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Research Article

Superstability of Some Pexider-Type Functional Equation

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We will investigate the superstability of the sine functional equation from the following Pexider-type functional equation $f(x+y)+g(x-y) = \lambda\cdot h(x)k(y)$ (λ : constant), which can be considered the mixed functional equation of the sine and cosine functions, the mixed functional equation of the hyperbolic sine and hyperbolic cosine functions, and the exponential-type functional equations.

1. Introduction

In 1940, Ulam [1] conjectured the stability problem. Next year, this problem was affirmatively solved by Hyers [2], which is through the following.

Let X and Y be Banach spaces with norm $\|\cdot\|$, respectively. If $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon, \quad \forall x, y \in X, \tag{1.1}$$

then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \varepsilon, \quad \forall x \in X. \tag{1.2}$$

The above result was generalized by Bourgin [3] and Aoki [4] in 1949 and 1950. In 1978 and 1982, Hyers' result was improved by Th. M. Rassias [5] and J. M. Rassias [6] which is that the condition bounded by the constant is replaced to the condition bounded by two variables, and thereafter it was improved moreover by Găvruţa [7] to the condition bounded by the function.

In 1979, Baker et al. [8] showed that if f is a function from a vector space to R satisfying

$$|f(x+y) - f(x)f(y)| \leq \varepsilon, \tag{1.3}$$

then either f is bounded or satisfies the exponential functional equation

$$f(x+y) = f(x)f(y). \tag{1.4}$$

This method is referred to as the superstability of the functional equation (1.4).

In this paper, let $(G,+)$ be a uniquely 2 divisible Abelian group, $\mathbb{C}$ the field of complex numbers, and $\mathbb{R}$ the field of real numbers, $\mathbb{R}_+$ the set of positive reals. Whenever we only deal with (C), $(G,+)$ needs the Abelian which is not 2-divisible.

We may assume that f, g, h and k are nonzero functions, λ, ε is a nonnegative real constant, and $\varphi : G \to \mathbb{R}_+$ is a mapping.

In 1980, the superstability of the cosine functional equation (also referred the d'Alembert functional equation)

$$f(x+y) + f(x-y) = 2f(x)f(y), \tag{C}$$

was investigated by Baker [9] with the following result: let $\varepsilon > 0$. If $f : G \to C$ satisfies

$$|f(x+y) + f(x-y) - 2f(x)f(y)| \leq \varepsilon, \tag{1.5}$$

then either $|f(x)| \leq (1+\sqrt{1+2\varepsilon})/2$ for all $x \in G$ or f is a solution of (C). Badora [10] in 1998, and Badora and Ger [11] in 2002 under the condition $|f(x+y)+f(x-y)-2f(x)f(y)| \leq \varepsilon,\ \varphi(x)$ or $\varphi(y)$, respectively. Also the stability of the d'Alembert functional equation is founded in papers [12–16].

In the present work, the stability question regarding a Pexider-type trigonometric functional equation as a generalization of the cosine equation (C) is investigated.

To be systematic, we first list all functional equations that are of interest here.

