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Integral reduction by unitarity method for two-loop amplitudes: a case study

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Abstract: In this paper, we generalize the unitarity method to two-loop diagrams and use it to discuss the master integrals of reduction. To test out method, we focus on the four-point double-box diagram as well as its related daughter diagrams, i.e., the double-triangle diagram and the triangle-box diagram. For later two kinds of diagrams, we have given complete analytical results in general $(4 - 2\epsilon)$-dimension.

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Contents

1 Introduction

Currently, the focus of high energy physics is the LHC experiment. To understand the experiment data, we need to evaluate scattering amplitudes to high accuracy level required by data. Thus for most processes, the one-loop evaluation becomes necessary. In last ten years, enormous progress has been made in the computation of one-loop scattering amplitudes (see, for example, the references [1–3] and citations in the papers). However, for some processes in modern colliders, such as the process $gg \to \gamma\gamma$ which is an important background for searching the Higgs boson at the LHC, one-loop amplitudes do not suffice since their leading-order terms begin at one loop. Thus next-to-leading order corrections require the computation of two-loop amplitudes [4–6].

The traditional method for amplitude calculation is through the Feynman diagram. This method is well organized and has clear physical picture. It has also been implemented into many computer programs. However, with increasing of loop level or the number of external particles, the complexity of computation increases dramatically. Thus even with the most powerful computer available, many interesting processes related to LHC experiments can not be dealt by the traditional method.

To solve the challenge, many new methods (see books [7–9]) have been developed, such as IBP (integrate-by-part) method [10–19] (some new developments, see [20–22]), differential equation method [23–30], MB (Mellin-Barnes) method [31–34], etc. Among these methods, the reduction method [35–41] is one of the most useful methods. More explicitly, the reduction of an amplitude means that any amplitude $\mathcal{A}$ can be expanded by bases (or "master integral") as

$$\mathcal{A} = \sum_i c_i \mathcal{A}_i \,, \tag{1.1}$$

with rational coefficients c_i. With this expansion, the amplitude calculation can be separated into two parts: (a) the evaluation of bases (or master integrals) at given loop order and (b) the determination of coefficients c_i for a particular process. For the former part, it can be done once for all and the results can be applied to any process. Thus in the practical application, the latter part, i.e., the determination of coefficients, becomes the central focus of all calculations.

Unitarity method is an ideal tool to determine coefficients [42–72]. With the expansion (1.1), if we perform unitarity cut on both sides, we will get

$$\Delta\mathcal{A} = \sum_i c_i \Delta\mathcal{A}_i \,. \tag{1.2}$$

So if both $\Delta\mathcal{A}$ and $\Delta\mathcal{A}_i$ can be evaluated analytically, and if different $\Delta\mathcal{A}_i$ has distinguishable analytic structure (which we will call the "signature" of basis under the unitarity cut), we can compare both sides of (1.2) to determine coefficients c_i, analogous to the fact that if two polynomials of x are equal, so are their coefficients of each term x^n. The unitarity method has been proven to be very successful in determining coefficients for one-loop amplitudes (see reviews [73, 74]). For some subsets of bases (such as box topology for one-loop and double-box topology for planar two-loop), more efficient method, the so called "generalized unitarity method" (or "maximum unitarity cut" or "leading singularity"), has been developed [48, 75–87].

The applicability of reduction method is based on the valid expression of expansion (1.1). Thus the determination of bases becomes the first issue. From recent study, it is realized that there are two kinds of bases: *the integrand bases and the master integrals.*[1] The integrand bases are algebraically independent rational functions before performing loop integration. For one-loop, the integrand bases have been determined by OPP [88]. For two-loop or more, the computational algebraic geometry method has been proposed to determine the integrand bases [89–99].

In general the number of integrand bases is larger than the number of master integrals, because after loop integration, some combinations of elements in integrand bases may vanish. For one-loop amplitudes, the difference between these two numbers is not very significant. For example, the number of master integral is one while the number of integrand bases is seven for triangle topology of renormalizable field theories [88]. However, for two-loop amplitudes, the difference could be huge. As we will show later, for double-triangle topology, there are only several master integrals, while the number of integrand bases is about one hundred for renormalizable field theories [98]. Thus the determination of master integrals for two-loop and higher-loop becomes necessary.

Although integrand bases can be determined systematically, the determination of master integrals is far from being completely solved. It is our attempt in this paper to find an efficient method to solve the problem.[2] Noticing that in the unitarity method, the action Δ in (1.2) is directly acting on the integrated results, thus if the left hand side $\Delta\mathcal{A}$ can be analytically integrated for arbitrary inputs, we can classify independent distinguishable analytic structures from these results. Each structure should correspond to one master integral.[3] By this attempt we can determine master integrals.

In this paper, taking double-box topology and its daughter topologies as examples, we generalize unitarity method to two-loop amplitudes and try to determine master integrals. Different from the maximal unitarity method [80], we cut only four propagators (the propagator with mixed loop momenta will not be touched). Comparing with maximal unitarity cut where solutions for loop momenta are complex number in general, our cut conditions guarantee the existence of real solutions for loop momenta, thus avoiding the affects from spurious integrations.

[1]To not confuse two kinds of bases, we use "master integrals" to denote the independent bases after integration.

[2]It is worth to notice that in reference [100], a very efficient way has been presented to count the number of master integrals although explicit expressions of these bases can not be determined.

[3]It is possible that two different master integrals have the same analytic structure for all physical unitarity cuts, but we do not consider this possibility in current paper. All our claims in this paper are true after neglecting above ambiguity.

This paper is organized as follows. In section 2 we review the one-loop unitarity method and then generalize the scheme to two-loop. For two-loop, two sub-one-loop phase space integrations should be evaluated. In section 3, we integrate the first sub-one-loop integration of triangle topology. The result is used in section 4, where integration over the second sub-one-loop of triangle topology is performed. Results obtained in this section allow us to determine master integrals for the topology $\mathcal{A}_{212}$. Results in section 3 is also used in section 5, where integration over the second sub-one-loop of box topology is performed, and the result can be used to determine master integrals for topology $\mathcal{A}_{213}$. In section 6, we briefly discuss master integrals of topology $\mathcal{A}_{313}$ since results are well known for this topology. Finally, in section 7, a short conclusion is given.

Technical details of calculation are presented in appendix. In appendix A, some useful formulae for phase space integration are summarized. In appendix B, the phase space integration is done for one-loop bubble, one-loop triangle and one-loop box topologies. In appendix C, details of an integration for topology $\mathcal{A}_{313}$ are discussed.

2 Setup

In this section, we present some general discussions about the calculation done in this paper. Firstly, we review how to do the phase space integration in unitarity method illustrated by one-loop example. Then we set up the framework in unitarity method for two-loop topologies which are the starting point of this paper.

2.1 Phase space integration

The unitarity method has been successfully applied to one-loop amplitudes [42–72] . Here we give a brief summary about the general $(4-2\epsilon)$-dimensional unitarity method [62–70], which will be used later. Through this paper we use the metric $\eta_{\mu\nu} = (+, -, \ldots, -)$ and QCD convention for spinors, i.e., $2k_i \cdot k_j \equiv \langle k_i|k_j\rangle\,[k_j|k_i]$.

For one-loop, the action Δ in (1.2) is realized by putting two internal propagators on-shell. More explicitly, let us consider the following most general input[4] with massless internal propagators[5]

$$\mathcal{A}_n^{(a)} \equiv \int d^{4-2\epsilon}\widehat{\ell}\,\mathcal{I}_n^{(a)} = \int d^{4-2\epsilon}\widehat{\ell}\,\frac{(2\widehat{\ell}\cdot T)^a}{\widehat{\ell}^2\prod_{i=1}^{n-1}(\widehat{\ell}-K_i)^2}\,, \tag{2.1}$$

where the inner momentum is in $(4-2\epsilon)$-dimensional space and all external momenta are in pure 4D space for our regularization scheme. The unitarity cut with intermediate flowing momentum K is given by putting $\widehat{\ell}^2$ and $(\widehat{\ell}-K)^2$ on-shell, and we get the expression

$$\Delta\mathcal{A}_n^{(a)} = \int d^{4-2\epsilon}\widehat{\ell}\,\frac{(2\widehat{\ell}\cdot T)^a\delta(\widehat{\ell}^2)\delta((\widehat{\ell}-K)^2)}{\prod_{i=1}^{n-2}(\widehat{\ell}-K_i)^2}\,. \tag{2.2}$$

[4]The most general expression for numerator will be $\sum_i\prod_j(\ell\cdot R_{ij})$. For each term $\prod_{j=1}^n(\ell\cdot R_{ij})$, we can construct $(\ell\cdot\widetilde{R}_i)^n$ with $\widetilde{R}_i=\sum_{j=1}^n y_jR_{ij}$. Thus if we know the result for numerator $(\ell\cdot\widetilde{R}_i)^n$, we can expand it into the polynomial of y_i and read out corresponding result for $\prod_{j=1}^n(\ell\cdot R_{ij})$.

[5]For simplicity we consider the massless propagators, but massive propagators can be dealt similarly.

With two delta-functions, the original $(4-2\epsilon)$-dimensional integration is reduced to $(2-2\epsilon)$-dimensional integration. To carry out the remaining integration, we decompose $\widehat{\ell}$ as $\widehat{\ell} = \widetilde{\ell} + \mu$, where $\widetilde{\ell}$ is the pure 4D part while μ is the (-2ϵ)-dimensional part [62–64], then the measure becomes

$$\int d^{4-2\epsilon}\widehat{\ell}\delta(\widehat{\ell}^2)\delta((\widehat{\ell}-K)^2)(\bullet) = \int d^{-2\epsilon}\mu \int d^4\widetilde{\ell}\delta(\widetilde{\ell}^2-\mu^2)\delta((\widetilde{\ell}-K)^2-\mu^2)(\bullet)\,. \tag{2.3}$$

Next, we split $\widetilde{\ell}$ into $\widetilde{\ell} = \ell + zK$ with $\ell^2 = 0$ to arrive

$$\begin{aligned}&\int d^4\widetilde{\ell}\delta(\widetilde{\ell}^2-\mu^2)\delta((\widetilde{\ell}-K)^2-\mu^2)(\bullet)\\ &= \int dz d^4\ell\delta(\ell^2)(2\ell\cdot K)\delta(z^2K^2+2z\ell\cdot K-\mu^2)\delta((1-2z)K^2-2\ell\cdot K)(\bullet)\,.\end{aligned} \tag{2.4}$$

Having the form (2.4), we can use the following well known result of spinor integration[6] [101]. Define null momentum as $\ell = t\lambda\widetilde{\lambda}$, then

$$\int d^4\ell\delta^+(\ell^2)(\bullet) = \int_0^{+\infty} tdt \int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right](\bullet)\,. \tag{2.5}$$

Substituting (2.5) back to (2.4), we can use remaining two delta-functions to fix t and z as

$$z = \frac{1-\sqrt{1-u}}{2}\,, \quad t = \frac{(1-2z)K^2}{\left\langle\lambda|K|\widetilde{\lambda}\right]}\,, \quad u \equiv \frac{4\mu^2}{K^2}\,. \tag{2.6}$$

After above simplification, the integral (2.2) is transformed to the following spinor form

$$\Delta\mathcal{A}_n^{(a)} = \int d^{-2\epsilon}\mu \int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] \frac{(-)^{n-2}[(1-2z)K^2]^{a-n+3}\left\langle\lambda|R|\widetilde{\lambda}\right]^a}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{a-n+4}\prod_{i=1}^{n-2}\left\langle\lambda|Q_i|\widetilde{\lambda}\right]}\,, \tag{2.7}$$

where

$$R \equiv T + \frac{z(2K\cdot T)}{(1-2z)K^2}K\,, \quad Q_i \equiv K_i + \frac{z(2K\cdot K_i)-K_i^2}{(1-2z)K^2}K\,. \tag{2.8}$$

To deal with the integral like $\int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] f(\lambda,\widetilde{\lambda})$ when $f(\lambda,\widetilde{\lambda})$ is a rational function, the first step is to find a function $g(\lambda,\widetilde{\lambda})$ satisfying

$$\int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] f(\lambda,\widetilde{\lambda}) = \int \langle\lambda|d\lambda\rangle \left[d\widetilde{\lambda}|\frac{\partial}{\partial\widetilde{\lambda}}\right] g(\lambda,\widetilde{\lambda})\,. \tag{2.9}$$

[6]For one-loop, we can take either positive light cone or negative light cone, where for negative light cone, the t-integration will be $\int_{-\infty}^0$. For two-loop, it can happen that if we take positive light cone for ℓ_1, then we need to take negative light cone for ℓ_2. However, the choice of light cone only gives an overall sign and does not affect $\lambda,\widetilde{\lambda}$ integration.

With $g(\lambda,\widetilde{\lambda})$, the integration is given algebraically by the sum of residues of holomorphic pole in $g(\lambda,\widetilde{\lambda})$ [55–57]. In appendix B, we summarize some general results of standard one-loop integrations using above technique. It is worth to mention that for two-loop, $f(\lambda,\widetilde{\lambda})$ might not be rational function. We will discuss how to deal with it later.

We also want to remark that under the framework of $(4-2\epsilon)$-dimensional unitarity method, coefficient of each master integrals will be polynomial of μ^2 (remembering the splitting $\widehat{\ell}=\widetilde{\ell}+\mu$). There are two ways to handle it. For the first way, one can further integrate $\int d^{-2\epsilon}\mu\ (\mu^2)^n$ to find coefficients depending on ϵ. For the second way, we just keep μ^2, but include the dimensional shifted master integrals [47, 102], such as

$$\mathcal{A}^{D=(4-2\epsilon)}[(\mu^2)^r]\equiv\int d^{-2\epsilon}\mu d^4\widetilde{\ell}\frac{(\mu^2)^r}{(\widetilde{\ell}^2-\mu^2)\prod_{i=1}^{n-1}((\ell-K_i)^2-\mu^2)}\,. \tag{2.10}$$

This is equivalent to

$$\mathcal{A}^{D=(4-2\epsilon)}[(\mu^2)^r]=-\epsilon(1-\epsilon)\ldots(r-1-\epsilon)\mathcal{A}^{D=(4+2r-2\epsilon)}[1]\,. \tag{2.11}$$

For one-loop, dimensional shifted master integrals are often used. In this paper we adapt the similar strategy, i.e., keeping the μ-part and introducing the dimensional shifted master integrals.

2.2 Generalizing to two-loop case

In this subsection, we set up unitarity method for two-loop amplitudes, particularly for the attempt of determining master integrals.

The first problem is to decide which propagators should be cut. There are three kinds of propagators: (1) propagators depending on $\widehat{\ell}_1$ only; (2) propagators depending on $\widehat{\ell}_2$ only; (3) propagators depending on both $\widehat{\ell}_1$ and $\widehat{\ell}_2$. In principle, we can cut any propagators, but for simplicity, in this paper we will cut propagators of the first two kinds. For our choice, we cut two propagators of the first kind and two propagators of the second kind. With this arrangement, for each loop it is exactly the familiar unitarity method in one-loop case.

Next we set up notation for two-loop integral. The two internal momenta are denoted as $\widehat{\ell}_1,\widehat{\ell}_2$ in $(4-2\epsilon)$-dimension, while all external momenta are in pure 4-dimension. We use n_1,n_2,n_{12} to denote the number of each kind of propagators respectively. Then a general integrand with massless propagators[7] can be represented by[8]

$$\mathcal{I}^{(a,b)}_{n_1n_{12}n_2}\equiv\frac{(2\widehat{\ell}_1\cdot T_1)^a(2\widehat{\ell}_2\cdot T_2)^b}{[\widehat{\ell}_1^2\prod_{i=1}^{n_1-1}(\widehat{\ell}_1-K_{1i})^2][\widehat{\ell}_2^2\prod_{j=1}^{n_2-1}(\widehat{\ell}_2-K_{2j})^2][(\widehat{\ell}_1+\widehat{\ell}_2)^2\prod_{t=1}^{n_{12}-1}(\widehat{\ell}_1+\widehat{\ell}_2-K_t)^2]}\,. \tag{2.12}$$

The unitarity cut action Δ is then given by[9]

$$\Delta\mathcal{A}=\int\prod_{i=1}^{2}d^{4-2\epsilon}\widehat{\ell}_i\left\{\mathcal{I}^{(a,b)}_{n_1n_{12}n_2}\prod_{i=1}^{2}\widehat{\ell}_i^2(\widehat{\ell}_i-K_{L_i})^2\right\}\prod_{i=1}^{2}\delta(\widehat{\ell}_i^2)\delta((\widehat{\ell}_i-K_{L_i})^2)\,. \tag{2.13}$$

[7]In this paper, we consider the massless case only. For inner propagators with masses, we will leave to further projects.

[8]In this paper, we use $\mathcal{I}$ for integrand and $\mathcal{A}$ for integral.

[9]We have neglected some overall factors in the definition of integration since it does not matter for our discussion.

JHEP06(2014)166

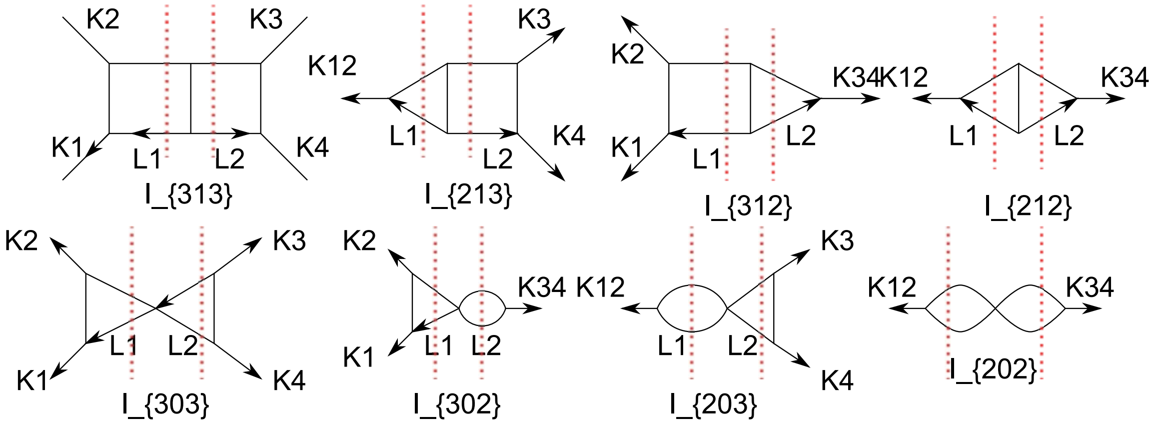

Figure 1. The unitarity cut of double box topology I_{313} as well as its seven daughter topologies. The dashed red lines indicate cuts.

Note that for cases studied in this paper, the left cut momentum $K_{L_1} = K_{12}$ is the same to the right cut momentum $K_{L_2} = K_{34}$ up to a sign, however we keep them independently so that it is possible to formulate them to more general situations for further investigations.

With above setup, we take a well studied example [20, 80–86], i.e., the four-point two-loop double-box ($\mathcal{A}_{313}$) integral as the target to apply the unitarity method and determine master integrals. The integrand is given by

$$\mathcal{I}_{313}^{(a,b)} = \frac{(2\widehat{\ell}_1 \cdot T_1)^a (2\widehat{\ell}_2 \cdot T_2)^b}{\widehat{\ell}_1^2(\widehat{\ell}_1 - K_1)^2(\widehat{\ell}_1 - K_{12})^2\widehat{\ell}_2^2(\widehat{\ell}_2 - K_4)^2(\widehat{\ell}_2 - K_{34})^2(\widehat{\ell}_1 + \widehat{\ell}_2)^2}\,, \tag{2.14}$$

and the four propagators to be cut are

$$\widehat{\ell}_1^2\,,\quad (\widehat{\ell}_1 - K_{12})^2\,,\quad \widehat{\ell}_2^2\,,\quad (\widehat{\ell}_2 - K_{34})^2\,,$$

where $K_{12} + K_{34} = 0$. With this choice of cuts, in order to completely understand the results, we also need to consider other topologies besides double-box. The other contributions come from those topologies by pinching one or more un-cut propagators of double-box, as shown in figure 1. There are three daughter topologies $\mathcal{I}_{213}, \mathcal{I}_{312}, \mathcal{I}_{303}$ by pinching one propagator. There are also three daughter topologies $\mathcal{I}_{212}, \mathcal{I}_{302}, \mathcal{I}_{203}$ by pinching two propagators. Finally there is only one daughter topology $\mathcal{I}_{202}$ by pinching three propagators. Among them, $\mathcal{I}_{303}, \mathcal{I}_{203}, \mathcal{I}_{302}, \mathcal{I}_{202}$ are direct products of two one-loop topologies, thus their signatures are well known (see appendix B). So in fact we need to examine two non-trivial topologies $\mathcal{I}_{212}, \mathcal{I}_{213}$ (by symmetry $\mathcal{I}_{312}$ is equivalent to $\mathcal{I}_{213}$) together with the mother topology $\mathcal{I}_{313}$. Integrand of these two additional topologies are given by

$$\begin{aligned}
\mathcal{I}_{212}^{(a,b)} &= \frac{(2\widehat{\ell}_1 \cdot T_1)^a (2\widehat{\ell}_2 \cdot T_2)^b}{\widehat{\ell}_1^2(\widehat{\ell}_1 - K_{12})^2\widehat{\ell}_2^2(\widehat{\ell}_2 - K_{34})^2(\widehat{\ell}_1 + \widehat{\ell}_2)^2}\,,\\
\mathcal{I}_{213}^{(a,b)} &= \frac{(2\widehat{\ell}_1 \cdot T_1)^a (2\widehat{\ell}_2 \cdot T_2)^b}{\widehat{\ell}_1^2(\widehat{\ell}_1 - K_{12})^2\widehat{\ell}_2^2(\widehat{\ell}_2 - K_4)^2(\widehat{\ell}_2 - K_{34})^2(\widehat{\ell}_1 + \widehat{\ell}_2)^2}\,.
\end{aligned} \tag{2.15}$$

In the following sections, we will study $\mathcal{I}_{212}$, $\mathcal{I}_{213}$ and $\mathcal{I}_{313}$ one by one, and our basic strategy will be to integrate one loop momentum $\widetilde{\ell}_1$ first while keeping $\widetilde{\ell}_2$ arbitrary. Then we analyze the integration of $\widetilde{\ell}_2$ based on the previous results.

