get simplified. The conformal block in this limit is also simplified [6–8, 78] and coincides with the PGL of the 1-point toric block [75–77].

JHEP02(2011)067

2.2 Dotsenko-Fateev integral

As explained in [78], the relevant Dotsenko-Fateev integral can be obtained by taking the PGL of the initial integral (1.1). The result is somewhat non-trivial [78]:

$$Z_{\rm DF}^{\rm pure}(\Lambda) = \Big\langle \Big\langle \det\left(1-\Lambda^4 U\otimes\widetilde{U}\right)^{2\beta} \Big\rangle_+^{\rm BGW} \Big\rangle_-^{\rm BGW} \tag{2.11}$$

where the averaging goes over two independent β-ensembles (labeled with the symbols $+,-$)

$$\Big\langle f(U)\Big\rangle_+^{\rm BGW} = \int\limits_{n_+\times n_+} \frac{[dU]_\beta}{{\rm Vol}_\beta(n_+)}\, f(U)\, Z_{\rm BGW}\Big(n_+ +\delta\Big|t_k\Big), \quad t_k = {\rm tr}\,(U^+)^k/k \tag{2.12}$$

$$\Big\langle f(\widetilde{U})\Big\rangle_-^{\rm BGW} = \int\limits_{n_-\times n_-} \frac{[d\widetilde{U}]_\beta}{{\rm Vol}_\beta(n_-)}\, f(\widetilde{U})\, Z_{\rm BGW}\Big(n_- +\delta\Big|\widetilde{t}_k\Big), \quad \widetilde{t}_k = {\rm tr}\,(\widetilde{U}^+)^k/k \tag{2.13}$$

with the β-deformed Brezin-Gross-Witten (BGW) partition function [178–181] in the role of integrand

$$Z_{\rm BGW}\Big(n\Big|t_k = {\rm tr}\,\Psi^k/k\Big) = \int\limits_{n\times n} \frac{[dU]_\beta}{{\rm Vol}_\beta(n)} e^{\beta\left({\rm tr}\,U^+ + {\rm tr}\,(U\Psi)\right)} \tag{2.14}$$

and with $\delta = (\beta-1)/\beta$. It is checked in the same paper [78] that Dotsenko-Fateev integral (2.11) reproduces correctly the first terms of the Λ-expansion of the conformal block.

2.3 The AGT conjecture

The AGT conjecture states that

$$Z_{\rm DF}^{\rm pure}(\Lambda) \;=\; Z_{\rm Nek}^{\rm pure}(\Lambda) \tag{2.15}$$

under the following identification of parameters:

$$\beta = \frac{-\epsilon_1}{\epsilon_2}, \qquad n_+ = \frac{2a}{\epsilon_1}, \qquad n_- = \frac{-2a}{\epsilon_1} \tag{2.16}$$

Let us prove this statement in the case of $\beta = 1$.

2.4 Proof of (2.15) at $\beta = 1$

We start from rewriting the determinant in eq. (2.11) in the exponential form:

$$\det\left(1-\Lambda^4 U\otimes\widetilde{U}\right)^{2\beta} = \exp\left(-2\beta\sum_{k=1}^{\infty}\frac{\Lambda^{4k}}{k}{\rm tr}\,U^k{\rm tr}\,\widetilde{U}^k\right) \tag{2.17}$$

Therefore, the Dotsenko-Fateev integral takes the form

$$Z_{\rm DF}^{\rm pure}(\Lambda) = \Bigg\langle \Bigg\langle \exp\left(-2\beta\sum_{k=1}^{\infty}\frac{\Lambda^{4k}}{k}{\rm tr}\,U^k{\rm tr}\,\widetilde{U}^k\right) \Bigg\rangle_+^{\rm BGW} \Bigg\rangle_-^{\rm BGW} \tag{2.18}$$

To expand this expression in characters, one can use the standard Cauchy-Stanley identity

$$\exp\left(\beta\sum_{k=1}^{\infty}kt_kt'_k\right)=\sum_R j_R(t)j_R(t') \tag{2.19}$$

where the sum is taken over all Young diagrams R and j_R are the normalized Jack polynomials, $j_R = J_R/||J_R||$ (see the appendix for the details) which at $\beta = 1$ coincide with the ordinary Schur polynomials:

$$j_R\Big|_{\beta=1}=\chi_R \tag{2.20}$$

The exponent in (2.18) contains -2β instead of $+\beta$; thus (2.19) is not directly applicable, instead one can use a trick: rewrite it in the following form:

$$\begin{aligned}
Z_{\rm DF}^{\rm pure}(\Lambda) &= \Big\langle\Big\langle \exp\left(-2\beta\sum_{k=1}^{\infty}\frac{\Lambda^{4k}}{k}\mathrm{tr}\,U^k\mathrm{tr}\,\widetilde{U}^k\right)\Big\rangle_+^{\rm BGW}\Big\rangle_-^{\rm BGW} = \\
&= \Big\langle\Big\langle \exp\left(\beta\sum_{k=1}^{\infty}\frac{\Lambda^{4k}}{k}(-\mathrm{tr}\,U^k)\mathrm{tr}\,\widetilde{U}^k\right)\exp\left(\beta\sum_{k=1}^{\infty}\frac{\Lambda^{4k}}{k}\mathrm{tr}\,U^k(-\mathrm{tr}\,\widetilde{U}^k)\right)\Big\rangle_+^{\rm BGW}\Big\rangle_-^{\rm BGW} = \\
&= \sum_{A,B}\Lambda^{4|A|+4|B|}\Big\langle j_A\big(-\mathrm{tr}\,U^k\big)j_B\big(\mathrm{tr}\,U^k\big)\Big\rangle_+^{\rm BGW}\Big\langle j_A\big(\mathrm{tr}\,\widetilde{U}^k\big)j_B\big(-\mathrm{tr}\,\widetilde{U}^k\big)\Big\rangle_-^{\rm BGW}
\end{aligned} \tag{2.21}$$

At $\beta = 1$, the r.h.s. is precisely the Nekrasov function, due to the pair-correlator identity (A.16):

$$\Big\langle \chi_A\big(-\mathrm{tr}\,U^k\big)\ \chi_B\big(\mathrm{tr}\,U^k\big)\Big\rangle_+^{\rm BGW}\ \Big\langle \chi_A\big(\mathrm{tr}\,\widetilde{U}^k\big)\ \chi_B\big(-\mathrm{tr}\,\widetilde{U}^k\big)\Big\rangle_-^{\rm BGW} = N_{A,B}^{\rm pure}\Big|_{\epsilon_1+\epsilon_2=0} \tag{2.22}$$

which is the pure gauge limit of (1.13) and is considered in more detail in the appendix, see eq. (A.16). Substituting this into (2.21), one obtains

$$Z_{\rm DF}^{\rm pure}(\Lambda)\Big|_{\beta=1} = Z_{\rm Nek}^{\rm pure}(\Lambda)\Big|_{\epsilon_1+\epsilon_2=0} \tag{2.23}$$

and this completes the proof.

It may even seem that the only non-trivial part of this calculation is the pair-correlator identity (2.22). However, the identity itself is nothing but a technical detail. Really important is a duality: the existence of the quadrilinear character expansion (2.21). Eq. (2.21) contains both a sum over A, B diagrams and an average over "+", "-" ensembles, and reduces either to the Nekrasov function (2.22) or to the Dotsenko-Fateev integral (2.11) if one evaluates either the double average or the double sum, respectively. This is a typical duality, only realized at a very simple algebraic level with the help of characters. Let us now include masses into our consideration.

JHEP02(2011)067

3 The case of SU(2) with 4 fundamentals

3.1 Nekrasov function

The Nekrasov function for this case has the form

$$Z_{\rm Nek}(q) \;=\; \sum_{A,B} N_{A,B}\; q^{|A|+|B|} \tag{3.1}$$

with coefficients

$$N_{A,B} = \frac{\prod_{k=1}^{4} f_A(\mu_k+a) f_A(\mu_k-a)}{g_{A,A}(0) g_{A,B}(2a) g_{B,A}(-2a) g_{B,B}(0)} \tag{3.2}$$

where in addition to contributions (2.2) of gauge fields, one now has matter contributions:

$$f_A(z) = \prod_{(i,j)\in A} \Big[z + \epsilon_1(i-1) + \epsilon_2(j-1) \Big] \tag{3.3}$$

A few first Nekrasov coefficients $N_{A,B}$ are written in eqs. (1.4)–(1.8).

3.2 Dotsenko-Fateev integral

The Dotsenko-Fateev integral for this case has the form (1.1), but as was noticed a while ago [73], for the purposes of q-expansion it is more convenient to rewrite this integral (of course, omitting the U(1) prefactors, which are irrelevant for comparison with the Nekrasov functions) as a double average:

$$Z_{\rm DF}(q) = \Bigg\langle \Bigg\langle \prod_{i=1}^{N_+} (1-qx_i)^{v_-} \prod_{j=1}^{N_-} (1-qy_j)^{v_+} \prod_{i=1}^{N_+}\prod_{j=1}^{N_-} (1-qx_iy_j)^{2\beta} \Bigg\rangle_+ \Bigg\rangle_- \tag{3.4}$$

where the averaging goes over two independent ensembles (labeled with symbols $+$ and $-$) of variables $x_1,\ldots,x_{N_+}$ and $y_1,\ldots,y_{N_-}$ ("eigenvalues" in matrix model terms) as follows:

$$\Big\langle f \Big\rangle_+ \;=\; \frac{1}{S_+} \int_0^1 dx_1 \ldots \int_0^1 dx_{N_+} \prod_{i<j} (x_i-x_j)^{2\beta} \prod_i x_i^{u_+} (x_i-1)^{v_+}\; f\big(x_1,\ldots,x_{N_+}\big) \tag{3.5}$$

$$\Big\langle f \Big\rangle_- \;=\; \frac{1}{S_-} \int_0^1 dy_1 \ldots \int_0^1 dy_{N_-} \prod_{i<j} (y_i-y_j)^{2\beta} \prod_i y_i^{u_-} (y_i-1)^{v_-}\; f\big(y_1,\ldots,y_{N_-}\big) \tag{3.6}$$

with the normalization constants

$$S_\pm = \int_{\gamma_\pm} dz_1\ldots dz_N \prod_{i<j} (z_i-z_j)^{2\beta} \prod_i z_i^{u_\pm} (z_i-1)^{v_\pm} \tag{3.7}$$

needed to satisfy $\Big\langle 1 \Big\rangle_+ = \Big\langle 1 \Big\rangle_- = 1$.

JHEP02(2011)067

3.3 The AGT conjecture

The AGT conjecture states that

$$Z_{\text{DF}}(q) \;=\; Z_{\text{Nek}}(q) \tag{3.8}$$

under the following identification of parameters:

$$\beta = \frac{-\epsilon_1}{\epsilon_2}, \qquad N_+ = \frac{a-\mu_2}{\epsilon_1}, \qquad N_- = \frac{-a-\mu_4}{\epsilon_1} \tag{3.9}$$

$$u_+ = \frac{\mu_1-\mu_2-\epsilon_1-\epsilon_2}{\epsilon_2}, \qquad u_- = \frac{\mu_3-\mu_4-\epsilon_1-\epsilon_2}{\epsilon_2} \tag{3.10}$$

$$v_+ = \frac{-\mu_1-\mu_2}{\epsilon_2}, \qquad v_- = \frac{-\mu_3-\mu_4}{\epsilon_2} \tag{3.11}$$

Let us prove this statement in the case of $\beta = 1$.

3.4 Proof at $\beta = 1$

The proof goes completely similar to the BGW case. Likewise, we start from rewriting the Dotsenko-Fateev integrand in an exponential form, and then use the Cauchy-Stanley identity to perform an expansion in the basis of Schur/Jack symmetric polynomials:

$$\begin{aligned}
&\prod_{i=1}^{N_+}(1-qx_i)^{v_-}\prod_{j=1}^{N_-}(1-qy_j)^{v_+}\prod_{i=1}^{N_+}\prod_{j=1}^{N_-}(1-qx_iy_j)^{2\beta} = \\
&\qquad = \exp\left(-\beta\sum_{k=1}^{\infty}\frac{q^k}{k}\widetilde{p}_k(p_k+v_+)\right)\exp\left(-\beta\sum_{k=1}^{\infty}\frac{q^k}{k}p_k(\widetilde{p}_k+v_-)\right) = \\
&\qquad = \sum_{A,B} q^{|A|+|B|} j_A\big(-p_k-v_+\big) j_B\big(p_k\big) j_A\big(\widetilde{p}_k\big) j_B\big(-\widetilde{p}_k-v_-\big)
\end{aligned} \tag{3.12}$$

where $p_k = \sum_i x_i^k$ and $\widetilde{p}_k = \sum_i y_i^k$. Therefore, the Dotsenko-Fateev integral takes the form

$$Z_{\text{DF}}(\Lambda) = \sum_{A,B} q^{|A|+|B|} \Big\langle j_A\big(-p_k-v_+\big)\; j_B\big(p_k\big)\Big\rangle_+ \Big\langle j_A\big(\widetilde{p}_k\big)\; j_B\big(-\widetilde{p}_k-v_-\big)\Big\rangle_- \tag{3.13}$$

At $\beta = 1$, the correlators at the r.h.s. precisely reproduce the Nekrasov function (see (A.12)):

$$\Big\langle \chi_A\big(-p_k-v_+\big)\; \chi_B\big(p_k\big)\Big\rangle_+ \Big\langle \chi_A\big(\widetilde{p}_k\big)\; \chi_B\big(-\widetilde{p}_k-v_-\big)\Big\rangle_- = N_{A,B}\Big|_{\epsilon_1+\epsilon_2=0} \tag{3.14}$$

Substituting this into (3.13), one obtains

$$Z_{\text{DF}}(\Lambda)\Big|_{\beta=1} = Z_{\text{Nek}}(\Lambda)\Big|_{\epsilon_1+\epsilon_2=0} \tag{3.15}$$

which completes the proof.

