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Induced l_∞ stability of fixed-point digital filters without overflow oscillations and instability due to finite word length effects

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Abstract

This article studies a new criterion for the induced l_∞ stability of fixed-point state-space digital filters without overflow oscillations and instability due to finite word length effects. The criterion not only guarantees exponential stability but also reduces the effect of external interference to an induced l_∞ norm constraint. We present a numerical example, which demonstrates the effectiveness of the proposed criterion.

1 Introduction

When designing a linear time-invariant digital filter using a fixed-point arithmetic, one encounters quantization and overflow nonlinearities. The presence of these nonlinearities may result in the instability of designed filters. The zero-input limit cycles, which are undesirable, may possibly occur due to these nonlinearities. The quantization and overflow nonlinear-ities may interact with each other. However, if the number of quantization steps is large, quantization effects may be decoupled when investigating the effects of overflow. Several researchers have studies stability criteria for digital filters employing saturation overflow arithmetic [1-8].

In the hardware implementation of high-order digital filters, they are usually split into some biquad filters before hardware implementation. Then, there may exist interferences between these biquad filters. These interferences lead to malfunction as well as destruction in the last [9,10]. However, most existing stability criteria for digital filters are only available under specific conditions without external interference, while in unfavorable environments with external interference, unfortunately, we cannot use these existing stability criteria any more. Therefore, it is important to study an alternative criterion that can overcome the shortcomings of existing stability criteria for digital filters.

There always exist model uncertainties and external disturbances in real physical systems. In recent years, this had led to an interest in the induced l_∞ approach [11,12]. The induced l_∞ approach is an effective tool to treat several dynamic systems because we can obtain general stability results using only inputs and outputs measurements. Now, the following question arises: is there an induced l_∞ stability condition for digital filters with external interference and saturation arithmetic? However, as far as we are aware, for the induced l_∞ stability for digital filters with external interference and saturation arithmetic, there are no results published in the literature so far.

This article studies a new stability criterion for fixed-point state-space digital filters with external interference and saturation arithmetic via the induced l_∞ stability approach. This criterion is a new contribution in fields of digital filters. Under the proposed criterion, the digital filter is exponentially stable and the induced l_∞ norm from the external interference to the state vector is reduced to an interference attenuation level. For a fixed scalar variable, we represent this criterion in terms of linear matrix inequalities (LMIs), which can be solved efficiently via existing numerical algorithms such as interior point algorithms [13,14].

This article is organized as follows. In Section 2, a new criterion for the induced l_∞ stability of fixed-point state-space digital filters is proposed. In Section 3, a numerical example is given, and finally, conclusions are presented in Section 4.

2 New induced l_∞ stability criterion

The digital filter under consideration is described by:

$$\begin{aligned} x(r+1) &= f(y(r)) + w(r) \\ &= [f_1(y_1(r)) f_2(y_2(r)) \cdots f_n(y_n(r))]^T + [w_1(r) w_2(r) \cdots w_n(r)]^T, \end{aligned} \tag{1}$$

$$\begin{aligned} y(r) &= [y_1(r) y_2(r) \cdots y_n(r)]^T \\ &= Ax(r), \end{aligned} \tag{2}$$

$$\begin{aligned} z(r) &= [z_1(r) z_2(r) \cdots z_p(r)]^T \\ &= Hx(r), \end{aligned} \tag{3}$$

where $x(r) \in R^n$ is the state vector, $z(r) \in R^p$ is a linear combination of the states, $w(r) \in R^n$ is the external interference, $A \in R^{n\times n}$ is the coefficient matrix, and $H \in R^{p\times n}$ is a known constant matrix. The following saturation nonlinearities:

$$f_i(y_i(r)) = \begin{cases} 1, & \text{if } y_i(r) > 1 \\ y_i(r), & \text{if } -1 \le y_i(r) \le 1 \\ -1, & \text{if } y_i(r) < -1 \end{cases} \tag{4}$$

are under consideration for $i = 1, 2, ..., n$. Note that the saturation nonlinearities are confined to the sector [0, 1], i.e.,

$$f_i(0) = 0, \quad 0 \le \frac{f_i(y_i(r))}{y_i(r)} \le 1, \quad i = 1, 2, ..., n. \tag{5}$$

In this article, given a level $\gamma > 0$, we find a new induced l_∞ stability criterion such that the digital filter (1)-(3) with $w(r) = 0$ is exponentially stable and

$$\sup_{r\ge 0} \left\{ z^T(r) z(r) \right\} < \gamma^2 \sup_{r\ge 0} \left\{ w^T(r) w(r) \right\} \tag{6}$$

under zero-initial conditions for all nonzero $w(r)$. The parameter γ is called the induced l_∞ norm bound or the interference attenuation level. In this case, the digital filter (1)-(3) is said to be exponentially stable with induced l_∞ performance γ.

