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‘Massless’ spin-$\frac{3}{2}$ fields in the de Sitter space

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Abstract In this paper, ‘massless’ spin-$\frac{3}{2}$ fields in the de Sitter space are considered. This work is a continuation of a previous paper devoted to the quantization of the de Sitter ‘massive’ spin-$\frac{3}{2}$ fields. Due to the appearance of gauge invariance and an indefinite metric, the covariant quantization of the ‘massless’ spin-$\frac{3}{2}$ fields requires an indecomposable representation of the de Sitter group. The gauge fixing corresponding to the simplest Gupta–Bleuler structure is used, and a gauge invariant field is discussed. The field equation is obtained by using the Casimir operator of the de Sitter group. The solutions are written in terms of the coordinate-independent de Sitter plane waves. Finally, the generalized two-point function is calculated.

1 Introduction

In the previous work [1], the ‘massive’ spin-$\frac{3}{2}$ fields in the de Sitter space have been studied. The term ‘massive’ refers to those de Sitter fields which unambiguously contract to the Minkowskian massive fields in the zero curvature limit. These fields are associated with the principal series of unitary irreducible representations of the de Sitter group $SO_0(1,4)$, with the Casimir operator eigenvalue $\langle Q_\nu^{(1)}\rangle = \nu^2 - \frac{3}{2}$, $\nu > \frac{3}{2}$ (or equivalently ‘mass’ $m_p^2 = H^2(\nu^2 - \frac{3}{2})$). The interpretation of m_p as a mass becomes possible by examining its null curvature limit. The quotation marks on ‘mass’ are used because of the absence of the intrinsic notion of mass in de Sitter relativity [2]. Indeed, the principal series of unitary irreducible representations admits a massive representation of the Poincaré group in the limit $H = 0$ [3–5].

In this paper, the ‘massless’ spin-$\frac{3}{2}$ fields in the de Sitter space are considered. ‘Massless’ is used by reference to the conformal invariance and propagation on the light-cone. They are associated with the discrete series of unitary irreducible representations of the de Sitter group. There are two unitary irreducible representations of the de Sitter group in the discrete series with a Minkowskian meaning in the null curvature limit, which are denoted by $\Pi^{\pm}_{\frac{3}{2},\frac{3}{2}}$ with the same Casimir operator eigenvalue $\langle Q^{(1)}\rangle = -\frac{5}{2}$.

The field equation of the ‘massless’ spin-$\frac{3}{2}$ fields is gauge invariant in the de Sitter space as well as the massless fields in the Minkowski space for $s \geq 1$. It is well known that the quantization of gauge invariant theories usually requires quantization *à la* Gupta–Bleuler [6,7]. It has been proved that the use of an indefinite metric is an unavoidable feature if one insists on preserving of causality (locality) and covariance in gauge quantum field theories [8]. This means that one cannot limit the state space of the massless fields (with $s \geq 1$) to Hilbert space; the appearance of states with negative or null norms is necessary for the quantization of gauge fields. Therefore, an indecomposable structure appears inevitable, where the physical states belong to a subspace V of the solutions, but where the field operator must be defined on a larger gauge dependent space V_λ (which contains negative norm states). The physical subspace V is invariant but not invariantly complemented in V_λ. The same feature is repeated in V, where one can find the invariant (but again not invariantly complemented) subspace of gauge solutions V_g. These gauge solutions have zero norms and are orthogonal to all the elements of V [9]. Consequently, one must eliminate them from the subspace V by considering the physical state space as the coset V/V_g.

For simplicity, the following units are used:

$$c = \hbar = 1, \quad [x^\alpha/H] = 1, \quad [M] = H,$$

where c, $\hbar$ and H are light velocity, Planck constant, and Hubble parameter, respectively. The paper is organized as follows: In Sect. 2, some of the useful notations of de Sitter space and unitary irreducible representations of the de Sitter group will be recalled. Section 3 is devoted to a derivation ofthe de Sitter spin-$\frac{3}{2}$ ‘massless’ field equation. In this section, we will show that the field equation is gauge invariant. We will

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adopt a very convenient value for the gauge fixing parameter λ. In this paper, we get the second-order wave equation by using the Casimir operator; subsequently it is converted to a first-order equation. There are two different equations for this case; one of those is considered in the context and the other is studied in the appendix. In Sect. 4, the solutions are calculated in terms of the coordinate-independent de Sitter plane waves. It will be shown that, for $\lambda = \frac{1}{2}$, the field solution has a simple form. In Sect. 5, we will define the two-point function $S(x, y)$ that satisfies the following conditions: (a) indefinite sesquilinear form, (b) locality, (c) covariance, and (d) normal analyticity. Normal analyticity allows us to define the two-point function $S(x, y)$ as the boundary value of the analytic function $\mathrm{S}(z_1, z_2)$ from the tube domains. The normal analyticity is related to the Hadamard condition that selects a unique vacuum state. $\mathrm{S}(z_1, z_2)$ is defined in terms of the spinor-vector de Sitter plane waves in their tube domains. Section 6 contains a brief conclusion and the outlook.

2 The de Sitter space notations

The de Sitter space is visualized as the hyperboloid described by the equation

$$X_H = \left\{ x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2} = -\frac{3}{\Lambda} \right\},$$

where $\eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1, -1, -1)$; $\alpha, \beta = 0, 1, \ldots, 4$, and Λ is a positive cosmological constant. The metric is

$$\mathrm{d}s^2 = \eta_{\alpha\beta} \mathrm{d}x^\alpha \mathrm{d}x^\beta |_{x^2=-H^{-2}} = g^{dS}_{\mu\nu} dX^\mu dX^\nu; \quad \mu, \nu = 0, 1, 2, 3,$$

where X^μ are the four local space-time coordinates on a de Sitter hyperboloid. A spinor-tensor field $\Psi_{\alpha_1,\ldots,\alpha_l}(x)$ on X_H can be viewed as an homogeneous function on $\mathbb{R}^5$ variables x^α with an arbitrary degree of homogeneity σ. Also it must satisfy the following conditions [10]:

$$x \cdot \partial \Psi = \sigma \Psi, \qquad \text{(homogeneity)},$$

$$x \cdot \Psi(x) = 0, \qquad \text{(transversality)}.$$

On the de Sitter space the tangential (or transverse) derivative is defined by

$$\partial^\top_\alpha = \theta_{\alpha\beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha x \cdot \partial, \quad x \cdot \partial^\top = 0, \tag{2.1}$$

where $\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ is transverse projection tensor ($\theta_{\alpha\beta}\, x^\alpha = \theta_{\alpha\beta}\, x^\beta = 0$).