4 Problems with generalization to $\beta \neq 1$

The basic puzzle of the AGT relation for $\beta \neq 1$ is a different structure of poles at the two sides of the equality. The conformal block has poles at zeroes of the Kac determinant, i.e. at $z = m\epsilon_1 + n\epsilon_2$ with $mn > 0$, while the poles of the particular Nekrasov functions $N_{AB}(z)$ (here $z = 2a$) occur also at $mn \leq 0$. Transition from the conformal blocks to the Selberg or BGW pair correlators of characters, exploited in the present paper, does not help: their poles are still at $mn > 0$, just as for the conformal blocks.

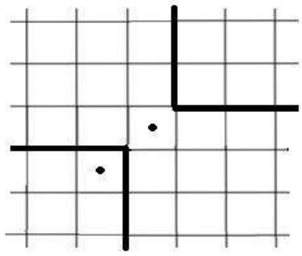


Figure 6. Poles of $N_{[1][]}(z)$.

In this section, figures 6-11 are used to illustrate the issue of poles. In these pictures, the square lattice represents the set of possible linear combinations $m\epsilon_1 + n\epsilon_2$, dots represent positions of poles, and the bold area in the top right corner (and its mirror image in the bottom left corner) represents the part of the lattice with $mn > 0$, where zeroes of the Kac determinant may be situated. The horizontal and vertical directions correspond to ϵ_1 and ϵ_2, respectively. The central cell of the lattice corresponds to the point $(m, n) = (0, 0)$.

Because of the problem of poles, it is unclear if it is at all possible to extend the relations like (1.13) and (2.22) to $\beta \neq 1$. What happens at $\beta = 1$ is that only the difference $m - n$ matters, and all the poles can be projected from the plane to a single line $z = (m - n)\epsilon_1$, and the difference between the sets with $mn > 0$ and $mn \leq 0$ disappears. This phenomenon at $\epsilon_1 + \epsilon_2 = 0$ is illustrated in figure 8.

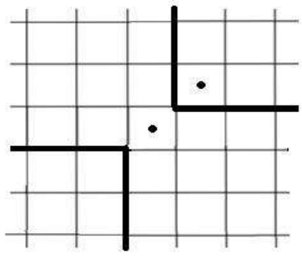


Figure 7. Poles of $N_{[][1]}(z)$.

Of course, the extra poles of the particular Nekrasov coefficients $N_{AB}(z)$ drop away from their sum, i.e. from the LMNS partition function, which is AGT-related to the conformal block. Thus the real puzzle is, what at all is the real role of the individual N_{AB}, i.e. why does the linear basis with the nicely factorizable coefficients (as functions of μ's and ϵ's) include extra poles in the z-variable. Anyhow, if $N_{AB}(z)$ are relevant, their Selberg or BGW interpretation is still missed when $\beta \neq 1$.

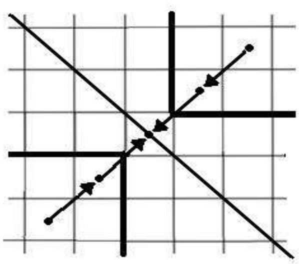


Figure 8. At $\epsilon_1 + \epsilon_2 = 0$, all poles with equal $m-n$ become indistinguishable.

In what follows we illustrate the problem at the first two levels of the Young diagram expansion. For this, in addition to explicit formulas for the Nekrasov functions in (1.4)–(1.9) and (2.4)–(2.9), one also needs explicit formulas for the pair correlators of the β-characters (i.e. the normalized Jack polynomials, see the appendix). These are listed in tables 2–4 and 7–9 for the BGW case. Actually we need only the entries of table 3, the other two are added to illustrate factorizability properties and to provide some data for the future study of alternatives to eq. (1.13): clearly, instead of

$$R_{AB}(z)R_{BA}(-z) = < j_A(p)j_B(-p) >_+ < j_B(p)j_A(-p) >_- \tag{4.1}$$

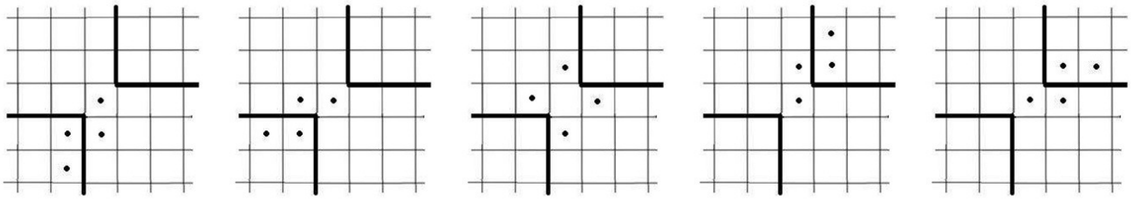


Figure 10. Poles of $N_{[2][]}, N_{[11][]}, N_{[1][1]}, N_{[][2]}$ and $N_{[][11]}$, respectively.

one could also use another types of correlators:

$$S_{AB}(z)Q_{AB}(-z) =< j_A(p)j_B(p) >_+ < j_A(-p)j_B(-p) >_- \tag{4.2}$$

or

$$Q_{AB}(z)S_{AB}(-z) =< j_A(-p)j_B(-p) >_+ < j_A(p)j_B(p) >_- \tag{4.3}$$

or any linear combination of the three. Tables 2 and 7 are devoted to correlators S_{AB}, tables 3 and 8 to R_{AB}, tables 4 and 9 to Q_{AB}.

Formulas for the Selberg correlators are more lengthy, but their properties are essentially the same, see tables 5 and 6. Actual examples below are given for the simpler BGW case, i.e. relevant for the AGT relation in the pure gauge limit.

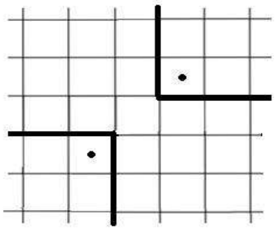


Figure 9. Poles of the sum $N_{[1][]} + N_{[][1]}$. The "extra" poles vanish.

Level 1. At level one, the relation looks like

$$N_{[1],[]}(z) + N_{[],[1]}(z) = -\frac{\epsilon_1\epsilon_2}{z(z-\epsilon_1-\epsilon_2)} - \frac{\epsilon_1\epsilon_2}{z(z+\epsilon_1+\epsilon_2)} =$$

$$= -\frac{2\epsilon_1\epsilon_2}{(z-\epsilon_1-\epsilon_2)(z+\epsilon_1+\epsilon_2)} = R_{[1],[]}(z)R_{[],[1]}(-z) + R_{[],[1]}(z)R_{[1],[]}(-z)\,. \tag{4.4}$$

The auxiliary poles, which are present at the particular Nekrasov coefficients, but disappear from the whole sum (see figure 10) in this case are represented by a single pole at $z = 0$. Moral: the individual $N_{10}(z)$ and $N_{01}(z)$ can not be expressed through $R_{10}(\pm z)$ and $R_{01}(\pm z)$, but their sum can, as shown by relation (4.4). At $\epsilon_1 = -\epsilon_2$, however, eq. (4.4) gets simplified: the l.h.s. sum $(N + N)$ and the r.h.s. sum $(RR + RR)$ become equal term by term! This is illustrated by

$$\begin{array}{c} \frac{1}{(00)(11)} + \frac{1}{(00)(-1,-1)} = \frac{2}{(11)(-1,-1)} \\ \downarrow \quad \beta = 1 \\ \frac{1}{(00)^2} + \frac{1}{(00)^2} = \frac{2}{(00)^2} \end{array} \tag{4.5}$$

in the abbreviated notation $(m, n) = (z - m\epsilon_1 - n\epsilon_2)$. The same phenomenon of transformation of a complicated equality between the whole sums at generic $\epsilon_1 + \epsilon_2$ into the very simple equality between individual terms at $\epsilon_1 + \epsilon_2 = 0$ persists at higher levels.

JHEP02(2011)067

This shows that, in fact, $\epsilon_1 = -\epsilon_2$ is a highly distinguished case. In this case, simply the passing to the basis of characters completely reveals the underlying structure behind the AGT relation, formulated in the present paper in terms of bilinear correlators in the Selberg models. The aim of this section is to stress that for general ϵ_1, ϵ_2 the relation between the Nekrasov functions and Selberg correlators is still missed and is probably more sophisticated. Finding such a relation would be crucial for development in this research direction.

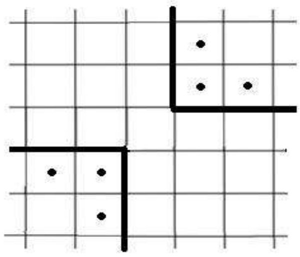


Figure 11. Poles of the sum $N_{[2][]} + N_{[11][]} + N_{[1][1]} + N_{[][2]} + N_{[][11]}$. The "extra" poles vanish.

Level 2. At level two, the relation looks like

$$\sum_{|A|+|B|=2} N_{A,B}(z) = (2.5) + (2.6) + (2.7) + (2.8) + (2.9) =$$

$$\begin{aligned} &= \frac{\epsilon_1^2\epsilon_2^2(2z^2 - 8\epsilon_1^2 - 17\epsilon_1\epsilon_2 - 8\epsilon_2^2)}{(z-2\epsilon_1-\epsilon_2)(z+\epsilon_2+\epsilon_1)(z+2\epsilon_2+\epsilon_1)(z+2\epsilon_1+\epsilon_2)(z-2\epsilon_2-\epsilon_1)(z-\epsilon_2-\epsilon_1)} = \\ &= R_{[1],[1]}(z)R_{[1],[1]}(-z) + R_{[2],[]}(z)R_{[],[2]}(-z) + R_{[],[2]}(z)R_{[2],[]}(-z) + \\ &\quad + R_{[1,1],[]}(z)R_{[],[1,1]}(-z) + R_{[],[1,1]}(z)R_{[1,1],[]}(-z) \end{aligned} \tag{4.6}$$

Again, this is a complicated relation, not quite expectable if one takes a look simply at the rational functions at the l.h.s. and the r.h.s.: this time, 5 auxiliary poles at $z = 0$, ϵ_1, ϵ_2, $-\epsilon_1$ and $-\epsilon_2$ disappear from the final sum (as illustrated in figure 11). In analogy with the level 1 case, at $\epsilon_1 + \epsilon_2 = 0$ the relation gets satisfied term by term and thus completely transparent:

$$\frac{\epsilon_1}{(00)(01)(11)(12)} + \frac{\epsilon_1}{(00)(0-1)(-1,-1)(-1,-2)} + \frac{2\epsilon_{12}}{(01)(0,-1)(1,0)(-1,0)} - \frac{\epsilon_2}{(00)(10)(11)(21)} - \frac{\epsilon_2}{(00)(-1,0)(-1,-1)(-2,-1)}$$

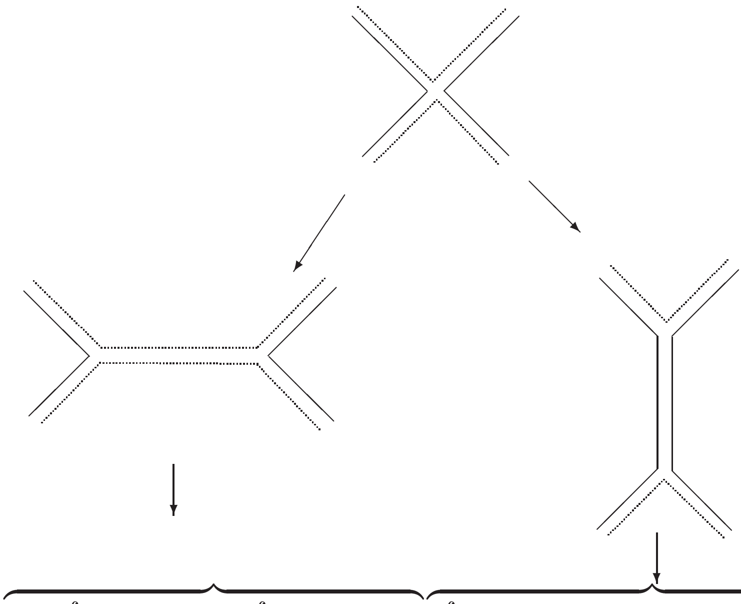
$$= \frac{\epsilon_1(23)(-2,-1) + \epsilon_1(21)(-2,-3) + 2\epsilon_{12}(22)(-2,-2) - \epsilon_2(32)(-1,-2) - \epsilon_2(1,2)(-3,-2)}{(11)(12)(21)(-1,-1)(-1,-2)(-2,-1)} \tag{4.7}$$

$$\downarrow \beta = 1$$

$$\frac{1}{(00)^2(01)^2} + \frac{1}{(00)^2(10)^2} + \frac{4}{(01)^2(10)^2} + \frac{1}{(00)^2(10)^2} + \frac{1}{(00)^2(01)^2} = \frac{(01)^2 + (10)^2 + 4(00)^2 + (10)^2 + (01)^2}{(00)^2(01)^2(10)^2}$$

Of course, instead of the combinations of correlators $R(z)R(-z)$ one could also use the combinations $S(z)Q(-z)$ or $Q(z)S(-z)$ or some linear combination like $S(z)Q(-z) + Q(z)S(-z)$. All these formulations are equivalent: each time there is a transcendental equality of sums of rational functions, which at $\epsilon_1 + \epsilon_2 = 0$ turns into a term-by-term equality.