Before ending this section, let us emphasize that in our framework, external momenta K_i, $i = 1, 2, 3, 4$ are arbitrary, i.e., they can be massive as well as massless. With this convention, it is also manifest that for I_{213} and I_{212} topologies, the momentum K_{12} is massive. One exception is that for I_{313} topology, because of the complexity of computation for general choice of K_i, we have restricted our discussions to the case $K_i^2 = 0$.

3 The $\widetilde{\ell}_1$-part integration ($n_1 = 2$)

In this section, we do the $\widetilde{\ell}_1$ integration. Using the standard method for one-loop amplitudes (reviewed in previous section as well as in appendix B) we get (see formula (2.12))

$$\Delta\mathcal{A}^{(a,b)}_{n_1 1 n_2} = \int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 d^4\widetilde{\ell}_2 \delta(\widetilde{\ell}_2^2 - \mu_2^2)\delta(K_{L_2}^2 - 2K_{L_2}\cdot\widetilde{\ell}_2)\frac{(2\widetilde{\ell}_2\cdot T_2)^b}{\prod_{j=1}^{n_2-2}((\widetilde{\ell}_2 - K_{2j})^2 - \mu_2^2)}$$
$$\int \langle\lambda_1|d\lambda_1\rangle \left[\widetilde{\lambda}_1|d\widetilde{\lambda}_1\right] \frac{(-)^{n_1-2}((1-2z_1)K_{L_1}^2)^{a-n_1+2}}{\left\langle\lambda_1|K_{L_1}|\widetilde{\lambda}_1\right]^{a-n_1+3}}$$
$$\frac{\left\langle\lambda_1|R_1|\widetilde{\lambda}_1\right]^a}{\left\langle\lambda_1|W_1|\widetilde{\lambda}_1\right]\prod_{i=1}^{n_1-2}\left\langle\lambda_1|Q_{1i}|\widetilde{\lambda}_1\right]}, \tag{3.1}$$

where various quantities are defined as

$$\begin{aligned}
R_1 &\equiv T_1 + \frac{z_1 2K_{L_1}\cdot T_1}{(1-2z_1)K_{L_1}^2}K_{L_1}\,,\\
Q_{1i} &\equiv K_{1i} + \frac{z_1(2K_{L_1}\cdot K_{1i}) - K_{1i}^2}{(1-2z_1)K_{L_1}^2}K_{L_1}\,,\\
W_1 &\equiv \widetilde{\ell}_2 + \frac{(\widetilde{\ell}_2^2 - \mu_2^2) - 2\mu_1\cdot\mu_2 + 2z_1\widetilde{\ell}_2\cdot K_{L_1}}{(1-2z_1)K_{L_1}^2}K_{L_1}\,,
\end{aligned} \tag{3.2}$$

with $z_1 = \frac{1-\sqrt{1-u_1}}{2}$ and $u_1 = \frac{4\mu_1^2}{K_{L_1}^2}$. The W_1 comes from the mixed propagator $(\widehat{\ell}_1 + \widehat{\ell}_2)^2$. Situations with non trivial topologies $\mathcal{A}_{313}$, $\mathcal{A}_{312}$, $\mathcal{A}_{213}$ and $\mathcal{A}_{212}$ are all included in the formula (3.1).

Let us apply our general framework to the specific case $n_1 = 2$. The general formula (3.1) now becomes

$$\Delta\mathcal{A}^{(a,b)}_{n_1 1 n_2}\Big|_{n_1=2} = \int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 d^4\widetilde{\ell}_2 \delta(\widetilde{\ell}_2^2 - \mu_2^2)\delta(K_{L_2}^2 - 2K_{L_2}\cdot\widetilde{\ell}_2)\frac{(2\widetilde{\ell}_2\cdot T_2)^b}{\prod_{j=1}^{n_2-2}((\widetilde{\ell}_2 - K_{2j})^2 - \mu_2^2)}$$
$$\int \langle\lambda_1|d\lambda_1\rangle \left[\widetilde{\lambda}_1|d\widetilde{\lambda}_1\right] \frac{((1-2z_1)K_{L_1}^2)^a}{\left\langle\lambda_1|K_{L_1}|\widetilde{\lambda}_1\right]^{a+1}}\frac{\left\langle\lambda_1|R_1|\widetilde{\lambda}_1\right]^a}{\left\langle\lambda_1|W_1|\widetilde{\lambda}_1\right]}\,. \tag{3.3}$$

The second line is nothing but the standard one-loop triangle integration (see appendix B). When $a = 0$, the integration gives the signature of triangle part. When $a \geq 1$, the integration can be decomposed into both triangle part and bubble part. We will evaluate contributions from these two parts separately.

3.1 The contribution to triangle part

The triangle signature. Based on our general formula of the standard one-loop triangle integration (B.8), the signature of the triangle part is

$$\mathcal{S}_{\rm tri} \equiv \frac{1}{\sqrt{\Delta_{W_1,K_{L_1}}}} \ln\left(\frac{W_1\cdot K_{L_1} - \sqrt{(W_1\cdot K_{L_1})^2 - W_1^2 K_{L_1}^2}}{W_1\cdot K_{L_1} + \sqrt{(W_1\cdot K_{L_1})^2 - W_1^2 K_{L_1}^2}}\right). \tag{3.4}$$

Imposing cut conditions for $\widetilde{\ell}_2$, i.e., $\delta(\widetilde{\ell}_2^2-\mu_2^2)$ and $\delta(K_{L_2}^2 - 2K_{L_2}\cdot\widetilde{\ell}_2)$ we can simplify it to

$$\mathcal{S}_{\rm tri} = \frac{1}{K_{L_1}^2\sqrt{1-u_2}} \ln\left(\frac{(4\mu_1\cdot\mu_2+K_{L_1}^2)+\sqrt{(1-u_1)(1-u_2)}K_{L_1}^2}{(4\mu_1\cdot\mu_2+K_{L_1}^2)-\sqrt{(1-u_1)(1-u_2)}K_{L_1}^2}\right) = \frac{1}{t_2 K_{L_1}^2}\ln\left(\frac{s+t_1t_2}{s-t_1t_2}\right), \tag{3.5}$$

where we have introduced

$$s = \frac{4\mu_1\cdot\mu_2 + K_{L_1}^2}{K_{L_1}^2}, \quad t_i = \sqrt{1-u_i}, \quad u_i = \frac{4\mu_i^2}{K_{L_i}^2}, \quad i=1,2. \tag{3.6}$$

One can observe that the signature part does not depend on $\widetilde{\ell}_2$. It is an important feature which makes $\Delta\mathcal{A}_{21n_2}^{(a,b)}$ easier to be treated.

The coefficient $\mathcal{C}_{3\to 3}^{(a)}$. Using (B.8) the expression is

$$\begin{aligned}\mathcal{C}_{3\to3}^{(a)} = \frac{(-)^a}{a!\Delta_{W_1,K_{L_1}}^a}\frac{d^a}{d\tau^a}\Big(&\tau^2W_1^2 + \tau(4W_1^2(R_1\cdot K_{L_1}) - 4(R_1\cdot W_1)(W_1\cdot K_{L_1})) + R_1^2\Delta_{W_1,K_{L_1}} \\ &+ (2R_1\cdot W_1)^2K_{L_1}^2 + (2R_1\cdot K_{L_1})^2W_1^2 - (2R_1\cdot W_1)(2R_1\cdot K_{L_1})(2W_1\cdot K_{L_1})\Big)^a\Big|_{\tau\to0}.\end{aligned} \tag{3.7}$$

Again, using cut conditions $\delta(\widetilde{\ell}_2^2-\mu_2^2)$ and $\delta(K_{L_2}^2 - 2K_{L_2}\cdot\widetilde{\ell}_2)$ we can do the following replacement

$$\widetilde{\ell}_2 \to \frac{(1-2z_2)K_{L_2}^2}{\langle\lambda_2|K_{L_2}|\widetilde{\lambda}_2]}\lambda_2\widetilde{\lambda}_2 + z_2K_{L_2} = \frac{(1-2z_2)K_{L_1}^2}{-\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2]}\lambda_2\widetilde{\lambda}_2 - z_2K_{L_1},$$

where $z_2 = \frac{1-t_2}{2}$. Since all derivatives act on τ only, such replacement will not affect the result. Some algebraic manipulation shows that the coefficients of different parts are given by

$$
\begin{aligned}
\tau^2 :& \quad \frac{(s^2 - t_1^2 t_2^2) K_{L_1}^2}{4t_1^2}\,, \\
\tau :& \frac{-t_2 K_{L_1}^2}{t_1 \left\langle \lambda_2 | K_{L_1} | \widetilde{\lambda}_2 \right]} \Big(t_2 (K_{L_1} \cdot T_1) \left\langle \lambda_2 | K_{L_1} | \widetilde{\lambda}_2 \right] + s(-(K_{L_1} \cdot T_1) \left\langle \lambda_2 | K_{L_1} | \widetilde{\lambda}_2 \right] \\
& \qquad\qquad\qquad\qquad + K_{L_1}^2 \left\langle \lambda_2 | T_1 | \widetilde{\lambda}_2 \right]) \Big)\,, \\
\tau^0 :& \frac{t_2^2 (K_{L_1}^2)^2}{\left\langle \lambda_2 | K_{L_1} | \widetilde{\lambda}_2 \right]^2} K_{L_1}^2 \left\langle \lambda_2 | (T_1 + y_1 K_{L_1}) | \widetilde{\lambda}_2 \right] \left\langle \lambda_2 | (T_1 + y_2 K_{L_1}) | \widetilde{\lambda}_2 \right]\,,
\end{aligned} \tag{3.8}
$$

with

$$
y_{1,2} = \frac{-(2T_1 \cdot K_{L_1}) \pm \sqrt{(2T_1 \cdot K_{L_1})^2 - 4 K_{L_1}^2 T_1^2}}{2 K_{L_1}^2}\,. \tag{3.9}
$$

To get non-zero contribution from $\frac{d^a}{d\tau^a}(\bullet)\Big|_{\tau\to 0}$, we only need to take terms with τ^a power. It means that terms with τ^2 in (3.8) will always appear with terms τ^0, therefore we can regroup

$$
\left\{ \tau^2 \frac{(s^2 - t_1^2 t_2^2) K_{L_1}^2}{4t_1^2} \right\} + \left\{ \frac{t_2^2 (K_{L_1}^2)^2}{\left\langle \lambda_2 | K_{L_1} | \widetilde{\lambda}_2 \right]^2} K_{L_1}^2 \left\langle \lambda_2 | (T_1 + y_1 K_{L_1}) | \widetilde{\lambda}_2 \right] \left\langle \lambda_2 | (T_1 + y_2 K_{L_1}) | \widetilde{\lambda}_2 \right] \right\}
$$

to

$$
\left\{ \tau^2 \frac{t_2 (s - t_1 t_2)(K_{L_1}^2)^2}{2t_1} \frac{\left\langle \lambda_2 | (T_1 + y_1 K_{L_1}) | \widetilde{\lambda}_2 \right]}{\left\langle \lambda_2 | K_{L_1} | \widetilde{\lambda}_2 \right]} \right\} + \left\{ \frac{t_2 (s + t_1 t_2)(K_{L_1}^2)^2}{2t_1} \frac{\left\langle \lambda_2 | (T_1 + y_2 K_{L_1}) | \widetilde{\lambda}_2 \right]}{\left\langle \lambda_2 | K_{L_1} | \widetilde{\lambda}_2 \right]} \right\}.
$$

Thus we can write

$$
\mathcal{C}_{3\to 3}^{(a)} = \frac{(-)^a (K_{L_1}^2)^a}{a! (t_1 t_2 K_{L_1}^2)^a} \frac{d^a}{d\tau^a} \frac{\left\langle \lambda_2 | \mathcal{F} | \widetilde{\lambda}_2 \right]^a}{\left\langle \lambda_2 | K_{L_1} | \widetilde{\lambda}_2 \right]^a} \Big|_{\tau \to 0}\,,
$$

where $\mathcal{F}$ is defined as

$$
\begin{aligned}
\mathcal{F} &= -\tau \left(t_2 \frac{K_{L_1} \cdot T_1}{K_{L_1}^2} K_{L_1} + s \Big(T_1 - \frac{(K_{L_1} \cdot T_1)}{K_{L_1}^2} K_{L_1} \Big) \right) \\
&\quad + \tau^2 \frac{s - t_1 t_2}{2} \Big(T_1 + y_1 K_{L_1} \Big) + \frac{s + t_1 t_2}{2} \Big(T_1 + y_2 K_{L_1} \Big)\,.
\end{aligned} \tag{3.10}
$$

Putting all results together, the triangle part becomes

$$
\mathcal{R}_{3\to 3}^{(a)} = \left(\frac{(-)^a (K_{L_1}^2)^a}{a! (t_1 t_2 K_{L_1}^2)^a} \frac{d^a}{d\tau^a} \frac{\left\langle \lambda_2 | \mathcal{F} | \widetilde{\lambda}_2 \right]^a}{\left\langle \lambda_2 | K_{L_1} | \widetilde{\lambda}_2 \right]^a} \Big|_{\tau \to 0} \right) \frac{1}{t_2 K_{L_1}^2} \ln \Big(\frac{s + t_1 t_2}{s - t_1 t_2} \Big)\,. \tag{3.11}
$$

To do the $\widetilde{\ell}_2$-part integration, it is more convenient to use above form before taking the derivative over τ.

3.2 The contribution to bubble part

Again we use results given in appendix B.

The $\mathcal{R}_{3\to 2}[i,m]$ term. Using (B.10), the typical term of triangle topology to bubble is

$$\begin{aligned}\mathcal{R}_{3\to 2}[i,m] = & \frac{(-)^{m+i}(K_{L_1}^2)^i}{i!(m+1)\sqrt{\Delta(W_1,K_{L_1})}^{m+2i+2}}\frac{d^i}{d\tau^i}\Big\{\big((2R_1\cdot P_2-\tau\langle P_1|R_1|P_2])^{m+1}\\ & (-x_2\langle P_2|R_1|P_1]-x_1\tau^2\langle P_1|R_1|P_2]+\tau(x_2(2R_1\cdot P_1)+x_1(2R_1\cdot P_2)))^i\big)\\ & +(-)^m\big((2R_1\cdot P_1-\tau\langle P_2|R_1|P_1])^{m+1}\\ & (-x_2\tau^2\langle P_2|R_1|P_1]-x_1\langle P_1|R_1|P_2]+\tau(x_2(2R_1\cdot P_1)+x_1(2R_1\cdot P_2)))^i\big)\Big\}\Big|_{\tau\to 0},\end{aligned} \tag{3.12}$$

where two null momenta P_1, P_2 are constructed as $P_i=W_1+x_iK_{L_1}$, with

$$x_1=\frac{s+t_1t_2}{2t_1},\quad x_2=\frac{s-t_1t_2}{2t_2}.$$

Again, to get non-zero contribution, $\langle P_1|R_1|P_2]$ and $\langle P_2|R_1|P_1]$ should always appear in pair. With a little calculations, one can see

$$\langle P_1|R_1|P_2]\langle P_2|R_1|P_1]=\mathcal{T}_1\mathcal{T}_2\,,\quad \mathcal{T}_i=\left(\frac{t_2K_{L_1}^2}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]}\right)\left\langle\lambda_2|(T_1+y_iK_{L_1})|\widetilde{\lambda}_2\right]\,, \tag{3.13}$$

where y_1 and y_2 are defined in (3.9). Thus we can take the following replacements

$$\langle P_1|R_1|P_2]\to\mathcal{T}_1\,,\quad \langle P_2|R_1|P_1]\to\mathcal{T}_2\,. \tag{3.14}$$

After such replacements we obtain

$$\begin{aligned}\mathcal{R}_{3\to 2}[i,m] = & \frac{(-)^{m+i}}{(m+1)i!K_{L_1}^2t_2^{i+1}t_1^{m+i+1}}\frac{d^i}{d\tau^i}\\ & \left\{\frac{\left\langle\lambda_2|T_1(-t_1-\tau t_1)+K_{L_1}(-\tau t_1y_1-(1-t_1)\frac{K_{L_1}\cdot T_1}{K_{L_1}^2})|\widetilde{\lambda}_2\right]^{m+1}}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^{m+i+1}}\right.\\ & \times\left\langle\lambda_2|T_1(-x_2t_1-x_1\tau^2t_1-s\tau)+K_{L_1}(-x_2t_1y_2-x_1t_1\tau^2y_1+\tau(s-t_2)\frac{K_{L_1}\cdot T_1}{K_{L_1}^2})|\widetilde{\lambda}_2\right]^i\\ & +(-)^m\frac{\left\langle\lambda_2|T_1(-t_1-\tau t_1)+K_{L_1}(-\tau t_1y_2+(1+t_1)\frac{K_{L_1}\cdot T_1}{K_{L_1}^2})|\widetilde{\lambda}_2\right]^{m+1}}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^{m+i+1}}\\ & \left.\times\left\langle\lambda_2|T_1(-x_2t_1\tau^2-x_1t_1-s\tau)+K_{L_1}(-x_2t_1\tau^2y_2-x_1t_1y_1+\tau(s-t_2)\frac{K_{L_1}\cdot T_1}{K_{L_1}^2})|\widetilde{\lambda}_2\right]^i\right\}\Bigg|_{\tau\to 0}.\end{aligned} \tag{3.15}$$

Above expression has the form $\left\langle\lambda_2\right|\bullet|\widetilde{\lambda}_2\big]\left\langle\lambda_2\right|\bullet|\widetilde{\lambda}_2\big]$. In order to use the results given in appendix B, we need to rewrite them by using

$$(A)^{m+1}(B)^i=\frac{i!}{(m+1+i)!}\frac{d^{m+1}}{d\tau_1^{m+1}}(\tau_1 A+B)^{m+1+i}\Big|_{\tau_1\to 0}.$$

So finally we have

$$\mathcal{R}_{3\to 2}[i,m]=\frac{(-)^{m+i}}{(m+1)(m+1+i)!K_{L_1}^2 t_2^{i+1}t_1^{m+i+1}}\left\{\frac{d^i}{d\tau^i}\frac{d^{m+1}}{d\tau_1^{m+1}}\frac{\left\langle\lambda_2|\mathcal{G}_2|\widetilde{\lambda}_2\right]^{m+i+1}+(-)^m\left\langle\lambda_2|\mathcal{G}_1|\widetilde{\lambda}_2\right]^{m+i+1}}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^{m+i+1}}\right\}_{\tau\to 0,\tau_1\to 0}, \tag{3.16}$$

where we have defined

$$\begin{aligned}\mathcal{G}_1&=T_1\left\{-t_1\tau_1-\frac{(s-t_1t_2)}{2}\tau^2-\frac{(s+t_1t_2)}{2}-\tau s-\tau\tau_1 t_1\right\}\\&\quad+K_{L_1}\left\{\tau_1\frac{K_{L_1}\cdot T_1}{K_{L_1}^2}(1+t_1)-\tau^2 y_2\frac{(s-t_1t_2)}{2}-\frac{(s+t_1t_2)}{2}y_1+\tau\frac{K_{L_1}\cdot T_1}{K_{L_1}^2}(s-t_2)-\tau\tau_1 t_1 y_2\right\},\\\mathcal{G}_2&=T_1\left\{-\tau_1 t_1-\tau\tau_1 t_1-\frac{(s-t_1t_2)}{2}-\tau^2\frac{(s+t_1t_2)}{2}-s\tau\right\}\\&\quad+K_{L_1}\left\{[-(1-t_1)\tau_1+\tau(s-t_2)]\frac{K_{L_1}\cdot T_1}{K_{L_1}^2}-\tau\tau_1 t_1 y_1-\frac{(s-t_1t_2)}{2}y_2-\tau^2\frac{(s+t_1t_2)}{2}y_1\right\}.\end{aligned} \tag{3.17}$$

3.3 The result for $n_1 = 2$ after $\widetilde{\ell}_1$-integration

Collecting results from triangle part and bubble part we obtain

$$\begin{aligned}\Delta\mathcal{A}^{(a,b)}_{n_1 1 n_2}\Big|_{n_1=2}&=\int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2\int d^4\widetilde{\ell}_2\delta(\widetilde{\ell}_2^2-\mu_2^2)\delta(K_{L_2}^2-2K_{L_2}\cdot\widetilde{\ell}_2)\frac{(2\widetilde{\ell}_2\cdot T_2)^b(t_1K_{L_1}^2)^a}{\prod_{j=1}^{n_2-2}((\widetilde{\ell}_2-K_{2j})^2-\mu_2^2)}\\&\left\{\frac{1}{K_{L_1}^2 t_2}\ln\left(\frac{(s+t_1t_2)}{(s-t_1t_2)}\right)\frac{(-)^a(K_{L_1}^2)^a}{a!(t_1t_2K_{L_1}^2)^a}\frac{d^a}{d\tau^a}\frac{\left\langle\lambda_2|\mathcal{F}|\widetilde{\lambda}_2\right]^a}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^a}\Big|_{\tau\to 0}\right.\\&\left.+\sum_{i=0}^{a-1}\frac{(-)^{a-1}}{(a-i)a!K_{L_1}^2 t_2^{i+1}t_1^a}\frac{d^i}{d\tau^i}\frac{d^{a-i}}{d\tau_1^{a-i}}\frac{\left\langle\lambda_2|\mathcal{G}_2|\widetilde{\lambda}_2\right]^a+(-)^{a-1-i}\left\langle\lambda_2|\mathcal{G}_1|\widetilde{\lambda}_2\right]^a}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^a}\Big|_{\tau\to 0,\tau_1\to 0}\right\},\end{aligned} \tag{3.18}$$

where s,t_1,t_2 are defined in (3.6), $\mathcal{F}$ in (3.10) and $\mathcal{G}_1,\mathcal{G}_2$ in (3.17).[10] The trick here is that instead of computing the operations $\frac{d^a}{d\tau^a}(\bullet)\Big|_{\tau\to 0}$ and $\frac{d^i}{d\tau^i}\frac{d^{a-i}}{d\tau_1^{a-i}}(\bullet)\Big|_{\tau\to 0,\tau_1\to 0}$, we will firstly do the $\widetilde{\ell}_2$-part integration.