The unitary irreducible representations of the 10-parameter group $\mathrm{SO}_0(1, 4)$ (connected component of the identity) of the de Sitter space, which is one of the two possible deformations of the Poincaré group (the other one being $\mathrm{SO}_0(2, 3)$), are characterized by the eigenvalues of the two Casimir operators $Q^{(1)}$ and $Q^{(2)}$. These operators, commuting with the group generators, are constant in each unitary irreducible representation. They read

$$Q^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta}, \quad Q^{(2)} = -W_\alpha$$
$$W^\alpha, \quad W_\alpha = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}, \tag{2.2}$$

where $\epsilon_{\alpha\beta\gamma\delta\eta}$ is the usual antisymmetrical tensor in $\mathbb{R}^5$ and $L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}$ is an infinitesimal generator. The orbital part $M_{\alpha\beta}$ is

$$M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) = -i(x_\alpha \partial^\top_\beta - x_\beta \partial^\top_\alpha). \tag{2.3}$$

In order to make precise the action of the spinorial part $S_{\alpha\beta}$ on a field tensor or spinor-tensor one must treat separately the integer and half-integer cases. Tensor fields of rank l, $\Psi_{\gamma_1,\ldots,\gamma_l}(x)$, show integer spin fields, and the spinorial action is [7]

$$S^{(l)}_{\alpha\beta} \Psi_{\gamma_1,\ldots,\gamma_l} = -i \sum_{i=1}^{l} \left(\eta_{\alpha\gamma_i} \Psi_{\gamma_1,\ldots,(\gamma_i\to\beta),\ldots,\gamma_l} - \eta_{\beta\gamma_i} \Psi_{\gamma_1,\ldots,(\gamma_i\to\alpha),\ldots,\gamma_l} \right), \tag{2.4}$$

where $(\gamma_i \to \beta)$ means γ_i index replaced with β. Half-integer spin fields with spin $s = l + \frac{1}{2}$ are represented by a four component spinor-tensor $\Psi^i_{\gamma_1,\ldots,\gamma_l}$ with spinor index $i = 1, 2, 3, 4$. The spinorial part now reads

$$S^{(s)}_{\alpha\beta} = S^{(l)}_{\alpha\beta} + S^{\left(\frac{1}{2}\right)}_{\alpha\beta}, \qquad \text{with} \qquad S^{\left(\frac{1}{2}\right)}_{\alpha\beta} = -\frac{i}{4} \left[\gamma_\alpha, \gamma_\beta\right],$$

and with the Dirac gamma matrices γ_α [11–13]

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta}, \qquad \gamma^{\alpha\dagger} = \gamma^0 \gamma^\alpha \gamma^0,$$

the useful representations, which are compatible with the group, are as follows:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^4 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$
$$\gamma^1 = \begin{pmatrix} 0 & i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \gamma^3 = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}, \tag{2.5}$$

where σ_i are Pauli matrices and I is a 2×2 unit matrix. The Casimir operators are simple to manipulate in ambient space notation. Since $Q^{(1)}$ is a second-order derivative operator, it is convenient to use for obtaining the field equation. In particular, it is easy to show that for a l-rank tensor field $\Psi_{\gamma_1,\ldots,\gamma_l}(x)$ one has

$$Q^{(1)}_l \Psi = Q^{(1)}_0 \Psi - 2\Sigma_1 \partial x.\Psi + 2\Sigma_1 x \partial.\Psi + 2\Sigma_2 \eta \Psi' - l(l+1)\Psi, \tag{2.6}$$

where

$$Q_l^{(1)} = -\frac{1}{2}L_{\alpha\beta}^{(l)}L^{\alpha\beta(l)} = -\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} - \frac{1}{2}S_{\alpha\beta}^{(l)}S^{\alpha\beta(l)} - M_{\alpha\beta}S^{\alpha\beta(l)}, \tag{2.7}$$

$$M_{\alpha\beta}S^{\alpha\beta(l)}\Psi(x) = 2\Sigma_1\partial x.\Psi - 2\Sigma_1 x\partial.\Psi - 2l\Psi, \tag{2.8}$$

$$\frac{1}{2}S_{\alpha\beta}^{(l)}S^{\alpha\beta(l)}\Psi = l(l+3)\Psi - 2\Sigma_2\eta\Psi', \tag{2.9}$$

$$Q_0^{(1)} = -\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta}, \tag{2.10}$$

Ψ' is the trace of the l-rank tensor $\Psi(x)$, and Σ_p is the non-normalized symmetrization operator:

$$\Psi'_{\alpha_1,\ldots,\alpha_{l-2}} = \eta^{\alpha_{l-1}\alpha_l}\Psi_{\alpha_1,\ldots,\alpha_{l-2}\alpha_{l-1}\alpha_l}, \tag{2.11}$$

$$(\Sigma_p AB)_{\alpha_1,\ldots,\alpha_l} = \sum_{i_1<i_2<\cdots<i_p} A_{\alpha_{i_1}\alpha_{i_2},\ldots,\alpha_{i_p}} \times B_{\alpha_1,\ldots,\not{\alpha}_{i_1},\ldots,\not{\alpha}_{i_2},\ldots,\not{\alpha}_{i_p},\ldots,\alpha_l}. \tag{2.12}$$

For half-integer spin fields with spin $s = l + \frac{1}{2}$, the $S_{\alpha\beta}^{(\frac{1}{2})}$ acts only upon the index i, and we have [13,14]

$$S_{\alpha\beta}^{(\frac{1}{2})}S^{\alpha\beta(l)}\Psi(x) = l\Psi(x) - \Sigma_1\gamma(\gamma\cdot\Psi(x)).$$

In this case, the Casimir operator is

$$Q_s^{(1)} = -\frac{1}{2}\left(M_{\alpha\beta}+S_{\alpha\beta}^{(l)}+S_{\alpha\beta}^{\left(\frac{1}{2}\right)}\right)\left(M^{\alpha\beta}+S^{\alpha\beta(l)}+S^{\alpha\beta\left(\frac{1}{2}\right)}\right) = Q_l^{(1)} - \frac{5}{2} + \frac{i}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta} - S_{\alpha\beta}^{(\frac{1}{2})}S^{\alpha\beta(l)}. \tag{2.13}$$

Then we obtain

$$Q_s^{(1)}\Psi(x) = \left(Q_l^{(1)} - l - \frac{5}{2} + \frac{i}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta}\right)\Psi(x) + \Sigma_1\gamma(\gamma.\Psi(x)), \tag{2.14}$$

or

$$Q_s^{(1)}\Psi(x) = \left(-\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} + \frac{i}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta} - l(l+2) - \frac{5}{2}\right)\Psi(x) - 2\Sigma_1\partial x.\Psi(x) + 2\Sigma_1 x\partial.\Psi(x) + 2\Sigma_2\eta\Psi'(x) + \Sigma_1\gamma(\gamma.\Psi(x)). \tag{2.15}$$

As you will see in the next section, the spin-$\frac{3}{2}$ field equation can be written in terms of the Casimir operator $Q^{(1)}$.

3 Field equation and Gauge transformation

3.1 Field equation

As previously mentioned, the operator $Q_{\frac{3}{2}}^{(1)}$ commutes with the group generators and consequently it is constant on each unitary irreducible representation. In fact, we can classify the spinor-vector unitary irreducible representations by using the eigenvalues of $Q^{(1)}$. The field equation can be written as

$$\left(Q_{\frac{3}{2}}^{(1)} - \langle Q_{\frac{3}{2}}^{(1)}\rangle\right)\Psi(x) = 0. \tag{3.1}$$

By Takahashi [15] and Dixmier [16], a general classification scheme for all the unitary irreducible representations of the de Sitter group is expressed and may be labeled by a pair of parameters (p, q) with $2p \in N$ and $q \in C$ as follows:

$$\langle Q^{(1)}\rangle = [-p(p+1) - (q+1)(q-2)],$$
$$\langle Q^{(2)}\rangle = [-p(p+1)q(q-1)]. \tag{3.2}$$

According to the possible values of p and q, two types of unitary irreducible representations of the spin-$\frac{3}{2}$ field are distinguished for the de Sitter group SO(1, 4) namely, the principal and the discrete series. More mathematical details of the group contraction and the relationship between the de Sitter and the Poincaré groups are given in [17,18]. The unitary irreducible representations of the spin-$\frac{3}{2}$ field relevant to the present work are as follows:

(i) The unitary irreducible representations $U^{\frac{3}{2},\nu}$ in the principal series where $p = s = \frac{3}{2}$ and $q = \frac{1}{2} + i\nu$ matching to the Casimir spectral values:

$$\langle Q_{\frac{3}{2}}^{(1)}\rangle = \nu^2 - \frac{3}{2}, \quad \nu \in \mathbb{R} \quad \nu > \frac{3}{2}.$$

Note that $U^{\frac{3}{2},\nu}$ and $U^{\frac{3}{2},-\nu}$ are equivalent.