$$f(x+y) + g(x-y) = \lambda h(x)h(y) \tag{P^{λ}_{fghh}}$$

$$f(x+y) + g(x-y) = \lambda f(x)h(y), \tag{P^{λ}_{fgfh}}$$

$$f(x+y) + g(x-y) = \lambda h(x)f(y), \tag{P^{λ}_{fghf}}$$

$$f(x+y)+g(x-y)=\lambda g(x)h(y), \tag{P^{λ}_{fggh}}$$

$$f(x+y)+g(x-y)=\lambda h(x)g(y), \tag{P^{λ}_{fghg}}$$

$$f(x+y)+g(x-y)=\lambda f(x)g(y), \tag{P^{λ}_{fgfg}}$$

$$f(x+y)+g(x-y)=\lambda g(x)f(y), \tag{P^{λ}_{fggf}}$$

$$f(x+y)+g(x-y)=\lambda f(x)f(y), \tag{P^{λ}_{fgff}}$$

$$f(x+y)+g(x-y)=\lambda g(x)g(y), \tag{P^{λ}_{fggg}}$$

$$f(x+y)+f(x-y)=\lambda g(x)h(y), \tag{P^{λ}_{ffgh}}$$

$$f(x+y)+f(x-y)=\lambda g(x)g(y), \tag{P^{λ}_{ffgg}}$$

$$f(x+y)+f(x-y)=\lambda f(x)g(y), \tag{C^{λ}_{fg}}$$

$$f(x+y)+f(x-y)=\lambda g(x)f(y), \tag{C^{λ}_{gf}}$$

$$f(x+y)+f(x-y)=\lambda f(x)f(y), \tag{C^{λ}}$$

$$f(x+y)+g(x-y)=2h(x)k(y), \tag{P_{fghk}}$$

$$f(x+y)+g(x-y)=2h(x)h(y), \tag{P_{fghh}}$$

$$f(x+y)+g(x-y)=2f(x)h(y), \tag{P_{fgfh}}$$

$$f(x+y)+g(x-y)=2h(x)f(y), \tag{P_{fghf}}$$

$$f(x+y)+g(x-y)=2g(x)h(y), \tag{P_{fggh}}$$

$$f(x+y)+g(x-y)=2h(x)g(y), \tag{P_{fghg}}$$

$$f(x+y)+g(x-y)=2f(x)g(y), \tag{P_{fgfg}}$$

$$f(x+y)+g(x-y)=2g(x)f(y), \tag{P_{fggf}}$$

$$f(x+y)+g(x-y)=2f(x)f(y), \tag{P_{fgff}}$$

$$f(x+y)+g(x-y)=2g(x)g(y), \tag{P_{fggg}}$$

$$f(x+y)+f(x-y)=2f(x)g(y), \tag{C_{fg}}$$

$$f(x+y)+f(x-y)=2g(x)f(y), \tag{C_{gf}}$$

$$f(x+y)+f(x-y)=2g(x)g(y), \tag{C_{gg}}$$

$$f(x+y)+f(x-y)=2g(x)h(y), \tag{C_{gh}}$$

$$f(x+y)+f(x-y)=2f(x). \tag{J_x}$$

The hyperbolic cosine function, hyperbolic sine function, hyperbolic trigonometric function, some exponential functions, and Jensen equation satisfy the above mentioned equations; therefore, they can also be called the *hyperbolic* cosine (sine, trigonometric) functional equation, exponential functional equation, and Jensen equation, respectively.

For example,

$$\begin{gathered}
\cosh(x+y)+\cosh(x-y)=2\cosh(x)\cosh(y),\\
\cosh(x+y)-\cosh(x-y)=2\sinh(x)\sinh(y),\\
\sinh(x+y)+\sinh(x-y)=2\sinh(x)\cosh(y),\\
\sinh(x+y)-\sinh(x-y)=2\cosh(x)\sinh(y),\\
\sinh^2\left(\frac{x+y}{2}\right)-\sinh^2\left(\frac{x-y}{2}\right)=\sinh(x)\sinh(y),\\
ca^{x+y}+ca^{x-y}=2\frac{ca^x}{2}\left(a^y+a^{-y}\right)=2ce^x\frac{a^y+a^{-y}}{2},\\
e^{x+y}+e^{x-y}=2\frac{e^x}{2}\left(e^y+e^{-y}\right)=2e^x\cosh(y),\\
(n(x+y)+c)+(n(x-y)+c)=2(nx+c):\text{for } f(x)=nx+c,
\end{gathered}\tag{1.6}$$

where a and c are constants.

The equation (C_{fg}) is referred to as the Wilson equation. In 2001, Kim and Kannappan [13] investigated the superstability related to the d'Alembert (C) and the Wilson functional equations (C_{fg}), (C_{gf}) under the condition bounded by constant. Kim has also improved the superstability of the generalized cosine type-functional equations (C_{gg}), and (P_{fgfg}), (P_{fggf}) in papers [14, 15, 17].

In particular, author Kim and Lee [18] investigated the superstability of (S) from the functional equation (C_{gh}) under the condition bounded by function, that is

(1) if $f, g, h : G \to C$ satisfies

$$\left|f(x+y)+f(x-y)-2g(x)h(y)\right|\le\varphi(x),\tag{1.7}$$

then either h is bounded or g satisfies (S);

(2) if $f, g, h : G \to C$ satisfies

$$\left|f(x+y)+f(x-y)-2g(x)h(y)\right|\le\varphi(y),\tag{1.8}$$

then either g is bounded or h satisfies (S).

In 1983, Cholewa [19] investigated the superstability of the sine functional equation

$$f(x)f(y)=f\left(\frac{x+y}{2}\right)^2-f\left(\frac{x-y}{2}\right)^2,\tag{S}$$

under the condition bounded by constant. Namely, if an unbounded function $f : G \to C$ satisfies

$$\left| f(x)f(y) - f\left(\frac{x+y}{2}\right)^2 + f\left(\frac{x-y}{2}\right)^2 \right| \le \varepsilon, \tag{1.9}$$

then it satisfies (S).