Higher levels. At higher levels, things get even more sophisticated. A new feature, which appears at this level of consideration, is that only the correlators S_{AB} remain factorized, while the correlators R_{AB} and Q_{AB} at $\beta \neq 1$ contain non-factorizable expressions in numerators. Thus, it becomes impossible to illustrate the phenomenon by using the shorthand notation (n, m). However, the phenomenon itself does not change: at $\epsilon_1 + \epsilon_2 = 0$, the relation between the Nekrasov side and the Selberg side becomes termwise.



$$\sum_{A,B} N_{AB} = \overbrace{\sum_{AB}\int_z \chi_A(z)\chi_B(z)\int_y \chi_A(y)\chi_B(y)} = \overbrace{\int_{z,y}\sum_A \chi_A(z)\chi_A(y)\sum_B \chi_B(z)\chi_B(y)} = \mathcal{B}(q)$$

Figure 12. The picture of Nekrasov functions/conformal block duality expressed by the Hubbard-Stratonovich type formula (5.1). The symbol $\int_z$ here denotes integration with the Selberg measure over variables z_i, and the symbol $\sum_A$ denotes summation over all Young diagrams A.

5 Conclusion

In this paper we succeeded in interpreting the AGT relation as the standard duality relation of the Hubbard-Stratonovich type, see figure 12:

$$\sum_{a,b}\left(\sum_i X_i^a X_i^b\right)\left(\sum_j X_j^a X_j^b\right) = \sum_{a,b,i,j} X_i^a X_i^b X_j^a X_j^b = \sum_{i,j}\left(\sum_a X_i^a X_j^a\right)\left(\sum_b X_i^b X_j^b\right) \tag{5.1}$$

The role of X_i^a is played by the $\mathrm{GL}(\infty)$ characters $\chi_A(p)$. This provides a very direct and conceptually clear proof of the AGT relation, very different from both the various formal proofs, suggested in [60, 63, 64, 72, 74, 81], and more transcendental projects like [59, 80] etc. Moreover, as a byproduct we found a new representation for the particular Nekrasov functions N_{AB} through the pair correlators of characters in relevant matrix models (like the Selberg or BGW ones).

Unfortunately, all this works so nicely only in the particular case of $\beta = 1$. The extra poles puzzle at $\beta \neq 1$, which we described above in s.4, remains unresolved and generalization of the duality interpretation to $\beta \neq 1$ is still missed. No representation of the individual Nekrasov functions $N_{AB}(z)$ in terms of pair correlators of characters is found for $\beta \neq 1$: they simply possess more poles than the known correlators.

At the same time, generalizations in other directions: from 4-point to generic conformal blocks (at least, spherical and elliptic) and from the U(2)/Virasoro symmetry to $\mathrm{U}(N)/W_N$

JHEP02(2011)067

seem straightforward. In both cases polylinear, rather than bilinear combinations of pair correlators are going to arise.

The technical base of our consideration is the further generalization of the Selberg/Kadell formulas from single to pair correlators of characters in the Selberg and BGW models, given by eqs. (A.12) and (A.16) respectively. We did not describe a proof of these formulas, it is straightforward done within the standard approaches (e.g., from the singularity analysis to the Ward identities). What deserves to be mentioned, these correlators are *different* from another important set of correlators recently considered in [81] (see also the end part of the appendix). The two most important differences are: (i) ours have poles, while the non-trivial part of those in [81] have only zeroes; (ii) ours remain non-trivial in the pure gauge limit (the Selberg correlators turn into the BGW ones), while the non-trivial part of those in [81] becomes trivial. An advantage of the correlators in [81] could be that they remain factorized for $\beta \neq 1$, just like our $R_{AB}(z)$, unfortunately, they are also not sufficient to describe the individual Nekrasov functions $N_{AB}(z)$ for $\beta \neq 1$.

To summarize, *the AGT relation is now clearly understood in two limits: for* $c = \infty$ [60], when conformal blocks become ordinary hypergeometric series, and *for* $c = 1$ when they possess the free fermion representation and, as we explained in the present paper, are related to the Nekrasov functions by the most naive duality transformation *a la* (5.1). An interpolation between these two extreme cases still remains to be found.

Acknowledgments

Our work is partly supported by Ministry of Education and Science of the Russian Federation under contract 02.740.11.0608, by RFBR grant 10-01-00536-a, by joint grants 09-02-90493-Ukr, 09-01-92440-CE, 09-02-91005-ANF, 10-02-92109-Yaf-a. The work of A.Morozov was also supported in part by CNRS.

A Averages of Jack polynomials in β-ensembles

The present section lists various known averages of the Jack polynomials in various β-ensembles, which possess remarkably simple factorization properties. In this paper, we demonstrated that, in some cases, averages of the Jack polynomials are directly related to the Nekrasov functions. Further progress in understanding of these averages may lead to a complete reformulation of the Nekrasov functions in terms of matrix model (β-ensemble) theory.

A.1 Jack polynomials

The Jack polynomials form a distinguished basis in the space of all symmetric polynomials. They are often used in two different versions: the "un-normalized" J_Y (see [78, appendix 1]

JHEP02(2011)067

for a general definition)

$$
\begin{aligned}
&J_1(p_k)=p_1\\
&J_2(p_k)=\frac{p_2+\beta p_{11}}{\beta+1}, \qquad\qquad J_{11}(p_k)=\frac{1}{2}(p_1^2-p_2)\\
&J_3(p_k)=\frac{2p_3+3\beta p_1p_2+\beta^2p_1^3}{(\beta+1)(\beta+2)},\ \ J_{21}(p_k)=\frac{(1-\beta)p_1p_2-p_3+\beta p_1^3}{(\beta+1)(\beta+2)},\ \ J_{111}(p_k)=\frac{1}{6}p_1^3-\frac{1}{2}p_1p_2+\frac{1}{3}p_3
\end{aligned}
$$

and the "normalized" $j_Y = \dfrac{J_Y}{||J_Y||}$, where $||J_Y||$ is a natural norm w.r.t. the orthogonality

$$
\Big\langle J_A\Big|J_B\Big\rangle = \delta_{AB}||J_A||^2, \qquad \Big\langle p_A\Big|p_B\Big\rangle \equiv \prod_j \frac{B_j}{\beta}\frac{\partial}{\partial p_{B_j}}\prod_i p_{A_i}\Bigg|_{p=0} \tag{A.1}
$$

Several first norms are given in the following table:

A	$\lvert\lvert J_A\rvert\rvert$
$[1]$	$\sqrt{\frac{1}{\beta}}$
$[2]$	$\sqrt{\frac{2}{\beta^2(\beta+1)}}$
$[1,1]$	$\sqrt{\frac{1+\beta}{2\beta}}$
$[3]$	$\sqrt{\frac{6}{\beta(\beta+1)(\beta+2)}}$
$[2,1]$	$\sqrt{\frac{\beta+2}{\beta(2\beta+1)}}$
$[1,1,1]$	$\sqrt{\frac{(1+\beta)(1+2\beta)}{6\beta^3}}$

A.2 β-ensembles

Here we consider just two β-ensembles. The *Selberg* averaging is defined as an integral

$$
\Big\langle f\Big\rangle^{\text{Selb}} = \frac{\int_0^1 dz_1\ldots\int_0^1 dz_N \prod_{i<j}(z_i-z_j)^2\prod_i z_i^u(z_i-1)^v\ f\big(z_1,\ldots,z_N\big)}{\int_0^1 dz_1\ldots\int_0^1 dz_N \prod_{i<j}(z_i-z_j)^2\prod_i z_i^u(z_i-1)^v} \tag{A.2}
$$

The *BGW* averaging is most simply defined as PGL [78] of the Selberg averaging:

$$
\Big\langle f\Big\rangle^{\text{BGW}} = \lim_{\substack{u,v,N\to\infty\\ u+v+2\beta N\equiv\beta n+\beta-1}} \left(\frac{\Big\langle f\Big\rangle^{\text{Selb}}}{(uN+\beta N^2)^{\deg f}}\right) \tag{A.3}
$$

or, equivalently, as the unitary integral average (2.14).

Let us describe various averages of the Jack polynomials in these ensembles.

A.3 1-Jack average

A.3.1 Selberg model

The average of single Jack polynomial in the Selberg model has the form

$$\Big\langle J_A(p)\Big\rangle^{\text{Selb}} = J_Y(\delta_{k,1})\frac{[N]_Y[u+N\beta+1-\beta]_Y}{[u+v+2N\beta+2-2\beta]_Y} \tag{A.4}$$

where the following notation is used:

$$[x]_Y = \prod_{(i,j)\in Y}\Big(x-\beta(i-1)+(j-1)\Big) \tag{A.5}$$

This Kadell formula is proved in [182, 183].

A.3.2 BGW model

The average of single Jack polynomial in the BGW model has the form

$$\Big\langle J_A(p)\Big\rangle^{\text{BGW}} = J_Y(\delta_{k,1})\frac{1}{[\beta n+1-\beta]_Y} \tag{A.6}$$

This formula directly follows from the PGL of the Kadell formula.

A.4 2-Jack average

A.4.1 Selberg model

The average of product of two Jack polynomials in the Selberg model is known in the form

$$\begin{aligned}\Big\langle J_A(p+w)J_B(p)\Big\rangle^{\text{Selb}} &= \frac{1}{\mathrm{Norm}_\beta(\mathrm{u,v,N})}\frac{[v+N\beta+1-\beta]_A[u+N\beta+1-\beta]_B}{[N\beta]_A[u+v+N\beta+2-2\beta]_B}\times \\ &\times\frac{\prod\limits_{i<j}^{N}\Big(A_i-A_j+(j-i)\beta\Big)_\beta\prod\limits_{i<j}^{N}\Big(B_i-B_j+(j-i)\beta\Big)_\beta}{\prod\limits_{i,j}^{N}\Big(\ u+v+2\beta N+2+A_i+B_j-(1+i+j)\beta\ \Big)_\beta}\end{aligned} \tag{A.7}$$

with the A,B-independent normalization constant $\mathrm{Norm}_\beta(u,v,N)$ determined from $\Big\langle 1\Big\rangle = 1$, and with the conventional Pochhammer symbol $(x)_\beta$ defined as

$$(x)_\beta = \frac{\Gamma(x+\beta)}{\Gamma(x)} = x(x+1)\dots(x+\beta-1) \tag{A.8}$$

As usual, A_i in (A.7) denotes the height of i-th coloumn in the diagram A. The shift $w=(v+1-\beta)/\beta$ is essential for the correlator to factorize. This formula is proved (at least for zero shift, $w=0$) in [184].

Note that eq. (A.7) contains only the heights of Young diagrams, A_i and B_j, while the Nekrasov functions contain also the heights of transposed diagrams like A'. The transposed diagrams can be obtained by the following identity:

$$j_A^{(\beta)}(-p/\beta) = (-1)^{|A|}j_{A'}^{(1/\beta)}(p) \tag{A.9}$$

which for $\beta=1$ turns into

$$\beta=1: \qquad \chi_A(-p) = (-1)^{|A|}\chi_{A'}(p) \tag{A.10}$$

and is used below in eq. (A.12).

A.4.2 BGW model

In the PGL only the denominator in the last factor survives in (A.7). The result can be written as

$$\big\langle j_A(p)j_B(p)\big\rangle^{\rm BGW} = \prod_{i=1}^{L_A}\frac{\Gamma\Big(z{+}1{-}(i{+}L_B)\beta\Big)}{\Gamma\Big(z{+}1{+}A_i{-}(i{+}L_B)\beta\Big)}\cdot\prod_{j=1}^{L_B}\frac{\Gamma\Big(z{+}1{-}(j{+}L_A)\beta\Big)}{\Gamma\Big(z{+}1{+}B_j{-}(j{+}L_A)\beta\Big)}\cdot$$
$$\cdot\prod_{i=1}^{L_A}\prod_{j=1}^{L_B}\frac{\Gamma\Big(z{+}1{-}(i{+}j{-}1)\beta\Big)}{\Gamma\Big(z{+}1{+}A_i{+}B_j{-}(i{+}j{-}1)\beta\Big)}\cdot\frac{\Gamma\Big(z{+}1{+}A_i{+}B_j{-}(i{+}j)\beta\Big)}{\Gamma\Big(z{+}1{-}(i{+}j)\beta\Big)} \tag{A.11}$$

where $z = \beta n$, and L_A, L_B are the maximal row lengths of the diagrams A and B: $A = \big(A_1 \geq \ldots \geq A_{L_A}\big)$, $B = \big(B_1 \geq \ldots \geq B_{L_B}\big)$.