(ii) The unitary irreducible representations $\Pi^{\pm}_{\frac{3}{2},q}$ of the discrete series, where $p = s = \frac{3}{2}$, correspond to

$$\langle Q_{\frac{3}{2}}^{(1)}\rangle = -\frac{5}{2}, \quad q = \frac{3}{2}, \quad \Pi^{\pm}_{\frac{3}{2},\frac{3}{2}}, \tag{3.3}$$

$$\langle Q_{\frac{3}{2}}^{(1)}\rangle = -\frac{3}{2}, \quad q = \frac{1}{2}, \quad \Pi^{\pm}_{\frac{3}{2},\frac{1}{2}}. \tag{3.4}$$

The physical content of the principal series and the discrete series representation from the point of view of a Minkowskian observer in the limit $H = 0$ have been expressed in [1].

The 'massless' spin-$\frac{3}{2}$ field in the de Sitter space corresponds to the discrete series $\Pi^{\pm}_{\frac{3}{2},\frac{3}{2}}$ and the field equation is

$$\left(Q_{\frac{3}{2}}^{(1)} + \frac{5}{2}\right)\Psi(x) = 0, \tag{3.5}$$

where

$$Q_{\frac{3}{2}}^{(1)}\Psi(x) = \left(-\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} + \frac{i}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta} - 3 - \frac{5}{2}\right)\Psi(x) - 2\partial x.\Psi(x) + 2x\partial.\Psi(x) + \gamma\gamma.\Psi(x). \tag{3.6}$$

If we compare this equation with the massive spin-$\frac{3}{2}$ field case (equation (3.7) in [1]) they are equivalent for the values $\nu = \pm i$, but these values are not admissible for the unitary irreducible representations of the de Sitter group in the principal series. It is natural to use the solutions of Eq. (3.1), which are already given for the massive case [1]. The corresponding spin-$\frac{3}{2}$ field solution can be written in the form [1]

$$\Psi_\alpha(x) = \frac{1}{\langle Q^{(1)}_{\frac{3}{2}}\rangle + \frac{5}{2}} \mathcal{E}_\alpha(x,\xi)\psi(x), \tag{3.7}$$

where $\mathcal{E}_\alpha(x,\xi)$ and $\psi(x)$ also contain constant terms involving the parameters p and q, not diverging for the specific values $p = q = \frac{3}{2}$, corresponding to the relation (3.3). One can simply see that the field solution for these values is singular (equation (4.12) in [1]). Clearly, the singularity appears only for the spin-$\frac{3}{2}$ massless field for representations $\Pi^{\pm}_{\frac{3}{2},\frac{3}{2}}$. This singularity appears due to the gauge invariance, which will be discussed in the next subsection. Therefore the subspace $\partial.\Psi = 0$ alone cannot be used for the construction of a quantum massless spin-$\frac{3}{2}$ field. One must solve the equation in a larger space, which includes the $\partial\cdot\Psi \neq 0$ types of solutions. As expected, one finds three main types of solutions: the general solutions which are not divergenceless, the divergencelessness type, and the gauge type solution.

3.2 Gauge transformation

The 'massless' spin-$\frac{3}{2}$ field, however, with the subsidiary conditions $\partial^\top.\Psi = 0 = \gamma.\Psi(x)$, is singular. This type of singularity is actually due to the divergencelessness condition needed to associate this field with a specific unitary irreducible representation of the de Sitter group. To solve this problem, the subsidiary conditions must be dropped. Then the field Eq. (3.5) is replaced with the following equation:

$$\left(Q^{(1)}_{\frac{3}{2}} + \frac{5}{2}\right)\Psi_\alpha(x) - D_{\frac{3}{2}\alpha}\partial^\top.\Psi(x) = 0, \tag{3.8}$$

where $D_{\frac{3}{2}\alpha} = -H^{-2}\partial^\top_\alpha - \gamma^\top_\alpha \not{x}$, $\not{x} = \gamma_\alpha x^\alpha$ and $\gamma^\top_\alpha = \theta_{\alpha\beta}\gamma^\beta$. One can show that this equation is invariant under a gauge transformation

$$\Psi_\alpha(x) \rightarrow \Psi'_\alpha(x) = \Psi_\alpha(x) + D_{\frac{3}{2}\alpha}\zeta, \tag{3.9}$$

where ζ is an arbitrary spinor field. For providing a gauge invariant the following identities are used:

$$Q^{(1)}_{\frac{3}{2}} D_{\frac{3}{2}} = D_{\frac{3}{2}} Q^{(1)}_{\frac{1}{2}}, \quad \partial^\top.D_{\frac{3}{2}}\zeta = Q^{(1)}_{\frac{1}{2}}\zeta + \frac{5}{2}\zeta.$$

Let us introduce a gauge fixing parameter λ. The wave equation now reads

$$\left(Q^{(1)}_{\frac{3}{2}} + \frac{5}{2}\right)\Psi(x) - \lambda D_{\frac{3}{2}\alpha}\partial^\top.\Psi(x) = 0, \tag{3.10}$$

the role of λ is just to fix the gauge spinor field ζ.

It is convenient and usual to continue this work with the first-order field equation for the spinor case. We can write Eq. (3.5) such as

$$\tilde{D}^\beta_\alpha D^\delta_\beta \Psi_\delta = 0, \quad \text{or} \quad D^\delta_\beta \tilde{D}^\beta_\alpha \Psi_\delta = 0, \tag{3.11}$$

where $\tilde{D}^\beta_\alpha$ and D^δ_β are defined by

$$\begin{cases} \tilde{D}^\delta_\beta = \not{x}\,\not{\partial}^\top \eta^\delta_\beta - 3\eta^\delta_\beta - x_\beta \not{x}\gamma^\delta, \\ D^\beta_\alpha = \not{x}\,\not{\partial}^\top \eta^\beta_\alpha - \eta^\beta_\alpha - x_\alpha \not{x}\gamma^\beta. \end{cases} \tag{3.12}$$

There are two possibilities for the first-order field equation as follows:

$$\tilde{D}^\delta_\alpha \Psi_\delta = 0, \tag{3.13}$$
$$D^\delta_\alpha \Psi_\delta = 0, \tag{3.14}$$

where for physical states, which means $\lambda = 0$ in Eq. (3.10), the two first-order equations (3.13) and (3.14) are equivalent, since their multiplication gives Eq. (3.5). But for unphysical states (gauge dependent states, i.e. $\lambda \neq 0$ and pure gauge states, i.e. $\lambda = 1$), the solutions of the two equations are different, although the physical parts or the central parts are equal in these cases. Equation (3.13) is considered in Sect. 7.1. Here we consider Eq. (3.14). This equation can be written in the simple form

$$\not{x}\,\not{\partial}^\top\Psi_\alpha(x) - x_\alpha \not{x}\,\not{\Psi} - \Psi_\alpha(x) = 0. \tag{3.15}$$