In Kim's work [20, 21], the superstability of sine functional equation from the generalized sine functional equations

$$f(x)g(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2, \tag{S_{fg}}$$

$$g(x)f(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2, \tag{S_{gf}}$$

$$g(x)h(y) = f\left(\frac{x+y}{2}\right)^2 - f\left(\frac{x-y}{2}\right)^2 \tag{S_{gh}}$$

was treated under the conditions bounded by constant and functions.

The aim of this paper is to investigate the transferred superstability for the sine functional equation from the following Pexider type functional equations:

$$f(x+y) + g(x-y) = \lambda \cdot h(x)k(y), \quad \lambda : \text{constant} \tag{P^{λ}_{fghk}}$$

on the abelian group. Furthermore, the obtained results can be extended to the Banach space.

Consequently, as corollaries, we can obtain 29×4 stability results concerned with the sine functional equation (S) and the Wilson-type equations (C^{λ}_{fg}) from 29 functional equations of the $P^{\lambda}, C^{\lambda}, P$, and C types from a selection of functions f, g, h, k in the order of variables $x+y, x-y, x, y$.

2. Superstability of the Sine Functional Equation from the Equation (P^{λ}_{fghk})

In this section, we will investigate the superstability related to the d'Alembert-type equation (C^{λ}) and Wilson-type equation (C^{λ}_{fg}), of the sine functional equation (S) from the Pexider type functional equation (P^{λ}_{fghk}).

Theorem 2.1. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda \cdot h(x)k(y)| \le \varphi(x), \quad \forall x, y \in G. \tag{2.1}$$

If k fails to be bounded, then

(i) h *satisfies* (S) *under one of the cases* $h(0) = 0$ *or* $f(-x) = -g(x)$*; and*

(ii) *In addition, if* k *satisfies* (C^{λ})*, then* h *and* k *are solutions of* $(C^{\lambda}_{fg}) := h(x+y) + h(x-y) = \lambda h(x)k(y)$.

Proof. Let k be unbounded solution of the inequality (3.12). Then, there exists a sequence $\{y_n\}$ in G such that $0 \neq |k(y_n)| \to \infty$ as $n \to \infty$.

(i) Taking $y = y_n$ in the inequality (3.12), dividing both sides by $|\lambda k(y_n)|$, and passing to the limit as $n \to \infty$, we obtain

$$h(x) = \lim_{n\to\infty} \frac{f(x+y_n) + g(x-y_n)}{\lambda \cdot k(y_n)}, \quad x \in G. \tag{2.2}$$

Replace y by $y + y_n$ and $-y + y_n$ in (3.12), we have

$$\begin{aligned} &\big| f(x + (y + y_n)) + g(x - (y + y_n)) - \lambda \cdot h(x)k(y + y_n) \\ &f(x + (-y + y_n)) + g(x - (-y + y_n)) - \lambda \cdot h(x)k(-y + y_n) \big| \le 2\varphi(x) \end{aligned} \tag{2.3}$$

so that

$$\begin{aligned} &\Bigg| \frac{f((x+y) + y_n) + g((x+y) - y_n)}{\lambda \cdot k(y_n)} \\ &+ \frac{f((x-y) + y_n) + g((x-y) - y_n)}{\lambda \cdot k(y_n)} - \lambda \cdot h(x) \cdot \frac{k(y+y_n) + k(-y+y_n)}{\lambda \cdot k(y_n)} \Bigg| \le \frac{2\varphi(x)}{\lambda \cdot |k(y_n)|} \end{aligned} \tag{2.4}$$

for all $x, y, y_n \in G$.

We conclude that, for every $y \in G$, there exists a limit function

$$l_k(y) := \lim_{n\to\infty} \frac{k(y+y_n) + k(-y+y_n)}{\lambda \cdot k(y_n)}, \tag{2.5}$$

where the function $l_k : G \to \mathbb{C}$ satisfies the equation

$$h(x+y) + h(x-y) = \lambda \cdot h(x) l_k(y), \quad \forall x, y \in G. \tag{2.6}$$