A.4.3 Case of $\beta = 1$

Formulas (A.7) and (A.11) can look similar to the Nekrasov ones, however, there is also an important difference: the heights of diagrams in denominator enter not in combinations like $A_i - j$, which would correspond to Arm- and Leg-lengths, but rather in combinations like $A_i - \beta j$: very much different from the Nekrasov side. Clearly, this difference disappears when $\beta = 1$, and above eq. (A.7) can be reduced to

$$\Big\langle\chi_A\big(-v-p_k\big)\ \chi_B\big(p_k\big)\Big\rangle^{\rm Selb} = \frac{(-1)^{|A|+|B|}[-v-N]_A[-u-v-N]_A[u+N]_A[N]_B}{G_{AA}(0)G_{AB}(-2N-u-v)G_{BA}(2N+u+v)G_{BB}(0)} \tag{A.12}$$

where the both sides are taken at $\beta = 1$, and the function G has the form

$$G_{AB}(x) = \prod_{(i,j)\in A}\Big(x+\beta{\rm Arm}_A(i,j)+{\rm Leg}_B(i,j)+1\Big) \tag{A.13}$$

Recalling that the gauge contribution to the Nekrasov functions has the form

$$g_{AB}(x) = G_{AB}(x)G_{AB}(x+\beta-1) \tag{A.14}$$

one can see that eq. (A.12) then directly implies that

$$N_{A,B} = \Big\langle\chi_A\big(-p_k-v_+\big)\ \chi_B\big(p_k\big)\Big\rangle_+\ \Big\langle\chi_A\big(p_k\big)\ \chi_B\big(-p_k-v_-\big)\Big\rangle_- \tag{A.15}$$

which is the main identity we use in section 3. Note that we wrote eq. (A.12), and its PGL eq. (A.16) below, in terms of $\chi_A(-p)$ instead of $\chi_A(p)$, because this is what we need to establish the relation with the DF integral. Eqs. (A.7) and (A.11) involve $\chi_A(p)$, but because of eq. (A.10), there is essentially no difference at $\beta = 1$.

JHEP02(2011)067

A.4.4 The PGL of the $\beta = 1$ case

From above eq. (A.12) it follows that

$$\Big\langle \chi_A\big(-p_k\big)\ \chi_B\big(p_k\big)\Big\rangle^{\rm BGW} = \frac{(-1)^{|A|+|B|}}{G_{AA}(0)G_{AB}(-n)G_{BA}(n)G_{BB}(0)} \tag{A.16}$$

where the both sides are taken at $\beta = 1$. Eq. (A.14) then implies a BGW counterpart of (A.15):

$$\Big\langle \chi_A\big(-\operatorname{tr} U^k\big)\ \chi_B\big(\operatorname{tr} U^k\big)\Big\rangle_+^{\rm BGW} \ \Big\langle \chi_A\big(\operatorname{tr} \widetilde{U}^k\big)\ \chi_B\big(-\operatorname{tr} \widetilde{U}^k\big)\Big\rangle_-^{\rm BGW} = N^{\rm pure}_{A,B}\Big|_{\epsilon_1+\epsilon_2=0} \tag{A.17}$$

which is the main identity we use in section 2.

A.5 BGW multiplication on Young diagrams

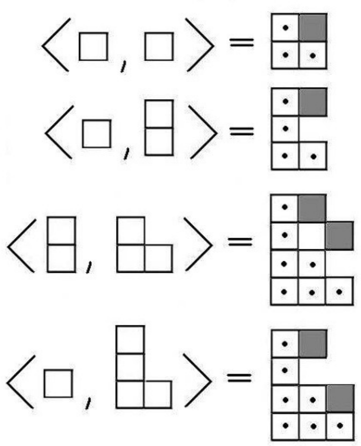


Figure 13. Pictorial representation of the BGW averages: poles are shown as white cells with dots, zeroes as grey cells.

Clearly all the poles and zeroes of (A.11) belong to the first quadrant (where also zeroes of the Kac determinant are located) and lie in a rectangular with the length $L_A + L_B$ and the height $H_A + H_B$, where $H_A = A_1$ is the maximal height of the Young diagram A.

A puzzling observation about these poles is that they always form a new Young diagram, in particular, there are no multiplicities. Zeroes lie over this newly emerging diagram (denoted $A \odot B$ in what follows) but form a rather strange configuration. This issue will be discussed elsewhere, here we just illustrate it with the two particular examples:

$$\big\langle j_1(p) j_1(p)\big\rangle^{\rm BGW} \sim \frac{(2,2)}{(1,1)(1,2)(2,1)} \tag{A.18}$$

$$\big\langle j_2(p) j_{21}(p)\big\rangle^{\rm BGW} \sim \frac{(4,2)(3,3)}{(1,1)(2,1)(3,1)(4,1)(1,2)(2,2)(1,3)} \tag{A.19}$$

where z-independent normalization prefactors are omitted. Poles (white cells with dots) and zeroes (grey cells) of these correlators are shown in figure 13. Note that the poles form a new Young diagram, a property not shared by the Nekrasov functions, and the same happens for all choices of A, B. Thus, the BGW averaging allows one to define a new amusing commutative "multiplication" $A, B \mapsto A \odot B$ on Young diagrams. Moreover, this multiplication seems to be associative! It does not, however, preserve the size-grading: $|A \odot B| \geq |A| + |B|$.

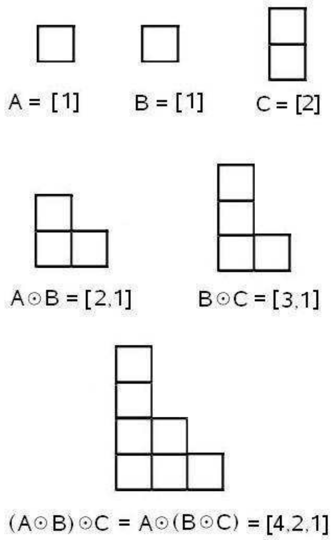


Figure 14. Multiplication on Young diagrams inspired by study of the BGW correlators. The law is simple: for any pair of Young diagrams A, B their "product" $A \odot B$ is equal to the Young diagram formed by positions of poles of the correlator $< J_A J_B >$ in the BGW model. All known examples suggest that this multiplication is associative.

A.6 Alternative 2-Jack average

A.6.1 Selberg model

A somewhat similar, still different 2-Jack correlator was recently considered in [81]. The main difference is the inverse powers $p_{-k} = \sum z_i^{-k}$ in the second argument:

$$\Big\langle J_A(v + p_k)\ J_B(p_{-k})\Big\rangle^{\text{Selb}} \sim G_{AB}(u + v + N\beta + 1 - \beta) G_{BA}(-u - v - N\beta + 2\beta - 2) \tag{A.20}$$

Note that G_{AB} functions appear here not in the denominator, but in the numerator, thus, the r.h.s. of (A.20) has no poles. The proportionality coefficient in (A.20) depends only on A and B independently: it is equal to

$$J_A(\delta_{k,1}) \left(-\frac{1}{\beta}\right)^{|A|} \frac{[v + N\beta + 1 - \beta]_A}{[u + v + 2N\beta + 2 - 2\beta]_A} \cdot J_B(\delta_{k,1}) \left(-\frac{1}{\beta}\right)^{|B|} \frac{[N\beta]_B}{[-u]_B} \tag{A.21}$$

In result, eq. (A.20) is of less direct use for the purpose of AGT proof than our eq. (A.12), however, ref. [81] suggests a more involved project with the use of this formula.

A.6.2 BGW model

The r.h.s. of (A.20) depends *not* on the BGW variable $z = \beta n = u + v + 2\beta N + 1 - \beta$, which is the only combination left finite in this limit. Thus, the r.h.s. of (A.20) becomes trivial in the PGL.

References

[1] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91** (2010) 167 [arXiv:0906.3219] [SPIRES].

[2] N. Wyllard, *A_{N-1} conformal Toda field theory correlation functions from conformal $N = 2$* SU(N) *quiver gauge theories*, JHEP **11** (2009) 002 [arXiv:0907.2189] [SPIRES].

[3] A. Mironov and A. Morozov, *On AGT relation in the case of $U(3)$*, Nucl. Phys. **B 825** (2010) 1 [arXiv:0908.2569] [SPIRES].

[4] A. Mironov and A. Morozov, *The power of Nekrasov functions*, Phys. Lett. **B 680** (2009) 188 [arXiv:0908.2190] [SPIRES].

[5] A. Marshakov, A. Mironov and A. Morozov, *On combinatorial expansions of conformal blocks*, Theor. Math. Phys. **164** (2010) 831 [arXiv:0907.3946] [SPIRES].

[6] D. Gaiotto, *Asymptotically free $N = 2$ theories and irregular conformal blocks*, arXiv:0908.0307 [SPIRES].

[7] A. Marshakov, A. Mironov and A. Morozov, *On non-conformal limit of the AGT relations*, Phys. Lett. **B 682** (2009) 125 [arXiv:0909.2052] [SPIRES].

[8] V. Alba and A. Morozov, *Non-conformal limit of AGT relation from the 1-point torus conformal block*, JETP Lett. **90** (2009) 708 [arXiv:0911.0363] [SPIRES].

[9] N. Drukker, D.R. Morrison and T. Okuda, *Loop operators and S-duality from curves on Riemann surfaces*, JHEP **09** (2009) 031 [arXiv:0907.2593] [SPIRES].

[10] A. Mironov, S. Mironov, A. Morozov and A. Morozov, *CFT exercises for the needs of AGT*, arXiv:0908.2064 [SPIRES].

[11] S.M. Iguri and C.A. Núñez, *Coulomb integrals and conformal blocks in the AdS_3-WZNW model*, JHEP **11** (2009) 090 [arXiv:0908.3460] [SPIRES].

[12] D.V. Nanopoulos and D. Xie, *On crossing symmmetry and modular invariance in conformal field theory and S duality in gauge theory*, Phys. Rev. **D 80** (2009) 105015 [arXiv:0908.4409] [SPIRES].

[13] D. Nanopoulos and D. Xie, *Hitchin equation, singularity and $N = 2$ superconformal field theories*, JHEP **03** (2010) 043 [arXiv:0911.1990] [SPIRES].

[14] D. Nanopoulos and D. Xie, *Hitchin equation, irregular singularity and $N = 2$ asymptotical free theories*, arXiv:1005.1350 [SPIRES].

[15] D. Nanopoulos and D. Xie, *$N = 2$ generalized superconformal quiver gauge theory*, arXiv:1006.3486 [SPIRES].

[16] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in $N = 2$ gauge theory and Liouville modular geometry*, JHEP **01** (2010) 113 [arXiv:0909.0945] [SPIRES].

[17] N. Drukker, J. Gomis, T. Okuda and J. Teschner, *Gauge theory loop operators and Liouville theory*, JHEP **02** (2010) 057 [arXiv:0909.1105] [SPIRES].

[18] A. Marshakov, A. Mironov and A. Morozov, *Zamolodchikov asymptotic formula and instanton expansion in $N = 2$ SUSY $N_f = 2N_c$ QCD*, JHEP **11** (2009) 048 [arXiv:0909.3338] [SPIRES].

JHEP02(2011)067

[19] R. Poghossian, *Recursion relations in CFT and $N = 2$ SYM theory*, JHEP **12** (2009) 038 [arXiv:0909.3412] [SPIRES].

[20] A. Gadde, E. Pomoni, L. Rastelli and S.S. Razamat, *S-duality and 2d topological QFT*, JHEP **03** (2010) 032 [arXiv:0910.2225] [SPIRES].

[21] L.F. Alday, F. Benini and Y. Tachikawa, *Liouville/Toda central charges from M5-branes*, Phys. Rev. Lett. **105** (2010) 141601 [arXiv:0909.4776] [SPIRES].

[22] S. Kanno, Y. Matsuo, S. Shiba and Y. Tachikawa, *$N = 2$ gauge theories and degenerate fields of Toda theory*, Phys. Rev. **D 81** (2010) 046004 [arXiv:0911.4787] [SPIRES].

[23] G. Bonelli and A. Tanzini, *Hitchin systems, $N = 2$ gauge theories and W-gravity*, Phys. Lett. **B 691** (2010) 111 [arXiv:0909.4031] [SPIRES].

[24] J.-F. Wu and Y. Zhou, *From Liouville to Chern-Simons, alternative realization of Wilson loop operators in AGT duality*, arXiv:0911.1922 [SPIRES].

[25] G. Giribet, *On triality in $N = 2$ SCFT with $N_f = 4$*, JHEP **01** (2010) 097 [arXiv:0912.1930] [SPIRES].

[26] V. Alba and A. Morozov, *Check of AGT relation for conformal blocks on sphere*, Nucl. Phys. **B 840** (2010) 441 [arXiv:0912.2535] [SPIRES].

[27] M. Fujita, Y. Hatsuda and T.-S. Tai, *Genus-one correction to asymptotically free Seiberg-Witten prepotential from Dijkgraaf-Vafa matrix model*, JHEP **03** (2010) 046 [arXiv:0912.2988] [SPIRES].

[28] M. Taki, *On AGT conjecture for pure super Yang-Mills and W-algebra*, arXiv:0912.4789 [SPIRES].

[29] M. Taki, *Surface operator, bubbling Calabi-Yau and AGT relation*, arXiv:1007.2524 [SPIRES].

[30] P. Sulkowski, *Matrix models for β-ensembles from Nekrasov partition functions*, JHEP **04** (2010) 063 [arXiv:0912.5476] [SPIRES].

[31] N. Nekrasov and E. Witten, *The Omega deformation, branes, integrability and Liouville theory*, JHEP **09** (2010) 092 [arXiv:1002.0888] [SPIRES].

[32] R. Santachiara and A. Tanzini, *Moore-Read fractional quantum Hall wavefunctions and* SU(2) *quiver gauge theories*, Phys. Rev. **D 82** (2010) 126006 [arXiv:1002.5017] [SPIRES].

[33] S. Yanagida, *Whittaker vectors of the Virasoro algebra in terms of Jack symmetric polynomial*, arXiv:1003.1049 [SPIRES].

[34] S. Yanagida, *Norms of logarithmic primaries of Virasoro algebra*, arXiv:1010.0528 [SPIRES].

[35] N. Drukker, D. Gaiotto and J. Gomis, *The virtue of defects in* 4D *gauge theories and* 2D *CFTs*, arXiv:1003.1112 [SPIRES].