In the following theorem, we present a new induced l_∞ stability criterion for digital filters.

Theorem 1. *For a given level $\gamma > 0$, if we assume that there exist a symmetric positive definite matrix P, a positive definite diagonal matrix M, positive scalars δ, λ, and μ such that*

$$\begin{bmatrix} \delta A^T A - P - \lambda P & MA & 0 \\ A^T M & P - \delta I - 2M & P \\ 0 & P & P - \gamma I \end{bmatrix} < 0, \tag{7}$$

$$\begin{bmatrix} \lambda P & 0 & H^T \\ 0 & (\gamma - \mu) I & 0 \\ H & 0 & \gamma I \end{bmatrix} > 0, \tag{8}$$

then the digital filter (1)-(3) is exponentially stable with induced l_∞ performance γ.

Proof. Consider the following Lyapunov function: $V(x(r)) = x^T(r)Px(r)$. Along the trajectory of the digital filter (1), we have

$$\begin{aligned} \Delta V(x(r)) &= V(x(r+1)) - V(x(r)) \\ &= \left[f(Ax(r)) + w(r)\right]^T P \left[f(Ax(r)) + w(r)\right] - x^T(r)Px(r) \\ &= f^T(Ax(r))Pf(Ax(r)) + f^T(Ax(r))Pw(r) + w^T(r)Pf(Ax(r)) + w^T(r)Pw(r) \\ &\quad - x^T(r)Px(r) + 2f^T(Ax(r))M[Ax(r) - f(Ax(r))] - 2f^T(y(r))M[y(r) - f(y(r))]. \end{aligned}$$

From (5), it is clear that

$$f^T(Ax(r))f(Ax(r)) = \left\| f(Ax(r)) \right\|^2 \leq \left\| Ax(r) \right\|^2 = (Ax(r))^T Ax(r). \tag{9}$$

Then, for a positive scalar δ, we have

$$\delta[x^T(r)A^T Ax(r) - f^T(Ax(r))f(Ax(r))] \geq 0. \tag{10}$$

If we use (10), we obtain a new upper bound for $\Delta V(x(r))$ as

$$\begin{aligned} \Delta V(x(r)) &\leq f^T(Ax(r))Pf(Ax(r)) + f^T(Ax(r))Pw(r) + w^T(r)Pf(Ax(r)) + w^T(r)Pw(r) \\ &\quad - x^T(r)Px(r) + 2f^T(Ax(r))M[Ax(r) - f(Ax(r))] - 2f^T(y(r))M[y(r) - f(y(r))] \\ &\quad + \delta[x^T(r)A^T Ax(r) - f^T(Ax(r))f(Ax(r))] \\ &= \begin{bmatrix} x(r) \\ f(Ax(r)) \\ w(r) \end{bmatrix}^T \begin{bmatrix} \delta A^T A - P + \lambda P & MA & 0 \\ A^T M & P - \delta I - 2M & P \\ 0 & P & P - \mu I \end{bmatrix} \begin{bmatrix} x(r) \\ f(Ax(r)) \\ w(r) \end{bmatrix} \\ &\quad - \lambda x^T(r)Px(r) + \mu w^T(r)w(r) + \Phi(r), \end{aligned} \tag{11}$$

where $\Phi(r) = -2f^T(y(r))M[y(r) - f(y(r))]$. Note that $\Phi(r)$ is nonpositive in view of (4). If the matrix inequality (7) is satisfied, we have

$$\begin{aligned} \Delta V(x(r)) &< -\lambda x^T(r)Px(r) + \mu w^T(r)w(r) \\ &= -\lambda V(x(r)) + \mu w^T(r)w(r). \end{aligned} \tag{12}$$

Hence $\Delta V(x(r)) < 0$ holds, whenever $V(x(r)) \geq \frac{\mu}{\lambda} w^T(r)w(r)$. Since $V(x(0)) = 0$ under the zero-initial condition, this shows that $V(x(r))$ cannot exceed the value $\frac{\mu}{\lambda} w^T(r)w(r)$

$$x^T(r)Px(r) = V(x(r)) < \frac{\mu}{\lambda} w^T(r)w(r) \tag{13}$$

for $r \geq 0$. It follows from (13) that

$$\begin{aligned} &\frac{1}{\gamma}x^T(r)H^T Hx(r) - \gamma w^T(r)w(r) \\ &= \frac{1}{\gamma}x^T(r)H^T Hx(r) - (\gamma - \mu)w^T(r)w(r) - \mu w^T(r)w(r) \\ &< \frac{1}{\gamma}x^T(r)H^T Hx(r) - (\gamma - \mu)w^T(r)w(r) - \lambda x^T(r)Px(r). \end{aligned} \tag{14}$$