For clarity it can be written as

$$\left(\text{Trpr}\,\not{x}\,\not{\partial}^\top - 1\right)\Psi_\alpha(x) = 0, \tag{3.16}$$

where (Trpr) means the transverse projection. We should recall that the massless spin-$\frac{3}{2}$ is singular when $\nu = \pm i$. This singularity refers to a gauge invariant, as we have this difficulty for all fields with spin $s \geq 1$. Now we rewrite Eq. (3.16) as

$$\left(\text{Trpr}\,\not{x}\,\not{\partial}^\top - 1\right)\Psi_\alpha(x) + D_{\frac{3}{2}\alpha}\not{x}\,\not{\Psi}(x) = 0, \tag{3.17}$$

which is invariant under a gauge transformation:

$$\Psi_\alpha(x) \longrightarrow \Psi'_\alpha(x) = \Psi_\alpha(x) + \partial^\top_\alpha\zeta, \tag{3.18}$$

ζ is an arbitrary spinor field. Let us introduce a gauge fixing parameter λ for this case. Now the wave equation reads

$$\left(\text{Trpr}\,\not{x}\,\not{\partial}^\top - 1\right)\Psi_\alpha(x) + \lambda D_{\frac{3}{2}\alpha}\not{x}\,\not{\Psi}(x) = 0. \tag{3.19}$$

The above equation under the gauge transformation (3.18) becomes

$$\left(\text{Trpr}\, \not{x}\, \not{\partial}^\top - 1\right) \Psi_\alpha(x) + \lambda D_{\frac{3}{2}\alpha} \not{x}\, \not{\Psi}(x) = (1-\lambda) D_{\frac{3}{2}\alpha} \not{x}\, \not{\partial}^\top \zeta. \quad (3.20)$$

With the choice of value $\lambda \neq 1$, the gauge field $\partial^\top_\alpha \zeta$ is put into the field equation only if ζ obeys

$$(\text{Trpr}\, \not{x}\, \not{\partial}^\top - 1)\partial^\top_\alpha \zeta = -D_{\frac{3}{2}\alpha} \not{x}\, \not{\partial}^\top \zeta = 0, \quad (3.21)$$

which means that the gauge field ζ is fixed. The adjoint $\Psi_\alpha(x)$ is defined as follows [11,13]:

$$\bar{\Psi}_\alpha(x) \equiv \Psi^\dagger_\alpha(x)\gamma^0\gamma^4,$$

satisfying the equation

$$\bar{\Psi}_\alpha \gamma^4 (\overleftarrow{\not{\partial}}^\top \not{x})\gamma^4 - \bar{\Psi}_\beta \gamma^4 (\gamma^\beta x_\alpha \not{x})\gamma^4 + \bar{\Psi}_\alpha + c\bar{\Psi}_\beta \gamma^4 \gamma^\beta \not{x} \overleftarrow{D}_{\frac{3}{2}\alpha} \gamma^4 = 0. \quad (3.22)$$

As stated in [6], ‘the appearance of the (Gupta–Bleuler) triplet seems to be universal in gauge theories, and crucial for ‘quantization’. The ambient space formalism will allow one to exhibit this triplet for the present field in exactly the same manner as it for the electromagnetic field.

Let us now define the Gupta–Bleuler triplet $V_g \subset V \subset V_\lambda$ carrying the indecomposable structure for the unitary irreducible representations of the de Sitter group appearing in our problem:

- The space V_λ is the space of all square integrable solutions of the field Eq. (3.19). It is λ dependent so that one can actually adopt an optimal value of λ which has a very simple form. In the next section, we will show that this value is $\lambda = \frac{1}{2}$ (more generally for a spin s field, we have $\lambda = (2/(2s+1))$ [7]).
- It contains a closed subspace V of solutions satisfying the conditions $\partial^\top \cdot \Psi = 0 = \gamma \cdot \Psi(x)$. This invariant subspace V is not invariantly complemented in V_λ. In view of Eq. (3.19), it is obviously λ independent.
- The subspace V_g of V consists of the gauge solutions of the form $\partial^\top_\alpha \zeta$ and $\not{\partial}^\top \zeta = 0 = Q_0 \zeta$. These are orthogonal to every element in V including themselves. They form an invariant subspace of V but admit no invariant complement in V.

The de Sitter group acts on the physical (or transverse) space V/V_g through the massless, helicity $\pm\frac{3}{2}$ unitary representation $\Pi^+_{\frac{3}{2},\frac{3}{2}} \bigoplus \Pi^-_{\frac{3}{2},\frac{3}{2}}$, which are called the central parts in this paper.

4 The de Sitter spin-$\frac{3}{2}$ plane waves

According to the de Sitter plane waves, which were presented by Bros et al. [19], the de Sitter–Dirac plane wave for a spinor field was calculated[13]. In this section, the spinor-vector solution can be written by using de Sitter–Dirac plane wave in terms of the following form:

$$\Psi_\alpha(x) = Z^\top_\alpha \psi_1 + D_{\frac{3}{2}\alpha}\psi_2 + \gamma^\top_\alpha \psi_3, \quad (4.1)$$

where Z is an arbitrary five-component constant vector field:

$$Z^\top_\alpha = \theta_{\alpha\beta} Z^\beta = Z_\alpha + H^2 x_\alpha x \cdot Z,\ \ x \cdot Z^\top = 0.$$

By putting Ψ_α in Eq. (3.19), we find that the spinor fields ψ_1, ψ_2, and ψ_3 must obey the following equations:

$$(\not{x}\, \not{\partial}^\top - 1)\psi_1 = 0, \quad (4.2)$$

$$\not{x}\, \not{\partial}^\top \psi_3 + \not{x}(\not{x}\, \not{\partial}^\top - 4)\psi_2 = \not{x} x.Z\psi_1, \quad (4.3)$$

$$2(1-2\lambda)\psi_3 + \not{x}\left[(1-\lambda)\not{x}\, \not{\partial}^\top - 2(1-2\lambda)\right]\psi_2 = \lambda \not{Z}^\top \psi_1. \quad (4.4)$$

For $\lambda = \frac{1}{2}$ Eq. (4.4) becomes

$$\not{\partial}^\top \psi_2 = -\not{Z}^\top \psi_1. \quad (4.5)$$

By multiplying Eq. (4.2) by $\not{\partial}^\top$, we have

$$Q_0\psi_1 = 2\psi_1.$$

Since the spinor fields ψ_1, ψ_2, and ψ_3 are homogeneous functions of the variable x with the same degree of homogeneity, by using Eq. (4.5), we obtain

$$Q_0\psi_2 = \left(4x.Z + 2Z.\not{\partial}^\top + \not{Z}^\top \not{x}\right)\psi_1, \quad (4.6)$$

and ψ_2 can be written in the following form:

$$\psi_2 = Q_0^{-1}\left(4x.Z + 2Z.\partial^\top + \not{Z}^\top \not{x}\right)\psi_1 + \psi_g, \quad (4.7)$$

where ψ_g is $\psi_g = \phi_g \mathbf{U}$ and $Q_0\phi_g = 0$. $\mathbf{U}$ is an arbitrary constant spinor field, and ϕ_g is a massless minimally coupled scalar field. By using the following identities:

$$Q_0 x.Z\psi_1 = -2x.Z\psi_1 - 2Z.\partial^\top \psi_1, \quad (4.8)$$

$$Q_0 Z.\partial^\top \psi_1 = 4x.Z\psi_1 + 4Z.\partial^\top \psi_1, \quad (4.9)$$

$$Q_0 \not{Z}^\top \not{x}\psi_1 = 2x.Z\psi_1 + 2Z.\partial^\top \psi_1, \quad (4.10)$$

and

$$Q_0\left(4x.Z + 2Z.\partial^\top + \not{Z}^\top \not{x}\right)\psi_1 = Q_0 \not{Z}^\top \not{x}\psi_1 = 2\not{Z}^\top \not{x}\psi_1, \quad (4.11)$$

we obtain

$$\psi_2 = \frac{1}{2}\not{Z}^\top \not{x}\psi_1 + \psi_g. \quad (4.12)$$

By replacing Eq. (4.12) in (4.3), we obtain (see Sect. 7.2)

$$\psi_3 = \not{x}\psi_g. \quad (4.13)$$

If Eqs. (4.12) and (4.13) are substituted in Eq. (4.1), it can be seen that in this choice of gauge $\lambda = \frac{1}{2}$, the spinor field ψ_g or equivalently ψ_3 is removed, and the solution can be written in the following simplest form:

$$\Psi_\alpha(x) = \mathcal{D}_\alpha \psi_1, \tag{4.14}$$

where

$$\mathcal{D}_\alpha = \frac{1}{2}\left[3Z_\alpha^\top + x.Z\partial_\alpha^\top - \gamma_\alpha^\top \not{Z}^\top\right], \tag{4.15}$$

and ψ_1 is the solution of the de Sitter–Dirac field equation. In the previous paper, the spinor field ψ_1 was explicitly calculated, and the solutions are given by [11,13]

$$(\psi_1)_1 = \mathcal{V}(x,\xi)(Hx.\xi)^{-3}, \tag{4.16}$$
$$(\psi_1)_2 = \mathcal{U}(\xi)(Hx.\xi)^{-1}, \tag{4.17}$$

where $\mathcal{V}(x,\xi) = \not{x}\ \not{\xi}\mathcal{V}(\xi)$ and

$$\xi \in \mathcal{C}^+ = \left\{\xi\ \ ; \eta_{\alpha\beta}\xi^\alpha\xi^\beta = (\xi^0)^2 - \vec{\xi}.\vec{\xi} - (\xi^4)^2 = 0,\ \xi^0 > 0\right\}.$$

The two spinors $\mathcal{V}(\xi)$ and $\mathcal{U}(\xi)$ are

$$\mathcal{U}^a(\xi) = \frac{\xi^0 - \vec{\xi}.\vec{\gamma}\gamma^0 + 1}{\sqrt{2(\xi^0+1)}}\mathcal{U}^a(\overset{o}{\xi}_+),$$
$$\mathcal{V}^a(\xi) = \frac{1}{\sqrt{2(\xi^0+1)}}\mathcal{U}^a(\overset{o}{\xi}_-), \quad a = 1,2, \tag{4.18}$$

where

$$\mathcal{U}_1(\overset{o}{\xi}_+) = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}, \quad \mathcal{U}_2(\overset{o}{\xi}_+) = \frac{1}{\sqrt{2}}\begin{pmatrix}\beta\\ \beta\end{pmatrix}, \tag{4.19}$$

$$\mathcal{U}_1(\overset{o}{\xi}_-) = \frac{1}{\sqrt{2}}\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}, \quad \mathcal{U}_2(\overset{o}{\xi}_-) = \frac{1}{\sqrt{2}}\begin{pmatrix}\beta\\ -\beta\end{pmatrix}, \tag{4.20}$$

with $\alpha = \begin{pmatrix}1\\0\end{pmatrix}$, $\beta = \begin{pmatrix}0\\1\end{pmatrix}$ and $\xi = \overset{o}{\xi}_\pm \equiv (1,\vec{0},\pm 1)$. Finally the two possible solutions for $\Psi_\alpha(x)$ are

$$\Psi^a_{1\alpha}(x) = \frac{1}{2}\left[3Z_\alpha^\top + x.Z\partial_\alpha^\top - \gamma_\alpha^\top \not{Z}^\top\right]\frac{\not{x}\ \not{\xi}}{x.\xi}\mathcal{V}(\xi)(Hx.\xi)^{-2}$$
$$\equiv \mathcal{V}_\alpha(x,\xi,Z)(Hx.\xi)^{-2}, \tag{4.21}$$

and

$$\Psi^a_{2\alpha}(x) = \frac{1}{2}\left[3Z_\alpha^\top + x.Z\partial_\alpha^\top - \gamma_\alpha^\top \not{Z}^\top\right]\mathcal{U}(\xi)(Hx.\xi)^{-1}$$
$$\equiv \mathcal{U}_\alpha(x,\xi,Z)(Hx.\xi)^{-1}. \tag{4.22}$$

By taking the derivation of the plane waves $(x.\xi)^\sigma$, the explicit forms of $\mathcal{U}_\alpha$ and $\mathcal{V}_\alpha$ are obtained in terms of ξ as follows:

$$\mathcal{V}_\alpha(x,\xi,Z) = \frac{1}{2}\left[3Z_\alpha^\top - 3\frac{x.Z}{x.\xi}\xi_\alpha^\top - x.Z\gamma_\alpha^\top - \gamma_\alpha^\top \not{Z}^\top\right]\frac{\not{x}\ \not{\xi}}{x.\xi}\mathcal{V}(\xi), \tag{4.23}$$

and

$$\mathcal{U}_\alpha(x,\xi,Z) = \frac{1}{2}\left[3Z_\alpha^\top - \frac{x.Z}{x.\xi}\xi_\alpha^\top - \gamma_\alpha^\top \not{Z}^\top\right]\mathcal{U}(\xi). \tag{4.24}$$

the spinor field ψ_1 satisfies the field equation

$$(Q_0 - 2)\psi_1 = 0. \tag{4.25}$$

It corresponds to the massless conformally invariant field equation [13,20].

The arbitrariness introduced with the constant vector Z will be removed by comparison of the solution with the Minkowskian limit. Unfortunately, our notations for the 'massless' conformally coupled scalar field are not adapted to the computation of the limit $H = 0$. It is due to the fact that contrary to the 'massive' case the values $\sigma = -1, -2$ are constant [21]. In order to obtain the behavior of the field solutions in the limit $H = 0$ (at least for the scalar part), one can use the global conformal coordinate system

$$x_H(X) = \left(x^0 = H^{-1}\sinh HX^0, \vec{x} = H^{-1}\frac{\vec{X}}{\parallel\vec{X}\parallel}\cosh HX^0, \times \sin H\parallel\vec{X}\parallel\ x^4 = H^{-1}\cosh HX^0\cos H\parallel\vec{X}\parallel\right), \tag{4.26}$$

where $X^0 = \rho, X^1 = \alpha, X^2 = \theta, X^3 = \phi$ [22,23]. The square-integrable solutions of the field equation are given by [24]

$$\phi(x) = \phi(\rho,\vec{v}) = \cos\rho\,\frac{e^{\pm i(L+1)\rho}}{\sqrt{L+1}}\,\mathrm{Y}_{Llm}(\vec{v}), \tag{4.27}$$