[36] F. Passerini, *Gauge theory Wilson loops and conformal Toda field theory*, JHEP **03** (2010) 125 [arXiv:1003.1151] [SPIRES].

[37] C. Kozcaz, S. Pasquetti and N. Wyllard, *A & B model approaches to surface operators and Toda theories*, JHEP **08** (2010) 042 [arXiv:1004.2025] [SPIRES].

[38] W. He and Y.-G. Miao, *Magnetic expansion of Nekrasov theory: the* SU(2) *pure gauge theory*, Phys. Rev. **D 82** (2010) 025020 [arXiv:1006.1214] [SPIRES].

JHEP02(2011)067

[39] W. He and Y.-G. Miao, *Mathieu equation and elliptic curve*, arXiv:1006.5185 [SPIRES].

[40] S. Kanno, Y. Matsuo and S. Shiba, *Analysis of correlation functions in Toda theory and AGT-W relation for* SU(3) *quiver*, *Phys. Rev.* **D 82** (2010) 066009 [arXiv:1007.0601] [SPIRES].

[41] H. Awata, H. Fuji, H. Kanno, M. Manabe and Y. Yamada, *Localization with a surface operator, irregular conformal blocks and open topological string*, arXiv:1008.0574 [SPIRES].

[42] C. Kozcaz, S. Pasquetti, F. Passerini and N. Wyllard, *Affine sl*(N) *conformal blocks from* $N = 2$ SU(N) *gauge theories*, arXiv:1008.1412 [SPIRES].

[43] H. Itoyama, T. Oota and N. Yonezawa, *Massive scaling limit of beta-deformed matrix model of Selberg type*, *Phys. Rev.* **D 82** (2010) 085031 [arXiv:1008.1861] [SPIRES].

[44] A. Braverman, B. Feigin, M. Finkelberg and L. Rybnikov, *A finite analog of the AGT relation I: finite W-algebras and quasimaps' spaces*, arXiv:1008.3655 [SPIRES].

[45] T.-S. Tai, *Triality in* SU(2) *Seiberg-Witten theory and Gauss hypergeometric function*, *Phys. Rev.* **D 82** (2010) 105007 [arXiv:1006.0471] [SPIRES].

[46] T.-S. Tai, *Uniformization, Calogero-Moser/Heun duality and Sutherland/bubbling pants*, *JHEP* **10** (2010) 107 [arXiv:1008.4332] [SPIRES].

[47] K. Maruyoshi and M. Taki, *Deformed prepotential, quantum integrable system and Liouville field theory*, *Nucl. Phys.* **B 841** (2010) 388 [arXiv:1006.4505] [SPIRES].

[48] M. Billó, L. Gallot, A. Lerda and I. Pesando, *F-theoretic vs microscopic description of a conformal* $N = 2$ *SYM theory*, *JHEP* **11** (2010) 041 [arXiv:1008.5240] [SPIRES].

[49] A. Brini, M. Mariño and S. Stevan, *The uses of the refined matrix model recursion*, arXiv:1010.1210 [SPIRES].

[50] M.C.N. Cheng, R. Dijkgraaf and C. Vafa, *Non-perturbative topological strings and conformal blocks*, arXiv:1010.4573 [SPIRES].

[51] N. Wyllard, *W-algebras and surface operators in* $N = 2$ *gauge theories*, arXiv:1011.0289 [SPIRES].

[52] Y. Yamada, *A quantum isomonodromy equation and its application to* $N = 2$ SU(N) *gauge theories*, *J. Phys.* **A 44** (2011) 055403 [arXiv:1011.0292] [SPIRES].

[53] N. Wyllard, *Instanton partition functions in* $N = 2$ SU(N) *gauge theories with a general surface operator and their W-algebra duals*, arXiv:1012.1355 [SPIRES].

[54] J. Wu, *Note on refined topological vertex, Jack polynomials and instanton counting*, arXiv:1012.2147 [SPIRES].

[55] N.A. Nekrasov and S.L. Shatashvili, *Quantization of integrable systems and four dimensional gauge theories*, arXiv:0908.4052 [SPIRES].

[56] A. Mironov and A. Morozov, *Nekrasov functions and exact Bohr-Sommerfeld integrals*, *JHEP* **04** (2010) 040 [arXiv:0910.5670] [SPIRES].

[57] A. Mironov and A. Morozov, *Nekrasov functions from exact BS periods: the case of* SU(N), *J. Phys.* **A 43** (2010) 195401 [arXiv:0911.2396] [SPIRES].

[58] A. Popolitov, *On relation between Nekrasov functions and BS periods in pure* SU(N) *case*, arXiv:1001.1407 [SPIRES].

[59] R. Dijkgraaf and C. Vafa, *Toda theories, matrix models, topological strings and $N = 2$ gauge systems*, arXiv:0909.2453 [SPIRES].

[60] A. Mironov and A. Morozov, *Proving AGT relations in the large-c limit*, Phys. Lett. **B 682** (2009) 118 [arXiv:0909.3531] [SPIRES].

[61] H. Awata and Y. Yamada, *Five-dimensional AGT Conjecture and the deformed Virasoro algebra*, JHEP **01** (2010) 125 [arXiv:0910.4431] [SPIRES].

[62] H. Awata and Y. Yamada, *Five-dimensional AGT Relation and the deformed β-ensemble*, Prog. Theor. Phys. **124** (2010) 227 [arXiv:1004.5122] [SPIRES].

[63] L. Hadasz, Z. Jaskolski and P. Suchanek, *Recursive representation of the torus 1-point conformal block*, JHEP **01** (2010) 063 [arXiv:0911.2353] [SPIRES].

[64] L. Hadasz, Z. Jaskolski and P. Suchanek, *Proving the AGT relation for $N_f = 0, 1, 2$ antifundamentals*, JHEP **06** (2010) 046 [arXiv:1004.1841] [SPIRES].

[65] H. Itoyama, K. Maruyoshi and T. Oota, *The quiver matrix model and 2d-4d conformal connection*, Prog. Theor. Phys. **123** (2010) 957 [arXiv:0911.4244] [SPIRES].

[66] T. Eguchi and K. Maruyoshi, *Penner type matrix model and Seiberg-Witten theory*, JHEP **02** (2010) 022 [arXiv:0911.4797] [SPIRES].

[67] T. Eguchi and K. Maruyoshi, *Seiberg-Witten theory, matrix model and AGT relation*, JHEP **07** (2010) 081 [arXiv:1006.0828] [SPIRES].

[68] R. Schiappa and N. Wyllard, *An A_r threesome: matrix models,* 2d *CFTs and* 4d $N = 2$ *gauge theories*, arXiv:0911.5337 [SPIRES].

[69] A. Mironov, A. Morozov and S. Shakirov, *Matrix model conjecture for exact BS periods and Nekrasov functions*, JHEP **02** (2010) 030 [arXiv:0911.5721] [SPIRES].

[70] A. Mironov, A. Morozov and S. Shakirov, *Conformal blocks as Dotsenko-Fateev integral discriminants*, Int. J. Mod. Phys. **A 25** (2010) 3173 [arXiv:1001.0563] [SPIRES].

[71] A. Mironov, A. Morozov and A. Morozov, *Matrix model version of AGT conjecture and generalized Selberg integrals*, Nucl. Phys. **B 843** (2011) 534 [arXiv:1003.5752] [SPIRES].

[72] V.A. Fateev and A.V. Litvinov, *On AGT conjecture*, JHEP **02** (2010) 014 [arXiv:0912.0504] [SPIRES].

[73] H. Itoyama and T. Oota, *Method of generating q-expansion coefficients for conformal block and $N = 2$ Nekrasov function by β-deformed matrix model*, Nucl. Phys. **B 838** (2010) 298 [arXiv:1003.2929] [SPIRES].

[74] S. Yanagida, *Five-dimensional* SU(2) *AGT conjecture and recursive formula of deformed Gaiotto state*, J. Math. Phys. **51** (2010) 123506 [arXiv:1005.0216] [SPIRES].

[75] K. Maruyoshi and F. Yagi, *Seiberg-Witten curve via generalized matrix model*, JHEP **01** (2011) 042 [arXiv:1009.5553] [SPIRES].

[76] A. Mironov, A. Morozov and S. Shakirov, *On 'Dotsenko-Fateev' representation of the toric conformal blocks*, J. Phys. **A 44** (2011) 085401 [arXiv:1010.1734] [SPIRES].

[77] G. Bonelli, K. Maruyoshi, A. Tanzini and F. Yagi, *Generalized matrix models and AGT correspondence at all genera*, arXiv:1011.5417 [SPIRES].

[78] A. Mironov, A. Morozov and S. Shakirov, *Brezin-Gross-Witten model as 'pure gauge' limit of Selberg integrals*, arXiv:1011.3481 [SPIRES].

JHEP02(2011)067

[79] A. Marshakov, A. Mironov and A. Morozov, *On AGT Relations with surface operator insertion and stationary limit of Beta-ensembles*, *Teor. Mat. Fiz.* **164** (2010) 1:3–27 [arXiv:1011.4491] [SPIRES].

[80] A. Mironov, A. Morozov and S. Shakirov, *Towards a proof of AGT conjecture by methods of matrix models*, arXiv:1011.5629 [SPIRES].

[81] V.A. Alba, V.A. Fateev, A.V. Litvinov and G.M. Tarnopolsky, *On combinatorial expansion of the conformal blocks arising from AGT conjecture*, arXiv:1012.1312 [SPIRES].

[82] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys.* **B 241** (1984) 333 [SPIRES];

[83] A. Zamolodchikov and Al. Zamolodchikov, *Conformal field theory and critical phenomena in 2d systems* (in Russian), MCNMO, Moscow, Russia (2009).

[84] A. Losev, N. Nekrasov and S.L. Shatashvili, *Issues in topological gauge theory*, *Nucl. Phys.* **B 534** (1998) 549 [hep-th/9711108] [SPIRES].

[85] A. Losev, N. Nekrasov and S.L. Shatashvili, *Testing Seiberg-Witten solution*, hep-th/9801061 [SPIRES].

[86] G.W. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs branches*, *Commun. Math. Phys.* **209** (2000) 97 [hep-th/9712241] [SPIRES].

[87] G.W. Moore, N. Nekrasov and S. Shatashvili, *D-particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77 [hep-th/9803265] [SPIRES].

[88] A. Gorsky, I. Krichever, A. Marshakov, A. Mironov and A. Morozov, *Integrability and Seiberg-Witten exact solution*, *Phys. Lett.* **B 355** (1995) 466 [hep-th/9505035] [SPIRES].

[89] E.J. Martinec, *Integrable structures in supersymmetric gauge and string theory*, *Phys. Lett.* **B 367** (1996) 91 [hep-th/9510204] [SPIRES].

[90] T. Nakatsu and K. Takasaki, *Whitham-Toda hierarchy and $N = 2$ supersymmetric Yang-Mills theory*, *Mod. Phys. Lett.* **A 11** (1996) 157 [hep-th/9509162] [SPIRES].

[91] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, *Nucl. Phys.* **B 460** (1996) 299 [hep-th/9510101] [SPIRES].

[92] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, *$N = 2$ Supersymmetric QCD and integrable spin chains: rational case $N_f < 2N_c$*, *Phys. Lett.* **B 380** (1996) 75 [hep-th/9603140] [SPIRES].

[93] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, *$N = 2$ Supersymmetric QCD and integrable spin chains: rational case $N_f < 2N_c$*, *Phys. Lett.* **B 380** (1996) 75 [hep-th/9603140] [SPIRES].

[94] A. Gorsky, A. Marshakov, A. Mironov and A. Morozov, *A note on spectral curve for the periodic homogeneous XYZ-spin chain*, hep-th/9604078 [SPIRES].

[95] A. Gorsky, S. Gukov and A. Mironov, *Multiscale $N = 2$ SUSY field theories, integrable systems and their stringy/brane origin. I*, *Nucl. Phys.* **B 517** (1998) 409 [hep-th/9707120] [SPIRES].

[96] A. Gorsky, S. Gukov and A. Mironov, *SUSY field theories, integrable systems and their stringy/brane origin. II*, *Nucl. Phys.* **B 518** (1998) 689 [hep-th/9710239] [SPIRES].

JHEP02(2011)067

[97] H.W. Braden, A. Marshakov, A. Mironov and A. Morozov, *On double-elliptic integrable systems. I: A duality argument for the case of* SU(2), hep-th/9906240 [SPIRES].

[98] H.W. Braden, A. Marshakov, A. Mironov and A. Morozov, *Seiberg-Witten theory for a non-trivial compactification from five to four dimensions*, Phys. Lett. **B 448** (1999) 195 [hep-th/9812078] [SPIRES].

[99] H.W. Braden, A. Marshakov, A. Mironov and A. Morozov, *On double-elliptic integrable systems. I: A duality argument for the case of* SU(2), hep-th/9906240 [SPIRES].

[100] A. Gorsky and A. Mironov, *Solutions to the reflection equation and integrable systems for* $N = 2$ *SQCD with classical groups*, Nucl. Phys. **B 550** (1999) 513 [hep-th/9902030] [SPIRES].

[101] A. Gorsky and A. Mironov, *Integrable many-body systems and gauge theories*, hep-th/0011197 [SPIRES].

[102] A. Mironov and A. Morozov, *Double elliptic systems: Problems and perspectives*, hep-th/0001168 [SPIRES].

[103] E.J. Martinec and N.P. Warner, *Integrable systems and supersymmetric gauge theory*, Nucl. Phys. **B 459** (1996) 97 [hep-th/9509161] [SPIRES].