[10] Do not confuse the t_2 here with the t_2-integration part of $\widetilde{\ell}_2$ as reviewed in (2.5).

For $\widetilde{\ell}_2$-integration, after the t_2-integration we are left with spinor integration given by[11]

$$\Delta\mathcal{A}^{(a,b)}_{n_1 1 n_2}\Big|_{n_1=2} = \int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 \int \langle\lambda_2|d\lambda_2\rangle\left[\widetilde{\lambda}_2|d\widetilde{\lambda}_2\right] \frac{(-)^{n_2+1}\left\langle\lambda_2|R_2|\widetilde{\lambda}_2\right]^b}{\prod_{j=1}^{n_2-2}\left\langle\lambda_2|Q_{2j}|\widetilde{\lambda}_2\right]\left\langle\lambda_2|K_{L_2}|\widetilde{\lambda}_2\right]^{2+b-(n_2-2)}}$$
$$\left\{\frac{(-)^a t_2^{b-(n_2-2)}(K_{L_1}^2)^{a+b-(n_2-2)}}{a!t_2^a}\ln\left(\frac{(s+t_1t_2)}{(s-t_1t_2)}\right)\frac{d^a}{d\tau^a}\frac{\left\langle\lambda_2|\mathcal{F}|\widetilde{\lambda}_2\right]^a}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^a}\Big|_{\tau\to 0}\right. \tag{3.19}$$
$$\left.+\sum_{i=0}^{a-1}\frac{(-)^{a-1}(K_{L_1}^2)^{a+b-(n_2-2)}t_2^{b-(n_2-2)}}{(a-i)a!t_2^i}\frac{d^i}{d\tau^i}\frac{d^{a-i}}{d\tau_1^{a-i}}\frac{\left\langle\lambda_2|\mathcal{G}_2|\widetilde{\lambda}_2\right]^a+(-)^{a-1-i}\left\langle\lambda_2|\mathcal{G}_1|\widetilde{\lambda}_2\right]^a}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^a}\Big|_{\tau\to 0,\tau_1\to 0}\right\},$$

where we have defined

$$R_2 \equiv T_2 + \frac{z_2 2K_{L_2}\cdot T_2}{(1-2z_2)K_{L_2}^2}K_{L_2}\,,\quad Q_{2j} \equiv K_{2j} + \frac{z_2(2K_{L_2}\cdot K_{2j}) - K_{2j}^2}{(1-2z_2)K_{L_2}^2}K_{L_2}\,. \tag{3.20}$$

4 The master integrals of $\mathcal{A}_{212}$ topology

With results of previous section, it is possible to discuss the master integrals of $\mathcal{A}_{212}$ topology in this section. To do so, we need to finish the spinor integration given in (3.19) with $n_2 = 2$, and attempt to identify the results. We will see that there are only (dimensional shifted) scalar master integrals.

4.1 The λ_2-integration for the case $n_2 = 2$

For the case $n_2 = 2$ the formula (3.19) becomes

$$\Delta\mathcal{A}^{(a,b)}_{212} = \int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 \int \langle\lambda_2|d\lambda_2\rangle\left[\widetilde{\lambda}_2|d\widetilde{\lambda}_2\right]\frac{(-)\left\langle\lambda_2|R_2|\widetilde{\lambda}_2\right]^b}{\left\langle\lambda_2|K_{L_2}|\widetilde{\lambda}_2\right]^{2+b}}$$
$$\left\{\frac{(-)^a(K_{L_1}^2)^{a+b}t_2^b}{a!t_2^a}\ln\left(\frac{(s+t_1t_2)}{(s-t_1t_2)}\right)\frac{d^a}{d\tau^a}\frac{\left\langle\lambda_2|\mathcal{F}|\widetilde{\lambda}_2\right]^a}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^a}\Big|_{\tau\to 0}\right. \tag{4.1}$$
$$\left.+\sum_{i=0}^{a-1}\frac{(-)^{a-1}(K_{L_1}^2)^{a+b}t_2^b}{(a-i)a!t_2^i}\frac{d^i}{d\tau^i}\frac{d^{a-i}}{d\tau_1^{a-i}}\frac{\left\langle\lambda_2|\mathcal{G}_2|\widetilde{\lambda}_2\right]^a+(-)^{a-1-i}\left\langle\lambda_2|\mathcal{G}_1|\widetilde{\lambda}_2\right]^a}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^a}\Big|_{\tau\to 0,\tau_1\to 0}\right\}.$$

For our momentum configuration, $K_{L_1} = -K_{L_2}$, thus we can combine denominator together to get a simpler expression. Terms of integrand can be classified into two parts, and we evaluate them one by one.

[11]There is an overall sign for t_2-integration since the momentum conservation forces $K_{L_1} = -K_{L_2}$, i.e., $\left\langle\lambda_2|K_{L_2}|\widetilde{\lambda}_2\right] < 0$.

First part. The first part can be rewritten as

$$\int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 \frac{(-)^{a+b+1}(K_{L_1}^2)^{a+b}t_2^b}{a!t_2^a}\ln\left(\frac{(s+t_1t_2)}{(s-t_1t_2)}\right)\frac{d^a}{d\tau^a}\frac{a!}{(a+b)!}\frac{d^b}{d\widetilde{\tau}^b}$$
$$\int \langle\lambda_2|d\lambda_2\rangle\left[\widetilde{\lambda}_2|d\widetilde{\lambda}_2\right]\frac{\left\langle\lambda_2|\widetilde{\tau}R_2+\mathcal{F}|\widetilde{\lambda}_2\right]^{a+b}}{\left\langle\lambda_2|K_{L_2}|\widetilde{\lambda}_2\right]^{a+b+2}}\Big|_{\tau\to 0,\widetilde{\tau}\to 0}. \tag{4.2}$$

The second line is the standard one-loop bubble integration, thus we can use the general formulae in appendix B.

Second part. The second part can be rewritten as

$$\int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 \sum_{i=0}^{a-1}\frac{(-)^{a+b}(K_{L_1}^2)^{a+b}t_2^b}{(a-i)(a+b)!t_2^i}\frac{d^b}{d\widetilde{\tau}^b}\frac{d^i}{d\tau^i}\frac{d^{a-i}}{d\tau_1^{a-i}} \tag{4.3}$$
$$\int \langle\lambda_2|d\lambda_2\rangle\left[\widetilde{\lambda}_2|d\widetilde{\lambda}_2\right]\frac{\left\langle\lambda_2|\widetilde{\tau}R_2+\mathcal{G}_2|\widetilde{\lambda}_2\right]^{a+b}+(-)^{a-1-i}\left\langle\lambda_2|\widetilde{\tau}R_2+\mathcal{G}_1|\widetilde{\lambda}_2\right]^{a+b}}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^{a+b+2}}\Big|_{\tau\to 0,\tau_1\to 0,\widetilde{\tau}\to 0}.$$

The second line is again the one-loop bubble integration. After finishing the integration over λ_2-part, we can take the derivative and the limit $\tau\to 0, \tau_1\to 0, \widetilde{\tau}\to 0$.

4.2 The result

Collecting all results together, we get an expression of the form

$$\Delta\mathcal{A}_{212}^{(a,b)} = \int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2\left\{f_{212\to 202}^{(a,b)}\mathcal{S}_{202}+f_{212\to 212}^{(a,b)}\mathcal{S}_{212}\right\}, \tag{4.4}$$

where we have defined

$$\mathcal{S}_{202}=-t_1t_2\,,\quad \mathcal{S}_{212}=\frac{1}{K_{L_1}^2}\ln\left(\frac{s+t_1t_2}{s-t_1t_2}\right). \tag{4.5}$$

Remind from appendix B that the signature of one-loop bubble is $\int d^{-2\epsilon}\mu(-\sqrt{1-u^2})$, thus the term $\mathcal{S}_{202}$ is the signature of topology $\mathcal{A}_{202}$ as the subscript indicates. For $\mathcal{S}_{212}$, since the factor $\ln\left(\frac{s+t_1t_2}{s-t_1t_2}\right)$ can not be factorized to a form where μ_1-part and μ_2-part are decoupled, it can not belong to the topology $\mathcal{A}_{n_10n_2}$. So it must be the signature of topology $\mathcal{A}_{212}$.

It is worth to mention that in the form (4.4), the dependence of a,b is completely encoded in the coefficients $f_{212\to 202}^{(a,b)}$ and $f_{212\to 212}^{(a,b)}$, while the signature (4.5) is universal. However, it does not mean master integral is just given by $a=b=0$. It could be true only when coefficients $f_{212\to 202}^{(a,b)}$ and $f_{212\to 212}^{(a,b)}$ satisfying the following two conditions: (1) they are polynomials of u_1,u_2 and s; (2) they are rational functions of external momentum K_{L_1}. More discussions will be given shortly after.

Having above general discussions, now we list coefficients for various a, b:

Coefficients $f_{212\to 212}$. Using expression given in appendix B, the analytic results for some levels of $a+b$ are given by

- $a+b=0,1$:

$$f^{(0,0)}_{212\to 212}=1\,,\quad f^{(1,0)}_{212\to 212}=T_1\cdot K_{L_1}\,,\quad f^{(0,1)}_{212\to 212}=-T_2\cdot K_{L_1}\,. \tag{4.6}$$

- $a+b=2$:

$$\begin{aligned}
f^{(1,1)}_{212\to 212}&=\frac{1}{3}\left(sK^2_{L_1}(T_1\cdot T_2)-(3+s)(K_{L_1}\cdot T_1)(K_{L_1}\cdot T_2)\right)\,,\\
f^{(2,0)}_{212\to 212}&=\frac{1}{3}\left((3+(1-u_1))(K_{L_1}\cdot T_1)^2-(1-u_1)K^2_{L_1}T^2_1\right)\,,\\
f^{(0,2)}_{212\to 212}&=\frac{1}{3}\left((3+(1-u_2))(K_{L_1}\cdot T_2)^2-(1-u_2)K^2_{L_1}T^2_2\right)\,.
\end{aligned} \tag{4.7}$$

- $a+b=3$:

$$\begin{aligned}
f^{(1,2)}_{212\to 212}&=\frac{1}{3}\left(-2sK^2_{L_1}(K_{L_1}\cdot T_2)(T_1\cdot T_2)+(K_{L_1}\cdot T_1)((3+2s+(1-u_2))(K_{L_1}\cdot T_2)^2\right.\\
&\quad\left.-(1-u_2)K^2_{L_1}T^2_2)\right)\,,\\
f^{(0,3)}_{212\to 212}&=-(1+(1-u_2))(K_{L_1}\cdot T_2)^3+(1-u_2)K^2_{L_1}(K_{L_1}\cdot T_2)T^2_2\,.
\end{aligned} \tag{4.8}$$

- $a+b=4$:

$$\begin{aligned}
f^{(2,2)}_{212\to 212}&=\frac{1}{15}\left\{2(-s(10+3s)+(1-u_2)(1-u_1))K^2_{L_1}(K_{L_1}\cdot T_1)(K_{L_1}\cdot T_2)(T_1\cdot T_2)\right.\\
&+(K_{L_1}\cdot T_1)^2((2s(10+s)+5(3+(1-u_1))+(5+(1-u_1))(1-u_2))(K_{L_1}\cdot T_2)^2\\
&+(s^2-(5+2(1-u_1))(1-u_2))K^2_{L_1}T^2_2)+K^2_{L_1}((s^2-(1-u_1)(5+2(1-u_2)))(K_{L_1}\cdot T_2)^2T^2_1\\
&\left.+K^2_{L_1}((3s^2-(1-u_1)(1-u_2))(T_1\cdot T_2)^2-(s^2-2(1-u_1)(1-u_2))T^2_1T^2_2))\right\}\,.
\end{aligned} \tag{4.9}$$

Coefficients $f_{212\to 202}$.

- $a=0$ or $b=0$: From our derivation, it can easily be seen that when $a=0$ or $b=0$, the coefficient must be zero, i.e.,

$$f^{(0,b)}_{212\to 202}=f^{(a,0)}_{212\to 202}=0\,. \tag{4.10}$$

JHEP06(2014)166

- Non-zero results:

$$
\begin{aligned}
f^{(1,1)}_{212\to 202} &= \frac{2}{3}\left(T_1\cdot T_2 - \frac{(K_{L_1}\cdot T_1)(K_{L_1}\cdot T_2)}{K^2_{L_1}}\right),\\
f^{(1,2)}_{212\to 202} &= \frac{4(K_{L_1}\cdot T_2)((K_{L_1}\cdot T_1)(K_{L_1}\cdot T_2)-K^2_{L_1}(T_1\cdot T_2))}{3K^2_{L_1}},\\
f^{(1,3)}_{212\to 202} &= \frac{2((K_{L_1}\cdot T_1)(K_{L_1}\cdot T_2)-K^2_{L_1}(T_1\cdot T_2))((5+(1-u_2))(K_{L_1}\cdot T_2)^2-(1-u_2)K^2_{L_1}T^2_2)}{-5K^2_{L_1}},\\
f^{(2,2)}_{212\to 202} &= \frac{2}{15K^2_{L_1}}\left\{-2(10+3s)K^2_{L_1}(K_{L_1}\cdot T_1)(K_{L_1}\cdot T_2)(T_1\cdot T_2)\right.\\
&\quad +(K_{L_1}\cdot T_1)^2(2(10+s)(K_{L_1}\cdot T_2)^2+sK^2_{L_1}T^2_2)+sK^2_{L_1}((K_{L_1}\cdot T_2)^2T^2_1\\
&\quad \left.+K^2_{L_1}(3(T_2\cdot T_1)^2-T^2_1T^2_2))\right\}.
\end{aligned}
\tag{4.11}
$$

4.3 Classification of master integrals

Now we need to analyze above results in order to determine master integrals. Firstly, noticing that $f^{(a,b)}_{212\to 212}$ and $f^{(a,b)}_{212\to 202}$ are polynomials of $T_1, T_2, \mu_1\cdot\mu_2, \mu_1^2, \mu_2^2$ as well as rational functions of external momentum K_{L_1}, thus we can write them more explicitly as

$$
\begin{aligned}
f^{(a,b)}_{212\to 212} &= \sum_{\kappa_0,\kappa_1,\kappa_2} f^{(a,b)}_{212\to 212;\mu_1,\ldots,\mu_a;\nu_1,\ldots,\nu_b} T_1^{\mu_1}\ldots T_1^{\mu_a}T_2^{\nu_1}\ldots T_2^{\nu_b}(\mu_1^2)^{\kappa_1}(\mu_2^2)^{\kappa_2}(\mu_1\cdot\mu_2)^{\kappa_0},\\
f^{(a,b)}_{212\to 202} &= \sum_{\kappa_0,\kappa_1,\kappa_2} f^{(a,b)}_{212\to 202;\mu_1,\ldots,\mu_a;\nu_1,\ldots,\nu_b} T_1^{\mu_1}\ldots T_1^{\mu_a}T_2^{\nu_1}\ldots T_2^{\nu_b}(\mu_1^2)^{\kappa_1}(\mu_2^2)^{\kappa_2}(\mu_1\cdot\mu_2)^{\kappa_0},
\end{aligned}
\tag{4.12}
$$

where the tensor coefficients $f^{(a,b)}_{212\to 212;\mu_1,\ldots,\mu_a;\nu_1,\ldots,\nu_b}$ are rational functions of external momentum K_{L_1} only. Putting it back we get

$$
\begin{aligned}
&\Delta\mathcal{A}^{(a,b)}_{212}\\
&=\sum_{\kappa_0,\kappa_1,\kappa_2} f^{(a,b)}_{212\to 202;\mu_1,\ldots,\mu_a;\nu_1,\ldots,\nu_b} T_1^{\mu_1}\ldots T_1^{\mu_a}T_2^{\nu_1}\ldots T_2^{\nu_b}\int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2(\mu_1^2)^{\kappa_1}(\mu_2^2)^{\kappa_2}(\mu_1\cdot\mu_2)^{\kappa_0}\mathcal{S}_{202}\\
&+\sum_{\kappa_0,\kappa_1,\kappa_2} f^{(a,b)}_{212\to 212;\mu_1,\ldots,\mu_a;\nu_1,\ldots,\nu_b} T_1^{\mu_1}\ldots T_1^{\mu_a}T_2^{\nu_1}\ldots T_2^{\nu_b}\int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2(\mu_1^2)^{\kappa_1}(\mu_2^2)^{\kappa_2}(\mu_1\cdot\mu_2)^{\kappa_0}\mathcal{S}_{212}.
\end{aligned}
\tag{4.13}
$$

The above expansion leads us to define following *dimensional shifted scalar master integrals*[12]

$$
\mathcal{B}^{(0,0)}_{202}[\kappa_0,\kappa_1,\kappa_2] \equiv \int d^{4-2\epsilon}\widehat{\ell}_1\int d^{4-2\epsilon}\widehat{\ell}_2\frac{(\mu_1^2)^{\kappa_1}(\mu_2^2)^{\kappa_2}(\mu_1\cdot\mu_2)^{\kappa_0}}{\widehat{\ell}_1^2(\widehat{\ell}_1-K_{L_1})^2\widehat{\ell}_2^2(\widehat{\ell}_2+K_{L_1})^2} \tag{4.14}
$$

and

$$
\mathcal{B}^{(0,0)}_{212}[\kappa_0,\kappa_1,\kappa_2] \equiv \int d^{4-2\epsilon}\widehat{\ell}_1\int d^{4-2\epsilon}\widehat{\ell}_2\frac{(\mu_1^2)^{\kappa_1}(\mu_2^2)^{\kappa_2}(\mu_1\cdot\mu_2)^{\kappa_0}}{\widehat{\ell}_1^2(\widehat{\ell}_1-K_{L_1})^2\widehat{\ell}_2^2(\widehat{\ell}_2+K_{L_1})^2(\widehat{\ell}_1+\widehat{\ell}_2)^2}. \tag{4.15}
$$

[12]As mentioned in the last paragraph of section 2, our all results and claims are valid when and only when $K^2_{L_1}\neq 0$, i.e., K_{L_1} is massive.

An important observation is that in the definition of $\mathcal{B}^{(0,0)}_{202}[\kappa_0,\kappa_1,\kappa_2]$, when $\kappa_0\neq 0$, we do have $\mu_1\cdot\mu_2$ in the numerator. Thus although there is no mixed propagator in the denominator, it contains information from the mother topology $\mathcal{A}_{212}$ where $\widehat{\ell}_1$ and $\widehat{\ell}_2$ are mixed.

With above definition, we find the following reduction hinted by unitarity method[13]

$$\begin{aligned}\mathcal{A}^{(a,b)}_{212}\to&\sum_{\kappa_0,\kappa_1,\kappa_2} f^{(a,b)}_{212\to 202;\mu_1,\ldots,\mu_a;\nu_1,\ldots,\nu_b}T_1^{\mu_1}\ldots T_1^{\mu_a}T_2^{\nu_1}\ldots T_2^{\nu_b}\mathcal{B}_{202}[\kappa_0,\kappa_1,\kappa_2]\\&+\sum_{\kappa_0,\kappa_1,\kappa_2} f^{(a,b)}_{212\to 212;\mu_1,\ldots,\mu_a;\nu_1,\ldots,\nu_b}T_1^{\mu_1}\ldots T_1^{\mu_a}T_2^{\nu_1}\ldots T_2^{\nu_b}\mathcal{B}_{212}[\kappa_0,\kappa_1,\kappa_2]\,.\end{aligned}\tag{4.16}$$

However, before claiming $\mathcal{B}_{212}[\kappa_0,\kappa_1,\kappa_2]$ are master integrals of the topology $\mathcal{A}_{212}$ studied in this paper, we need to notice that in general T_i could have four independent choices in 4D, i.e., e_i, $i=1,2,3,4$ as the momentum bases for Lorentz momenta. So if master integrals have non-trivial dependence of T_i in the numerator, we should be careful to identify bases. This happens to topologies $\mathcal{A}_{213}$ and $\mathcal{A}_{313}$. However, for the current topology $\mathcal{A}_{212}$, the expressions $\mathcal{B}_{212}[\kappa_0,\kappa_1,\kappa_2]$ are *scalar integrals*, i.e., the numerator of master integrals does not depend on any external momenta T_i.

Now we count the number of master integrals. For pure 4D case, we can take the limit $\mu_1^2,\mu_2^2,\mu_1\cdot\mu_2\to 0$, thus there is only one master integral, with $\kappa_i=0$, $i=0,1,2$. In [98] it is found that for planar double-triangle (i.e., the topology $\mathcal{A}_{212}$), the number of integrand bases is 111 under the renormalizable conditions in pure 4D. For general $(4-2\epsilon)$-dimension, if we set constraint $\sum_{i=0,1,2}\kappa_i\leq 3$ (i.e., the sum of the power of ℓ_1,ℓ_2 in the numerator is less than or equal to 6) for well-behaved quantum field theories, the number of master integrals is 20.