The matrix inequality (8) gives

$$\frac{1}{\gamma}\begin{bmatrix} H^T \\ 0 \end{bmatrix}[H0] < \begin{bmatrix} \lambda P & 0 \\ 0 & (\gamma - \mu)I \end{bmatrix}. \tag{15}$$

Pre- and post-multiplying (15) by $[x^T(r)\ w^T(r)]$ and $[x^T(r)\ w^T(r)]^T$, respectively, yields

$$\frac{1}{\gamma}x^T(r)H^T Hx(r) - (\gamma - \mu)w^T(r)w(r) - \lambda x^T(r)Px(r) < 0, \tag{16}$$

which ensures

$$\frac{1}{\gamma}x^T(r)H^T Hx(r) - \gamma w^T(r)w(r) < 0 \tag{17}$$

from (14). Thus, we have

$$\begin{aligned} z^T(r)z(r) &= x^T(r)H^T Hx(r) \\ &< \gamma^2 w^T(r)w(r). \end{aligned} \tag{18}$$

Taking the supremum over $r \geq 0$ leads to (6).

Next, we show that, under the conditions (7) and (8), the filter (1)-(3) with $w(r) = 0$ is exponentially stable. $V(x(r))$ satisfies the following Rayleigh inequality [15]:

$$\lambda_{\min}(P)\|x(r)\|^2 \leq V(x(r)) \leq \lambda_{\max}(P)\|x(r)\|^2, \tag{19}$$

where $\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ are the maximum and minimum eigenvalues of the matrix. When $w(r) = 0$, we have

$$\Delta V(x(r)) < -\lambda V(x(r)) = -\lambda x^T(r)Px(r) \leq -\lambda_{\min}(P)\|x(r)\|^2 \tag{20}$$

from (12). According to Theorem 3.1 of [16], (19) and (20) guarantee the exponential stability of the digital filter (1)-(3). This completes the proof.

Corollary 1. *If $w(r)$ is bounded as $w^T(r)w(r) < \chi$, the conditions (7) and (8) guarantee that $x(r)$ is bounded as*

$$\|x(r)\| < \sqrt{\frac{\mu\chi}{\lambda_{\min}(P)\lambda}}, \quad r \geq 0. \tag{21}$$

Proof. From (13), we have

$$\lambda_{\min}(P)\|x(r)\|^2 \leq x^T(r)Px(r) = V(x(r)) < \frac{\mu}{\lambda}\chi. \tag{22}$$

Thus, we obtain the relation (21). This completes the proof.

Remark 1. *Recently, an $\mathcal{H}_\infty$(or energy-to-energy) stability criterion for fixed-point state-space digital filters with saturation arithmetic and external interference was* *proposed in* [17]. *In contrast to this study, the induced l_∞ stability criterion can handle the worst-case peak value of the state vector for all bounded peak values of the disturbance signals.*

Remark 2. *For a fixed positive scalar λ, (7) and (8) are LMIs. We can apply various convex optimization algorithms to check whether these LMIs are feasible. In order to solve these LMIs, this article used MATLAB LMI Control Toolbox* [14].

Remark 3. *The l_∞ induced norm* [11,12]*is defined as*

$$\|T_{zw}\|_{l_\infty} = \frac{\sqrt{\sup_{r\geq 0}\left\{z^T(r)z(r)\right\}}}{\sqrt{\sup_{r\geq 0}\left\{w^T(r)w(r)\right\}}}$$

where T_{zw} is a transfer function matrix from $w(r)$ to $z(r)$. For a given level $\gamma > 0$, $\|T_{zw}\|_{l_\infty} < \gamma$can be restated in the equivalent form (6). If we define

$$L(r) = \frac{\sup_{0\leq k\leq r}\{z^T(k)z(k)\}}{\sup_{0\leq k\leq r}\{w^T(k)w(k)\}}, \tag{23}$$

the relation (6) can be represented by $L(\infty) < \gamma^2$. In the following section, through the plot of $L(r)$, $L(\infty) < \gamma^2$ is verified.