[104] A. Gorsky and A. Marshakov, *Towards effective topological gauge theories on spectral curves*, Phys. Lett. **B 375** (1996) 127 [hep-th/9510224] [SPIRES].

[105] H. Itoyama and A. Morozov, *Integrability and Seiberg-Witten theory: curves and periods*, Nucl. Phys. **B 477** (1996) 855 [hep-th/9511126] [SPIRES].

[106] H. Itoyama and A. Morozov, *Prepotential and the Seiberg-Witten theory*, Nucl. Phys. **B 491** (1997) 529 [hep-th/9512161] [SPIRES].

[107] H. Itoyama and A. Morozov, *Integrability and Seiberg-Witten theory*, hep-th/9601168 [SPIRES].

[108] E. D'Hoker, I.M. Krichever and D.H. Phong, *The effective prepotential of* $N = 2$ *supersymmetric* $\mathrm{SU}(N(c))$ *gauge theories*, Nucl. Phys. **B 489** (1997) 179 [hep-th/9609041] [SPIRES].

[109] E. D'Hoker, I.M. Krichever and D.H. Phong, *The effective prepotential of* $N = 2$ *supersymmetric* $SO(N(c))$ *and* $Sp(N(c))$ *gauge theories*, Nucl. Phys. **B 489** (1997) 211 [hep-th/9609145] [SPIRES].

[110] N. Nekrasov, *Five dimensional gauge theories and relativistic integrable systems*, Nucl. Phys. **B 531** (1998) 323 [hep-th/9609219] [SPIRES].

[111] A. Marshakov and A. Mironov, 5*d and* 6*d supersymmetric gauge theories: Prepotentials from integrable systems*, Nucl. Phys. **B 518** (1998) 59 [hep-th/9711156] [SPIRES].

[112] A. Marshakov, *Seiberg-Witten theory and integrable systems*, World Scientific, Singapore (1999).

[113] A. Mironov and A. Morozov, *Commuting Hamiltonians from Seiberg-Witten Theta-functions*, Phys. Lett. **B 475** (2000) 71 [hep-th/9912088] [SPIRES].

[114] H.W. Braden and A. Marshakov, *Singular phases of Seiberg-Witten integrable systems: Weak and strong coupling*, Nucl. Phys. **B 595** (2001) 417 [hep-th/0009060] [SPIRES].

[115] H.W. Braden and A. Marshakov, *Singular phases of Seiberg-Witten integrable systems: Weak and strong coupling*, *Nucl. Phys.* **B 595** (2001) 417 [hep-th/0009060] [SPIRES].

[116] N.A. Nekrasov and S.L. Shatashvili, *Supersymmetric vacua and Bethe ansatz*, *Nucl. Phys. Proc. Suppl.* **192-193** (2009) 91 [arXiv:0901.4744] [SPIRES].

[117] N.A. Nekrasov and S.L. Shatashvili, *Quantum integrability and supersymmetric vacua*, *Prog. Theor. Phys. Suppl.* **177** (2009) 105 [arXiv:0901.4748] [SPIRES].

[118] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099] [SPIRES].

[119] N. Seiberg and E. Witten, *Monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087] [SPIRES].

[120] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, *Matrix models of $2-D$ gravity and Toda theory*, *Nucl. Phys.* **B 357** (1991) 565 [SPIRES].

[121] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, *Unification of all string models with $C < 1$*, *Phys. Lett.* **B 275** (1992) 311 [hep-th/9111037] [SPIRES].

[122] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, *Towards unified theory of $2-D$ gravity*, *Nucl. Phys.* **B 380** (1992) 181 [hep-th/9201013] [SPIRES].

[123] S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Generalized Kontsevich model versus Toda hierarchy and discrete matrix models*, *Nucl. Phys.* **B 397** (1993) 339 [hep-th/9203043] [SPIRES].

[124] A. Morozov, *String theory: what is it?*, *Phys. Usp.* **35** (1992) 671.

[125] A. Morozov, *Integrability and matrix models*, *Phys. Usp.* **37** (1994) 1 [hep-th/9303139] [SPIRES].

[126] A. Morozov, *Matrix models as integrable systems*, hep-th/9502091 [SPIRES].

[127] A. Morozov, *Challenges of matrix models*, hep-th/0502010 [SPIRES].

[128] A. Mironov, *$2-D$ gravity and matrix models. 1. $2-D$ gravity*, *Int. J. Mod. Phys.* **A 9** (1994) 4355 [hep-th/9312212] [SPIRES].

[129] A. Mironov, *Matrix models of two-dimensional gravity*, *Phys. Part. Nucl.* **33** (2002) 537 [SPIRES].

[130] A. Mironov, *Quantum deformations of tau functions, bilinear identities and representation theory*, hep-th/9409190 [SPIRES].

[131] L. Chekhov and A. Mironov, *Matrix models vs. Seiberg-Witten/Whitham theories*, *Phys. Lett.* **B 552** (2003) 293 [hep-th/0209085] [SPIRES].

[132] V.A. Kazakov and A. Marshakov, *Complex curve of the two matrix model and its tau-function*, *J. Phys.* **A 36** (2003) 3107 [hep-th/0211236] [SPIRES].

[133] L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, *DV and WDVV*, *Phys. Lett.* **B 562** (2003) 323 [hep-th/0301071] [SPIRES].

[134] L. Chekhov, A. Marshakov, A. Mironov and D. Vasiliev, *Complex geometry of matrix models*, *Proc. Steklov Inst. Math.* **251** (2005) 254 [hep-th/0506075] [SPIRES].

[135] A.S. Alexandrov, A. Mironov and A. Morozov, *Unified description of correlators in non-Gaussian phases of Hermitean matrix model*, *Int. J. Mod. Phys.* **A 21** (2006) 2481 [hep-th/0412099] [SPIRES].

JHEP02(2011)067

[136] A.S. Alexandrov, A. Mironov and A. Morozov, *Solving Virasoro constraints in matrix models*, *Fortsch. Phys.* **53** (2005) 512 [hep-th/0412205] [SPIRES].

[137] A. Mironov, *Matrix models vs. matrix integrals*, *Theor. Math. Phys.* **146** (2006) 63 [hep-th/0506158] [SPIRES].

[138] B. Eynard and N. Orantin, *Invariants of algebraic curves and topological expansion*, math-ph/0702045 [SPIRES].

[139] V.S. Dotsenko and V.A. Fateev, *Conformal algebra and multipoint correlation functions in 2D statistical models*, *Nucl. Phys.* **B 240** (1984) 312 [SPIRES].

[140] A. Gerasimov, A. Morozov, M. Olshanetsky, A. Marshakov and S.L. Shatashvili, *Wess-Zumino-Witten model as a theory of free fields*, *Int. J. Mod. Phys.* **A 5** (1990) 2495 [SPIRES].

[141] A. Gerasimov, A. Marshakov and A. Morozov, *Free field representation of parafermions and related coset models*, *Nucl. Phys.* **B 328** (1989) 664 [*Theor. Math. Phys.* **83** (1990) 466] [SPIRES].

[142] A. Gerasimov, A. Marshakov and A. Morozov, *Hamiltonian reduction of Wess-Zumino-Witten theory from the point of view of bosonization*, *Phys. Lett.* **B 236** (1990) 269 [*Sov. J. Nucl. Phys.* **51** (1990) 371] [SPIRES].

[143] P. Di Francesco, M. Gaudin, C. Itzykson and F. Lesage, *Laughlin's wave functions, Coulomb gases and expansions of the discriminant*, *Int. J. Mod. Phys.* **A 9** (1994) 4257 [hep-th/9401163] [SPIRES].

[144] A. Zabrodin, *Random matrices and Laplacian growth* arXiv:0907.4929.

[145] S. Shakirov, *Exact solution for mean energy of 2d Dyson gas at $\beta = 1$*, *Phys. Lett.* **A 375** (2011) 984 [arXiv:0912.5520] [SPIRES].

[146] A. Morozov and S. Shakirov, *The matrix model version of AGT conjecture and CIV-DV prepotential*, *JHEP* **08** (2010) 066 [arXiv:1004.2917] [SPIRES].

[147] L. Chekhov, *Logarithmic potential beta-ensembles and Feynman graphs*, arXiv:1009.5940 [SPIRES].

[148] A. Marshakov, A. Mironov and A. Morozov, *Generalized matrix models as conformal field theories: Discrete case*, *Phys. Lett.* **B 265** (1991) 99 [SPIRES].

[149] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and S. Pakuliak, *Conformal matrix models as an alternative to conventional multimatrix models*, *Nucl. Phys.* **B 404** (1993) 717 [hep-th/9208044] [SPIRES].

[150] H. Awata, Y. Matsuo, S. Odake and J. Shiraishi, *A note on Calogero-Sutherland model, $W(n)$ singular vectors and generalized matrix models*, *Soryushiron Kenkyu.* **91** (1995) A69 [hep-th/9503028] [SPIRES].

[151] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [hep-th/0206161] [SPIRES].

[152] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238 [SPIRES].

[153] F. Fucito, J.F. Morales and R. Poghossian, *Instantons on quivers and orientifolds*, *JHEP* **10** (2004) 037 [hep-th/0408090] [SPIRES].

[154] A.S. Alexandrov, A. Mironov and A. Morozov, *Partition functions of matrix models as the first special functions of string theory. I: Finite size Hermitean 1-matrix model*, *Int. J. Mod. Phys.* **A 19** (2004) 4127 [hep-th/0310113] [SPIRES].

[155] A.S. Alexandrov, A. Mironov and A. Morozov, *M-theory of matrix models*, *Teor. Mat. Fiz.* **150** (2007) 179 [hep-th/0605171] [SPIRES].

[156] A.S. Alexandrov, A. Mironov and A. Morozov, *Instantons and merons in matrix models*, *Physica* **D 235** (2007) 126 [hep-th/0608228] [SPIRES].

[157] A. Alexandrov, A. Mironov and A. Morozov, *BGWM as second constituent of complex matrix model*, *JHEP* **12** (2009) 053 [arXiv:0906.3305] [SPIRES].

[158] A.S. Alexandrov, A. Mironov, A. Morozov and P. Putrov, *Partition functions of matrix models as the first special functions of string theory. II. Kontsevich model*, *Int. J. Mod. Phys.* **A 24** (2009) 4939 [arXiv:0811.2825] [SPIRES].

[159] B. Eynard, *Topological expansion for the 1-hermitian matrix model correlation functions*, *JHEP* **11** (2004) 031 [hep-th/0407261] [SPIRES].

[160] L. Chekhov and B. Eynard, *Hermitean matrix model free energy: Feynman graph technique for all genera*, *JHEP* **03** (2006) 014 [hep-th/0504116] [SPIRES].

[161] L. Chekhov and B. Eynard, *Matrix eigenvalue model: Feynman graph technique for all genera*, *JHEP* **12** (2006) 026 [math-ph/0604014] [SPIRES].

[162] N. Orantin, *Symplectic invariants, Virasoro constraints and Givental decomposition*, arXiv:0808.0635 [SPIRES].

[163] I. Kostov and N. Orantin, *CFT and topological recursion*, arXiv:1006.2028 [SPIRES].

[164] L.O. Chekhov, B. Eynard and O. Marchal, *Topological expansion of beta-ensemble model and quantum algebraic geometry in the sectorwise approach*, arXiv:1009.6007 [SPIRES].

[165] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, *Invent. Math.* **85** (1986) 457.

[166] S.K. Lando and A.K. Zvonkin, *Embedded graphs*, Max-Plank-Institut für Mathematik, Preprint Series 2001, MPIM2001-63 **63** (2001).

[167] A. Morozov and S. Shakirov, *Exact 2-point function in Hermitian matrix model*, *JHEP* **12** (2009) 003 [arXiv:0906.0036] [SPIRES].

[168] A. Morozov and S. Shakirov, *From Brezin-Hikami to Harer-Zagier formulas for Gaussian correlators*, arXiv:1007.4100 [SPIRES].

[169] D. Bellisai, F. Fucito, A. Tanzini and G. Travaglini, *Multi-instantons, supersymmetry and topological field theories*, *Phys. Lett.* **B 480** (2000) 365 [hep-th/0002110] [SPIRES].

[170] U. Bruzzo, F. Fucito, A. Tanzini and G. Travaglini, *On the multi-instanton measure for super Yang-Mills theories*, *Nucl. Phys.* **B 611** (2001) 205 [hep-th/0008225] [SPIRES].

[171] U. Bruzzo, F. Fucito, J.F. Morales and A. Tanzini, *Multi-instanton calculus and equivariant cohomology*, *JHEP* **05** (2003) 054 [hep-th/0211108] [SPIRES].

[172] U. Bruzzo and F. Fucito, *Superlocalization formulas and supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 678** (2004) 638 [math-ph/0310036] [SPIRES].

[173] A. Morozov and S. Shakirov, *Generation of matrix models by W-operators*, *JHEP* **04** (2009) 064 [arXiv:0902.2627] [SPIRES].

[174] A. Alexandrov, *Matrix models for random partitions*, arXiv:1005.5715 [SPIRES].

[175] A.Y. A. . M. Morozov, ITEP, *Unitary integrals and related matrix models*, *Theor. Math. Phys.* **162** (2010) 1 [arXiv:0906.3518] [SPIRES].

[176] A.B. Balantekin, *Character expansions in physics*, *AIP Conf. Proc.* **1323** (2010) 1 [arXiv:1011.3859] [SPIRES].

[177] A. Alexandrov, A. Mironov and A. Morozov, *Cut-and-join operators, matrix models and characters*, to appear.

