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A numerical approach for the solution of a class of singular boundary value problems arising in physiology

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Abstract

The purpose of this paper is to introduce a novel approach based on the operational matrix of orthonormal Bernoulli polynomial for the numerical solution of the class of singular second-order boundary value problems that arise in physiology. The main thrust of this approach is to decompose the domain of the problem into two subintervals. The singularity, which lies in the first subinterval, is removed via the application of an operational matrix procedure based on differentiating that is applied to surmount the singularity. Then, in the second subdomain, which is outside the vicinity of the singularity, the resulting problem is treated via employing the proposed basis. The performance of the numerical scheme is assessed and tested on specific test problems. The oxygen diffusion problem in spherical cells and a nonlinear heat-conduction model of the human head are discussed as illustrative examples. The numerical outcomes indicate that the method yields highly accurate results and is computationally more efficient than the existing ones.

MSC: 34L30; 34L16

Keywords: orthonormal Bernoulli polynomials; differential operational matrix; physiology problems

1 Introduction

The aim of this paper is to introduce a new approach for the numerical solution of the following class of singular boundary value problems (SBVP) arising in physiology:

$$y'' + \frac{m}{x}y' = f(x,y) \tag{1}$$

defined on the interval $[0,1]$ and subject to the following boundary conditions:

$$\alpha_1 y(0) + \beta_1 y'(0) = \gamma_1, \tag{2}$$

$$\alpha_2 y(1) + \beta_2 y'(1) = \gamma_2. \tag{3}$$

We assume that $f(x,y)$ is continuous, $\frac{\partial f}{\partial x}$ exists and it is continuous on the domain $[0,1]$. SBVP (1)-(3) arises in important applications, for different values of $m = 0, 1, 2$ and certain linear and nonlinear functions $f(x,y)$. For instance, SBVP (1)-(3) with $m = 2$ and

$$f(x,y) = \frac{ny}{y+k}, \quad n > 0, k > 0$$

arises in the modeling of steady state oxygen diffusion in a spherical cell with Michaelis-Menten uptake kinetics (see [1, 2]). Another case of physical significance is when $m = 2$ and

$$f(x,y) = -le^{-lky}, \quad l > 0, k > 0,$$

which occurs in the formulation of the distribution of heat sources in the human head (see [3, 4]).

In recent years, finding numerical solutions of singular differential equations, particularly those arising in physiology, has been focused by several authors. Ravi Kanth and Bhattacharya [5] used a quasi-linearization technique to reduce a class of nonlinear SBVP arising in physiology to a sequence of linear problems; the resulting set of differential equations is modified at the singular point, then spline collocation is utilized to obtain the numerical solution. Pandey and Singh [6] described a finite difference method based on a uniform mesh for the solution of a class of SBVP arising in physiology; it was shown that the method is of second-order accuracy under quite general conditions. Caglar *et al.* [7] used B-spline functions to develop a numerical method for computing approximations to the solution of nonlinear SBVP associated with physiological science. The original differential equation was modified at the singular point, then the boundary value problems were treated by using the B-spline approximation. Asaithambi and Garner [8] presented a numerical technique for obtaining pointwise bounds for the solution of a class of nonlinear boundary-value problems appearing in physiology. Gustaffsson [9] presented a numerical method for solving SBVP. Ravi Kanth and Reddy [10] presented a numerical method for solving a two-point boundary value problem in the interval $[0,1]$ with regular singularity at $x = 0$. Ravi Kanth and Reddy [11] presented a numerical method for singular two-point boundary value problems via Chebyshev economization. A number of papers discussed the existence of solutions for the given problem, for instance, existence and uniqueness of the solution of SBVP (1)-(3) for the special case $m = 2$, $\alpha_1 = \alpha_2 = \gamma$ and $\beta = 1$ were given in [12]. In the past decade, there has been a great deal of interest [13–21] in applying the decomposition method for solving a wide range of nonlinear equations, including algebraic, differential, partial-differential, differential-delay and integro-differential equations.

This paper is organized as follows. In Section 2, we are going to introduce Bernoulli polynomials and their properties; also we will show the operational matrix of derivative for orthonormal Bernoulli polynomials. In Section 3, the operational matrix of derivative of the proposed basis together with collocation method are used to reduce the nonlinear singular ordinary differential equation to a nonlinear algebraic equation that can be solved by Newton's method. Section 4 illustrates some applied models to show the convergence, accuracy and advantage of the proposed method and compares it with some other existing method. Finally Section 5 concludes the paper.