3 Numerical example

Consider a second-order filter (1)-(3) with

$$A = \begin{bmatrix} 0.25 & 0.5 \\ -0.5 & 0.8 \end{bmatrix}, \quad H = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad w(r) = 0.5\begin{bmatrix} \cos(2r) \\ 2\sin(r) \end{bmatrix}. \tag{24}$$

Let the induced l_∞ performance be specified by $\gamma = 0.3$. In addition, we fix $\lambda = 1$. Solving (7) and (8) by the convex optimization technique of MATLAB software gives

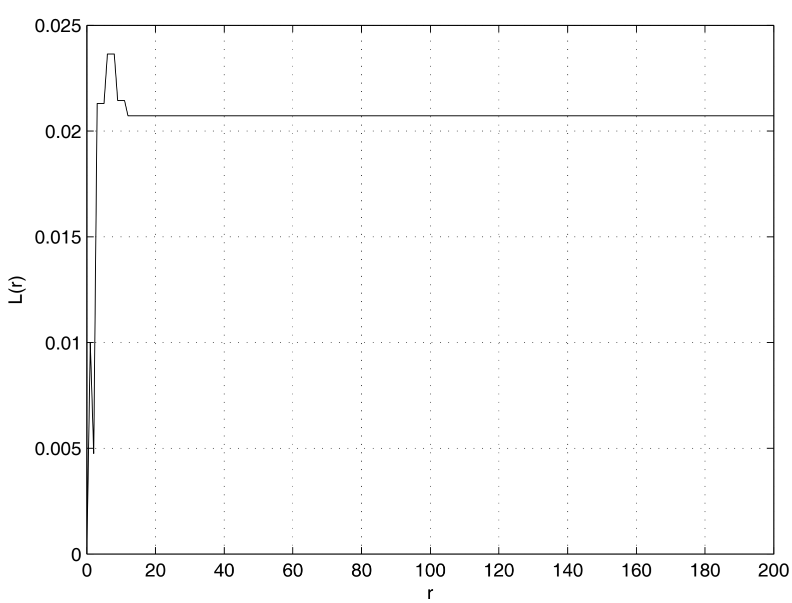

Figure 1 The plot of *L*(*r*).

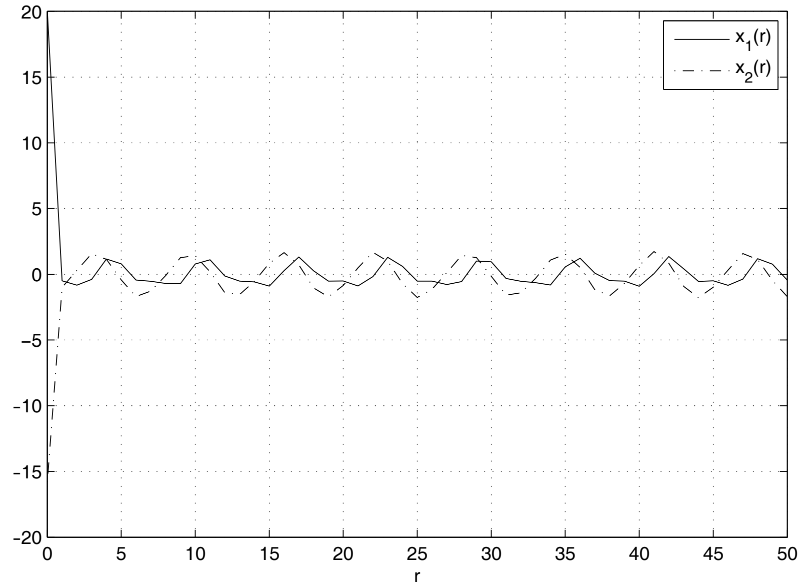

Figure 2 State trajectories.

Figure 1 shows the plot of $L(r)$, which is defined in (23). This figure verifies $L(\infty) < \gamma^2 = 0.09$, which means that the induced l_∞ norm from $w(r)$ to $z(r)$ is reduced within the induced l_∞ norm bound γ. Figure 2 represents the state trajectories of the digital filter (1)-(3) when (x_1(0), x_2(0)) = (20,-15.8). It is clear that stability criteria in existing studies [1-8] fail in the filter given by (1)-(3) with the parameters (24). On the other hand, the proposed criterion (7) and (8) verifies the exponential stability result with induced l_∞ performance in this example.

4 Conclusion

This article studies a new criterion for the induced l_∞ stability of fixed-point state-space digital filters with external interference and saturation overflow arithmetic. It is shown that the criterion can ensure to reduce the effect of the external interference to a prescribed attenuation level. Thus, it can overcome the disadvantages of existing stability criteria. For a fixed scalar variable, this criterion is represented in terms of LMIs and, hence, computationally tractable. Finally, a numerical example shows the usefulness of the proposed stability criterion.

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Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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