[178] E. Brézin and D.J. Gross, *The external field problem in the large-N limit of QCD*, *Phys. Lett.* **B 97** (1980) 120 [SPIRES].

[179] D.J. Gross and E. Witten, *Possible third order phase transition in the large-N lattice gauge theory*, *Phys. Rev.* **D 21** (1980) 446 [SPIRES].

[180] A. Mironov, A. Morozov and G.W. Semenoff, *Unitary matrix integrals in the framework of generalized Kontsevich model. 1. Brezin-Gross-Witten model*, *Int. J. Mod. Phys.* **A 11** (1996) 5031 [hep-th/9404005] [SPIRES].

[181] A. Alexandrov, A. Mironov and A. Morozov, *BGWM as second constituent of complex matrix model*, *JHEP* **12** (2009) 053 [arXiv:0906.3305] [SPIRES].

[182] J.Kaneko, *Selberg integrals and hypergeometric functions associated with Jack polynomials*, SIAM. J. Math. Anal. **24** (1993) 1086.

[183] K.W.J. Kadell, *The Selberg-Jack symmetric functions*, Adv. Math. **130** (1997) 33.

[184] K.W.J. Kadell, *An integral for the product of two Selberg-Jack symmetric functions*, *Compositio Math.* **87** (1993) 5.

JHEP02(2011)067

A	B	$S_{AB} = \Big\langle j_A(p_k)\ j_B(p_k) \Big\rangle_{\pm}^{\text{BGW}}$	The set of factors	At $(\epsilon_1, \epsilon_2) = (\hbar, -\hbar)$
$[1]$	$[]$	$\dfrac{g}{z - \epsilon_2 - \epsilon_1}$	$g \cdot \dfrac{\varnothing}{(1,1)}$	$\dfrac{\hbar}{z}$
$[]$	$[1]$	$\dfrac{g}{z - \epsilon_2 - \epsilon_1}$	$g \cdot \dfrac{\varnothing}{(1,1)}$	$\dfrac{\hbar}{z}$
$[2]$	$[]$	$\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{-g^2}{(z - \epsilon_2 - \epsilon_1)(z - \epsilon_1 - 2\epsilon_2)}$	$-g^2\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{\varnothing}{(1,1)(1,2)}$	$\dfrac{\hbar^2}{2z(z+\hbar)}$
$[1,1]$	$[]$	$\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{-g^2}{(z - \epsilon_2 - \epsilon_1)(z - 2\epsilon_1 - \epsilon_2)}$	$-g^2\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{\varnothing}{(1,1)(2,1)}$	$\dfrac{\hbar^2}{2z(z-\hbar)}$
$[1]$	$[1]$	$\dfrac{-g^2(z - 2\epsilon_1 - 2\epsilon_2)}{(z - 2\epsilon_1 - \epsilon_2)(z - \epsilon_1 - 2\epsilon_2)(z - \epsilon_2 - \epsilon_1)}$	$-g^2 \cdot \dfrac{(2,2)}{(1,1)(2,1)(1,2)}$	$\dfrac{\hbar^2}{(z-\hbar)(z+\hbar)}$
$[]$	$[2]$	$\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{-g^2}{(z - \epsilon_2 - \epsilon_1)(z - \epsilon_1 - 2\epsilon_2)}$	$-g^2\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{\varnothing}{(1,1)(1,2)}$	$\dfrac{\hbar^2}{2z(z+\hbar)}$
$[]$	$[1,1]$	$\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{-g^2}{(z - \epsilon_2 - \epsilon_1)(z - 2\epsilon_1 - \epsilon_2)}$	$-g^2\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{\varnothing}{(1,1)(2,1)}$	$\dfrac{\hbar^2}{2z(z-\hbar)}$

Table 1. Correlators S_{AB} in the "+" and "-" BGW models, at levels 1 and 2. Here $z \equiv \pm 2a$, $g = \sqrt{-\epsilon_1\epsilon_2}$ and $\epsilon_{12} = \epsilon_1 - \epsilon_2$.

A	B	$R_{AB} = \left\langle j_A\big(-p_k\big)\ j_B\big(p_k\big)\right\rangle_{\pm}^{\text{BGW}}$	The set of factors	At $(\epsilon_1, \epsilon_2) = (\hbar, -\hbar)$
[1]	[]	$\dfrac{-g}{z-\epsilon_1-\epsilon_2}$	$-g \cdot \dfrac{\varnothing}{(1,1)}$	$\dfrac{-\hbar}{z}$
[]	[1]	$\dfrac{g}{z-\epsilon_1-\epsilon_2}$	$g \cdot \dfrac{\varnothing}{(1,1)}$	$\dfrac{\hbar}{z}$
[2]	[]	$\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{-g^2(z-2\epsilon_1-3\epsilon_2)}{(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$-g^2\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{(2,3)}{(1,1)(2,1)(1,2)}$	$\dfrac{\hbar^2}{2z(z-\hbar)}$
[1, 1]	[]	$\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{-g^2(z-3\epsilon_1-2\epsilon_2)}{(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$-g^2\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{(3,2)}{(1,1)(2,1)(1,2)}$	$\dfrac{\hbar^2}{2z(z+\hbar)}$
[1]	[1]	$\dfrac{g^2(z-2\epsilon_1-2\epsilon_2)}{(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$g^2 \cdot \dfrac{(2,2)}{(1,1)(2,1)(1,2)}$	$\dfrac{-\hbar^2}{(z-\hbar)(z+\hbar)}$
[]	[2]	$\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{-g^2}{(z-\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)}$	$-g^2\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{\varnothing}{(1,1)(1,2)}$	$\dfrac{\hbar^2}{2z(z+\hbar)}$
[]	[1, 1]	$\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{-g^2}{(z-\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)}$	$-g^2\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{\varnothing}{(1,1)(2,1)}$	$\dfrac{\hbar^2}{2z(z-\hbar)}$

Table 2. Correlators R_{AB} in the "+" and "-" BGW models, at levels 1 and 2. Here $z \equiv \pm 2a$, $g = \sqrt{-\epsilon_1\epsilon_2}$ and $\epsilon_{12} = \epsilon_1 - \epsilon_2$.

A	B	$Q_{AB} = \left\langle j_A\big(-p_k\big)\ j_B\big(-p_k\big)\right\rangle_{\pm}^{\mathrm{BGW}}$	The set of factors	At $(\epsilon_1, \epsilon_2) = (\hbar, -\hbar)$
$[1]$	$[]$	$\dfrac{-g}{z-\epsilon_1-\epsilon_2}$	$-g \cdot \dfrac{\varnothing}{(1,1)}$	$\dfrac{-\hbar}{z}$
$[]$	$[1]$	$\dfrac{-g}{z-\epsilon_1-\epsilon_2}$	$-g \cdot \dfrac{\varnothing}{(1,1)}$	$\dfrac{-\hbar}{z}$
$[2]$	$[]$	$\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{-g^2(z-2\epsilon_1-3\epsilon_2)}{(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$-g^2\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{(2,3)}{(1,1)(2,1)(1,2)}$	$\dfrac{\hbar^2}{2z(z-\hbar)}$
$[1,1]$	$[]$	$\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{-g^2(z-3\epsilon_1-2\epsilon_2)}{(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$-g^2\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{(3,2)}{(1,1)(2,1)(1,2)}$	$\dfrac{\hbar^2}{2z(z+\hbar)}$
$[1]$	$[1]$	$\dfrac{-g^2(z-2\epsilon_1-2\epsilon_2)}{(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$-g^2\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{(2,2)}{(1,1)(2,1)(1,2)}$	$\dfrac{\hbar^2}{(z-\hbar)(z+\hbar)}$
$[]$	$[2]$	$\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{-g^2(z-2\epsilon_1-3\epsilon_2)}{(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$-g^2\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{(2,3)}{(1,1)(2,1)(1,2)}$	$\dfrac{\hbar^2}{2z(z-\hbar)}$
$[]$	$[1,1]$	$\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{-g^2(-3\epsilon_1-2\epsilon_2+z)}{2(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$-g^2\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{(3,2)}{(1,1)(2,1)(1,2)}$	$\dfrac{\hbar^2}{2z(z+\hbar)}$

Table 3. Correlators Q_{AB} in the "+" and "-" BGW models, at levels 1 and 2. Here $z \equiv \pm 2a$, $g = \sqrt{-\epsilon_1\epsilon_2}$ and $\epsilon_{12} = \epsilon_1 - \epsilon_2$.

A	B	$\Big\langle j_A\big(-w-p_k\big)\; j_B\big(-p_k\big)\Big\rangle_{\pm}^{\text{Selb}}$
$[1]$	$[]$	$\frac{1}{\sqrt{\beta}}\frac{(v+2-2\beta+u+N\beta)(N\beta-\beta+1+v)}{(u+v+2N\beta+2-2\beta)}$
$[]$	$[1]$	$\frac{1}{\sqrt{\beta}}\frac{(u+N\beta-\beta+1)N\beta}{u+v+2N\beta+2-2\beta}$
$[2]$	$[]$	$\frac{1}{\sqrt{2\beta(\beta+1)}}\frac{(v+2-\beta+N\beta)(v+1-\beta+N\beta)(v+3+u-2\beta+N\beta)(v+2-2\beta+N\beta+u)}{(u+v+2N\beta+2-2\beta)(u+v+2N\beta+3-2\beta)}$
$[1,1]$	$[]$	$\frac{1}{\sqrt{2\beta^2(\beta+1)}}\frac{(v+1-2\beta+N\beta)(v+1+N\beta-\beta)(v+2+u-3\beta+N\beta)(v+2+u-2\beta+N\beta)}{(u+v+2N\beta+2-2\beta)(u+v+2N\beta+2-3\beta)}$
$[1]$	$[1]$	$\frac{1}{\beta}\frac{(v+1+N\beta-\beta)(v+2+u-2\beta+N\beta)(v+3-3\beta+2N\beta+u)(u+N\beta-\beta+1)N\beta}{(u+v+2N\beta+3-2\beta)(u+v+2N\beta+2-2\beta)(u+v+2N\beta+2-3\beta)}$
$[]$	$[2]$	$\frac{1}{\sqrt{2\beta(\beta+1)}}\frac{N\beta(N\beta+1)(u+N\beta-\beta+1)(u+N\beta-\beta+2)}{(u+v+2N\beta+2-2\beta)(u+v+2N\beta+3-2\beta)}$
$[]$	$[1,1]$	$\frac{1}{\sqrt{2\beta^2(\beta+1)}}\frac{N\beta(N\beta-1)(u+N\beta-\beta+1)(u+N\beta-2\beta+1)}{(u+v+2N\beta+2-2\beta)(u+v+2N\beta+2-3\beta)}$

Table 4. The table of correlators in the "+" and "-" Selberg models, at levels 1 and 2. The correlators are written in matrix model notations, where u, v are the parameters of the Selberg potential, N is the number of eigenvalues, and $\beta = -\epsilon_1/\epsilon_2$ is the Van-der-Monde power. The shift w, which is essential for the correlators to be completely factorizable, are equal to $w = (v+1-\beta)/\beta$.

A	B	$\left\langle j_A\left(w+p_k\right)\ j_B\left(p_{-k}\right)\right\rangle_{\pm}^{\text{Selb}}$
[1]	[]	$\frac{1}{\sqrt{\beta}}\frac{(1+u+v+N\beta-\beta)N\beta}{u}$
[]	[1]	$\frac{1}{\sqrt{\beta}}\frac{(v+1-\beta+N\beta)(v+u+N\beta+2-2\beta)}{\beta(u+v+2N\beta+2-2\beta)}$
[2]	[]	$\frac{1}{\sqrt{2\beta(\beta+1)}}\frac{N\beta(N\beta+1)(1+u+v+N\beta-\beta)(u+v+N\beta-\beta)}{u(-1+u)}$
[1, 1]	[]	$\frac{1}{\sqrt{2\beta^2(\beta+1)}}\frac{N\beta(N\beta-\beta)(1+u+v+N\beta)(1+u+v+N\beta-\beta)}{2u(u+\beta)}$
[1]	[1]	$\frac{1}{\beta}\frac{(v+1-\beta+N\beta)(u+v+1+N\beta-2\beta)(2+u+v+N\beta-\beta)N\beta}{\beta u(u+v+2N\beta+2-2\beta)}$
[]	[2]	$\frac{1}{\sqrt{2\beta(\beta+1)}}\frac{(v+1-\beta+N\beta)(2+v+N\beta-\beta)(v+u+N\beta+2-2\beta)(u+v+3+N\beta-2\beta)}{(u+v+2N\beta+2-2\beta)(3+u+v+2N\beta-2\beta)}$
[]	[1, 1]	$\frac{1}{\sqrt{2\beta^2(\beta+1)}}\frac{(v+1-\beta+N\beta)(1+v+N\beta-2\beta)(u+v+2+N\beta-3\beta)(v+u+N\beta+2-2\beta)}{(u+v+2N\beta+2-2\beta)(2+u+v+2N\beta-3\beta)}$

Table 5. The Selberg correlators considered in [...] — with inverse p_{-k} in the second Jack polynomial — at levels $|A|+|B|=1$ and $|A|+|B|=2$. Again, u, v are the parameters of the Selberg potential, N is the number of eigenvalues, and $\beta=-\epsilon_1/\epsilon_2$ is the Van-der-Monde power. The shift w, which is essential for the correlators to be completely factorizable, equals $w=(v+1-\beta)/\beta$.