Applying the case $h(0) = 0$ in (2.6), it implies that h is odd. Keeping this in mind, by means of (2.6), we infer the equality

$$\begin{aligned} h(x+y)^2 - h(x-y)^2 &= \lambda \cdot h(x) l_k(y) [h(x+y) - h(x-y)] \\ &= h(x)[h(x+2y) - h(x-2y)] \\ &= h(x)[h(2y+x) + h(2y-x)] \\ &= \lambda \cdot h(x) h(2y) l_k(x). \end{aligned} \tag{2.7}$$

Putting $y = x$ in (2.6), we get the equation

$$h(2x) = \lambda \cdot h(x) l_k(x), \quad x \in G. \tag{2.8}$$

This, in return, leads to the equation

$$h(x+y)^2 - h(x-y)^2 = h(2x)h(2y) \tag{2.9}$$

valid for all $x, y \in G$ which, in the light of the unique 2divisibility of G, states nothing else but (S).

In the particular case $f(-x) = -g(x)$, it is enough to show that $h(0) = 0$. Suppose that this is not the case.

Putting $x = 0$ in (3.12), due to $h(0) \neq 0$ and $f(-x) = -g(x)$, we obtain the inequality

$$|k(y)| \leq \frac{\varphi(0)}{\lambda \cdot |h(0)|}, \quad y \in G. \tag{2.10}$$

This inequality means that k is globally bounded, which is a contradiction. Thus, since the claimed $h(0) = 0$ holds, we know that h satisfies (S).

(ii) In the case k satisfies (C^{λ}), the limit l_k states nothing else but k, so, from (2.6), h and k validate (C^{λ}_{fg}). □

Theorem 2.2. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda \cdot h(x)k(y)| \leq \varphi(y) \quad \forall x, y \in G. \tag{2.11}$$

If h fails to be bounded, then

(i) *k satisfies (S) under one of the cases $k(0) = 0$ or $f(x) = -g(x)$*

(ii) *in addition, if h satisfies (C^{λ}), then k and h are solutions of the equation of $(C^{\lambda}_{gf}) := k(x+y) + k(x-y) = \lambda h(x)k(y)$.*

Proof. (i) Taking $x = x_n$ in the inequality (2.11), dividing both sides by $|\lambda \cdot h(x_n)|$, and passing to the limit as $n \to \infty$, we obtain that

$$k(y) = \lim_{n \to \infty} \frac{f(x_n + y) + g(x_n - y)}{\lambda \cdot h(x_n)}, \quad x \in G. \tag{2.12}$$

Replace x by $x_n + x$ and $x_n - x$ in (2.11) divide by $\lambda \cdot h(x_n)$; then it gives us the existence of the limit function

$$l_h(x) := \lim_{n \to \infty} \frac{h(x_n + x) + h(x_n - x)}{\lambda \cdot h(x_n)}, \tag{2.13}$$

where the function $l_h : G \to \mathbb{C}$ satisfies the equation

$$k(x+y) + k(-x+y) = \lambda \cdot l_h(x) k(y), \quad \forall x, y \in G. \tag{2.14}$$

Applying the case $k(0) = 0$ in (2.14), it implies that k is odd.

A similar procedure to that applied after (2.6) of Theorem 2.1 in (2.14) allows us to show that k satisfies (S).

The case $f(x) = -g(x)$ is also the same as the reason for Theorem 2.1.

(ii) In the case h satisfies (C^{λ}), the limit l_h states nothing else but h, so, from (2.14), k and h validate (C^{λ}_{fg}). □

The following corollaries followly immediate from the Theorems 2.1 and 2.2.

Corollary 2.3. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$\left|f(x+y) + g(x-y) - \lambda \cdot h(x)k(y)\right| \le \min\{\phi(x), \phi(y)\}, \quad \forall x, y \in G. \tag{2.15}$$

(a) *If k fails to be bounded, then*

(i) *h satisfies (S) under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$, and*

(ii) *in addition, if h satisfies (C^{λ}), then h and k are solutions of $(C^{\lambda}_{fg}) := h(x+y) + h(x-y) = \lambda h(x)k(y)$.*

(b) *If h fails to be bounded, then*

(iii) *k satisfies (S) under one of the cases $k(0) = 0$ or $f(x) = -g(x)$, and*

(iv) *in addition, if h satisfies (C^{λ}), then h and k are solutions of $(C^{\lambda}_{gf}) := k(x+y) + k(x-y) = \lambda h(x)k(y)$.*

Corollary 2.4. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$\left|f(x+y) + g(x-y) - \lambda \cdot h(x)k(y)\right| \le \varepsilon, \quad \forall x, y \in G. \tag{2.16}$$