5 The master integrals of $\mathcal{A}_{213}$ topology

Encouraged by the results in previous section, in this section we determine the master integrals of $\mathcal{A}_{213}$ topology. As it will be shown shortly after, new features will appear.

5.1 λ_2-integration for the case $n_2=3$

For $n_2=3$ the general formula (3.19) becomes (for simplicity, we will drop "$\tau_i\to 0$" from now on)

$$\begin{aligned}\Delta\mathcal{A}^{(a,b)}_{213}=&\int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2\int\langle\lambda_2|d\lambda_2\rangle\left[\widetilde{\lambda}_2|d\widetilde{\lambda}_2\right]\frac{\left\langle\lambda_2|R_2|\widetilde{\lambda}_2\right]^b}{\left\langle\lambda_2|Q_2|\widetilde{\lambda}_2\right]\left\langle\lambda_2|K_{L_2}|\widetilde{\lambda}_2\right]^{b+1}}\\&\left\{\frac{(-)^a t_2^{b-1}(K_{L_1}^2)^{a+b-1}}{a!t_2^a}\ln\left(\frac{s+t_1t_2}{s-t_1t_2}\right)\frac{d^a}{d\tau^a}\frac{\left\langle\lambda_2|\mathcal{F}|\widetilde{\lambda}_2\right]^a}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^a}\right.\\&\left.+\sum_{i=0}^{a-1}\frac{(-)^{a-1}(K_{L_1}^2)^{a+b-1}t_2^{b-1}}{(a-i)a!t_2^i}\frac{d^i}{d\tau^i}\frac{d^{a-i}}{d\tau_1^{a-i}}\frac{\left\langle\lambda_2|\mathcal{G}_2|\widetilde{\lambda}_2\right]^a+(-)^{a-1-i}\left\langle\lambda_2|\mathcal{G}_1|\widetilde{\lambda}_2\right]^a}{\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^a}\right\}.\end{aligned}\tag{5.1}$$

[13]For some topologies, such as $\mathcal{A}_{112}$, since they are not detectable by our choice of unitarity cuts, we can not find their coefficients.

Again, using $K_{L_1} = -K_{L_2}$ we can simplify the denominator. There are also two parts we need to compute.

First part. The first part of remaining integration can be rewritten as

$$\int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 \frac{(-)^{a+b+1} t_2^{b-1} (K_{L_1}^2)^{a+b-1}}{(a+b)! t_2^a} \ln\left(\frac{(s+t_1t_2)}{(s-t_1t_2)}\right) \frac{d^a}{d\tau^a}\frac{d^b}{d\widetilde{\tau}^b}$$
$$\int \langle\lambda_2|d\lambda_2\rangle \left[\widetilde{\lambda}_2|d\widetilde{\lambda}_2\right] \frac{\left\langle\lambda_2|\widetilde{\tau}R_2 + \mathcal{F}|\widetilde{\lambda}_2\right]^{a+b}}{\left\langle\lambda_2|Q_2|\widetilde{\lambda}_2\right]\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^{a+b+1}}. \tag{5.2}$$

The second line is the standard one-loop triangle integration. The one-loop triangle can be reduced to triangle part and bubble part, thus they can be interpreted as contributions from topologies $\mathcal{A}_{213}$ and $\mathcal{A}_{212}$.

Second part. The second part can be written as

$$\int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 \sum_{i=0}^{a-1} \frac{(-)^{a+b}(K_{L_1}^2)^{a+b-1} t_2^{b-1}}{(a-i)(a+b)! t_2^i} \frac{d^i}{d\tau^i}\frac{d^{a-i}}{d\tau_1^{a-i}}\frac{d^b}{d\widetilde{\tau}^b}$$
$$\int \langle\lambda_2|d\lambda_2\rangle \left[\widetilde{\lambda}_2|d\widetilde{\lambda}_2\right] \frac{\left\langle\lambda_2|\widetilde{\tau}R_2 + \mathcal{G}_2|\widetilde{\lambda}_2\right]^{a+b} + (-)^{a-1-i}\left\langle\lambda_2|\widetilde{\tau}R_2 + \mathcal{G}_1|\widetilde{\lambda}_2\right]^{a+b}}{\left\langle\lambda_2|Q_2|\widetilde{\lambda}_2\right]\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^{a+b+1}}. \tag{5.3}$$

The second line is again the standard triangle integration which contain contributions from topologies $\mathcal{A}_{203}$ and $\mathcal{A}_{202}$.

5.2 Overview of results

Collecting all results together, we get an expression of the form

$$\Delta\mathcal{A}_{213}^{(a,b)} = \int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 \left\{ f_{213\to 213}^{(a,b)}\mathcal{S}_{213} + f_{213\to 212}^{(a,b)}\mathcal{S}_{212} + f_{213\to 203}^{(a,b)}\mathcal{S}_{203} + f_{213\to 202}^{(a,b)}\mathcal{S}_{202} \right\}, \tag{5.4}$$

where $\mathcal{S}_{202}$ and $\mathcal{S}_{212}$ have been defined in (4.5) and two new signatures are

$$\mathcal{S}_{203} = \frac{t_1}{2\sqrt{(K_4\cdot K_{L_1})^2 - K_{L_1}^2 K_4^2}} \ln\left(\frac{K_4^2 + K_4\cdot K_{L_1} - t_2\sqrt{(K_4\cdot K_{L_1})^2 - K_{L_1}^2K_4^2}}{K_4^2 + K_4\cdot K_{L_1} + t_2\sqrt{(K_4\cdot K_{L_1})^2 - K_{L_1}^2K_4^2}}\right),$$
$$\mathcal{S}_{213} = \frac{-\ln\left(\frac{s+t_1t_2}{s-t_1t_2}\right)}{2t_2K_{L_1}^2\sqrt{(K_4\cdot K_{L_1})^2 - K_{L_1}^2 K_4^2}} \ln\left(\frac{K_4^2 + K_4\cdot K_{L_1} - t_2\sqrt{(K_4\cdot K_{L_1})^2 - K_{L_1}^2K_4^2}}{K_4^2 + K_4\cdot K_{L_1} + t_2\sqrt{(K_4\cdot K_{L_1})^2 - K_{L_1}^2K_4^2}}\right). \tag{5.5}$$

There are a few remarks for expression (5.4). Firstly it is easy to see that the signature $\mathcal{S}_{203}$ is the direct product of signatures of one-loop bubble and one-loop triangle. Secondly there are two logarithms in the signature $\mathcal{S}_{213}$: one depends on both μ_1, μ_2 and the other only depends on μ_2. Pictorially, the first logarithm is related to the mixed propagator $(\ell_1+\ell_2)^2$ while the second logarithm is related to the right hand side sub-triangle.

Thirdly, all dependence of a, b are inside coefficients f while signatures are universal. However, unlike in the expression (4.4) where coefficients f are all rational functions of external momenta and polynomials of s, u_1, u_2, here we find that the coefficients f are in general not polynomials of s, u_1, u_2. In fact, factor $t_2 = \sqrt{1-u_2}$ will appear in denominators. Such behavior can not be explained by dimensional shifted master integral. Instead, we must regard it as the signature of new master integral. Because of such complexity, when talking about *the signature of a master integral for $\mathcal{A}_{213}$ topology*, we should treat all coefficients together in a list $\{f^{(a,b)}_{213\to 213}, f^{(a,b)}_{213\to 212}, f^{(a,b)}_{213\to 203}, f^{(a,b)}_{213\to 202}\}$ as a single object. More explicitly we will write the expression (5.4) as

$$\Delta\mathcal{A}^{(a,b)}_{213} \equiv \{f^{(a,b)}_{213\to 213}, f^{(a,b)}_{213\to 212}, f^{(a,b)}_{213\to 203}, f^{(a,b)}_{213\to 202}\}\,. \tag{5.6}$$

The reduction of $\Delta\mathcal{A}^{(a,b)}_{213}$ is to write it as the linear combination $\sum_i C_i\{a_{i1}, a_{i2}, a_{i3}, a_{i4}\}$ where $\{a_{i1}, a_{i2}, a_{i3}, a_{i4}\}$ is the signature of i-th master integral. In this notation, we can rewrite the signatures of previously discussed master integrals as

$$\Delta\mathcal{A}^{(0,0)}_{212} = \{0,1,0,0\}\,, \quad \Delta\mathcal{A}^{(0,0)}_{203} = \{0,0,1,0\}\,, \quad \Delta\mathcal{A}^{(0,0)}_{202} = \{0,0,0,1\}\,. \tag{5.7}$$

Having above general remarks, now we present explicit results.

5.3 The result of $a = 0$

We list results for $a = 0$ with various b. Noticing that $a = 0$ implies $f^{(0,b)}_{213\to 203} = 0$ and $f^{(0,b)}_{213\to 202} = 0$, we will focus on the first two coefficients only.

The case $b = 0$. It is easy to see that

$$\Delta\mathcal{A}^{(0,0)}_{213} = \{1,0,0,0\}\,. \tag{5.8}$$

Since it can not be written as the linear combination of three master integrals in (5.7), it must indicate a new master integral. In other words, $\mathcal{A}^{(0,0)}_{213}$ is an master integral with signature (5.8).

The case $b = 1$. The result is

$$\begin{aligned}\Delta\mathcal{A}^{(0,1)}_{213} = \Bigg\{ &\frac{(K_4^2 + K_4\cdot K_{L_1})(K_4\cdot T_2)K_{L_1}^2 - K_4^2(K_4\cdot K_{L_1} + K_{L_1}^2)(K_{L_1}\cdot T_2)}{K_4^2 K_{L_1}^2 - (K_4\cdot K_{L_1})^2}, \\ &\frac{(K_4\cdot K_{L_1})(T_2\cdot K_{L_1}) - K_{L_1}^2(K_4\cdot T_2)}{K_4^2 K_{L_1}^2 - (K_4\cdot K_{L_1})^2}, 0, 0\Bigg\}\,. \end{aligned} \tag{5.9}$$

Thus, at least for our choice of unitarity cuts, $\Delta\mathcal{A}^{(0,1)}_{213}$ can be written as the linear combination of signatures $\Delta\mathcal{A}^{(0,0)}_{213}$ and $\Delta\mathcal{A}^{(0,0)}_{212}$ with rational coefficients of external momenta.

For other b's. We have calculated cases $b = 2$ and $b = 3$. Again we find that $\Delta\mathcal{A}^{(0,b)}_{213}$ can be written as linear combinations of signatures $\Delta\mathcal{A}^{(0,0)}_{213}$ and $\Delta\mathcal{A}^{(0,0)}_{212}$, with coefficients being rational functions of external momenta and polynomials of s, u_1, u_2. The explicit expressions are too long to write down here. When s, u_1, u_2 appear in the results, we should include dimensional shifted master integrals too.

5.4 The result of $a = 1$

In this case a non-trivial phenomenon appears, and we will show how to explain it.

The case $b = 0$. Calculation yields

$$f^{(1,0)}_{213\to 213} = \frac{s}{t_2^2} f^{(1,0)}_{213\to 213;s^1} + f^{(1,0)}_{213\to 213;s^0}\,, \tag{5.10}$$

where

$$\begin{aligned} f^{(1,0)}_{213\to 213;s^1} &= \frac{(K_4^2 + K_4\cdot K_{L_1})[(K_4\cdot T_1)K_{L_1}^2 - (K_4\cdot K_{L_1})(K_{L_1}\cdot T_1)]}{(K_4^2 K_{L_1}^2 - (K_4\cdot K_{L_1})^2)}\,,\\ f^{(1,0)}_{213\to 213;s^0} &= K_{L_1}\cdot T_1\,. \end{aligned}$$

Although the s^0-part can be explained by the signature $\Delta\mathcal{A}^{(0,0)}_{213}$, the s^1-part with factor $\frac{s}{t_2^2}$ can not because the appearance of $t_2^2 = (1 - u_2)$ in the denominator. Thus factor $\frac{s}{t_2^2}$ indicates a new master integral.

Besides $f^{(1,0)}_{213\to 213}$, other coefficients are given by

$$\begin{aligned} f^{(1,0)}_{213\to 212} &= \frac{s[-K_{L_1}^2(K_4\cdot T_1) + (K_4\cdot K_{L_1})(T_1\cdot K_{L_1})]}{t_2^2(-K_4^2K_{L_1}^2 + (K_4\cdot K_{L_1})^2)}\,,\\ f^{(1,0)}_{213\to 203} &= \frac{-2(K_4^2 + K_4\cdot K_{L_1})[K_{L_1}^2(K_4\cdot T_1) - (K_4\cdot K_{L_1})(T_1\cdot K_{L_1})]}{t_2^2K_{L_1}^2(K_4^2K_{L_1}^2 - (K_4\cdot K_{L_1})^2)}\,,\\ f^{(1,0)}_{213\to 202} &= \frac{2[-K_{L_1}^2(K_4\cdot T_1) + (K_4\cdot K_{L_1})(T_1\cdot K_{L_1})]}{t_2^2K_{L_1}^2(-K_4^2K_{L_1}^2 + (K_4\cdot K_{L_1})^2)}\,. \end{aligned} \tag{5.11}$$

Again, because of the factor $\frac{1}{t_2^2}$, they can not be explained by signatures (5.7). Thus we have the first non-trivial example of signatures where all four components are non-zero

$$\Delta\mathcal{A}^{(1,0)}_{213} = \{f^{(1,0)}_{213\to 213}, f^{(1,0)}_{213\to 212}, f^{(1,0)}_{213\to 203}, f^{(1,0)}_{213\to 202}\}\,. \tag{5.12}$$

The case $b = 1$. All coefficients $\{f^{(1,1)}_{213\to 213}, f^{(1,1)}_{213\to 212}, f^{(1,1)}_{213\to 203}, f^{(1,1)}_{213\to 202}\}$ have $\frac{1}{t_2^2}$ dependence. However, all these $\frac{1}{t_2^2}$ factors can be absorbed into $\Delta\mathcal{A}^{(1,0)}_{213}$. More explicitly, we found the following decomposition

$$\Delta\mathcal{A}^{(1,1)}_{213} = a_{11\to 00}\Delta\mathcal{A}^{(0,0)}_{213} + a_{11\to 10}\Delta\mathcal{A}^{(1,0)}_{213} + b_{11\to 00}\Delta\mathcal{A}^{(0,0)}_{212} + d_{11\to 00}\Delta\mathcal{A}^{(0,0)}_{202}\,, \tag{5.13}$$

where

$$a_{11\to 10} = \frac{(1-u_2)K_{L_1}^2((K_4\cdot K_{L_1})^2 - K_4^2K_{L_1}^2)\Sigma_1 + (K_4^2 + K_4\cdot K_{L_1})\Sigma_2}{2(K_4^2+K_4\cdot K_{L_1})((K_4\cdot K_{L_1})^2 - K_4^2K_{L_1}^2)[(K_4\cdot T_1)K_{L_1}^2 - (K_4\cdot K_{L_1})(T_1\cdot K_{L_1})]},$$

$$a_{11\to 00} = \frac{(K_{L_1}\cdot T_1)[-(K_4\cdot T_2)K_{L_1}^2(K_4^2+K_4\cdot K_{L_1}) + K_4^2(K_4\cdot K_{L_1}+K_{L_1}^2)(K_{L_1}\cdot T_2)]}{((K_4\cdot K_{L_1})^2 - K_4^2K_{L_1}^2)} - (K_{L_1}\cdot T_1)a_{11\to 10},$$

$$b_{11\to 00} = \frac{1}{2(K_4\cdot K_{L_1}+K_4^2)(-(K_4\cdot K_{L_1})^2+K_4^2K_{L_1}^2)}\{2(K_4\cdot K_{L_1}+K_4^2)K_{L_1}\cdot T_1 (-K_{L_1}^2(K_4\cdot T_2)+(K_4\cdot K_{L_1})(K_{L_1}\cdot T_2))+sK_{L_1}^2(K_{L_1}\cdot T_1(K_4\cdot K_{L_1}(K_4\cdot T_2)-K_4^2(K_{L_1}\cdot T_2)) +K_4\cdot T_1(-K_{L_1}^2(K_4\cdot T_2)+K_4\cdot K_{L_1}(K_{L_1}\cdot T_2)+T_1\cdot T_2(-(K_4\cdot K_{L_1})^2+K_4^2K_{L_1}^2))\},$$

$$d_{11\to 00} = \frac{K_{L_1}\cdot T_1(K_4\cdot K_{L_1}(K_4\cdot T_2)-K_4^2(K_{L_1}\cdot T_2))+K_4\cdot T_1(-K_{L_1}^2(K_4\cdot T_2)+K_4\cdot K_{L_1}(K_{L_1}\cdot T_2))}{(K_4\cdot K_{L_1}+K_4^2)(-(K_4\cdot K_{L_1})^2+K_4^2K_{L_1}^2)} + \frac{T_1\cdot T_2}{(K_4\cdot K_{L_1}+K_4^2)},$$

with

$$\Sigma_1 = (K_{L_1}\cdot T_1)(-(K_4\cdot K_{L_1})(K_4\cdot T_2)+K_4^2(K_{L_1}\cdot T_2)) + K_4\cdot T_1((K_4\cdot T_2)K_{L_1}^2 - (K_4\cdot K_{L_1})(T_2\cdot K_{L_1})) + ((K_4\cdot K_{L_1})^2 - K_4^2K_{L_1}^2)T_1\cdot T_2,$$

$$\Sigma_2 = (K_4^2)^2K_{L_1}^2(-(K_{L_1}\cdot T_1)(K_{L_1}\cdot T_2)+K_{L_1}^2(T_1\cdot T_2)) + (K_4\cdot K_{L_1})K_{L_1}^2[K_4\cdot T_1(-3(K_4\cdot T_2)K_{L_1}^2 + (K_4\cdot K_{L_1})(K_{L_1}\cdot T_2)) + K_4\cdot K_{L_1}(3(K_4\cdot T_2)(K_{L_1}\cdot T_1) - (K_4\cdot K_{L_1})(T_1\cdot T_2))] + K_4^2(K_4\cdot T_1K_{L_1}^2(-3K_4\cdot T_2K_{L_1}^2 + (3K_4\cdot K_{L_1}+2K_{L_1}^2)K_{L_1}\cdot T_2) + (K_4\cdot K_{L_1}) ((K_{L_1}\cdot T_1)(3K_{L_1}^2(K_4\cdot T_2 - K_{L_1}\cdot T_2) - 2(K_4\cdot K_{L_1})(K_{L_1}\cdot T_2)) + K_{L_1}^2(-K_4\cdot K_{L_1}+K_{L_1}^2)T_1\cdot T_2).$$

Since above four coefficients are rational functions of external momenta and polynomials of u_2, we can claim that $\mathcal{A}_{213}^{(1,1)}$ is not a master integral at least for our choice of unitarity cuts.

There are some details we want to remark. The coefficient $a_{11\to 00}$ is a polynomial of T_1 and T_2 with degree one while coefficient $a_{11\to 10}$ is a polynomial of T_2 with degree one but rational function of T_1. More accurately, both the denominator and the numerator of $a_{11\to 10}$ are polynomials of T_1 with degree one. It is against the intuition since T_1 should not appear in the denominator. However, this subtlety is resolved if one notice that the first component $f_{213\to 213;s^1}^{(1,0)}$ of $\Delta\mathcal{A}_{213}^{(1,0)}$ contains exactly the same factor $[-(K_4\cdot T_1)K_{L_1}^2+(K_4\cdot K_{L_1})(K_{L_1}\cdot T_1)]$ in its numerator, so it cancels the same factor in denominator of $a_{11\to 10}$.

The case $b = 2$. The whole expression is too long to write down, thus we present only the general feature. Again although all coefficients contain factor $\frac{1}{t_2^2}$, the whole result can be expanded like the one (5.13) with coefficients as rational functions of external momenta and polynomials of s, u_1, u_2. Thus $\mathcal{A}_{213}^{(1,2)}$ is not a new master integral.

5.5 The result of $a = 2$

We will encounter similar phenomenon as in the case $a = 1$. To get rid of tedious expressions, we will present only the main features.

The case $b = 0$. The coefficient $f^{(2,0)}_{213\to 213}$ has the following form

$$f^{(2,0)}_{213\to 213} = \frac{s^2}{t_2^4} g_{0;1} + \frac{s^2}{t_2^2} g_{0;2} + \frac{s}{t_2^2} g_{0;3} + \frac{1}{t_2^2} g_{0;4} + g_{0;5}\,, \tag{5.14}$$

where $g_{0;i}$'s are polynomials of u_1 and rational functions of external momenta (similar for all other coefficients such as h, i, j in this subsection). The appearance of $g_{0;1}$-part and $g_{0;4}$-part can not be counted by signatures $\Delta\mathcal{A}^{(0,0)}_{213}$ and $\Delta\mathcal{A}^{(1,0)}_{213}$, thus we should take $\mathcal{A}^{(2,0)}_{213}$ as a new master integral. For other coefficients, we have

$$\begin{aligned} f^{(2,0)}_{213\to 212} &= \frac{s^2}{t_2^4} h_{0;1} + \frac{s}{t_2^2} h_{0;2} + \frac{1}{t_2^2} h_{0;3}\,, \\ f^{(2,0)}_{213\to 203} &= \frac{s}{t_2^4} i_{0;1} + \frac{s}{t_2^2} i_{0;2} + \frac{1}{t_2^2} i_{0;3}\,, \\ f^{(2,0)}_{213\to 202} &= \frac{s}{t_2^4} j_{0;1} + \frac{1}{t_2^2} j_{0;2}\,. \end{aligned} \tag{5.15}$$

The signature of the new master integral can be represented by

$$\Delta\mathcal{A}^{(2,0)}_{213} = \{f^{(2,0)}_{213\to 213}, f^{(2,0)}_{213\to 212}, f^{(2,0)}_{213\to 203}, f^{(2,0)}_{213\to 202}\}\,. \tag{5.16}$$

The case of $b = 1$. The behavior of various coefficients are

$$\begin{aligned} f^{(2,1)}_{213\to 213} &= \frac{s^2}{t_2^4} g_{1;1} + \frac{s^2(g_{1;2;0} + t_2^2 g_{1;2;1})}{t_2^2} + \frac{s(g_{1;3;0} + t_2^2 g_{1;3;1})}{t_2^2} + \frac{1}{t_2^2} g_{1;4} + g_{1;5}\,, \\ f^{(2,1)}_{213\to 212} &= \frac{s^2(h_{1;1;0} + t_2^2 h_{1;1;1})}{t_2^4} + \frac{s}{t_2^2} h_{1;2} + \frac{1}{t_2^2}(h_{1;3;0} + t_2^2 h_{1;3;1})\,, \\ f^{(2,1)}_{213\to 203} &= \frac{s}{t_2^4} i_{1;1} + \frac{s}{t_2^2} i_{1;2} + \frac{1}{t_2^2}(i_{1;3;0} + t_2^2 i_{1;3;1})\,, \\ f^{(2,1)}_{213\to 202} &= \frac{s(j_{1;1;0} + t_2^2 j_{1;1;1})}{t_2^4} + \frac{1}{t_2^2} j_{1;2}\,, \end{aligned} \tag{5.17}$$

where the integer n in $g_{1;m;n}$ denotes the power of t_2^2, and similar for $h,\ i,\ j$.