A	B	$S_{AB} = \left\langle j_A(p_k)\ j_B(p_k)\right\rangle_{\pm}^{\text{BGW}}$	At $(\epsilon_1, \epsilon_2) = (\hbar, -\hbar)$
$[3]$	$[]$	$\sqrt{\frac{\epsilon_1^2}{6\epsilon_{12}\epsilon_{122}}} \cdot \frac{-g^3}{(z-\epsilon_2-\epsilon_1)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-3\epsilon_2)}$	$\frac{\hbar^3}{6z(z+\hbar)(z+2\hbar)}$
$[2,1]$	$[]$	$\sqrt{\frac{-\epsilon_1\epsilon_2}{\epsilon_{112}\epsilon_{122}}} \cdot \frac{-g^3}{(z-\epsilon_2-\epsilon_1)(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)}$	$\frac{\hbar^3}{3z(z-\hbar)(z+\hbar)}$
$[1,1,1]$	$[]$	$\sqrt{\frac{\epsilon_2^2}{6\epsilon_{12}\epsilon_{112}}} \cdot \frac{-g^3}{(z-\epsilon_2-\epsilon_1)(z-2\epsilon_1-\epsilon_2)(z-3\epsilon_1-\epsilon_2)}$	$\frac{\hbar^3}{6z(z-\hbar)(z-2\hbar)}$
$[2]$	$[1]$	$\sqrt{\frac{\epsilon_1}{2\epsilon_{12}}} \cdot \frac{-g^3(z-2\epsilon_1-3\epsilon_2)}{(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-3\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_2-\epsilon_1)}$	$\frac{\hbar^3}{2z(z-\hbar)(z+2\hbar)}$
$[1,1]$	$[1]$	$\sqrt{\frac{-\epsilon_2}{2\epsilon_{12}}} \cdot \frac{-g^3(z-3\epsilon_1-2\epsilon_2)}{(z-\epsilon_1-2\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_2-\epsilon_1)}$	$\frac{\hbar^3}{2z(z+\hbar)(z-2\hbar)}$
$[1]$	$[2]$	$\sqrt{\frac{\epsilon_1}{2\epsilon_{12}}} \cdot \frac{-g^3(z-2\epsilon_1-3\epsilon_2)}{(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-3\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_2-\epsilon_1)}$	$\frac{\hbar^3}{2z(z-\hbar)(z+2\hbar)}$
$[1]$	$[1,1]$	$\sqrt{\frac{-\epsilon_2}{2\epsilon_{12}}} \cdot \frac{-g^3(z-3\epsilon_1-2\epsilon_2)}{(z-\epsilon_1-2\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_2-\epsilon_1)}$	$\frac{\hbar^3}{2z(z+\hbar)(z-2\hbar)}$
$[]$	$[3]$	$\sqrt{\frac{\epsilon_1^2}{6\epsilon_{12}\epsilon_{122}}} \cdot \frac{-g^3}{(z-\epsilon_2-\epsilon_1)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-3\epsilon_2)}$	$\frac{\hbar^3}{6z(z+\hbar)(z+2\hbar)}$
$[]$	$[2,1]$	$\sqrt{\frac{-\epsilon_1\epsilon_2}{\epsilon_{112}\epsilon_{122}}} \cdot \frac{-g^3}{(z-\epsilon_2-\epsilon_1)(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)}$	$\frac{\hbar^3}{3z(z-\hbar)(z+\hbar)}$
$[]$	$[1,1,1]$	$\sqrt{\frac{\epsilon_2^2}{6\epsilon_{12}\epsilon_{112}}} \cdot \frac{-g^3}{(z-\epsilon_2-\epsilon_1)(z-2\epsilon_1-\epsilon_2)(z-3\epsilon_1-\epsilon_2)}$	$\frac{\hbar^3}{6z(z-\hbar)(z-2\hbar)}$

Table 6. Correlators S_{AB} in the "+" and "-" BGW models, level 3. Here $z \equiv \pm 2a$, $g = \sqrt{-\epsilon_1\epsilon_2}$, $\epsilon_{12} = \epsilon_1 - \epsilon_2$, $\epsilon_{112} = 2\epsilon_1 - \epsilon_2$, $\epsilon_{122} = \epsilon_1 - 2\epsilon_2$.

A	B	$R_{AB} = \Big\langle j_A\big(-p_k\big)\; j_B\big(p_k\big)\Big\rangle_{\pm}^{\mathrm{BGW}}$	At $(\epsilon_1, \epsilon_2) = (\hbar, -\hbar)$
$[3]$	$[]$	$\sqrt{\dfrac{\epsilon_1^2}{6\epsilon_{12}\epsilon_{122}}} \cdot \dfrac{g^3(6\epsilon_1^2 + 23\epsilon_1\epsilon_2 - 5\epsilon_1 z + 19\epsilon_2^2 - 8\epsilon_2 z + z^2)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\dfrac{-\hbar^3}{6z(z-\hbar)(z-2\hbar)}$
$[2,1]$	$[]$	$\sqrt{\dfrac{-\epsilon_1\epsilon_2}{\epsilon_{112}\epsilon_{122}}} \cdot \dfrac{g^3(9\epsilon_1^2 - 6\epsilon_1 z + 22\epsilon_1\epsilon_2 + 9\epsilon_2^2 + z^2 - 6\epsilon_2 z)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\dfrac{-\hbar^3}{3z(z-\hbar)(z+\hbar)}$
$[1,1,1]$	$[]$	$\sqrt{\dfrac{\epsilon_2^2}{6\epsilon_{12}\epsilon_{112}}} \cdot \dfrac{g^3(19\epsilon_1^2 + 23\epsilon_1\epsilon_2 - 8\epsilon_1 z + z^2 - 5\epsilon_2 z + 6\epsilon_2^2)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\dfrac{-\hbar^3}{6z(z+\hbar)(z+2\hbar)}$
$[2]$	$[1]$	$\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{-g^3(6\epsilon_1^2 - 5\epsilon_1 z + 17\epsilon_1\epsilon_2 + 9\epsilon_2^2 + z^2 - 6\epsilon_2 z)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\dfrac{\hbar^3}{2z(z+\hbar)(z-2\hbar)}$
$[1,1]$	$[1]$	$\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{-g^3(9\epsilon_1^2 - 6\epsilon_1 z + 17\epsilon_1\epsilon_2 + 6\epsilon_2^2 - 5\epsilon_2 z + z^2)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\dfrac{\hbar^3}{2z(z-\hbar)(z+2\hbar)}$
$[1]$	$[2]$	$\sqrt{\dfrac{\epsilon_1}{2\epsilon_{12}}} \cdot \dfrac{g^3(-2\epsilon_1 - 3\epsilon_2 + z)}{(z-\epsilon_1-3\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\dfrac{-\hbar^3}{2z(z-\hbar)(z+2\hbar)}$
$[1]$	$[1,1]$	$\sqrt{\dfrac{-\epsilon_2}{2\epsilon_{12}}} \cdot \dfrac{g^3(-3\epsilon_1 - 2\epsilon_2 + z)}{(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\dfrac{-\hbar^3}{2z(z+\hbar)(z-2\hbar)}$
$[]$	$[3]$	$\sqrt{\dfrac{\epsilon_1^2}{6\epsilon_{12}\epsilon_{122}}} \cdot \dfrac{-g^3}{(z-\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-3\epsilon_2)}$	$\dfrac{\hbar^3}{6z(z+\hbar)(z+2\hbar)}$
$[]$	$[2,1]$	$\sqrt{\dfrac{-\epsilon_1\epsilon_2}{\epsilon_{112}\epsilon_{122}}} \cdot \dfrac{-g^3}{(z-\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-2\epsilon_1-\epsilon_2)}$	$\dfrac{\hbar^3}{3z(z+\hbar)(z-\hbar)}$
$[]$	$[1,1,1]$	$\sqrt{\dfrac{\epsilon_2^2}{6\epsilon_{12}\epsilon_{112}}} \cdot \dfrac{-g^3}{(z-\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-3\epsilon_1-\epsilon_2)}$	$\dfrac{\hbar^3}{6z(z-\hbar)(z-2\hbar)}$

Table 7. Correlators R_{AB} in the "+" and "-" BGW models, level 3. Here $z \equiv \pm 2a$, $g = \sqrt{-\epsilon_1\epsilon_2}$, $\epsilon_{12} = \epsilon_1 - \epsilon_2$, $\epsilon_{112} = 2\epsilon_1 - \epsilon_2$, $\epsilon_{122} = \epsilon_1 - 2\epsilon_2$.

A	B	$Q_{AB} = \left\langle j_A\big(-p_k\big)\ j_B\big(-p_k\big)\right\rangle_{\pm}^{\mathrm{BGW}}$	At $(\epsilon_1,\epsilon_2)=(\hbar,-\hbar)$
[3]	[]	$\sqrt{\frac{\epsilon_1^2}{6\epsilon_{12}\epsilon_{122}}}\cdot\frac{g^3(6\epsilon_1^2+23\epsilon_1\epsilon_2-5\epsilon_1 z-8\epsilon_2 z+19\epsilon_2^2+z^2)}{(z-\epsilon_1-3\epsilon_2)(-\epsilon_2+\epsilon_1)(z-3\epsilon_1-\epsilon_2)(-2\epsilon_2+\epsilon_1)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{6z(z-\hbar)(z-2\hbar)}$
[2, 1]	[]	$\sqrt{\frac{-\epsilon_1\epsilon_2}{\epsilon_{112}\epsilon_{122}}}\cdot\frac{g^3(9\epsilon_1^2+22\epsilon_1\epsilon_2-6\epsilon_1 z+9\epsilon_2^2+z^2-6\epsilon_2 z)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(-\epsilon_2+2\epsilon_1)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{3z(z+\hbar)(z-\hbar)}$
[1,1,1]	[]	$\sqrt{\frac{\epsilon_2^2}{6\epsilon_{12}\epsilon_{112}}}\cdot\frac{g^3(19\epsilon_1^2-8\epsilon_1 z+23\epsilon_1\epsilon_2+z^2+6\epsilon_2^2-5\epsilon_2 z)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{6z(z+\hbar)(z+2\hbar)}$
[2]	[1]	$\sqrt{\frac{\epsilon_1}{2\epsilon_{12}}}\cdot\frac{g^3(6\epsilon_1^2+17\epsilon_1\epsilon_2-5\epsilon_1 z+z^2-6\epsilon_2 z+9\epsilon_2^2)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(-\epsilon_2+\epsilon_1)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{2z(z+\hbar)(z-2\hbar)}$
[1, 1]	[1]	$\sqrt{\frac{-\epsilon_2}{2\epsilon_{12}}}\cdot\frac{g^3(9\epsilon_1^2+17\epsilon_1\epsilon_2-6\epsilon_1 z-5\epsilon_2 z+z^2+6\epsilon_2^2)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{2z(z-\hbar)(z+2\hbar)}$
[1]	[2]	$\sqrt{\frac{\epsilon_1}{2\epsilon_{12}}}\cdot\frac{g^3(6\epsilon_1^2-5\epsilon_1 z+17\epsilon_1\epsilon_2+9\epsilon_2^2+z^2-6\epsilon_2 z)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(-\epsilon_2+\epsilon_1)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{2z(z+\hbar)(z-2\hbar)}$
[1]	[1, 1]	$\sqrt{\frac{-\epsilon_2}{2\epsilon_{12}}}\cdot\frac{g^3(9\epsilon_1^2-6z\epsilon_1+17\epsilon_1\epsilon_2+6\epsilon_2^2-5z\epsilon_2+z^2)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{2z(z-\hbar)(z+2\hbar)}$
[]	[3]	$\sqrt{\frac{\epsilon_1^2}{6\epsilon_{12}\epsilon_{122}}}\cdot\frac{g^3(6\epsilon_1^2+23\epsilon_1\epsilon_2-5\epsilon_1 z+z^2-8\epsilon_2 z+19\epsilon_2^2)}{(z-\epsilon_1-3\epsilon_2)(-\epsilon_2+\epsilon_1)(z-3\epsilon_1-\epsilon_2)(-2\epsilon_2+\epsilon_1)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{6z(z-\hbar)(z-2\hbar)}$
[]	[2, 1]	$\sqrt{\frac{-\epsilon_1\epsilon_2}{\epsilon_{112}\epsilon_{122}}}\cdot\frac{g^3(9\epsilon_1^2+22\epsilon_1\epsilon_2-6\epsilon_1 z+9\epsilon_2^2+z^2-6z\epsilon_2)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(-\epsilon_2+2\epsilon_1)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{3z(z+\hbar)(z-\hbar)}$
[]	[1,1,1]	$\sqrt{\frac{\epsilon_2^2}{6\epsilon_{12}\epsilon_{112}}}\cdot\frac{g^3(19\epsilon_1^2+23\epsilon_1\epsilon_2-8z\epsilon_1-5z\epsilon_2+z^2+6\epsilon_2^2)}{(z-\epsilon_1-3\epsilon_2)(z-3\epsilon_1-\epsilon_2)(z-2\epsilon_1-\epsilon_2)(z-\epsilon_1-2\epsilon_2)(z-\epsilon_1-\epsilon_2)}$	$\frac{-\hbar^3}{6z(z+\hbar)(z+2\hbar)}$

Table 8. Correlators Q_{AB} in the "+" and "-" BGW models, level 3. Here $z\equiv\pm 2a$, $g=\sqrt{-\epsilon_1\epsilon_2}$, $\epsilon_{12}=\epsilon_1-\epsilon_2$, $\epsilon_{112}=2\epsilon_1-\epsilon_2$, $\epsilon_{122}=\epsilon_1-2\epsilon_2$.