(a) *If k fails to be bounded, then*

(i) *h satisfies (S) under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$, and*

(ii) *in addition, if k satisfies (C^{λ}), then h and k are solutions of $(C^{\lambda}_{fg}) := h(x+y) + h(x-y) = \lambda h(x)k(y)$.*

(b) *If h fails to be bounded, then*

(iii) *k satisfies (S) under one of the cases $k(0) = 0$ or $f(x) = -g(x)$, and*

(iv) *in addition, if h satisfies (C^{λ}), then h and k are solutions of $(C^{\lambda}_{gf}) := k(x+y) + k(x-y) = \lambda h(x)k(y)$.*

3. Applications in the Reduced Equations

3.1. Corollaries of the Equations Reduced to Three Unknown Functions

Replacing k by one of the functions f, g, h in all the results of the Section 2 and exchanging each functions f, g, h in the above equations, we then obtain P^{λ}, C^{λ} types 14 equations.

We will only illustrate the results for the cases of (P^{λ}_{fghh}), (P^{λ}_{fgfh}) in the obtained equations. The other cases are similar to these; thus their illustrations will be omitted.

Corollary 3.1. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda \cdot h(x)h(y)| \leq \begin{cases} \varphi(x) & \text{or} \\ \varphi(y) & \text{or} \\ \min\{\varphi(x), \varphi(y)\} & \text{or} \\ \varepsilon \end{cases} \quad \forall x, y \in G. \tag{3.1}$$

If h fails to be bounded, then, under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$, h satisfies (S).

Corollary 3.2. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda \cdot f(x)h(y)| \leq \varphi(x), \quad \forall x, y \in G. \tag{3.2}$$

If h fails to be bounded, then

(i) *f satisfies (S) under one of the cases $f(0) = 0$ or $f(-x) = -g(x)$, and*

(ii) *in addition, if h satisfies (C^{λ}), then f and h are solutions of $(C_{fg}) := f(x+y) + f(x-y) = \lambda \cdot f(x)h(y)$.*

Corollary 3.3. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda \cdot f(x)h(y)| \leq \varphi(y), \quad \forall x, y \in G. \tag{3.3}$$

If f fails to be bounded, then

(i) *h satisfies (S) under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$, and*

(ii) *in addition, if f satisfies (C^{λ}), then h and f are solutions of $(C^{\lambda}_{gf}) := h(x+y) + h(x-y) = \lambda \cdot f(x)h(y)$.*

Corollary 3.4. *Suppose that $f, g, h : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - \lambda \cdot f(x)h(y)| \leq \min\{\varphi(x), \varphi(y)\}, \quad \forall x, y \in G. \tag{3.4}$$

(a) *If h fails to be bounded, then*

(i) f *satisfies* (S) *under one of the cases* $f(0)=0$ *or* $f(-x)=-g(x)$*, and*

(ii) *in addition, if* h *satisfies* (C^{λ})*, then* f *and* h *are solutions of* $(C_{fg}){:=}\ f(x+y)+f(x-y)=\lambda\cdot f(x)h(y)$.

(b) *If f fails to be bounded, then*

(i) h *satisfies* (S) *under one of the cases* $h(0)=0$ *or* $f(-x)=-g(x)$*, and*

(ii) *in addition, if* f *satisfies* (C^{λ})*, then* h *and* f *are solutions of* $(C^{\lambda}_{gf}) := h(x+y)+h(x-y)=\lambda\cdot f(x)h(y)$.

Corollary 3.5. *Suppose that* $f,g,h : G\to\mathbb{C}$ *satisfy the inequality*

$$\left|f(x+y)+g(x-y)-\lambda\cdot f(x)h(y)\right|\le\varepsilon,\quad\forall x,y\in G. \tag{3.5}$$

(a) *If h fails to be bounded, then*

(i) f *satisfies* (S) *under one of the cases* $f(0)=0$ *or* $f(-x)=-g(x)$*, and*

(ii) *in addition, if* h *satisfies* (C^{λ})*, then* f *and* h *are solutions of* $(C_{fg}) := f(x+y)+f(x-y)=\lambda\cdot f(x)h(y)$.

(b) *If f fails to be bounded, then*

(i) h *satisfies* (S) *under one of the cases* $h(0)=0$ *or* $f(-x)=-g(x)$*, and*

(ii) *in addition, if* f *satisfies* (C^{λ})*, then* h *and* f *are solutions of* $(C^{\lambda}_{gf}) := h(x+y)+h(x-y)=\lambda\cdot f(x)h(y)$.