where $\mathrm{Y}_{Llm}(\vec{v})$ are the hyperspherical harmonics on $S^3 = \{v^i \in R^4 | v_1^2 + v_2^2 + v_3^2 + v_4^2 = r^2\}$. The unitary irreducible representations $\Pi^{\pm}_{\frac{3}{2},\frac{3}{2}}$ of the de Sitter group correspond to the unitary irreducible representations of the Poincaré group, in the limit $H = 0$. Since in this limit the wave solutions are equivalent to the solutions of the Minkowskian space, the numbers of the polarization states can be fixed in the same way as in the Minkowskian counterpart. It can be shown that in the $H = 0$, $L \to \infty$ limit and with

$$\rho = Ht,\ \alpha = Hr;\ HL = k_0 = |\vec{k}|, \quad \text{with} \quad \theta,\varphi \quad \text{unchanged}, \tag{4.28}$$

the functions (4.26) become, when suitably rescaled, the usual massless spherical waves (with $k^2 = (k^0)^2 - (\vec{k})^2 = 0$) [25]. With these coordinates, the de Sitter spinor-vector field solutions read

$$\Psi_\mu(\rho,\vec{v})=V^{-1}(1\pm i\gamma^4)\frac{\partial x^\alpha}{\partial X^\mu}\frac{1}{2}\left[3Z_\alpha^\top+x.Z\partial_\alpha^\top-\gamma_\alpha^\top \not{Z}^\top\right]$$
$$\times\cos\rho\frac{e^{\pm i(L+1)\rho}}{\sqrt{L+1}}\mathrm{Y}_{Llm}(\text{vecv}), \tag{4.29}$$

where V is a spinor transformation matrix from ambient space notation to intrinsic coordinate which is defined in [26].

The plane waves $(x\cdot\xi)^\sigma$ are singular at $x\cdot\xi=0$ and they are not globally defined due to the ambiguity concerning the phase factor. In contrast with the Minkowskian exponentials plane wave, these waves are singular on three-dimensional light-like manifolds and can at first be defined only on suitable halves of X_H. We will need an appropriate $i\epsilon$-prescription (indicated below) to obtain global waves; for details see [20]. For a complete determination, one may consider the solution in the complex de Sitter space-time $X_H^{(c)}$. The complex de Sitter space-time is defined as [20]

$$X_H^{(c)}=\left\{z=x+iy\in\mathbb{C}^5;\ \ \eta_{\alpha\beta}z^\alpha z^\beta=(z^0)^2-\vec{z}.\vec{z}-(z^4)^2 = -H^{-2}\right\}$$
$$=\left\{(x,y)\in\mathbb{R}^5\times\mathbb{R}^5;\ \ x^2-y^2=-H^{-2},\ x.y=0\right\}, \tag{4.30}$$

where $T^\pm=\mathbb{R}^5+iV^\pm$ and V^+ (resp. V^-) stems from the causal structure on X_H:

$$V^\pm=\left\{x\in\mathbb{R}^5;\ \ x^0\gtrless\sqrt{\parallel\vec{x}\parallel^2+(x^4)^2}\right\}. \tag{4.31}$$

We introduce their respective intersections with $X_H^{(c)}$,

$$\mathcal{T}^\pm=T^\pm\cap X_H^{(c)}. \tag{4.32}$$

which will be called forward and backward tubes of the complex de Sitter space $X_H^{(c)}$. Then we define the above 'tuboid' $X_H^{(c)}\times X_H^{(c)}$ by

$$\mathcal{T}_{12}=\{(z,z');\ \ z\in\mathcal{T}^+,z'\in\mathcal{T}^-\}. \tag{4.33}$$

More details are given in [20]. When z varies in $\mathcal{T}^+$ (or $\mathcal{T}^-$) and ξ lies in the positive cone $\mathcal{C}^+$:

$$\xi\in\mathcal{C}^+=\{\xi\in\mathcal{C};\ \xi^0>0\}.$$

The sign of the imaginary part of $(z.\xi)$ is fixed, so the plane-wave solutions are globally defined. The phase is chosen such that

$$\text{boundary value of }(z.\xi)^\sigma\ |_{x.\xi>0}>0. \tag{4.34}$$

Finally we have

$$\Psi_{1\alpha}(z)=\mathcal{U}_\alpha^{(\lambda)}(z,\xi)(Hz\cdot\xi)^{-1}, \tag{4.35}$$
$$\Psi_{2\alpha}(z)=\mathcal{V}_\alpha^{(\lambda)}(z,\xi)(Hz\cdot\xi)^{-3}, \tag{4.36}$$

where $z\in X_H^{(c)}$ and $\xi\in\mathcal{C}^+$ these solutions are globally defined in the de Sitter hyperboloid and they are independent of the choice of the intrinsic coordinate.

5 Two-point function

The two-point function of the 'massless' spin-$\frac{3}{2}$ field is defined as

$$S_{\alpha\alpha'}^{i\bar{j}}(x,x')=\left\langle\Omega\mid\Psi_\alpha^i(x)\overline{\Psi}_{\alpha'}^{\bar{j}}(x')\mid\Omega\right\rangle, \tag{5.1}$$

where $x,x'\epsilon X_H$. This function is a solution of the wave equation (3.19) with respect to x and the wave equation (3.22) with respect to x'. In Sect. 7.3, the solution of Eq. (3.22) is obtained and here we consider Eq. (3.19). The solution can be found in terms of the two-point function of the spinor field, which was calculated in the previous paper [13].

By using the recurrence formula (4.1), we define

$$S_{\alpha\alpha'}(x,x')=\theta_\alpha.\theta'_{\alpha'}S_1(x,x')-D_{\frac{3}{2}\alpha}S_2(x,x')\gamma^4\overleftarrow{D}'_{\frac{3}{2}\alpha'}\gamma^4$$
$$-\gamma_\alpha^\top S_3(x,x')\gamma^4\gamma'^\top_{\alpha'}\gamma^4. \tag{5.2}$$

By imposing the two-point function $S_{\alpha\alpha'}$ to obey Eq. (3.19) and by using the identities of Eqs. (4.2)–(4.4), S_1, S_2, and S_3 must be satisfied by the following equations:

$$(\not{x}\not{\partial}^\top-1)S_1(x,x')=0, \tag{5.3}$$
$$\not{x}\not{\partial}^\top S_3(x,x')\gamma^4\gamma'^\top_{\alpha'}\gamma^4+\not{x}(\not{x}\not{\partial}^\top-4)$$
$$\times S_2(x,x')\gamma^4\overleftarrow{D}'_{\frac{3}{2}\alpha'}\gamma^4=\not{x}x.\theta'S_1(x,x'), \tag{5.4}$$
$$2(1-2\lambda)S_3(x,x')\gamma^4\gamma'^\top_{\alpha'}\gamma^4+\not{x}\left[(1-\lambda)\not{x}\not{\partial}^\top-2(1-2\lambda)\right]$$
$$\times S_2(x,x')\gamma^4\overleftarrow{D}'_{\frac{3}{2}\alpha'}\gamma^4=\lambda\gamma^\top.\theta'S_1(x,x'). \tag{5.5}$$

For $\lambda=\frac{1}{2}$, Eq. (5.5) becomes

$$\not{\partial}^\top S_2(x,x')\gamma^4\overleftarrow{D}'_{\frac{3}{2}\alpha'}\gamma^4=\gamma^\top.\theta'S_1(x,x'), \tag{5.6}$$

and we know that S_1 is a solution of Eq. (5.3) and it is given by [13]

$$S_1(x,x')=\frac{1}{4\pi}\left[3P_{-1}^{(7)}(x.x')\not{x}-P_{-3}^{(7)}(x.x')\not{x}'\right]\gamma^4. \tag{5.7}$$