We found the following expansion

$$\begin{aligned} \Delta\mathcal{A}^{(2,1)}_{213} &= a_{21\to 20}\Delta\mathcal{A}^{(2,0)}_{213} + a_{21\to 10}\Delta\mathcal{A}^{(1,0)}_{213} + a_{21\to 00}\Delta\mathcal{A}^{(0,0)}_{213} + b_{21\to 00}\Delta\mathcal{A}^{(0,0)}_{212} \\ &\quad + c_{21\to 00}\Delta\mathcal{A}^{(0,0)}_{203} + d_{21\to 00}\Delta\mathcal{A}^{(0,0)}_{202}\,, \end{aligned} \tag{5.18}$$

where coefficients are rational functions of external momenta and polynomials of s, u_1, u_2. Thus $\mathcal{A}^{(2,1)}_{213}$ is not a new master integral.

5.6 Classification of master integrals

With above results, we can classify the master integrals of $\mathcal{A}_{213}$ topology. Before doing so, we want to emphasize that in our calculations, momenta K_3, K_4 can be massive or massless, while $K_{L_1} = -K_3 - K_4$ is massive.

JHEP06(2014)166

Having shown that coefficients such as $a_{21\to 00}$ are polynomials of $\mu_1\cdot\mu_2, \mu_1^2, \mu_2^2$ and rational functions of external momenta, we can expand them, for example

$$a_{21\to 00} = \sum_{\kappa_0,\kappa_1,\kappa_2} a^{(a,b)}_{21\to 00}(\mu_1^2)^{\kappa_1}(\mu_2^2)^{\kappa_2}(\mu_1\cdot\mu_2)^{\kappa_0}\,, \tag{5.19}$$

where the tensor coefficients a are rational functions of external momenta. This expansion leads us to define the following dimensional shifted integrals

$$\mathcal{B}_{213;a}[\kappa_0,\kappa_1,\kappa_2;T_1] \equiv \int d^{4-2\epsilon}\widehat{\ell}_1 \int d^{4-2\epsilon}\widehat{\ell}_2 \frac{(\mu_1^2)^{\kappa_1}(\mu_2^2)^{\kappa_2}(\mu_1\cdot\mu_2)^{\kappa_0}(\widehat{\ell}_1\cdot T_1)^a}{\widehat{\ell}_1^2(\widehat{\ell}_1-K_{L_1})^2\widehat{\ell}_2^2(\widehat{\ell}_2-K_4)^2(\widehat{\ell}_2+K_{L_1})^2(\widehat{\ell}_1+\widehat{\ell}_2)^2}\,. \tag{5.20}$$

Unlike the scalar basis $\mathcal{B}_{212}[\kappa_0,\kappa_1,\kappa_2]$ for $\mathcal{A}_{212}$ topology, the basis $\mathcal{B}_{213;a}[\kappa_0,\kappa_1,\kappa_2]$ depends on T_1 explicitly. Since T_1 is a 4-dimensional Lorentz vector, there are four independent choices and we need to clarify if different choice of T_1 gives new independent master integrals.

To discuss this problem we expand $T_1 = \sum_{i=1}^4 x_i e_i$. The momentum bases e_i are constructed as follows. Using K_4, K_{L_1} we can construct two null momenta $P_i = K_4 + w_i K_{L_1}$ with $w_i = \frac{-K_4\cdot K_{L_1}\pm\sqrt{(K_{L_1}\cdot K_4)^2 - K_4^2K_{L_1}^2}}{K_{L_1}^2}$, thus the momentum bases can be taken as

$$e_1 = K_4\,,\quad e_2 = K_{L_1}\,,\quad e_3 = |P_1\rangle\,|P_2]\,,\quad e_4 = |P_2\rangle\,|P_1]\,. \tag{5.21}$$

The case $a=0$. For $a=0$, since T_1 does not appear, only **scalar integral** exist. Thus the independent master integrals are $\mathcal{B}_{213;0}[\kappa_0,\kappa_1,\kappa_2]$.

The case $a=1$. We set $T_1 = e_i$ for $i=1,2,3,4$ in the expressions $f^{(1,0)}_{213\to 213}$, $f^{(1,0)}_{213\to 212}$, $f^{(1,0)}_{213\to 203}$, $f^{(1,0)}_{213\to 202}$, and found that:

- (1) For $T_1 = e_3$ or $T_1 = e_4$ we have

$$\{f^{(1,0)}_{213\to 213}, f^{(1,0)}_{213\to 212}, f^{(1,0)}_{213\to 203}, f^{(1,0)}_{213\to 202}\} = \{0,0,0,0\}\,. \tag{5.22}$$

  It can be shown that $T_1 = e_{3,4}$ are spurious and the integrations are zero.

- (2) For $T_1 = K_{L_1}$, we find

$$\{f^{(1,0)}_{213\to 213}, f^{(1,0)}_{213\to 212}, f^{(1,0)}_{213\to 203}, f^{(1,0)}_{213\to 202}\} = \{K_{L_1}^2,0,0,0\}\,. \tag{5.23}$$

  It is, in fact, equivalent to expressions $\mathcal{B}_{213;0}[\kappa_0,\kappa_1,\kappa_2]$ and does not give new master integrals.

- (3) For $T_1 = K_4$, we find

$$\begin{aligned}&\left\{f^{(1,0)}_{213\to 213}, f^{(1,0)}_{213\to 212}, f^{(1,0)}_{213\to 203}, f^{(1,0)}_{213\to 202}\right\}\\ &=\left\{\frac{-s(K_4^2+K_4\cdot K_{L_1})+t_2^2K_4\cdot K_{L_1}}{t_2^2}, \frac{s}{t_2^2}, -\frac{2(K_4^2+K_4\cdot K_{L_1})}{t_2^2K_{L_1}^2}, \frac{2}{t_2^2K_{L_1}^2}\right\}\,,\end{aligned} \tag{5.24}$$

  which is the true new master integral.

Conclusion: for $a = 1$, the master integrals are given by $\mathcal{B}_{213;1}[\kappa_0, \kappa_1, \kappa_2; K_4]$.

The case $a = 2$. There are ten possible combinations $(\widehat{\ell}_1 \cdot e_i)(\widehat{\ell}_1 \cdot e_j)$. With the explicit result we found that

- (1) For the following six combinations

$$(e_i, e_j) = (e_3, e_3)\,,\ (e_4, e_4)\,,\ (e_1, e_3)\,,\ (e_2, e_3)\,,\ (e_1, e_4)\,,\ (e_2, e_4)\,, \tag{5.25}$$

  the coefficients are $\{0, 0, 0, 0\}$. In fact, integrations for these six cases are zero.

- (2) For $(e_i, e_j) = (e_2, e_2)$ the list of coefficients is $\{2(K_{L_1}^2)^2, 0, 0, 0\}$. It is equivalent to the expressions $\mathcal{B}_{213;0}[\kappa_0, \kappa_1, \kappa_2]$. Therefore it does not give new master integrals.

- (3) For $(e_i, e_j) = (e_1, e_2)$ the list of coefficients is

$$2K_{L_1}^2 \left\{ \frac{-s(K_4^2 + K_4 \cdot K_{L_1}) + t_2^2 K_4 \cdot K_{L_1}}{t_2^2}, \frac{s}{t_2^2}, -\frac{2(K_4^2 + K_4 \cdot K_{L_1})}{t_2^2 K_{L_1}^2}, \frac{2}{t_2^2 K_{L_1}^2} \right\}\,, \tag{5.26}$$

  which is proportional to (5.24) by a factor $2K_{L_1}^2$. Therefore it can be reduced to expressions $\mathcal{B}_{213;1}[\kappa_0, \kappa_1, \kappa_2]$, and dose not give new master integrals.

- (4) For $(e_i, e_j) = (e_1, e_1)$ and $(e_i, e_j) = (e_3, e_4)$ the list is non-trivial. However, it can be checked that

$$\begin{aligned}
&\{f^{(2,0)}_{213\to 213}, f^{(2,0)}_{213\to 212}, f^{(2,0)}_{213\to 203}, f^{(2,0)}_{213\to 202}\}|_{(e_i,e_j)=(e_1,e_1)} \\
&= \{f^{(2,0)}_{213\to 213}, f^{(2,0)}_{213\to 212}, f^{(2,0)}_{213\to 203}, f^{(2,0)}_{213\to 202}\}|_{(e_i,e_j)=(e_3,e_4)} \\
&\quad + 2(K_4 \cdot K_{L_1})\{f^{(1,0)}_{213\to 213}, f^{(1,0)}_{213\to 212}, f^{(1,0)}_{213\to 203}, f^{(1,0)}_{213\to 202}\} \\
&\quad + ((t_1^2 - 1)(K_4 \cdot K_{L_1})^2 - t_1^2 K_4^2 K_{L_1}^2)\{f^{(0,0)}_{213\to 213}, f^{(0,0)}_{213\to 212}, f^{(0,0)}_{213\to 203}, f^{(0,0)}_{213\to 202}\}\,.
\end{aligned} \tag{5.27}$$

  Thus we can take either one (but only one of them) as the master integral. We choose the combination $(e_i, e_j) = (e_1, e_1)$ to be a new master integral.

Conclusion: for $a = 2$, master integrals can be chosen as $\mathcal{B}_{213;2}[\kappa_0, \kappa_1, \kappa_2; K_4]$.

For general a. Although we have not done explicit calculations for $a \geq 3$, we expect for each a there are new integrals $\mathcal{B}_{213;a}[\kappa_0, \kappa_1, \kappa_2; K_4]$.

The number of master integrals. To finish this section, let us count the number of master integrals. For pure 4D, we just need to set $\mu_1 \cdot \mu_2, \mu_1^2, \mu_2^2$ to zero. In this case, the factor $\frac{1}{t_2^n} \to 1$. In other words, there is only one master integral

$$\int d^{4-2\epsilon}\widehat{\ell}_1 \int d^{4-2\epsilon}\widehat{\ell}_2 \frac{1}{\widehat{\ell}_1^2(\widehat{\ell}_1 - K_{L_1})^2 \widehat{\ell}_2^2(\widehat{\ell}_2 - K_4)^2(\widehat{\ell}_2 + K_{L_1})^2(\widehat{\ell}_1 + \widehat{\ell}_2)^2}\,. \tag{5.28}$$

It is useful to compare it with about 70 elements in the integrand bases found in [98] under renormalizable conditions.

For general $(4-2\epsilon)$-dimension, renormalizable conditions can be roughly given by $2\kappa_1 + a \leq 3$, $\kappa_2 \leq 2$. Under these two conditions, we find 48 master integrals.[14]

6 The integral basis of $\mathcal{A}_{313}$ topology

In this section we turn to the topology $\mathcal{A}_{313}$. This topology has been extensively studied by various methods, such as IBP method [20] and maximum unitarity cut method [80–86], and master integrals have been determined [20]. To determine these master integrals using our method, we need to integrate the following expression

$$\Delta\mathcal{A}_{313}^{(a,b)} = \int d^{-2\epsilon}\mu_1 d^{-2\epsilon}\mu_2 \int d^4\widetilde{\ell}_2\ \delta(\widetilde{\ell}_2^2 - \mu_2^2)\delta(K_{L_2}^2 - 2K_{L_2}\cdot\widetilde{\ell}_2)\frac{(2\widetilde{\ell}_2\cdot T_2)^b}{((\widetilde{\ell}_2 - K_4)^2 - \mu_2^2)}$$
$$\int \langle\lambda_1|d\lambda_1\rangle\left[\widetilde{\lambda}_1|d\widetilde{\lambda}_1\right]\frac{-((1-2z_1)K_{L_1}^2)^{a-1}}{\left\langle\lambda_1|K_{L_1}|\widetilde{\lambda}_1\right]^a}\frac{\left\langle\lambda_1|R_1|\widetilde{\lambda}_1\right]^a}{\left\langle\lambda_1|W_1|\widetilde{\lambda}_1\right]\left\langle\lambda_1|Q_1|\widetilde{\lambda}_1\right]}, \tag{6.1}$$

with $K_{L_1} = K_1 + K_2$. For general situation, the integration is very complicated and we postpone it to future study. In this paper, we take the following simplification. Firstly we take all out-going momenta $K_i^2 = 0$ $(i = 1,\ldots,4)$ (unlike the topologies $\mathcal{A}_{212}$ and $\mathcal{A}_{213}$ where K_i can be massive or massless). Secondly, based on the known results of master integrals, we focus on the specific case $a = 0$ and $T_2 = K_1$.

In order to make expressions compact we define some new parameters as[15]

$$s \equiv s_{12}\,, \quad m \equiv \frac{s_{14} - s_{13}}{s_{12}}\,, \quad \chi \equiv \frac{s_{14}}{s_{12}} = \frac{m-1}{2}\,. \tag{6.2}$$

For physical unitarity cut, momentum configuration requires $s_{12} > 0$, $s_{13} < 0$ and $s_{14} < 0$. So we have

$$-1 < m < 1\,, \quad -1 < \chi < 0 \tag{6.3}$$

by momentum conservation $s_{12} + s_{13} + s_{14} = 0$. Furthermore, we define the regularization parameters γ_i as

$$\gamma \equiv \frac{1 + \nu_1\cdot\nu_2}{\sqrt{1-\nu_1^2}\sqrt{1-\nu_2^2}}\,, \quad \gamma_i \equiv \frac{1}{\sqrt{1-\nu_i^2}}\,, \quad i = 1,2\,, \tag{6.4}$$

where the dimensionless extra-dimensional vector ν_i is defined as $\nu_i \equiv 2\mu_i/\sqrt{s}\,,\ i = 1,2$.

[14] From explicit expressions of (5.10), especially the coefficient $f^{(1,0)}_{213\to 213;s^1}$, one can see that putting $T_1 = K_4$, $f^{(1,0)}_{213\to 213;s^1}$ is not zero no matter K_4 is massive or massless. Based on this observation, we believe that our counting of the number of master integrals is independent of K_4, K_3.

[15] It is worth to notice that s in this section is different from s in (3.6) of section 3.

Under the simplification $a = 0$, the integration over λ_1-part is trivial. Using (B.21) in appendix B we can get

$$\Delta\mathcal{A}_{313}^{(0,b)} = \int d\mu_i \int \langle\lambda_2|d\lambda_2\rangle \left[\widetilde{\lambda}_2|d\widetilde{\lambda}_2\right] \frac{\gamma_1}{s^2}\Big(-\frac{s}{\gamma_2}\Big)^{b-1} \frac{\left\langle\lambda_2|R_2|\widetilde{\lambda}_2\right]^b}{\left\langle\lambda_2|Q_2|\widetilde{\lambda}_2\right]\left\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2\right]^{b+1}}$$
$$\frac{1}{\sqrt{\frac{\langle\lambda_2|\widetilde{K}_1|\widetilde{\lambda}_2]^2}{\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2]^2}-\frac{\beta^2}{4}}} \ln\left(\frac{\frac{\langle\lambda_2|\widetilde{K}_1|\widetilde{\lambda}_2]}{\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2]}+\sqrt{\frac{\langle\lambda_2|\widetilde{K}_1|\widetilde{\lambda}_2]^2}{\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2]^2}-\frac{\beta^2}{4}}}{\frac{\langle\lambda_2|\widetilde{K}_1|\widetilde{\lambda}_2]}{\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2]}-\sqrt{\frac{\langle\lambda_2|\widetilde{K}_1|\widetilde{\lambda}_2]^2}{\langle\lambda_2|K_{L_1}|\widetilde{\lambda}_2]^2}-\frac{\beta^2}{4}}}\right), \tag{6.5}$$

where

$$\beta^2 = (\gamma^2-1)(\gamma^2-1)\,.$$

An important feature is that the signature after λ_1-integration depends on ℓ_2 explicitly, which is different from the signature in (3.5). Because of this, the integration over λ_2 becomes very complicated. One way to overcome is to use

$$\frac{1}{b}\log\frac{a+b}{a-b} = \int_0^1 dx\left(\frac{1}{a+xb}+\frac{1}{a-xb}\right) = \int_0^1 dx\frac{2a}{a^2-x^2b^2}\,. \tag{6.6}$$

Thus the logarithmic part in (6.5) becomes rational function of ℓ_2 and we can use the same strategy as in previous sections. However, for the current simple situation, we can use another method. After expanding the spinor variables as

$$|\lambda_2\rangle = |k_2\rangle + z\,|k_1\rangle\ ,\quad \left|\widetilde{\lambda}_2\right] = |k_2] + \overline{z}\,|k_1]\ ,\quad \langle\lambda_2|d\lambda_2\rangle\left[\widetilde{\lambda}_2|d\widetilde{\lambda}_2\right] = -s dz d\overline{z}\,, \tag{6.7}$$

the integration becomes an integration over complex plane

$$\Delta\mathcal{A}_{313}^{(0,b)} = \int d\mu_i \int |dzd\overline{z}|(\bullet) = \int d\mu_i \int_0^{+\infty} rdr\int_0^{2\pi} d\theta(\bullet)\,,\quad z = re^{i\theta}\,. \tag{6.8}$$

θ-integration. The θ-dependent part of (6.5) is given by

$$\int_0^{2\pi} d\theta\frac{\Big(\langle K_2|R_2|K_2] + r^2\langle K_1|R_2|K_1] + re^{i\theta}\langle K_1|R_2|K_2] + re^{-i\theta}\langle K_2|R_2|K_1]\Big)^b}{(s_{24}-\widetilde{t}_2 s_{12}) + r^2(s_{14}-\widetilde{t}_2 s_{12}) + re^{i\theta}\langle K_1|K_4|K_2] + re^{-i\theta}\langle K_2|K_4|K_1]}\,, \tag{6.9}$$

with $\widetilde{t}_2 = \frac{\gamma_2-1}{2}$. Setting $x = e^{i\theta}$ the integral becomes a circle contour integration with radius one

$$\oint_{|x|=1} dx\frac{\Big(x\langle K_2|R_2|K_2] + xr^2\langle K_1|R_2|K_1] + rx^2\langle K_1|R_2|K_2] + r\langle K_2|R_2|K_1]\Big)^b}{ix^b\Big(x(s_{24}-\widetilde{t}_2 s_{12}) + xr^2(s_{14}-\widetilde{t}_2 s_{12}) + rx^2\langle K_1|K_4|K_2] + r\langle K_2|K_4|K_1]\Big)}\,. \tag{6.10}$$

There are three poles in total. The first one is $x = 0$ when $b \neq 0$ for general R_2. The other two are roots of the quadratic polynomial in denominator

$$x_{1,2} = \frac{-\Big(s_{24} - s_{14} + (r^2+1)(s_{14} - \widetilde{t}_2 s_{12})\Big) \pm \sqrt{\Delta}}{2r \langle K_1|K_4|K_2]}, \tag{6.11}$$

where

$$\Delta = \Big(-s_{12} + (r^2+1)(s_{14} - \widetilde{t}_2 s_{12})\Big)^2 + 4 s_{12} s_{14}(r^2+1)(1+\widetilde{t}_2). \tag{6.12}$$

It is easy to check that $|x_1 x_2| = 1$. Thus one root is inside the integration contour and the other is outside. The kinematic conditions $s_{12} > 0, s_{24} < 0, s_{14} < 0$ ensure that x_1 is the one inside. The residue at the pole x_1 is

$$\frac{1}{i\sqrt{\Delta}} \Big(\langle K_2|R_2|K_2] + r^2 \langle K_1|R_2|K_1] + rx \langle K_1|R_2|K_2] + rx^{-1} \langle K_2|R_2|K_1] \Big)^b_{x=x_1}. \tag{6.13}$$

The case ($T_2 = K_1$). Under our simplification, we set $T_2 = K_1$, thus $\langle K_1|R_2|K_2] = 0$ and $\langle K_2|R_2|K_1] = 0$. Because of this, there is no pole at $x = 0$ in (6.10). Thus after the θ-integration, (6.5) is reduced to

$$\begin{aligned}\Delta\mathcal{A}^{(0,b)}_{313} &= \int d\mu_i \; \frac{\gamma_1}{2s^2} \left(-\frac{s}{\gamma_2}\right)^{b-1} \int_0^{+\infty} dr^2 \frac{1}{\sqrt{\left(\frac{\alpha-1}{2} + \frac{1}{(1+r^2)}\right)^2 - \frac{\beta^2}{4}}} \\ &\quad \left(\ln \frac{\left(\frac{\alpha-1}{2} + \frac{1}{1+r^2}\right) + \sqrt{\left(\frac{\alpha-1}{2} + \frac{1}{1+r^2}\right)^2 - \frac{\beta^2}{4}}}{\left(\frac{\alpha-1}{2} + \frac{1}{1+r^2}\right) - \sqrt{\left(\frac{\alpha-1}{2} + \frac{1}{1+r^2}\right)^2 - \frac{\beta^2}{4}}} \right) \frac{1}{1+r^2} \left(\frac{\gamma_2 - 1}{2} + \frac{1}{1+r^2} \right)^b \\ &\quad \frac{1}{\sqrt{\left((r^2+1)\left(\chi - \frac{\gamma_2-1}{2}\right) - 1\right)^2 + 4(r^2+1)\chi\left(1 + \frac{\gamma_2-1}{2}\right)}},\end{aligned} \tag{6.14}$$

in which

$$\alpha = \gamma\gamma_1, \quad \beta = \sqrt{(\gamma^2-1)(\gamma_1^2-1)}.$$

Defining $u = \frac{1-r^2}{1+r^2}$ we arrive

$$\begin{aligned}\Delta\mathcal{A}^{(0,b)}_{313} &= \int d\mu_i \; \frac{\gamma_1}{2s^2}\left(-\frac{s}{2\gamma_2}\right)^{b-1} \int_{-1}^{+1} du \frac{(u+\gamma_2)^b}{\sqrt{(u+m\gamma_2)^2 + (1-m^2)(\gamma_2^2-1)}} \\ &\quad \frac{1}{\sqrt{(u+\alpha)^2 - \beta^2}} \ln \frac{(u+\alpha) + \sqrt{(u+\alpha)^2 - \beta^2}}{(u+\alpha) - \sqrt{(u+\alpha)^2 - \beta^2}}.\end{aligned} \tag{6.15}$$

An important observation from (6.15) is that $\mathcal{D}^{(0,b)}_{313}/(\gamma_1\gamma_2)$ has the symmetry $\gamma \leftrightarrow \gamma_1$ as well as the symmetry $\gamma_2 \leftrightarrow \gamma_1$ for $b = 0$ by the topology.