2 Definition of Bernoulli polynomials

In this section, we introduce Bernoulli polynomials and their properties such as differentiation, integral means conditions, *etc.* Obviously, Bernoulli polynomials are not orthonormal polynomials, we orthonormal these polynomials by Gram-Schmidt algorithm. The operational matrix constructed by this new basis is sparser than the operational matrix which is made by Bernoulli polynomial, which can be the advantage of normalization of Bernoulli polynomials. Also, by this new basis, we construct a new approach to solve SBVP, which can get better approximation for numerical solution of these equations in comparison with other methods.

2.1 Bernoulli polynomials

The recurrence formula for Bernoulli polynomials of degree n is defined as [22]

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k(x)=(n+1)x^n, \tag{4}$$

where

$$\begin{aligned}
&B_0(x)=1,\\
&B_1(x)=x-\frac{1}{2},\\
&B_2(x)=x^2-x+\frac{1}{6},\\
&B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x,\\
&\vdots
\end{aligned}$$

If in Eq. (4) n varies from 0 to N, we have the following matrix equation $MB(x)=X$, such that

$$\begin{aligned}
&M=\begin{pmatrix}
\binom{1}{0} & 0 & \cdots & 0\\
\frac{1}{2}\binom{1}{0} & \frac{1}{2}\binom{2}{1} & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
\frac{1}{N+1}\binom{N+1}{0} & \frac{1}{N+1}\binom{N+1}{1} & \cdots & \frac{1}{N+1}\binom{N+1}{N}
\end{pmatrix},\\
&B(x)=\left[B_0(x),B_1(x),\ldots,B_N(x)\right]^T,\\
&X=\left[1,x,\ldots,x^N\right]^T.
\end{aligned}$$

Since M is a lower triangular matrix with nonzero diagonal elements, it is nonsingular and hence M^{-1} exists. Thus, the Bernoulli polynomials in vector form can be given directly from

$$B(x)=M^{-1}X. \tag{5}$$

By using Gram-Schmidt algorithm, we obtain orthonormal Bernoulli polynomials to construct a new basis, this new basis is introduced by $OB_n(x)$. For instance, for $n=4$,

$$B_4(x)=x^4-2x^3+x^2-\frac{1}{30},$$

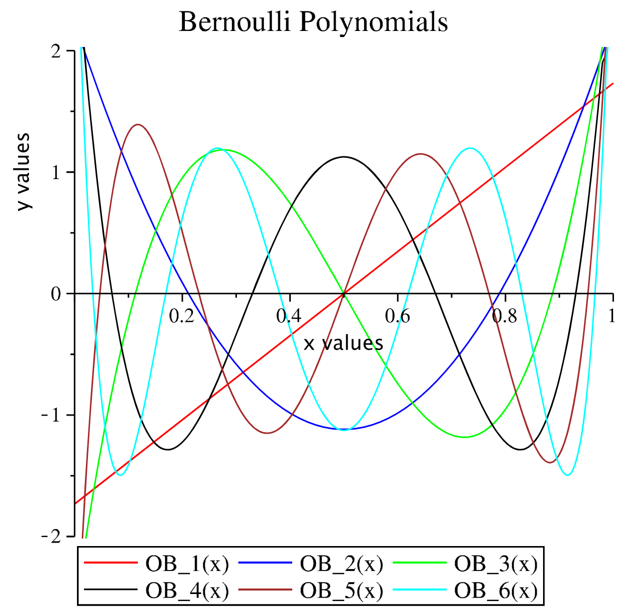

Figure 1 The graph of the first six of orthonormal Bernoulli polynomials.

and

$$OB_4(x) = 210x^4 - 420x^3 + 270x^2 - 60x + 3.$$

Thus, the proposed basis is

$$\{OB_0(x), OB_1(x), \dots, OB_n(x)\}.$$

So, we can approximate the functions by using this basis. The orthonormal Bernoulli polynomials considered in this paper have special properties and applications in different fields of mathematics apart from analytic theory of numbers to the classical and numerical analysis [23, 24]. In the following, we mention some properties of the Bernoulli polynomials which will be of fundamental importance in the sequel.

- Property 1 (Integral means conditions):

$$\int_0^1 OB_i(x)\,dx = 0, \quad i = 1, 2, \dots, n.$$

- Property 2 (Norm):

$$\|OB_i(x)\|_2 = 1, \quad i = 1, 2, \dots, n.$$

In Figure 1 the behavior of several orthonormal Bernoulli polynomials in the interval $[0, 1]$ is shown. The property of $\int_0^1 OB_i(x)\,dx = 0$ could be observed geometrically.