Remark 3.6. As the above corollaries, we obtain the stability results of $12\times 4(\varphi(x), \varphi(y), \min\{\varphi(x),\varphi(y)\},\varepsilon)$ numbers for 12 equations by choosing f,g,h, and λ, namely, which are the following: (P^{λ}_{fghf}), (P^{λ}_{fggh}), (P^{λ}_{fghg}), (P^{λ}_{fgfg}), (P^{λ}_{fggf}), (P^{λ}_{fgff}), (P^{λ}_{fggg}), (P^{λ}_{ffgh}), (P^{λ}_{ffgg}), (C^{λ}_{fg}), (C^{λ}_{gf}), and (C^{λ}).

3.2. Applications of the Case $\lambda=2$ in (P^{λ}_{fghk})

Let us apply the case $\lambda=2$ in (P^{λ}_{fghk}) and all P^{λ}-type equations considered in the Sections 2 and Sec 3.1. Then, we obtain the P-type equations

$$f(x+y)+g(x-y)=2\cdot h(x)k(y), \tag{P_{fghk}}$$

and (P^{λ}_{fghh}), (P^{λ}_{fgfh}), (P^{λ}_{fghf}), (P^{λ}_{fggh}), (P^{λ}_{fghg}), (P^{λ}_{fgfg}), (P^{λ}_{fggf}), (P^{λ}_{fgff}), (P^{λ}_{fggg}), and C- and J-type (C_{fg}), (C_{gf}), (C_{gg}), (C_{gh}), (C), and (J_x), which are concerned with the (hyperbolic) cosine, sine, exponential functions, and Jensen equation.

In papers (Aczél [22], Aczél and Dhombres [23], Kannappan [24, 25], and Kim and Kannappan [13]), we can find that the Wilson equation and the sine equations can be represented by the composition of a homomorphism. By applying these results, we also obtain, additionally, the explicit solutions of the considered functional equations.

Corollary 3.7. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - 2h(x)k(y)| \le \varphi(x) \quad \forall x, y \in G. \tag{3.6}$$

If k fails to be bounded, then

(i) *h satisfies (S) under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$, and h is of the form*

$$h(x) = A(x) \quad or \quad h(x) = c(E(x) - E^*(x)), \tag{3.7}$$

where $A : G \to C$ is an additive function, $c \in \mathbb{C}$, $E : G \to \mathbb{C}^*$ is a homomorphism and $E^* = 1/E(x)$,

(ii) *in addition, if k satisfies (C), then h and k are solutions of (C_{fg}) and h, k are given by*

$$k(x) = \frac{E(x) + E^*(x)}{2}, \qquad h(x) = c(E(x) - E^*(x)) + \frac{d(E(x) + E^*(x))}{2}, \tag{3.8}$$

where $c, d \in \mathbb{C}$, E and E^* are as in (i).

Proof. The proof of the Corollary is enough from Theorem 2.1 except for the solution. However, they are immediate from the following:

(i) appealing to the solutions of (S) in [2, page 153] (see also [24, 25]), the explicit shapes of h are as stated in the statement of the theorem. This completes the proof of the Corollary,

(ii) the given explicit solutions are taken from [24, 25] (page 148) (see also [22, 23]). □

Corollary 3.8. *Suppose that $f, g, h, k : G \to \mathbb{C}$ satisfy the inequality*

$$|f(x+y) + g(x-y) - 2h(x)k(y)| \le \varphi(y) \quad \forall x, y \in G. \tag{3.9}$$

If h fails to be bounded, then

(i) *k satisfies (S) under one of the cases $h(0) = 0$ or $f(-x) = -g(x)$, and k is of the form*

$$k(x) = A(x) \quad or \quad k(x) = c(E(x) - E^*(x)), \tag{3.10}$$

where $A : G \to C$ is an additive function, $c \in \mathbb{C}$, $E : G \to \mathbb{C}^*$ is a homomorphism and $E^* = 1/E(x)$, and

(ii) *in addition, if* h *satisfies* (C)*, then* k *and* h *are solutions of* (C_{fg}) *and* k, h *are given by*

$$h(x) = \frac{E(x) + E^*(x)}{2}, \qquad k(x) = c(E(x) - E^*(x)) + \frac{d(E(x) + E^*(x))}{2}, \tag{3.11}$$

where $c, d \in \mathbb{C}$*,* E *and* E^* *are as in (i).*

Corollary 3.9. *Suppose that* $f, g, h, k : G \to \mathbb{C}$ *satisfy the inequality*

$$|f(x + y) + g(x - y) - 2h(x)k(y)| \le \begin{cases} \min\{\varphi(x), \varphi(y)\} & \text{or} \\ \varepsilon \end{cases} \quad \forall x, y \in G. \tag{3.12}$$