JHEP06(2014)166

Since we use the dimensional shifted bases, the μ_i part is kept and we will focus on $\mathcal{D}_{313}^{(0,b)}$ after the u-integration, i.e.,

$$\Delta\mathcal{A}_{313}^{(0,b)} \equiv \int d\mu_i\, \mathcal{D}_{313}^{(0,b)}\,. \tag{6.16}$$

We found it hard to integrate over u and get analytic results. However, in the general $(4-2\epsilon)$-dimensional framework, we can treat μ_i^2 and $\mu_1\cdot\mu_2$ as small parameters and take series expansion around $\mu_i^2 \to 0$. It is equivalent to taking the series expansion around $\gamma_i \to 1$. The details of calculation can be found in appendix C. Up to the leading order, result for $a=0, b=0$ is given by

$$\begin{aligned}\mathcal{D}_{313}^{(0,0)} = \frac{1}{s^3\, 2\chi}\Bigg[&\ln\left(\frac{-2\chi}{\gamma-1}\right)\ln\left(\frac{-2\chi}{\gamma_1-1}\right)+\ln\left(\frac{-2\chi}{\gamma-1}\right)\ln\left(\frac{-2\chi}{\gamma_2-1}\right)+\ln\left(\frac{-2\chi}{\gamma_1-1}\right)\ln\left(\frac{-2\chi}{\gamma_2-1}\right)\\ &+2\mathsf{Li}_2(1+\chi)-\frac{\pi^2}{3}\Bigg]\,.\end{aligned} \tag{6.17}$$

An important check for the result (6.17) is that it has the S_3 permutation symmetry among $\gamma_1, \gamma_2, \gamma$. The terms $\ln(-\chi)$ and $\mathsf{Li}_2(1+\chi)$ do not show up for topologies $\mathcal{A}_{212}$ and $\mathcal{A}_{213}$, thus they belong to the signature of $\mathcal{A}_{313}$. For $b=1$, the result is

$$\mathcal{D}_{313}^{(0,1)} = \chi\, s\mathcal{D}_{313}^{(0,0)} + \mathcal{D}_{312}^{(0,0)} - \frac{1}{s^2}\ln\left(\frac{-2\chi}{\gamma-1}\right)\ln\left(\frac{-2\chi}{\gamma_1-1}\right). \tag{6.18}$$

The extra term $-\frac{1}{s^2}\ln\left(\frac{-2\chi}{\gamma-1}\right)\ln\left(\frac{-2\chi}{\gamma_1-1}\right)$ in (6.18) indicates that comparing to $\mathcal{D}_{313}^{(0,0)}$, $\mathcal{D}_{313}^{(0,1)}$ should be taken as a new master integral. For $b=2$, the result is

$$\mathcal{D}_{313}^{(0,2)} = \chi\, s\mathcal{D}_{313}^{(0,1)} + \frac{2\chi+1}{s}\mathcal{D}_{202}^{(0,0)} - \frac{2\chi+1}{2}\mathcal{D}_{212}^{(0,0)} - \frac{2\chi+1}{2}\mathcal{D}_{302}^{(0,0)} - \frac{2\chi}{s}\ln(-\chi)\,. \tag{6.19}$$

For this result, there are a few things we want to discuss. Firstly, the same coefficient $-\frac{2\chi+1}{2}$ appears for $\mathcal{D}_{212}^{(0,0)}$ and $\mathcal{D}_{302}^{(0,0)}$, which is the consequence of symmetry $\gamma \leftrightarrow \gamma_1$ in (6.15). Secondly, the appearance of term $\ln(-\chi)$ is quite intriguing. There are several possible interpretations:

- Under the general $(4-2\epsilon)$-dimensional framework, $\mathcal{D}_{313}^{(0,2)}$ could be considered as a new master integral.

- From the result in [20], $\mathcal{A}_{313}^{(0,2)}$ can be written as linear combinations of master integrals $\mathcal{A}_{313}^{(0,0)}$ and $\mathcal{A}_{313}^{(0,1)}$. However, the coefficients depend on ϵ. Then $\epsilon\Delta\mathcal{A}_{313}^{(0,0)}$ and $\epsilon\Delta\mathcal{A}_{313}^{(0,1)}$ could contribute to finite terms, such as $\ln(-\chi)$, under the unitarity cut.

- In fact, $\ln\left(\frac{-2\chi}{\gamma_i-1}\right)$ is the result given by unitarity cut channel K_{12} of one-loop massless box (K_1, K_2, K_3, K_4) up to zero-order of $(\gamma_i - 1)$. It may indicate some connection with one-loop box diagram.

Finally for $b=3$ we found

$$\mathcal{D}^{(0,3)}_{313} = \chi^2\ s^2\mathcal{D}^{(0,1)}_{313} + \Big(\frac{5}{2}\chi^2+\chi-\frac{1}{4}\Big)\mathcal{D}^{(0,0)}_{202} - s\Big(\frac{3}{2}\chi^2+\frac{1}{2}\chi-\frac{1}{4}\Big)\Big(\mathcal{D}^{(0,0)}_{212}+\mathcal{D}^{(0,0)}_{302}\Big)$$
$$-3\chi^2\ln(-\chi)\,. \tag{6.20}$$

It is obvious that $\mathcal{D}^{(0,3)}_{313}$ can be written as linear combination of $\mathcal{D}^{(0,i)}_{313}$, $i=0,1,2$ (as well as lower topologies) with rational functions of χ, s.

7 Conclusion

In this paper we applied the unitarity method to two-loop diagrams to determine their master integrals. Two propagators for each loop are cut while mixed propagators are untouched. Integrations for the reduced phase space have been done in the spinor form analytically. Based on these results, analytical structures have been identified and master integrals have been determined.

To demonstrate, we applied our method to investigate the double-box topology and its daughters, with appropriate choice of cut momenta and kinematic region. For the $\mathcal{A}_{212}$ topology with K_{L_1} massive, we found that there is only one scalar master integral for the pure 4D case, while for general $(4-2\epsilon)$-dimension, if we use the dimensional shifted bases, there are 20 scalar master integrals under good renormalizability conditions. For the $\mathcal{A}_{213}$ topology with K_{L_1} massive (K_3, K_4 can be massive or massless), there is also only one scalar master integral for the pure 4D case, but for the $(4-2\epsilon)$-dimension, scalar master integrals are not enough even considering the dimensional shifted bases. We found that there are 48 dimensional-shifted master integrals for renoramalizable theories. For the $\mathcal{A}_{313}$ topology, it is difficult to get an exact expression for general $(4-2\epsilon)$-dimension case. Thus we only considered a specific case $\mathcal{A}^{(0,b)}_{313}$ with $T_2=K_1$ and $K_i^2=0$, $i=1,2,3,4$. We presented results to the zeroth-order and found three master integrals for general $(4-2\epsilon)$-dimension if we do not allow coefficients depending on ϵ.

Based on the method demonstrated in this paper, several possible directions can be done in the future. Firstly, for the $\mathcal{A}_{313}$ topology, the exact result for the specific case $a=0$ is still missing. The general value of a should also be considered. Secondly, topologies discussed in this paper are not the most general cases. The most general configurations are those that each vertex has external momenta attached as well as massive propagators. Results of these more general cases are necessary. Thirdly, to obtain a complete set of master integrals, we need to investigate other topologies classified in [98]. Finally, besides determining master integrals, the unitarity method is also powerful for finding rational coefficients of bases in the reduction. We expect that, after the complete set of master integrals being obtained, such method can be useful for practical two-loop calculations.[16]

A Some useful formulae

In this section, we present some useful formulae appearing in various calculations in the paper.

[16] See also a very interesting new method [103].

Total derivative. For the holomorphic anomaly method, it is important to write an expression into the total derivative form. Here we list results for two typical inputs:

$$\frac{[\ell|d\ell]\,[\eta|\ell]^n}{\langle\ell|P|\ell]^{n+2}} = [d\ell|\partial\ell]\left(\frac{1}{(n+1)\,\langle\ell|P|\eta]}\frac{[\eta|\ell]^{n+1}}{\langle\ell|P|\ell]^{n+1}}\right), \tag{A.1}$$

and

$$\frac{[\ell|d\ell]}{\langle\ell|P|\ell]\,\langle\ell|Q|\ell]} = \frac{1}{\langle\ell|PQ|\ell\rangle}\,[d\ell|\partial\ell]\ln\left(\frac{\langle\ell|P|\ell]}{\langle\ell|Q|\ell]}\right). \tag{A.2}$$

Pole of $\langle\ell|QK|\ell\rangle$. In the calculation, we will meet pole of the form $\langle\ell|QK|\ell\rangle$ frequently. It contains two poles and we need to separate them. If both Q, K are massless we can write it as $\langle\ell|Q\rangle\,[Q|K]\,\langle K|\ell\rangle$. If at least one of them is massive, for example K, we can construct two massless momenta as $P_i = Q + x_i K$, $i = 1, 2$, where

$$x_{1,2} = \frac{-2Q\cdot K \pm \sqrt{\Delta}}{2K^2}, \quad \Delta = (2Q\cdot K)^2 - 4Q^2K^2\,. \tag{A.3}$$

Using this we have

$$\langle\ell|QK|\ell\rangle = \frac{\langle\ell|P_1\rangle\,[P_1|P_2]\,\langle\ell|P_2\rangle}{(x_1 - x_2)}, \tag{A.4}$$

and

$$\begin{aligned} Q &= \frac{x_1P_2 - x_2P_1}{x_1 - x_2}, & K &= \frac{P_1 - P_2}{x_1 - x_2}, & 2P_1\cdot P_2 &= \frac{-\Delta}{K^2}, \\ x_1x_2 &= \frac{Q^2}{K^2}, & x_1 + x_2 &= \frac{-2Q\cdot K}{K^2}, & x_1 - x_2 &= \frac{\sqrt{\Delta}}{K^2}\,. \end{aligned} \tag{A.5}$$

Residue of high order pole. Poles we met are often not single poles. To read out residues of poles with high order we can do as follows. Using the expression

$$\frac{1}{\langle\ell\;(\eta-\tau s)\rangle^n} = \frac{d^{n-1}}{d\tau^{n-1}}\left(\frac{1}{(n-1)!\,\langle\ell\;s\rangle^{n-1}}\frac{1}{\langle\ell\;(\eta-\tau s)\rangle}\right)\Big|_{\tau\to 0} \tag{A.6}$$

with arbitrary auxiliary spinor $|s\rangle$, the residue of function $\frac{1}{\langle\ell\;\eta\rangle^n}\frac{N(|\ell\rangle,|\ell])}{D(|\ell\rangle,|\ell])}$ is then given by

$$\frac{d^{n-1}}{d\tau^{n-1}}\left(\frac{1}{(n-1)!\,\langle\eta\;s\rangle^{n-1}}\frac{N(|\eta-\tau s\rangle\,,|\eta])}{D(|\eta-\tau s\rangle\,,|\eta])}\right)\Big|_{\tau\to 0}. \tag{A.7}$$

It is very important to emphasize that the $|\ell]$ part has been set to $|\eta]$, while the $|\ell\rangle$ is replaced by $(|\eta\rangle - \tau\,|s\rangle)$ before taking the derivative.

Evaluation of $\langle P_1|R|P_2]\langle P_2|S|P_1]$. We often encounter expression $\langle P_1|R|P_2]\langle P_2|S|P_1]$, which can be evaluated as

$$\begin{aligned} \langle P_1|R|P_2]\,\langle P_2|S|P_1] &= \mathrm{tr}\left(\frac{1-\gamma_5}{2}\not{P}_1\not{R}\not{P}_2\not{S}\right) \\ &= 2(P_1\cdot R)(P_2\cdot S) + 2(P_1\cdot S)(P_2\cdot R) - 2(P_1\cdot P_2)(R\cdot S) \\ &\quad -2i\epsilon(P_1RP_2S)\,, \end{aligned} \tag{A.8}$$

where $\epsilon(P_1RP_2S)$ denotes $\epsilon_{\mu\nu\rho\sigma}P_1^\mu R^\nu P_2^\rho S^\sigma$. To evaluate $\epsilon(P_1RP_2S)^2$, a simple way is to consider

$$\langle P_1|R|P_2]\,\langle P_2|S|P_1]\,\langle P_1|S|P_2]\,\langle P_2|R|P_1]\;.$$

B Standard one-loop integrations

In this section we list some standard one-loop results. We focus on the following standard integral [65–70]

$$\mathcal{R}_n^{(a)} \equiv \int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] \frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^a}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{a+4-n} \prod_{i=1}^{n-2} \left\langle\lambda|Q_i|\widetilde{\lambda}\right]}, \tag{B.1}$$

which is the integration in (2.7). In our application, we only need cases $n = 2, 3, 4$.

B.1 The bubble integration

When $n = 2$, we have $\Delta\mathcal{A}_2^{(a)} = \int d^{-2\epsilon}\mu[(1-2z)K^2]^{a+1}\mathcal{R}_2^{(a)}$ with

$$\begin{aligned}\mathcal{R}_2^{(a)} &= \int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] \frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^a}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{a+2}} \\ &= \int \langle\lambda|d\lambda\rangle \left[d\widetilde{\lambda}|\frac{\partial}{\partial\widetilde{\lambda}}\right] \frac{1}{(a+1)\langle\lambda|RK|\lambda\rangle} \frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^{a+1}}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{a+1}},\end{aligned} \tag{B.2}$$

where (A.1) has been used. For the pole $\langle\lambda|RK|\lambda\rangle$, we use the construction given in appendix A to read out two poles $\langle\lambda|P_1\rangle$ and $\langle\lambda|P_2\rangle$ with $P_i = R + x_iK$ (see (A.3)). For the first pole $|\lambda\rangle = |P_1\rangle$, the residue is

$$\frac{(x_1 - x_2)}{(a+1)(-2P_1 \cdot P_2)}(-x_1)^{a+1},$$

while for the second pole $|\lambda\rangle = |P_2\rangle$, the residue is

$$\frac{(x_1 - x_2)}{(a+1)(2P_1 \cdot P_2)}(-x_2)^{a+1}.$$

Putting them together we obtain

$$\begin{aligned}\mathcal{R}_2^{(a)} &= \frac{1}{(a+1)\sqrt{\Delta_{R,K}}}((-x_1)^{a+1} - (-x_2)^{a+1}), \\ \Delta\mathcal{A}_2^{(a)} &= \int d^{-2\epsilon}\mu[(1-2z)K^2]^{a+1}\mathcal{R}_2^{(a)},\end{aligned} \tag{B.3}$$

where

$$\Delta_{R,K} = (2R\cdot K)^2 - 4R^2K^2, \quad x_1 = \frac{-2R\cdot K + \sqrt{\Delta_{R,K}}}{2K^2}, \quad x_2 = \frac{-2R\cdot K - \sqrt{\Delta_{R,K}}}{2K^2}.$$

Let us give a few examples:

$$\begin{aligned}\Delta\mathcal{A}_2^{(a=0)} &= \int d^{-2\epsilon}\mu(-\sqrt{1-u}), \\ \Delta\mathcal{A}_2^{(a=1)} &= \int d^{-2\epsilon}\mu(-\sqrt{1-u})\{K\cdot T\}, \\ \Delta\mathcal{A}_2^{(a=2)} &= \int d^{-2\epsilon}\mu(-\sqrt{1-u})\{\frac{(4-u)(K\cdot T)^2 + (-1+u)K^2T^2}{3}\}.\end{aligned} \tag{B.4}$$

The case $a = 0$ gives the analytic signature $\mathcal{S}_{\rm bub} = (-\sqrt{1-u})$ for the one-loop scalar bubble basis. For cases $a = 1, 2$, the part inside the curly bracket is indeed polynomial of u.

B.2 The triangle integration

For the case $n = 3$, we can split the integrand as follows

$$
\begin{aligned}
\mathcal{R}_3^{(a)} &= \int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] \frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^a}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{a+1}\left\langle\lambda|Q|\widetilde{\lambda}\right]} \\
&= \int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] \left\{ \left(\frac{\langle\lambda|RQ|\lambda\rangle}{\langle\lambda|KQ|\lambda\rangle}\right)^a \frac{1}{\left\langle\lambda|K|\widetilde{\lambda}\right]\left\langle\lambda|Q|\widetilde{\lambda}\right]} \right. \\
&\qquad \left. + \sum_{i=0}^{a-1} \frac{\langle\lambda|RK|\lambda\rangle}{\langle\lambda|QK|\lambda\rangle} \left(\frac{\langle\lambda|RQ|\lambda\rangle}{\langle\lambda|KQ|\lambda\rangle}\right)^i \frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^{a-1-i}}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{a+1-i}} \right\}. \qquad \text{(B.5)}
\end{aligned}
$$

After the splitting, the first term inside the big bracket produces the signature of triangle, while the second term produces the signature of bubble. Thus we have the following two standard integrations.

B.2.1 Triangle-to-triangle part

For the first term, writing into total derivative we have

$$
\mathcal{R}_{3\to 3}^{(a)} = \int \langle\lambda|d\lambda\rangle \left[d\widetilde{\lambda}|\frac{\partial}{\partial\widetilde{\lambda}}\right] \frac{(-)^a\langle\lambda|RQ|\lambda\rangle^a}{\langle\lambda|QK|\lambda\rangle^{a+1}} \ln\left(\frac{\left\langle\lambda|Q|\widetilde{\lambda}\right]}{\left\langle\lambda|K|\widetilde{\lambda}\right]}\right). \qquad \text{(B.6)}
$$

The pole is given by factor $\langle\lambda|QK|\lambda\rangle^{a+1}$. Using results in appendix A, for the pole $\eta = P_1$ with auxiliary spinor $s = P_2$ the residue is

$$
\frac{(-)^a(x_1-x_2)^{a+1}}{[P_1|P_2]^{a+1}} \ln(-x_1) \frac{d^a}{d\tau^a}\left(\frac{\langle P_1-\tau P_2|RQ|P_1-\tau P_2\rangle^a}{a!\,\langle P_1|P_2\rangle^{2a+1}}\right)\Big|_{\tau\to 0}.
$$

For the pole $\eta = P_2$ with auxiliary spinor $s = P_1$ the residue is

$$
\frac{(-)^a(x_1-x_2)^{a+1}}{[P_1|P_2]^{a+1}} \ln(-x_2) \frac{d^a}{d\tau^a}\left(\frac{\langle P_2-\tau P_1|RQ|P_2-\tau P_1\rangle^a}{a!\,\langle P_2|P_1\rangle^{2a+1}}\right)\Big|_{\tau\to 0}.
$$

One can observe that the derivative part is in fact the same for both contributions after taking the limit $\tau \to 0$. Thus the sum of two contributions is

$$
\frac{(-)^a(x_1-x_2)^{a+1}\ln\frac{x_1}{x_2}}{[P_1|P_2]^{a+1}\,a!\,\langle P_1|P_2\rangle^{2a+1}} \frac{d^a}{d\tau^a}\langle P_1-\tau P_2|RQ|P_1-\tau P_2\rangle^a\Big|_{\tau\to 0}.
$$

After some manipulation, we finally have

$$
\mathcal{R}_{3\to 3}^{(a)} = \mathcal{C}_{3\to 3}^{(a)}\,\mathcal{S}_{\rm tri}\,, \qquad \text{(B.7)}
$$

where $\mathcal{S}_{\text{tri}}$ is the signature of triangle and $\mathcal{C}^{(a)}_{3\to 3}$ is the corresponding coefficient:

$$\begin{aligned}\mathcal{S}_{\text{tri}} &\equiv \frac{1}{\sqrt{\Delta_{Q,K}}}\ln\left(\frac{Q\cdot K-\sqrt{(Q\cdot K)^2-Q^2K^2}}{Q\cdot K+\sqrt{(Q\cdot K)^2-Q^2K^2}}\right),\\ \mathcal{C}^{(a)}_{3\to 3} &= \frac{(-)^a}{a!\Delta^a_{Q,K}}\frac{d^a}{d\tau^a}\left(+\tau(4Q^2(R\cdot K)-4(R\cdot Q)(Q\cdot K))+\tau^2(Q^2)\right. \qquad \text{(B.8)}\\ &\quad \left.+(R^2\Delta_{Q,K}+(2R\cdot Q)^2K^2+(2R\cdot K)^2Q^2-(2R\cdot Q)(2R\cdot K)(2Q\cdot K))\right)^a|_{\tau\to 0}\,.\end{aligned}$$

The $a=0$ case gives the result for standard scalar triangle and other a's, give the corresponding coefficients under the reduction. One can verify that the coefficients are indeed rational functions.