2.2 Function approximation

Suppose $H = L^2[0,1]$ and $\{OB_0(x), OB_1(x), \dots, OB_n(x)\} \subset H$, also

$$Y = \operatorname{span}\{OB_0(x), OB_1(x), \dots, OB_n(x)\}$$

and $f(x)$ is an arbitrary element in H. Since Y is a finite dimensional vector space, $f(x)$ has the unique best approximation [25] out of Y such as $f_0(x) \in Y$, that is,

$$\forall y(x) \in Y, \quad \|f(x) - f_0(x)\| \leq \|f(x) - y(x)\|.$$

Since $f_0(x) \in Y$, there exist the unique coefficients $c_0, c_1, \dots, c_n$ such that

$$f(x) \simeq f_0(x) = \sum_{i=0}^{n} c_i OB_i(x) = C^T OB(x), \tag{6}$$

where

$$OB(x) = [OB_0(x), OB_1(x), \dots, OB_n(x)] \tag{7}$$

and

$$C = [c_0, c_1, \dots, c_n]^T, \tag{8}$$

where the constant coefficient c_i for $i = 1, 2, \dots, n$ is

$$c_i = \frac{\langle f(x), OB_i(x)\rangle}{\langle OB_i(x), OB_i(x)\rangle}. \tag{9}$$

2.3 Error bounds

In this section, the error bound and convergence are established by the following lemma.

Theorem 1 [25] *Let H be a Hilbert space and W be a closed subspace of H such that $\dim W < \infty$ and $\{w_1, w_2, \dots, w_n\}$ is any basis for W. Let g be an arbitrary element in H and g_0 be the unique best approximation to g out of W. Then*

$$\|g - g_0\| = G_g,$$

where

$$G_g = \sqrt{\frac{F(g, w_1, \dots, w_n)}{F(w_1, \dots, w_n)}},$$

and F is introduced in [25].

Lemma 1 [26] *Suppose that $g \in C^{m+1}$ is an $m+1$ times continuously differentiable function such that $g = \sum_{j=1}^{n} g_j$, and let $Y = \operatorname{span}\{OB_0(x), OB_1(x), \dots, OB_n(x)\}$. If $C_j^T OB_j(x)$ is the best approximation to g_j from Y, then $C^T OB(x)$ approximates g with the following error bound*:

$$\|g - C^T OB(x)\|_2 \leq \frac{\delta}{n^{m+1}(m+1)!\sqrt{2m+3}}, \quad \delta = \max_{x\in[0,1]} |g^{m+1}(x)|.$$

Proof The Taylor expansion for the function $g_j(x)$ is

$$g_j(x) \simeq \tilde{g}_j(x) = g_j\left(\frac{j-1}{n}\right) + g_j'\left(\frac{j-1}{n}\right)\left(x - \frac{j-1}{n}\right) + g_j''\left(\frac{j-1}{n}\right)\frac{(x-\frac{j-1}{n})^2}{2!} + \cdots$$
$$+ g_j^{(m)}\left(\frac{j-1}{n}\right)\frac{(x-\frac{j-1}{n})^m}{m!}, \quad \frac{j-1}{n} \le x < \frac{j}{n},$$

for which it is known that

$$\left|g_j(x) - \tilde{g}_j(x)\right| \le \left|g^{(m+1)}(\eta)\right| \frac{(x-\frac{j-1}{n})^{m+1}}{(m+1)!}, \quad \eta \in \left[\frac{j-1}{n}, \frac{j}{n}\right], j = 1, 2, \ldots, n. \tag{10}$$

Since $C_j^T OB_j(x)$ is the best approximation to g_j from Y and $\tilde{g}_j \in Y$, using Eq. (10), we have

$$\begin{aligned}
\left\|g_j - C_j^T OB_j(x)\right\|_2^2 \le \|g_j - \tilde{g}_j\|_2^2 &= \int_{(j-1)/n}^{j/n} \left|g_j(x) - \tilde{g}_j(x)\right|^2 dx \\
&\le \int_{(j-1)/n}^{j/n} \left(\frac{|g^{(m+1)}(\eta)|(x-\frac{j-1}{n})^{m+1}}{(m+1)!}\right)^2 dx \\
&\le \left[\frac{\delta}{(m+1)!}\right]^2 \int_{(j-1)/n}^{j/n} \left(x - \frac{j-1}{n}\right)^{2m+2} dx \\
&= \left[\frac{\delta}{(m+1)!}\right]^2 \frac{1}{n^{2m+3}(2m+3)}.
\end{aligned}$$