(a) *If* k *fails to be bounded, then*

(i) h *satisfies* (S) *under one of the cases* $h(0) = 0$ *or* $f(-x) = -g(x)$*, and* h *is of the formh*$(x) = A(x)$ *or* $h(x) = c(E(x) - E^*(x))$*, and where* $A : G \to C$ *is an additive function,* $c \in \mathbb{C}$*,* $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^* = 1/E(x)$*.*

(ii) *in addition, if* k *satisfies* (C)*, then* h *and* k *are solutions of* (C_{fg}) *and* h, k *are given by*

$$k(x) = \frac{E(x) + E^*(x)}{2}, \qquad h(x) = c(E(x) - E^*(x)) + \frac{d(E(x) + E^*(x))}{2}, \tag{3.13}$$

where $c, d \in \mathbb{C}$*,* E *and* E^* *are as in (i).*

(b) *If* h *fails to be bounded, then*

(i) k *satisfies* (S) *under one of the cases* $h(0) = 0$ *or* $f(-x) = -g(x)$*, and* k *is of the form*

$$k(x) = A(x) \quad \text{or} \quad k(x) = c(E(x) - E^*(x)), \tag{3.14}$$

where $A : G \to C$ *is an additive function,* $c \in \mathbb{C}$*,* $E : G \to \mathbb{C}^*$ *is a homomorphism and* $E^* = 1/E(x)$*, and*

(ii) *in addition, if* h *satisfies* (C)*, then* k *and* h *are solutions of* (C_{fg}) *and* k, h *are given by*

$$h(x) = \frac{E(x) + E^*(x)}{2}, \qquad k(x) = c(E(x) - E^*(x)) + \frac{d(E(x) + E^*(x))}{2}, \tag{3.15}$$

where $c, d \in \mathbb{C}$*,* E *and* E^* *are as in (i).*

Remark 3.10. Applying the case $\lambda = 2$ in the first paragraph of the Section 3.2 implies the above 15 equations. Therefore, as the above Corollaries 3.7, 3.8, and 3.9, we can obtain additionally the stability results of $14 \times 4(\varphi(x), \varphi(y), \min\{\varphi(x), \varphi(y)\}, \varepsilon)$ numbers for the other 14 equations. Some, which excepted the explicit solutions represented by composition of a homomorphism, in the obtained results are found in papers [7, 11, 13–15, 17].

4. Extension to the Banach Space

In all the results presented in Sections 2 and 3, the range of functions on the Abelian group can be extended to the semisimple commutative Banach space. We will represent just for the main equation (P^{λ}_{fghk}).

Theorem 4.1. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach space. Assume that $f, g, h, k : G \to E$ satisfy one of each inequalities*

$$\|f(x+y)+g(x-y)-\lambda\cdot h(x)k(y)\| \le \varphi(x), \tag{4.1}$$

$$\|f(x+y)+g(x-y)-\lambda\cdot h(x)k(y)\| \le \varphi(y) \tag{4.2}$$

for all $x, y \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$.

(a) case (4.1)

Suppose that $x^* \circ k$ fails to be bounded, then

(i) *h satisfies (S) under one of the cases $x^* \circ h(0) = 0$ or $x^* \circ f(-x) = -x^* \circ g(x)$, and*

(ii) *in addition, if k satisfies (C^{λ}), then h and k are solutions of (C^{λ}_{fg}).*

(b) Case (4.2)

Suppose that $x^* \circ h$ fails to be bounded, then

(iii) *k satisfies (S) under one of the cases $x^* \circ k(0) = 0$ or $x^* \circ f(-x) = -x^* \circ g(x)$, and*

(iv) *in addition, if $x^* \circ h$ satisfies (C^{λ}), then h and k are solutions of (C^{λ}_{fg}).*

Proof. For (i) of (a), assume that (4.1) holds and arbitrarily fixes a linear multiplicative functional $x^* \in E^*$. As is well known, we have $\|x^*\| = 1$; hence, for every $x, y \in G$, we have

$$\begin{aligned}\varphi(x) &\ge \|f(x+y)+g(x-y)-\lambda\cdot h(x)k(y)\| \\ &= \sup_{\|y^*\|=1} \left|y^*(f(x+y)+g(x-y)-\lambda\cdot h(x)k(y))\right| \\ &\ge \left|x^*(f(x+y))+x^*(g(x-y))-\lambda\cdot x^*(h(x))x^*(k(y))\right|,\end{aligned} \tag{4.3}$$

which states that the superpositions $x^*\circ f$, $x^*\circ g$, $x^*\circ h$, and $x^*\circ k$ yield a solution of inequality (2.1) in Theorem 2.1. Since, by assumption, the superposition $x^* \circ k$ with $x^* \circ h(0) = 0$ is unbounded, an appeal to Theorem 2.1 shows that the two results hold.