B.2.2 Triangle-to-bubble part

The typical term in (B.5) for triangle-to-bubble part is

$$\begin{aligned}\mathcal{R}_{3\to 2}[i,n] &\equiv \int \langle\lambda|d\lambda\rangle\left[\widetilde{\lambda}|d\widetilde{\lambda}\right]\frac{\langle\lambda|RK|\lambda\rangle}{\langle\lambda|QK|\lambda\rangle}\left(\frac{\langle\lambda|RQ|\lambda\rangle}{\langle\lambda|KQ|\lambda\rangle}\right)^i\frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^n}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{n+2}}\\ &= \int\langle\lambda|d\lambda\rangle\left[d\widetilde{\lambda}|\frac{\partial}{\partial\widetilde{\lambda}}\right]\frac{(-)^i\langle\lambda|RQ|\lambda\rangle^i}{(n+1)\langle\lambda|QK|\lambda\rangle^{i+1}}\frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^{n+1}}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{n+1}}\,. \qquad \text{(B.9)}\end{aligned}$$

The residue of pole $\langle\lambda|QK|\lambda\rangle^{i+1}$ can be read out as in previous subsubsection and we get

$$\begin{aligned}\mathcal{R}_{3\to 2}[i,n] = \frac{(-)^{n+i}(K^2)^i}{i!(n+1)\sqrt{\Delta}^{n+2i+2}}\frac{d^i}{d\tau^i}\Big\{&\left((2R\cdot P_2-\tau\langle P_1|R|P_2])^{n+1}\right. \qquad \text{(B.10)}\\ &\left.(-x_2\langle P_2|R|P_1]-x_1\tau^2\langle P_1|R|P_2]+\tau(x_2(2R\cdot P_1)+x_1(2R\cdot P_2)))^i\right)\\ &+(-)^n\left((2R\cdot P_1-\tau\langle P_2|R|P_1])^{n+1}\right.\\ &\left.(-x_2\tau^2\langle P_2|R|P_1]-x_1\langle P_1|R|P_2]+\tau(x_2(2R\cdot P_1)+x_1(2R\cdot P_2)))^i\right)\Big\}\Big|_{\tau\to 0}\,.\end{aligned}$$

To get a Lorentz contracted form, we need to use the following key fact: *to have non-zero contribution, factors $\langle P_1|R|P_2]$ and $\langle P_2|R|P_1]$ should always appear in pair.* Thus we can transfer (B.10) to

$$\begin{aligned}\mathcal{R}_{3\to 2}[i,n] = \frac{(-)^{n+i}(K^2)^i}{i!(n+1)\sqrt{\Delta}^{n+2i+2}}\frac{d^i}{d\tau^i}\Big\{&\left((2R\cdot P_2-\tau)^{n+1}\right. \qquad \text{(B.11)}\\ &\left.(-x_2\langle P_2|R|P_1]\langle P_1|R|P_2]-x_1\tau^2+\tau(x_2(2R\cdot P_1)+x_1(2R\cdot P_2)))^i\right)\\ &+(-)^n\left((2R\cdot P_1-\tau)^{n+1}\right.\\ &\left.(-x_2\tau^2-x_1\langle P_1|R|P_2]\langle P_2|R|P_1]+\tau(x_2(2R\cdot P_1)+x_1(2R\cdot P_2)))^i\right)\Big\}\Big|_{\tau\to 0}\,,\end{aligned}$$

where

$$\langle P_2|R|P_1]\langle P_1|R|P_2] = \frac{R^2\Delta}{K^2}+(2R\cdot Q)^2+(2R\cdot K)^2\frac{Q^2}{K^2}-\frac{(2R\cdot Q)(2R\cdot K)(2Q\cdot K)}{K^2}\,. \qquad \text{(B.12)}$$

JHEP06(2014)166

Thus the contribution for the triangle-to-bubble part is given by

$$\mathcal{R}^{(a)}_{3\to 2} = \sum_{i=0}^{a-1} \mathcal{R}_{3\to 2}[i, a-1-i] \,. \tag{B.13}$$

Putting two parts together, we get

$$\mathcal{R}^{(a)}_{3} = \mathcal{R}^{(a)}_{3\to 3} + \mathcal{R}^{(a)}_{3\to 2} \,. \tag{B.14}$$

B.3 The box integration

The box integration is given by

$$\mathcal{R}^{(a)}_{4} = \int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] \frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^a}{\left\langle\lambda|K|\widetilde{\lambda}\right]^a \left\langle\lambda|Q_1|\widetilde{\lambda}\right] \left\langle\lambda|Q_2|\widetilde{\lambda}\right]} \,. \tag{B.15}$$

After splitting, we have the part producing signatures of box and triangle

$$\begin{aligned} \int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] \Bigg\{ & \frac{-\langle\lambda|RQ_1|\lambda\rangle}{\langle\lambda|Q_1Q_2|\lambda\rangle} \left(\frac{\langle\lambda|RQ_1|\lambda\rangle}{\langle\lambda|KQ_1|\lambda\rangle}\right)^{a-1} \frac{1}{\left\langle\lambda|K|\widetilde{\lambda}\right] \left\langle\lambda|Q_1|\widetilde{\lambda}\right]} \\ & + \frac{\langle\lambda|RQ_2|\lambda\rangle}{\langle\lambda|Q_1Q_2|\lambda\rangle} \left(\frac{\langle\lambda|RQ_2|\lambda\rangle}{\langle\lambda|KQ_2|\lambda\rangle}\right)^{a-1} \frac{1}{\left\langle\lambda|K|\widetilde{\lambda}\right] \left\langle\lambda|Q_2|\widetilde{\lambda}\right]} \Bigg\} \,, \end{aligned} \tag{B.16}$$

and the part producing the signature of bubble

$$\begin{aligned} \int \langle\lambda|d\lambda\rangle \left[\widetilde{\lambda}|d\widetilde{\lambda}\right] \Bigg\{ & \frac{\langle\lambda|RQ_2|\lambda\rangle}{\langle\lambda|Q_1Q_2|\lambda\rangle} \sum_{i=0}^{a-2} \frac{\langle\lambda|RK|\lambda\rangle}{\langle\lambda|Q_2K|\lambda\rangle} \left(\frac{\langle\lambda|RQ_2|\lambda\rangle}{\langle\lambda|KQ_2|\lambda\rangle}\right)^{i} \frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^{a-2-i}}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{a-i}} \\ & + \frac{-\langle\lambda|RQ_1|\lambda\rangle}{\langle\lambda|Q_1Q_2|\lambda\rangle} \sum_{i=0}^{a-2} \frac{\langle\lambda|RK|\lambda\rangle}{\langle\lambda|Q_1K|\lambda\rangle} \left(\frac{\langle\lambda|RQ_1|\lambda\rangle}{\langle\lambda|KQ_1|\lambda\rangle}\right)^{i} \frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^{a-2-i}}{\left\langle\lambda|K|\widetilde{\lambda}\right]^{a-i}} \Bigg\} \,. \end{aligned} \tag{B.17}$$

Now we can evaluate various parts one by one.

B.3.1 The box-to-box part

This part comes from pole $\langle\lambda|Q_1Q_2|\lambda\rangle$ in (B.16). Using $Q_1 + x_iQ_2$ to construct two null momenta P_i, we get the residue

$$\frac{(x_1 - x_2)}{[P_1|P_2]\,\langle P_1|P_2\rangle} \left(\frac{\langle P_1|R|P_2]}{\langle P_1|K|P_2]}\right)^a \ln(-x_1) + \frac{-(x_1 - x_2)}{[P_1|P_2]\,\langle P_1|P_2\rangle} \left(\frac{\langle P_2|R|P_1]}{\langle P_2|K|P_1]}\right)^a \ln(-x_2) \,,$$

which can be written as

$$\begin{aligned} & \frac{1}{2} \frac{(x_1 - x_2)}{[P_1|P_2]\,\langle P_1|P_2\rangle} \ln\left(\frac{x_1}{x_2}\right) \left[\left(\frac{\langle P_1|R|P_2]}{\langle P_1|K|P_2]}\right)^a + \left(\frac{\langle P_2|R|P_1]}{\langle P_2|K|P_1]}\right)^a\right] \\ & + \frac{1}{2} \frac{(x_1 - x_2)}{[P_1|P_2]\,\langle P_1|P_2\rangle} \ln(x_1x_2) \left[\left(\frac{\langle P_1|R|P_2]}{\langle P_1|K|P_2]}\right)^a - \left(\frac{\langle P_2|R|P_1]}{\langle P_2|K|P_1]}\right)^a\right] \,. \end{aligned} \tag{B.18}$$

JHEP06(2014)166

Box part. The first term in (B.18) produces the signature of box

$$\mathcal{S}_{\text{box}} = \frac{1}{\sqrt{(2Q_1\cdot Q_2)^2 - 4Q_1^2Q_2^2}} \ln \frac{Q_1\cdot Q_2 - \sqrt{(Q_1\cdot Q_2)^2 - Q_1^2Q_2^2}}{Q_1\cdot Q_2 + \sqrt{(Q_1\cdot Q_2)^2 - Q_1^2Q_2^2}}\,, \tag{B.19}$$

as well as the coefficient

$$\mathcal{C}^{(a)}_{4\to4} = \frac{1}{2}\left[\left(\frac{\langle P_1|R|P_2]}{\langle P_1|K|P_2]}\right)^a + \left(\frac{\langle P_2|R|P_1]}{\langle P_2|K|P_1]}\right)^a\right]. \tag{B.20}$$

Thus we have

$$\mathcal{R}^{(a)}_{4\to4} = \mathcal{C}^{(a)}_{4\to4}\,\mathcal{S}_{\text{box}}\,. \tag{B.21}$$

It can be shown that there is a recursion relation

$$\mathcal{R}^{(a+1)}_{4\to4} = \frac{T_2}{T_1}\mathcal{R}^{(a)}_{4\to4} - \frac{T_3}{T_1}\mathcal{R}^{(a-1)}_{4\to4}\,, \tag{B.22}$$

with

$$\mathcal{C}^{(0)}_{4\to4} = 1\,,\quad \mathcal{C}^{(1)}_{4\to4} = \frac{T_2}{2T_1}\,, \tag{B.23}$$

where

$$\begin{aligned}
T_1 &= 4\left[(Q_1\cdot K)^2 + \frac{Q_1^2}{Q_2^2}(Q_2\cdot K)^2 - \frac{2Q_1\cdot Q_2}{Q_2^2}(Q_1\cdot K)(Q_2\cdot K)\right] + K^2\frac{((2Q_1\cdot Q_2)^2 - 4Q_1^2Q_2^2)}{Q_2^2}\,,\\
T_2 &= \frac{8(R\cdot K)((Q_1\cdot Q_2)^2 - Q_1^2Q_2^2)}{Q_2^2} + 8(R\cdot Q_1)(K\cdot Q_1) + 8(R\cdot Q_2)(K\cdot Q_2)\frac{Q_1^2}{Q_2^2}\\
&\quad - 8\frac{(Q_1\cdot Q_2)}{Q_2^2}((R\cdot Q_1)(K\cdot Q_2) + (R\cdot Q_2)(K\cdot Q_1))\,,\\
T_3 &= 4[(Q_1\cdot R)^2 + \frac{Q_1^2}{Q_2^2}(Q_2\cdot R)^2 - \frac{2Q_1\cdot Q_2}{Q_2^2}(Q_1\cdot R)(Q_2\cdot R)] + R^2\frac{((2Q_1\cdot Q_2)^2 - 4Q_1^2Q_2^2)}{Q_2^2}\,.
\end{aligned}$$

Triangle part. The second term in (B.18) produces the signature of triangle. Using

$$\ln(x_1x_2) = \ln\frac{Q_1^2}{Q_2^2} = \ln\frac{Q_1^2}{K^2} - \ln\frac{Q_2^2}{K^2}\,,$$

the second term in (B.18) can be rewritten as

$$\begin{aligned}
&\frac{1}{2}\ln\frac{Q_1^2}{K^2}\left\{\frac{1}{\langle\lambda|Q_1Q_2|\lambda\rangle}\left(\frac{\langle\lambda|RQ_1|\lambda\rangle}{\langle\lambda|KQ_1|\lambda\rangle}\right)^a\right\}_{\text{Residue of }\langle\lambda|Q_1Q_2|\lambda\rangle}\\
&+\frac{1}{2}\ln\frac{Q_2^2}{K^2}\left\{\frac{-1}{\langle\lambda|Q_1Q_2|\lambda\rangle}\left(\frac{\langle\lambda|RQ_2|\lambda\rangle}{\langle\lambda|KQ_2|\lambda\rangle}\right)^a\right\}_{\text{Residue of }\langle\lambda|Q_1Q_2|\lambda\rangle}\,.
\end{aligned} \tag{B.24}$$

We will combine (B.24) with results in the next subsubsection to produce the complete triangle part.

B.3.2 The box-to-triangle part

Since Q_1 and Q_2 are symmetric, we will focus on the triangle constructed by K, Q_1. The contribution comes from the first term of (B.16). This term contains two kinds of poles: $\langle\lambda|Q_1Q_2|\lambda\rangle$ and $\langle\lambda|KQ|\lambda\rangle$. The contribution of pole $\langle\lambda|Q_1Q_2|\lambda\rangle$ has been evaluated in previous subsubsection. For the second pole, after writing it into total derivative, it is $\frac{1}{\langle\lambda|KQ|\lambda\rangle^a}$. Using Q_1, K to construct two null momenta P_1, P_2, the residue is given by two parts. The first part contains $\ln(x_1x_2)$ (which is nothing but $\ln\frac{Q_1^2}{K^2}$) and is given by

$$\frac{1}{2}\ln\frac{Q_1^2}{K^2}\left\{\frac{1}{\langle\lambda|Q_1Q_2|\lambda\rangle}\left(\frac{\langle\lambda|RQ_1|\lambda\rangle}{\langle\lambda|KQ_1|\lambda\rangle}\right)^a\right\}_{\text{Residue of }\langle\lambda|KQ_1|\lambda\rangle^a}. \tag{B.25}$$

It cancels the first term of (B.24), since the sum of all residues of a holomorphic function is zero.[17] The second part contains $\ln(x_1/x_2)$ which is the signature of triangle. The contribution can be written as

$$\mathcal{R}^{(a)}_{4\to3}(Q_1) = \mathcal{C}^{(a)}_{4\to3}(Q_1)\ \mathcal{S}_{\text{tri}}(Q_1,K)\,, \tag{B.26}$$

where

$$\mathcal{S}_{\text{tri}}(Q_1,K) = \frac{1}{\sqrt{(2Q_1\cdot K)^2-4Q_1^2K^2}}\ln\frac{Q_1\cdot K-\sqrt{(Q_1\cdot K)^2-Q_1^2K^2}}{Q_1\cdot K+\sqrt{(Q_1\cdot K)^2-Q_1^2K^2}}\,,$$

and

$$\begin{aligned}\mathcal{C}^{(a)}_{4\to3}(Q_1) =& \frac{(-)^{a-1}}{(a-1)!}\left(\frac{K^2}{4((Q_1\cdot K)^2-K^2Q_1^2)}\right)^{a-1} \\ &\left\{\frac{d^{a-1}}{d\tau^{a-1}}\left(\frac{(-\tau^2x_2\langle P_2|R|P_1]-x_1\langle P_1|R|P_2]+\tau(x_2\langle P_1|R|P_1]+x_1\langle P_2|R|P_2]))^a}{(-\tau^2x_2\langle P_2|Q_2|P_1]-x_1\langle P_1|Q_2|P_2]+\tau(x_2\langle P_1|Q_2|P_1]+x_1\langle P_2|Q_2|P_2]))}\right)\right. \\ &\left.+\frac{d^{a-1}}{d\tau^{a-1}}\left(\frac{(-x_2\langle P_2|R|P_1]-x_1\tau^2\langle P_1|R|P_2]+\tau(x_2\langle P_1|R|P_1]+x_1\langle P_2|R|P_2]))^a}{(-x_2\langle P_2|Q_2|P_1]-x_1\tau^2\langle P_1|Q_2|P_2]+\tau(x_2\langle P_1|Q_2|P_1]+x_1\langle P_2|Q_2|P_2]))}\right)\right\}\Big|_{\tau\to0}.\end{aligned} \tag{B.27}$$

To write the spinor form to the Lorentz contracted form, we can take similar manipulation as the one from (B.10) to (B.11). The result is

$$\begin{aligned}\mathcal{C}^{(a)}_{4\to3}(Q_1) =& \frac{(-)^{a-1}}{(a-1)!}\left(\frac{K^2}{4((Q_1\cdot K)^2-Q_1^2K^2)}\right)^{a-1} \\ &\frac{d^{a-1}}{d\tau^{a-1}}\frac{2K^2T_6(K^2T_7+K^2T_5T_6\tau+Q_1^2T_6T_7\tau^2)\left(1+\frac{Q_1^2}{K^2}T_6\tau^2+T_4\tau\right)^a}{(Q_1^2T_6T_7\tau^2+K^2(T_7+T_5T_6\tau))^2-T_8^2(K^2-Q_1^2T_6\tau^2)^2}\Big|_{\tau\to0}\,,\end{aligned} \tag{B.28}$$

where we have defined

$$\begin{aligned}T_4 &= \frac{4(R\cdot K)Q_1^2-4(R\cdot Q_1)(K\cdot Q_1)}{K^2}\,,\quad T_5=\frac{4(Q_2\cdot K)Q_1^2-4(Q_2\cdot Q_1)(K\cdot Q_1)}{K^2}\,,\\ T_6 &= \frac{R^2\Delta}{K^2}+4(R\cdot Q_1)^2+\frac{4Q_1^2(R\cdot K)^2}{K^2}-\frac{8(R\cdot Q_1)(R\cdot K)(Q_1\cdot K)}{K^2}\,,\\ T_7 &= 2(P_1\cdot Q_2)(P_2\cdot R)+2(P_1\cdot R)(Q_2\cdot P_2)-2(P_1\cdot P_2)(Q_2\cdot R)\,,\\ T_8 &= \frac{4i\epsilon(Q_1Q_2KR)\sqrt{(K\cdot Q_1)^2-K^2Q_1^2}}{K^2}\,.\end{aligned}$$

It is worth to mention that T_8 appears as T_8^2, thus the Levi-Civita symbol has been removed.

[17] It is worth to notice that by power counting, infinity does not contribute residue.

B.3.3 The box-to-bubble part

Having finished the computation of (B.16), we turn to the (B.17). The total result can be expressed as

$$\mathcal{R}^{(a)}_{4\to 2} = \sum_{i=0}^{a-2} \mathcal{R}_{4\to 2}(Q_1)[i, a-1-i] + \{Q_1 \leftrightarrow Q_2\} \,, \tag{B.29}$$

where the typical term is

$$\mathcal{R}_{4\to 2}(Q_1)[i, m] = \left\{ \frac{1}{\langle\lambda|Q_1Q_2|\lambda\rangle} \left(\frac{\langle\lambda|RQ_1|\lambda\rangle}{\langle\lambda|KQ_1|\lambda\rangle}\right)^{i+1} \frac{1}{m} \frac{\left\langle\lambda|R|\widetilde{\lambda}\right]^m}{\left\langle\lambda|K|\widetilde{\lambda}\right]^m} \right\}_{\text{residue}} . \tag{B.30}$$

There are three poles for this part: $\langle\lambda|Q_1Q_2|\lambda\rangle$, $\langle\lambda|Q_1K|\lambda\rangle$ and $\langle\lambda|Q_2K|\lambda\rangle$. The contribution from $\langle\lambda|Q_1Q_2|\lambda\rangle$ is zero when summing up the two lines in (B.17). For the remaining two poles, because of the symmetry $Q_1 \leftrightarrow Q_2$, we will focus on $\mathcal{R}_{4\to 2}(Q_1)[i, m]$ only.