S_2 and S_3 are given by

$$S_2(x,x')\gamma^4\overleftarrow{D}'_{\frac{3}{2}\alpha'}\gamma^4=-\frac{1}{2}x.\theta'S_1+S_g,$$
$$S_3(x,x')\gamma^4\gamma'^\top_{\alpha'}\gamma^4=-\not{x}S_g. \tag{5.8}$$

Finally, the two-point function is obtained similarly to Eq. (4.14) as

$$S_{\alpha\alpha'}(x,x')=D_{\alpha\alpha'}(x,\partial^\top;x',\partial'^\top)S_1(x,x'), \tag{5.9}$$

where

$$D_{\alpha\alpha'}=\frac{1}{2}\left[3\theta_\alpha\cdot\theta'_{\alpha'}+x\cdot\theta'_\alpha\partial_\alpha^\top+\gamma_\alpha^\top\gamma\cdot\theta'\right], \tag{5.10}$$

this function satisfies the following conditions.

(a) **Indefinite sesquilinear form**: For any spinor-vector test function $f_\alpha \in \mathcal{D}(X_H)$, we have an indefinite sesquilinear form that is defined by

$$\int_{X_H \times X_H} \bar{f}_i^{\alpha}(x) S^{i\bar{j}}_{\alpha\alpha'}(x,x') f^{\alpha'}_{\bar{j}}(x') \mathrm{d}\sigma(x)\mathrm{d}\sigma(x'), \tag{5.11}$$

where $\bar{f}$ is the adjoint of f, and $\mathrm{d}\sigma(x)$ denotes the de Sitter-invariant measure on X_H [20]. $\mathcal{D}(X_H)$ is the space of C^∞ spinor-vector functions with compact support in X_H and with values in $\mathbb{C}^5$.

(b) **Covariance** We have

$$\Lambda^\alpha_\beta \Lambda^{\alpha'}_{\beta'} g^{-1} S_{\alpha\alpha'}\big(\Lambda(g)x, \Lambda(g)x'\big) i(g) = S_{\beta\beta'}(x,x'), \tag{5.12}$$

where $\Lambda \in \mathrm{SO}_0(1,4)$, $g \in Sp(2,2)$ and $g\gamma^\alpha g^{-1} = \Lambda^\alpha_\beta \gamma^\beta$. $i(g)$ is the group involution defined by

$$i(g) = -\gamma^4 g \gamma^4. \tag{5.13}$$

(c) **Locality** For every space-like separated pair (x,x'), *i.e.* $x \cdot x' > -H^{-2}$,

$$S^{i\bar{j}}_{\alpha\alpha'}(x,x') = -S^{\bar{j}i}_{\alpha'\alpha}(x',x), \tag{5.14}$$

where $S^{\bar{j}i}_{\alpha'\alpha}(x',x) = \langle \Omega \mid \overline{\Psi}^{\bar{j}}_{\alpha'}(x') \Psi^i_\alpha(x) \mid \Omega\rangle$.

(d) **Transversality** We have

$$x \cdot S(x,x') = 0 = x' \cdot S(x,x'). \tag{5.15}$$

(e) **Normal analyticity** $S_{\alpha\alpha'}(x,x')$ is the boundary value (in the distributional sense) of an analytic function $\mathrm{S}_{\alpha\alpha'}(z,z')$.
$\mathrm{S}_{\alpha\alpha'}(z,z')$ is maximally analytic, i.e., it can be analytically continued to the 'cut domain' [13,20]:

$$\Delta = \left\{(z,z') \in X_H^{(c)} \times X_H^{(c)} \ : \ (z-z')^2 \le 0\right\}.$$

The two-point function $S_{\alpha\alpha'}(x,x')$ is the boundary value of $\mathrm{S}_{\alpha\alpha'}(z,z')$ from $\mathcal{T}_{12}$ and the 'permuted two-point function' $S_{\alpha'\alpha}(x',x)$ is the boundary value of $\mathrm{S}_{\alpha\alpha'}(z,z')$ from the domain

$$\mathcal{T}_{21} = \{(z,z'); \quad z' \in \mathcal{T}^+, z \in \mathcal{T}^-\}.$$

6 Conclusions

We have studied the 'massless' spin-$\frac{3}{2}$ fields in the de Sitter space-time in the ambient space formalism. This formalism is independent from the chosen coordinate system. Gauge and conformal invariances are the properties of 'massless' fields. Gauge invariance and the Gupta–Bleuler triplet are discussed. We have shown that the field equation of 'massless' spin-$\frac{3}{2}$ fields is gauge invariant. The two-point function is calculated. We saw that the spinor field ψ_1 in Eq. (4.25) is conformally invariant. Conformal invariance of the spinor-vector field will be considered in the forthcoming paper [27].

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7 Appendix

7.1 The field equation (3.13)

In this appendix the field Eq. (3.13) is considered:

$$\tilde{D}^\delta_\alpha \Psi_\delta = 0. \tag{7.1}$$

The solutions of Eq. (7.1) obviously satisfy the field Eq. (3.5). This equation can be written in the simple form

$$\not{x}\,\not{\partial}^\top \Psi_\alpha(x) - x_\alpha \not{x}\,\not{\Psi} - 3\Psi_\alpha(x) = 0. \tag{7.2}$$

For clarity it can be written as

$$\Big(\mathrm{Trpr}\,\not{x}\,\not{\partial}^\top - 3\Big)\,\Psi_\alpha(x) = 0. \tag{7.3}$$

Now we rewrite Eq. (7.3) as

$$\Big(\mathrm{Trpr}\,\not{x}\,\not{\partial}^\top - 3\Big)\,\Psi_\alpha(x) - \partial^\top_\alpha \not{x}\,\not{\Psi}(x) = 0, \tag{7.4}$$

which is invariant under the gauge transformation

$$\Psi_\alpha(x) \longrightarrow \Psi'_\alpha(x) = \Psi_\alpha(x) + D_{\frac{3}{2}\alpha}\zeta, \tag{7.5}$$

where ζ is an arbitrary spinor field. Let us introduce a gauge fixing parameter λ for this case. Now the wave equation reads

$$\Big(\mathrm{Trpr}\,\not{x}\,\not{\partial}^\top - 3\Big)\,\Psi_\alpha(x) - \lambda\partial^\top_\alpha \not{x}\,\not{\Psi}(x) = 0. \tag{7.6}$$

The above equation under the gauge transformation (7.5) becomes

$$\Big(\mathrm{Trpr}\,\not{x}\,\not{\partial}^\top - 3\Big)\,\Psi_\alpha(x) - \lambda\partial^\top_\alpha \not{x}\,\not{\Psi}(x)$$
$$= (\lambda - 1)\partial^\top_\alpha \not{x}\gamma^\beta\, D_{\frac{3}{2}\beta}\zeta. \tag{7.7}$$