Now,

$$\left\|g - C^T OB(x)\right\|_2^2 \le \sum_{j=1}^{n} \left\|g_j - C_j^T OB_j(x)\right\|_2^2 \le \frac{\delta^2}{n^{2m+2}[(m+1)!]^2(2m+3)}.$$

By taking the square roots, we have the above bound. □

2.4 Operational matrix of derivative

The derivative of the vector $OB(t)$ can be expressed by

$$\frac{d(OB(t))}{dt} = \mathbf{D}^{(1)} OB(t), \tag{11}$$

where $\mathbf{D}^{(1)}$ is the $(n+1) \times (n+1)$ operational matrix of derivative and its elements are

$$\mathbf{D}^{(1)} = (d_{ij}) = \begin{cases} 2\sqrt{2i-1}\sqrt{2j-1}, & i > j \text{ and } i + j = 2l + 1, \\ 0, & \text{otherwise} \end{cases}$$

for each $l \in \mathbb{N}$. For example, for $n = 5$, we have

$$\mathbf{D}^{(1)} = 2\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{5}\sqrt{3} & 0 & 0 & 0 & 0 \\ \sqrt{7} & 0 & \sqrt{7}\sqrt{5} & 0 & 0 & 0 \\ 0 & \sqrt{9}\sqrt{3} & 0 & \sqrt{9}\sqrt{7} & 0 & 0 \\ \sqrt{11} & 0 & \sqrt{11}\sqrt{5} & 0 & \sqrt{11}\sqrt{7} & 0 \end{pmatrix}.$$

By using Eq. (11), it is clear that

$$\frac{d^n OB(x)}{dx^n} = \left(\mathbf{D}^{(1)}\right)^n OB(x), \tag{12}$$

where $n \in \mathbb{N}$ and the superscript in $\mathbf{D}^{(1)}$ denotes matrix powers. Thus

$$\mathbf{D}^{(n)} = \left(\mathbf{D}^{(1)}\right)^n, \quad n = 1, 2, \ldots. \tag{13}$$

The presented operational matrix is a sparse matrix, so it can reduce the error of solving Eq. (1).

3 Implementation of operational matrix of orthonormal Bernoulli polynomials on physiology problems

In this section we solve nonlinear singular boundary value problem of the form Eq. (1) with the mixed conditions Eq. (2) and Eq. (3) by using the operational matrix of derivative [27] based on orthonormal Bernoulli polynomials. From Eq. (6) we can approximate an unknown function as

$$y(x) = C^T OB(x), \tag{14}$$

where $OB(x)$ and C are defined in Eqs. (7)-(8). By using Eqs. (11)-(12) we have

$$y'(x) = C^T OB'(x) = C^T \mathbf{D}^{(1)} OB(x), \tag{15}$$

and

$$y''(x) = C^T OB''(x) = C^T \left(\mathbf{D}^{(1)}\right)^2 OB(x). \tag{16}$$

By substituting Eqs. (14)-(16) in Eq. (1), we have

$$C^T \left(\mathbf{D}^{(1)}\right)^2 OB(x) + \left(\frac{m}{x}\right) C^T \mathbf{D}^{(1)} OB(x) = f\left(x, C^T OB(x)\right). \tag{17}$$

Also, by using Eqs. (2)-(3) and Eqs. (14)-(15), we have

$$\alpha_1 C^T OB(0) + \beta_1 C^T \mathbf{D}^{(1)} OB(0) = \gamma_1, \tag{18}$$

$$\alpha_2 C^T OB(1) + \beta_2 C^T \mathbf{D}^{(1)} OB(1) = \gamma_2. \tag{19}$$

Eqs. (18)-(19) give two linear equations. Since the total unknowns for vector C in Eq. (14) is $(n+1)$, we collocate Eq. (17) in $(n-1)$ Newton-Cotes points x_i in the interval $[0,1]$, then we have

$$C^T \left(\mathbf{D}^{(1)}\right)^2 OB(x_i) + \left(\frac{m}{x_i}\right) C^T \mathbf{D}^{(1)} OB(x) = f\left(x_i, C^T OB(x_i)\right) \tag{20}$$

for $i = 1, \ldots, n-1$. Now the resulting Eqs. (18)-(20) generate a system of $(n+1)$ nonlinear equations in $(n+1)$ unknown. We used the `Maple 15` software to solve this nonlinear system.

4 Illustrative examples

To show the efficiency of the proposed method, we implement it on three nonlinear singular boundary problems that arise in real physiology applications. Our results are compared with the results in Refs. [28, 29]. The austerity of our method in implementation in analogy to other existing methods and its trusty answers is considerable.