First, the superposition $x^* \circ h$ solves (S), that is

$$(x^*\circ h)\left(\frac{x+y}{2}\right)^2 - (x^*\circ h)\left(\frac{x-y}{2}\right)^2 = (x^*\circ h)(x)(x^*\circ h)(y). \tag{4.4}$$

Since x^* is a linear multiplicative functional, we get

$$x^*\left(h\left(\frac{x+y}{2}\right)^2 - h\left(\frac{x-y}{2}\right)^2 - h(x)h(y)\right) = 0. \tag{4.5}$$

Hence an unrestricted choice of x^* implies that

$$h\left(\frac{x+y}{2}\right)^2 - h\left(\frac{x-y}{2}\right)^2 - h(x)h(y) \in \bigcap\{\ker x^* : x^* \in E^*\}. \tag{4.6}$$

Since the space E is semisimple, $\bigcap\{\ker x^* : x^* \in E^*\} = 0$, which means that h satisfies the claimed equation (S).

For second case $(x^* \circ f)(-x) = -(x^* \circ g)(x)$, it is enough to show that $(x^* \circ h)(0) = 0$, which can be easily check as Theorem 2.1. Hence, the proof (i) of (a) is completed.

For (ii) of (a), as (i) of (a), an appeal to Theorem 2.1 shows that if $x^* \circ k$ satisfies (C^λ), then $x^* \circ h$ and $x^* \circ k$ are solutions of the Wilson-type equation

$$(x^* \circ h)(x+y) + (x^* \circ h)(x-y) = \lambda(x^* \circ h)(x)(x^* \circ k)(y). \tag{4.7}$$

This means by a linear multiplicativity of x^* that

$$\mathfrak{D}C^{\lambda}_{hk}(x,y) := h(x+y) + h(x-y) - \lambda h(x)k(y) \tag{4.8}$$

falls into the kernel of x^*. As the above process, since x^* is a linear multiplicative, we obtain

$$\mathfrak{D}C^{\lambda}_{hk}(x,y) = 0, \quad \forall x,y \in G \tag{4.9}$$

as claimed.

(b) the case (4.2) also runs along the proof of case (4.1). □

Theorem 4.2. *Let $(E, \|\cdot\|)$ be a semisimple commutative Banach space. Assume that $f, g, h, k : G \to E$ satisfy one of each inequalities*

$$\|f(x+y) + g(x-y) - \lambda \cdot h(x)k(y)\| \le \begin{cases} \min\{\varphi(x), \varphi(y)\} & \text{or} \\ \varepsilon \end{cases} \quad \forall x,y \in G. \tag{4.10}$$

for all $x, y \in G$. For an arbitrary linear multiplicative functional $x^* \in E^*$,

(a) *suppose that $x^* \circ k$ fails to be bounded, then*

(i) *h satisfies (S) under one of the cases $x^* \circ h(0) = 0$ or $x^* \circ f(-x) = -x^* \circ g(x)$, and*

(ii) *in addition, if k satisfies (C^λ), then h and k are solutions of (C^λ_{fg}).*

(b) *Suppose that $x^* \circ h$ fails to be bounded, then*

(iii) *k satisfies (S) under one of the cases $x^* \circ k(0) = 0$ or $x^* \circ f(-x) = -x^* \circ g(x)$, and*

(iv) *in addition, if $x^* \circ h$ satisfies (C^λ), then h and k are solutions of (C^λ_{fg}).*

Remark 4.3. As in the Remark 3.10, we can apply all results of the Sections 2 and 3 to the Banach space.

Namely, we obtain the stability results of $14 \times 4(\varphi(x), \varphi(y), \min\{\varphi(x), \varphi(y)\}, \varepsilon)$ numbers for the other 14 equations except for (P^λ_{fghk}). Some of them are found in papers [7, 11, 13–15, 17, 18].

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