With the choice of a value $\lambda \neq 1$, the gauge field $D_{\frac{3}{2}\alpha}\zeta$ is solved for the field equation only if ζ obeys

$$\partial^\top_\alpha \not{x}\gamma^\beta D_{\frac{3}{2}\beta}\zeta = (\mathrm{Trpr}\,\not{x}\,\not{\partial}^\top - 3) D_{\frac{3}{2}\alpha}\zeta = 0, \tag{7.8}$$

which means that the gauge field ζ is fixed. The adjoint field $\bar{\Psi}_\alpha$ satisfies the equation

$$\bar{\Psi}_\alpha \gamma^4 \overleftarrow{\not{\partial}}^\top \not{x} \gamma^4 - \bar{\Psi}_\beta \gamma^4 (\gamma^\beta x_\alpha \not{x}) \gamma^4 + 3\bar{\Psi}_\alpha$$
$$-\lambda \bar{\Psi}_\beta \gamma^4 \gamma^\beta \not{x} \overleftarrow{\partial}^\top_\alpha \gamma^4 = 0. \tag{7.9}$$

By putting Ψ_α in Eq. (7.6), we find that the spinor fields ψ_1, ψ_2, and ψ_3 must obey the following equations:

$$(\not{x}\not{\partial}^\top - 3)\psi_1 = 0, \tag{7.10}$$
$$(\lambda - 1)[4\not{x} + \not{\partial}^\top]\psi_2 + 2(1-2\lambda)\psi_3 = \lambda \not{Z}^\top \psi_1, \tag{7.11}$$
$$(\lambda - 1)[4\not{x} + \not{\partial}^\top]\psi_2 + [\not{x}\not{\partial}^\top - 2(2\lambda + 1)]\psi_3$$
$$= (\not{x} x.Z + \lambda \not{Z}^\top)\psi_1. \tag{7.12}$$

By multiplying Eq. (7.10) by $\not{\partial}^\top$, we have

$$Q_0 \psi_1 = 0.$$

For $\lambda = \frac{1}{2}$ Eq. (7.11) becomes

$$\not{\partial}^\top \psi_2 + 4\not{x}\psi_2 = -\not{Z}^\top \psi_1, \tag{7.13}$$

and Eq. (7.12) by using Eq. (7.13) becomes

$$\not{x}\not{\partial}^\top \psi_3 - 4\psi_3 = \not{x} x \cdot Z\psi_1. \tag{7.14}$$

Using Eq. (7.13) and multiplying it by $\not{\partial}^\top$, we obtain

$$Q_0\psi_2 = -4\psi_2 - \not{Z}^\top \not{x}\psi_1 + 2Z\cdot\partial^\top\psi_1 + 4Z\cdot x\psi_1, \tag{7.15}$$

and by using Eq. (7.14), we obtain

$$\not{x} Q_0 \psi_3 = -4\not{x}\psi_3 + 2x\cdot Z\psi_1 + \not{Z}^\top \not{x}\psi_1. \tag{7.16}$$

Since the homogenized coefficient is the same for the functions ψ_1, ψ_2, and ψ_3, we have

$$Q_0\psi_2 = 0,$$
$$Q_0\psi_3 = 0.$$

Then we have

$$\psi_2 = \frac{1}{4}(-\not{Z}^\top \not{x}\psi_1 + 2Z\cdot\partial^\top\psi_1 + 4x\cdot Z\psi_1), \tag{7.17}$$
$$\not{x}\psi_3 = \frac{1}{4}(2x\cdot Z\psi_1 + \not{Z}^\top \not{x}\psi_1). \tag{7.18}$$

So the solution be written as follows:

$$\Psi_\alpha = \left(-\frac{1}{4}Z^\top_\alpha - \frac{5}{4}\gamma^\top_\alpha \not{x} x\cdot Z + \frac{1}{4}\not{Z}^\top \not{x}\partial^\top_\alpha - \frac{1}{2}\partial^\top_\alpha Z\cdot\partial^\top \right.$$
$$\left. -x\cdot Z\partial^\top_\alpha - \frac{1}{2}\gamma^\top_\alpha \not{x} Z\cdot\partial^\top + \frac{1}{4}\not{Z}^\top \gamma^\top_\alpha\right)\psi_1. \tag{7.19}$$

7.2 The proof of Eq. (4.13)

By replacing Eq. (4.12) in (4.3) we have

$$\not{x}\not{\partial}^\top\psi_3 + \not{x}(\not{x}\not{\partial}^\top - 4)\left(\frac{1}{2}\not{Z}^\top\not{x}\psi_1 + \psi_g\right) = \not{x} x.Z\psi_1. \tag{7.20}$$

Multiplying Eq. (7.20) by $\not{x}$, we get

$$\not{\partial}^\top\psi_3 + \frac{1}{2}\not{x}\not{\partial}^\top\not{Z}^\top\not{x}\psi_1 - 2\not{Z}^\top\not{x}\psi_1 - 4\psi_g = x.Z\psi_1. \tag{7.21}$$

By using the equation

$$\not{\partial}^\top\not{Z}^\top\not{x}\psi_1 = 4\not{x} x.Z\psi_1 - 2\not{Z}^\top\psi_1 + 2\not{x} Z.\partial^\top\psi_1, \tag{7.22}$$

and the following conditions:

$$x.Z\psi_1 = -\not{Z}^\top\not{x}\psi_1\,, \quad \not{\partial}^\top\psi_g = 0,$$

we have

$$\not{\partial}^\top\psi_3 = 4\psi_g. \tag{7.23}$$

Finally ψ_3 is obtained as follows:

$$\psi_3 = \not{x}\psi_g. \tag{7.24}$$

7.3 Two-point function

Here, the two-point function is calculated with respect to x', which satisfies Eq. (3.22). In this case, we obtain

$$S_1(x,x')\gamma^4(\overleftarrow{\not{\partial}'^\top}\not{x} - 1)\gamma^4 = 0, \tag{7.25}$$
$$\gamma^\top_\alpha S_3(x,x')\gamma^4\overleftarrow{\not{\partial}'^\top}\not{x}\gamma^4 + D_{\frac{3}{2}\alpha}S_2(x,x')\gamma^4(\overleftarrow{\not{\partial}'^\top}\not{x} - 4)$$
$$\times\not{x}\gamma^4 = S_1(x,x')\gamma^4\not{x} x.\theta'\gamma^4, \tag{7.26}$$
$$2(1-2\lambda)\gamma^\top_\alpha S_3(x,x')$$
$$+D_{\frac{3}{2}\alpha}S_2(x,x')\gamma^4\left[(1-\lambda)\overleftarrow{\not{\partial}'^\top}\not{x} - 2(1-2\lambda)\right]\not{x}\gamma^4$$
$$= \lambda S_1(x,x')\gamma^4\gamma^\top.\theta'\gamma^4. \tag{7.27}$$

Therefore, S_2 and S_3 are given by

$$D_{\frac{3}{2}\alpha}S_2(x,x') = -S_1\frac{1}{2}\gamma^4 x'.\theta\gamma^4 - S_g,$$
$$\gamma^\top_\alpha S_3(x,x') = -S_g\gamma^4\not{x}\gamma^4. \tag{7.28}$$

Finally the two-point function in this case (for $\lambda = \frac{1}{2}$) is

$$S_{\alpha\alpha'}(x,x') = S_1(x,x')\overleftarrow{D}_{\alpha\alpha'}(x,\partial^\top;x',\partial'^\top), \tag{7.29}$$

where

$$D_{\alpha\alpha'}(x,\partial^\top;x',\partial'^\top) = \frac{1}{2}\left[3\theta_\alpha\cdot\theta'_{\alpha'} + \overleftarrow{\not{\partial}'^\top} x\cdot\theta'_\alpha + \gamma^4\gamma^\top_{\alpha'}\gamma\cdot\theta\gamma^4\right].$$

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