Example 1 Consider the following oxygen diffusion problem:

$$y''(x) + \frac{2}{x}y'(x) = \frac{0.76129y}{y + 0.03119},$$

with boundary conditions

$$y'(0) = 0, \qquad 5y(1) + y'(1) = 5.$$

Table 1 shows the numerical results for various number of meshes, and present method solutions are compared with the results in Refs. [28] and [30].

Example 2 Consider this problem that coincides by heat conduction model of the human head

$$y''(x) + \frac{2}{x}y'(x) = -e^{-y},$$

we consider the solution of this problem with conditions as follows:

$$y'(0) = 0, \qquad y(1) + y'(1) = 0.$$

Table 2 illustrates the results for this example by the proposed method alongside numerical solutions for this example that were given in Refs. [27] and [29].

Example 3 Consider the following singular two-point boundary value problem:

$$y''(x) + \frac{1}{x}y'(x) = -e^{y},$$

$$y'(0) = 0, \qquad y(1) = 0,$$

Table 1 Approximate solutions for Example 1

x	Present method with $n = 14$	Method in [28] with $n = 20$	Method in [30] with $n = 60$
0.0	0.82848328186193	0.82848329481355	0.82848327295802
0.1	0.82970609243390	0.82970609688790	0.82970607521884
0.2	0.83337473359110	0.83337473804308	0.83337471691089
0.3	0.83948991395381	0.83948991833986	0.83948989814383
0.4	0.84805278499617	0.84805278876051	0.84805277036165
0.5	0.85906492716933	0.85906492753032	0.85906491397434
0.6	0.87252831995828	0.87252831569855	0.87252830841853
0.7	0.88844530562329	0.88844529949702	0.88844529589927
0.8	0.90681854806690	0.90681854179965	0.90681854026297
0.9	0.92765098836568	0.92765098305256	0.92765098252660
1.0	0.95094579849657	0.95094579480523	0.95094579461056

Table 2 Approximate solutions for Example 2

x	Present method with $n = 14$	Method in [27] with forth-order	Method in [29]
0.0	0.3675152742	0.3675181074	0.3675169710
0.1	0.3663623292	0.3663637561	0.3663623697
0.2	0.3628940661	0.3628959378	0.3628941066
0.3	0.3570975457	0.3570991429	0.3570975842
0.4	0.3489484206	0.3489499903	0.3489484612
0.5	0.3384121487	0.3384136581	0.3384121893
0.6	0.3254435224	0.3254450019	0.3254435631
0.7	0.3099860402	0.3099878567	0.3099860810
0.8	0.2919711030	0.2919789654	0.2919711440
0.9	0.2713170101	0.2713185637	0.2713170512
1.0	0.2479277233	0.2479292837	0.2479277646

Table 3 Numerical errors for Example 3

x	Present method with $n = 10$	Present method with $n = 14$	Approach II [28] with $n = 20$
0.0	3.77×10^{-8}	6.72×10^{-8}	2.00×10^{-6}
0.1	1.05×10^{-7}	6.69×10^{-8}	1.99×10^{-6}
0.2	6.33×10^{-9}	7.87×10^{-9}	1.97×10^{-6}
0.3	5.91×10^{-8}	6.92×10^{-9}	1.94×10^{-6}
0.4	2.12×10^{-7}	2.87×10^{-8}	1.83×10^{-6}
0.5	1.00×10^{-8}	7.40×10^{-10}	1.78×10^{-6}
0.6	5.36×10^{-7}	6.32×10^{-8}	1.67×10^{-6}
0.7	4.25×10^{-8}	6.95×10^{-8}	1.34×10^{-6}
0.8	8.32×10^{-7}	3.38×10^{-9}	9.20×10^{-7}
0.9	4.67×10^{-8}	7.85×10^{-8}	4.57×10^{-7}
1.0	6.42×10^{-8}	6.63×10^{-8}	0

with the exact solution $y(x) = 2\ln(\frac{c+1}{cx^2+1})$, where $c = 3-2\sqrt{2}$. Table 3 shows numerical errors of this example in analogy to errors for this example in [28].

5 Conclusion

This paper presents a new approach based on the operational matrix of derivative of the orthonormal Bernoulli polynomials for the numerical solution of a class of singular boundary value problems arising in physiology. By use of orthonormal Bernoulli polynomials as basis and the operational matrix of derivative of these functions, we convert such problems to an algebraic system. The implementation of current approach in analogy to existing methods is more convenient and the accuracy is high, and we can execute this method in a computer in a speedy manner with minimum CPU time used. The numerical applied models that have been presented in the paper are compared with other methods.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this work equally and read and approved the final version of the manuscript.

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