We use $Q_1,\ K$ to construct two null momenta $P_1,\ P_2$ and get residue

$$\begin{aligned}
&\mathcal{R}_{4\to 2}(Q_1)[i, m] \\
&= \frac{(-)^i (K^2)^i}{i! m (\sqrt{\Delta})^{m+2i+1}} \frac{d^i}{d\tau^i} \left\{ \frac{(\tau T_4 - \tau^2 x_2 \langle P_2|R|P_1] - x_1 \langle P_1|R|P_2])^{i+1} (\langle P_1|R|P_1] - \tau \langle P_2|R|P_1])^m}{\tau T_5 - \tau^2 x_2 \langle P_2|Q_2|P_1] - x_1 \langle P_1|Q_2|P_2]} \right. \\
&\quad \left. + (-)^{m+1} \frac{(\tau T_4 - x_2 \langle P_2|R|P_1] - \tau^2 x_1 \langle P_1|R|P_2])^{i+1} (\langle P_2|R|P_2] - \tau \langle P_1|R|P_2])^m}{\tau T_5 - x_2 \langle P_2|Q_2|P_1] - \tau^2 x_1 \langle P_1|Q_2|P_2]} \right\} \Bigg|_{\tau\to 0} .
\end{aligned} \tag{B.31}$$

We can rewrite the expression to the following Lorentz contracted form

$$\begin{aligned}
\mathcal{R}_{4\to 2}(Q_1)[i, a] = &\frac{(-)^i (K^2)^i}{i! a (\sqrt{\Delta})^{a+2i+1}} \frac{d^i}{d\tau^i} \left\{ \frac{(\tau T_4 + \tau^2 \frac{Q_1^2}{K^2} T_6 + 1)^{i+1} (2Q_1\cdot R + 2x_1 K\cdot R + \tau x_1 T_6)^a}{\tau T_5 + \tau^2 \frac{Q_1^2}{K^2}(T_7 - T_8) + \frac{T_7+T_8}{T_6}} \right. \\
&\left. + (-)^{a+1} \frac{(\tau T_4 + \tau^2 \frac{Q_1^2}{K^2} T_6 + 1)^{i+1} (2Q_1\cdot R + 2x_2 K\cdot R + \tau x_2 T_6)^a}{\tau T_5 + \tau^2 \frac{Q_1^2}{K^2}(T_7 + T_8) + \frac{T_7-T_8}{T_6}} \right\} .
\end{aligned} \tag{B.32}$$

One can verify that T_8 will appear as T_8^2 after summing $\mathcal{R}_{4\to 2}(Q_1)[i, a]$ and $\mathcal{R}_{4\to 2}(Q_2)[i, a]$, thus the Levi-Civita symbol does not appear in the final result.

C The integration for topology $\mathcal{A}_{313}$

It is hard to get the explicit result for (6.15). In this appendix we develop a method to find approximate expressions. Technically the case $b = 0$ is the most complicated one, while the $b \geq 1$ cases can be reduced to the case $b = 0$ plus some simple integration. Before working out the integration case by case, we give two explicit integrations

$$\int_{-1}^{+1} \frac{du}{\sqrt{(u + m\gamma_2)^2 + (1 - m^2)(\gamma_2^2 - 1)}} = \ln\left(\frac{\gamma_2 + 1}{\gamma_2 - 1}\right) , \tag{C.1}$$

and

$$\int_{-1}^{+1} \frac{du}{\sqrt{(u+\alpha)^2-\beta^2}} \ln \frac{(u+\alpha)+\sqrt{(u+\alpha)^2-\beta^2}}{(u+\alpha)-\sqrt{(u+\alpha)^2-\beta^2}} = \ln\Big(\frac{\gamma+1}{\gamma-1}\Big) \ln\Big(\frac{\gamma_1+1}{\gamma_1-1}\Big), \qquad (C.2)$$

where we have used the conditions $-1 < m < 1$ and $\gamma_i \geq 1$. These two results are useful for our further discussion.

C.1 Pure 4D solution in the case $b = 0$

From (6.15) we get that $\mathcal{D}_{313}^{(0,0)}$ is

$$-\frac{\gamma_1\gamma_2}{s^3}\int_{-1}^{+1} du \frac{1}{\sqrt{(u+m\gamma_2)^2+\xi^2}} \frac{1}{\sqrt{(u+\alpha)^2-\beta^2}} \ln \frac{(u+\alpha)+\sqrt{(u+\alpha)^2-\beta^2}}{(u+\alpha)-\sqrt{(u+\alpha)^2-\beta^2}}, \quad (C.3)$$

where $\xi^2 = (1-m^2)(\gamma_2^2-1)$ is positive. In the pure 4D, $\xi \to 0$, so we need to study the limit behavior at $\xi \to 0$. If we expand

$$f(u) \equiv \frac{1}{\sqrt{(u+\alpha)^2-\beta^2}} \ln \frac{(u+\alpha)+\sqrt{(u+\alpha)^2-\beta^2}}{(u+\alpha)-\sqrt{(u+\alpha)^2-\beta^2}} = \sum_{n=0}^{+\infty} f_n u^n \qquad (C.4)$$

in the region $[-1,+1]$, where $f(u)$ is positive and convergent uniformly, we will have (ignoring the factor $(-\gamma_1\gamma_2/s^3)$)

$$\mathcal{D}_{313}^{(0,0)} = \sum_{n=0}^{+\infty} f_n \sum_{k=0}^{n} C_n^k (-m\gamma_2)^{n-k} \int_{m\gamma_2-1}^{m\gamma_2+1} du \frac{u^k}{\sqrt{u^2+\xi^2}} \qquad (C.5)$$

after shifting of u. Now we introduce a series of functions defined as

$$H_n(a,b) = \int_0^b dx\, \frac{x^n}{\sqrt{x^2+a^2}}, \quad n \geq 0, \qquad (C.6)$$

with integer n. It is easy to figure out the answers

$$H_{n=2m} = \frac{(-)^m a^n}{2^n} C_n^m \ln\Big(\frac{\sqrt{a^2+b^2}+b}{a}\Big) + \frac{a^n}{2^n}\sum_{k=0}^{m-1} \frac{(-)^k C_n^k}{n-2k}\left[\Big(\frac{\sqrt{a^2+b^2}+b}{a}\Big)^{n-2k} - \Big(\frac{\sqrt{a^2+b}-b}{a}\Big)^{n-2k}\right],$$

$$H_{n=2m+1} = \frac{a^n}{2\sqrt{\pi}}\Gamma\Big(-\frac{n}{2}\Big)\Gamma\Big(\frac{n+1}{2}\Big) + \frac{a^n}{2^n}\sum_{k=0}^{m} \frac{(-)^k C_n^k}{n-2k}\left[\Big(\frac{\sqrt{a^2+b^2}+b}{a}\Big)^{n-2k} + \Big(\frac{\sqrt{a^2+b^2}-b}{a}\Big)^{n-2k}\right].$$

For the limit $a \to 0$, it is easy to see that only in the case $n = 0$ it is divergent and we have

$$\lim_{a\to 0} H_0(a,b) = \ln\Big(\frac{\sqrt{a^2+b^2}+b}{a}\Big)\Big|_{a\to 0}, \quad \lim_{a\to 0} H_n(a,b) = \frac{b^n}{n}, \quad n \geq 1. \qquad (C.7)$$

Using this observation the expression (C.5) can be separated into the divergent part and the finite part. The divergent part is

$$\sum_{n=0}^{+\infty} f_n (-m\gamma_2)^n \Big(H_0(\xi, m\gamma_2+1) - H_0(\xi, m\gamma_2-1)\Big) = f(-m\gamma_2) \ln\Big(\frac{\gamma_2+1}{\gamma_2-1}\Big) \qquad (C.8)$$

by using the conditions $\gamma_2 > 1$ and $-1 < m < 1$, where function f is defined in (C.4). Under the 4D limit, *the divergent term* is

$$\mathcal{D}^{(0,0)}_{313}|_{\rm div} = \frac{1}{s^3}\,\frac{1}{2\chi}\ln\left(\frac{\gamma_2+1}{\gamma_2-1}\right)\left[2\ln(-\chi)+\ln\left(\frac{\gamma+1}{\gamma-1}\right)+\ln\left(\frac{\gamma_1+1}{\gamma_1-1}\right)\right], \tag{C.9}$$

where we have recovered the missing factor. If we consider $\mathcal{D}^{(0,0)}_{313}$ as a series of γ_2, this is just the first (divergent) term. The finite part of the expression (C.5) is

$$\sum_{n=0}^{+\infty} f_n \sum_{k=1}^{n} C_n^k(-m\gamma_2)^{n-k}\Big(H_k(\xi,m\gamma_2+1)-H_k(\xi,m\gamma_2-1)\Big)\,. \tag{C.10}$$

Under the pure 4D limit, using (C.7) it becomes

$$\sum_{n=0}^{+\infty} f_n \sum_{k=1}^{n} \frac{C_n^k}{k}(-m\gamma_2)^{n-k}\Big((m\gamma_2+1)^k+(m\gamma_2-1)^k\Big)\,. \tag{C.11}$$

We can use parameterizing method to sum up above awesome form. If we define

$$G(x)\equiv\sum_{k=1}^{n}\frac{C_n^k}{k}(-m\gamma_2)^{n-k}\Big((m\gamma_2+x)^k+(m\gamma_2-x)^k\Big)\,, \tag{C.12}$$

then $G(x)$ satisfies the differential equation

$$\frac{\partial G}{\partial x}=g(x)-g(-x)\,,\quad g(x)=\frac{x^n-\rho^n}{x-\rho}\,,\quad \rho\equiv -m\gamma_2\,. \tag{C.13}$$

Obviously,

$$G(1)-G(0)=\int_0^{+1}dx\ g(x)+\int_0^{-1}dx\ g(x)\,, \tag{C.14}$$

where $G(1)$ is the result we want to find. To compute $G(0)$, we define new function

$$\widetilde{G}(0,x)=2\sum_{k=1}^{n}\frac{C_n^k}{k}(-m\gamma_2)^{n-k}(m\gamma_2\ x)^k\,,\quad \widetilde{G}(0,1)=G(0),\ \widetilde{G}(0,0)=0\,. \tag{C.15}$$

Using the same method, we can find the differential equation for $\widetilde{G}(0,x)$. After some variable replacement we get

$$G(0)=\widetilde{G}(0,1)=2\int_\rho^0 g(x)\ dx\,. \tag{C.16}$$

Combining (C.14) with (C.16) and exchanging the integration and the summation

$$\sum_{n=0}^{+\infty} f_n\int dx\ \frac{x^n-\rho^n}{x-\rho}=\int dx\ \frac{f(x)-f(\rho)}{x-\rho}\,, \tag{C.17}$$

finally *the finite term* can be written as

$$\mathcal{D}_{313}^{(0,0)}|_{\text{finite}} = \frac{-\gamma_1\gamma_2}{s^3}\Big(\int_\rho^{+1} + \int_\rho^{-1}\Big)dx\ \frac{f(x)-f(\rho)}{x-\rho}\,. \tag{C.18}$$

Now we focus on the indefinite integration. If we change the integration variable as

$$\cosh y \equiv \frac{\alpha + x}{\beta}\,, \quad \cosh y_\pm \equiv \frac{\alpha \pm 1}{\beta}\,, \quad \cosh y_0 \equiv \frac{\alpha - m\gamma_2}{\beta}\,, \tag{C.19}$$

then

$$\int dx\ \frac{f(x)}{x-\rho} = \frac{2}{\beta}\int dy\ \frac{y}{\cosh y - \cosh y_0}\,. \tag{C.20}$$

After integration by parts it becomes

$$\frac{2}{\beta\sinh y_0}\left[-\frac{y^2}{2} + y\ln\frac{1-e^{-(y_0-y)}}{1-e^{-(y_0+y)}} + \mathsf{Li}_2\Big(e^{-(y_0-y)}\Big) + \mathsf{Li}_2\Big(e^{-(y_0+y)}\Big)\right], \tag{C.21}$$

in which $\mathsf{Li}_s(z)$ is the polylogarithm. Combining with the other part, the whole indefinite integral of (C.18) can be written as

$$\begin{aligned} F(y) \equiv \frac{2}{\beta\sinh y_0}\bigg[&\mathsf{Li}_2\Big(e^{-(y_0-y)}\Big) + \mathsf{Li}_2\Big(e^{-(y_0+y)}\Big) - \frac{y^2}{2} \\ &- (y_0-y)\ln\Big(1-e^{-(y_0-y)}\Big) - (y_0+y)\ln\Big(1-e^{-(y_0+y)}\Big)\bigg]. \end{aligned} \tag{C.22}$$

Thus $\mathcal{D}_{313}^{(0,0)}|_{\text{finite}}$ is given by $F(y_+) + F(y_-) - 2F(y_0)$ up to an overall factor. Above calculations are done for pure 4D limit of γ_2. After taking the pure 4D limit of γ_1, γ we finally reach

$$\begin{aligned} \mathcal{D}_{313}^{(0,0)} = \frac{1}{s^3\ 2\chi}\bigg[&\ln\Big(\frac{-2\chi}{\gamma-1}\Big)\ln\Big(\frac{-2\chi}{\gamma_1-1}\Big) + \ln\Big(\frac{-2\chi}{\gamma-1}\Big)\ln\Big(\frac{-2\chi}{\gamma_2-1}\Big) + \ln\Big(\frac{-2\chi}{\gamma_1-1}\Big)\ln\Big(\frac{-2\chi}{\gamma_2-1}\Big) \\ &+ 2\ \mathsf{Li}_2(1+\chi) - \frac{\pi^2}{3}\bigg], \end{aligned} \tag{C.23}$$

after combining with the divergent term (C.9).

C.2 Pure 4D solution in the case $b \geq 1$

For the case $b = 1$ we can define a combination of $(b = 1)$ and $(b = 0)$ as

$$\mathcal{D}_{313}^{(0,1)} - \chi\ s\mathcal{D}_{313}^{(0,0)} = \frac{\gamma_1}{2s^2}\int_{-1}^{+1} du \frac{u + m\gamma_2}{\sqrt{(u+m\gamma_2)^2 + \xi^2}}\ f(u)\,, \tag{C.24}$$

and again $f(u)$ defined in (C.4). Since

$$\left|\frac{u + m\gamma_2}{\sqrt{(u+m\gamma_2)^2+\xi^2}}\right| \leq 1 \tag{C.25}$$

and $f(u) > 0$ in the whole integration zone, using the result (C.2) we have

$$\mathcal{D}_{313}^{(0,1)} - \chi\ s\mathcal{D}_{313}^{(0,0)} \ < \ \frac{\gamma_1}{2s^2} \ln\left(\frac{\gamma+1}{\gamma-1}\right) \ln\left(\frac{\gamma_1+1}{\gamma_1-1}\right). \tag{C.26}$$

It means that as a function of γ_2, above combination is finite under the limit $\gamma_2 \to 1$. Thus to our zero-order (i.e., the pure 4D case), we can just set $\gamma_2 = 1$ before doing the integration and get

$$\mathcal{D}_{313}^{(0,1)} - \chi\ s\mathcal{D}_{313}^{(0,0)} = \frac{\gamma_1}{2s^2} \int_{-1}^{+1} du \frac{u+m}{|u+m|} f(u) = \frac{\gamma_1}{2s^2} \left(\int_{-m}^{+1} + \int_{-m}^{-1} \right) du\ f(u)\,. \tag{C.27}$$

Taking the same integration variable replacement as in (C.19), this integral can be worked out easily. Then in the limit $\gamma_2 \to 1$, $\mathcal{D}_{313}^{(0,1)} - \chi\ s\mathcal{D}_{313}^{(0,0)}$ is equal to

$$\frac{\gamma_1}{2s^2}\left[\ln \frac{\gamma\gamma_1 - m + \alpha_m}{(\gamma-1)(\gamma_1-1)} \ln \frac{\gamma\gamma_1 - m - \alpha_m}{(\gamma-1)(\gamma_1-1)} + \ln \frac{\gamma\gamma_1 - m + \alpha_m}{(\gamma+1)(\gamma_1-1)} \ln \frac{\gamma\gamma_1 - m - \alpha_m}{(\gamma+1)(\gamma_1-1)} \right], \tag{C.28}$$

where

$$\alpha_m = \sqrt{\gamma^2 + \gamma_1^2 + m^2 - 2\gamma\gamma_1 m - 1}\,. \tag{C.29}$$

It is worth to point out that when we take $m \to 1$, this result reduces to

$$\frac{\gamma_1}{2s^2} \ln\left(\frac{\gamma+1}{\gamma-1}\right) \ln\left(\frac{\gamma_1+1}{\gamma_1-1}\right), \tag{C.30}$$

which is just the explicit result of $\mathcal{D}_{312}^{(0,0)}$. To keep only zero-order results, we take $\gamma, \gamma_1 \to 1$ further in (C.28) and find that in the pure 4D

$$\mathcal{D}_{313}^{(0,1)} = \chi\ s\mathcal{D}_{313}^{(0,0)} + \mathcal{D}_{312}^{(0,0)} - \frac{1}{s^2} \ln\left(\frac{-2\chi}{\gamma-1}\right) \ln\left(\frac{-2\chi}{\gamma_1-1}\right). \tag{C.31}$$

For the case $b = 2, 3$ we will not show the computation details again. The main point is that in the first step, we choose a proper linear combination of $\mathcal{D}_{313}^{(0,b)}$ to make the integrand having the form

$$\frac{(u+m\gamma_2)^b}{\sqrt{(u+m\gamma_2)^2 + \xi^2}}\ f(u)\,. \tag{C.32}$$

For $b = 2$ we should choose the combination as

$$\mathcal{D}_{313}^{(0,2)} - 2\chi\ s\mathcal{D}_{313}^{(0,1)} + \chi^2\ s^2\mathcal{D}_{313}^{(0,0)}\,, \tag{C.33}$$

and for $b = 3$ it is

$$\mathcal{D}_{313}^{(0,3)} - 3\chi\ s\mathcal{D}_{313}^{(0,2)} + 3\chi^2\ s^2\mathcal{D}_{313}^{(0,1)} - \chi^3\ s^3\mathcal{D}_{313}^{(0,0)}\,. \tag{C.34}$$

Then we can prove that these combinations are convergent at $\gamma_2 \to 1$ just as in the case $b = 1$. Thus we can take $\gamma_2 = 1$ before integrating those combinations. The second step is to integrate these dramatically simplified integrands. In this step the most efficient way is to use the variable replacement in (C.19) and we can integrate quickly. For example in the case $b = 2$, combination $\mathcal{D}_{313}^{(0,2)} - \chi\, s\mathcal{D}_{313}^{(0,1)}$ is given by

$$-\frac{\gamma_1}{2s}\left(\gamma \ln\left(\frac{\gamma_1+1}{\gamma_1-1}\right) + \gamma_1 \ln\left(\frac{\gamma+1}{\gamma-1}\right) - 2\alpha_m \ln\frac{\gamma\gamma_1 - m + \alpha_m}{\beta} - 2m\right), \tag{C.35}$$

and the corresponding expressions for $b = 3$ are even longer. To find the approximate results in the pure 4D, in the last step we take limit $\gamma, \gamma_1 \to 1$. Carrying out these steps, finally we get

$$\mathcal{D}_{313}^{(0,2)} = \chi\, s\mathcal{D}_{313}^{(0,1)} + \frac{2\chi+1}{s}\mathcal{D}_{202}^{(0,0)} - \frac{2\chi+1}{2}\mathcal{D}_{212}^{(0,0)} - \frac{2\chi+1}{2}\mathcal{D}_{302}^{(0,0)} - \frac{2\chi}{s}\ln(-\chi)\,, \tag{C.36}$$

$$\mathcal{D}_{313}^{(0,3)} = \chi^2\, s^2\mathcal{D}_{313}^{(0,1)} + \left(\frac{5}{2}\chi^2 + \chi - \frac{1}{4}\right)\mathcal{D}_{202}^{(0,0)} - \left(\frac{3}{2}\chi^2 + \frac{1}{2}\chi - \frac{1}{4}\right)\left(s\mathcal{D}_{212}^{(0,0)} + s\mathcal{D}_{302}^{(0,0)}\right) - 3\chi^2 \ln(-\chi)\,.$$

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References

[1] NLO Multileg Working Group collaboration, Z. Bern et al., *The NLO multileg working group: Summary report*, arXiv:0803.0494 [inSPIRE].

[2] SM and NLO Multileg Working Group collaborations, J.R. Andersen et al., *The SM and NLO Multileg Working Group: Summary report*, arXiv:1003.1241 [inSPIRE].

[3] SM and NLO MULTILEG and SM MC Working Groups collaborations, J. Alcaraz Maestre et al., *The SM and NLO Multileg and SM MC Working Groups: Summary Report*, arXiv:1203.6803 [inSPIRE].

[4] E.L. Berger, E. Braaten and R.D. Field, *Large p_T Production of Single and Double Photons in Proton Proton and Pion-Proton Collisions*, Nucl. Phys. **B 239** (1984) 52 [inSPIRE].

[5] P. Aurenche, A. Douiri, R. Baier, M. Fontannaz and D. Schiff, *Large p_T Double Photon Production in Hadronic Collisions: Beyond Leading Logarithm QCD Calculation*, *Z. Phys.* **C 29** (1985) 459 [INSPIRE].

[6] R.K. Ellis, I. Hinchliffe, M. Soldate and J.J. van der Bij, *Higgs Decay to tau+ tau-: A Possible Signature of Intermediate Mass Higgs Bosons at the SSC*, *Nucl. Phys.* **B 297** (1988) 221 [INSPIRE].

[7] V.A. Smirnov, *Evaluating Feynman integrals*, *Springer Tracts Mod. Phys.* **211** (2004) 1.

[8] V.A. Smirnov, *Feynman integral calculus*, Springer, Berlin Germany (2006).

[9] V.A. Smirnov, *Analytic tools for Feynman integrals*, *Springer Tracts Mod. Phys.* **250** (2012) 1.

[10] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate β-functions in 4 Loops*, *Nucl. Phys.* **B 192** (1981) 159 [INSPIRE].

[11] O.V. Tarasov, *Reduction of Feynman graph amplitudes to a minimal set of basic integrals*, *Acta Phys. Polon.* **B 29** (1998) 2655 [hep-ph/9812250] [INSPIRE].

[12] Z. Bern, L.J. Dixon and D.A. Kosower, *A two loop four gluon helicity amplitude in QCD*, *JHEP* **01** (2000) 027 [hep-ph/0001001] [INSPIRE].

[13] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two-loop QCD corrections to the scattering of massless distinct quarks*, *Nucl. Phys.* **B 601** (2001) 318 [hep-ph/0010212] [INSPIRE].

[14] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two loop QCD corrections to massless identical quark scattering*, *Nucl. Phys.* **B 601** (2001) 341 [hep-ph/0011094] [INSPIRE].

[15] E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two loop QCD corrections to gluon-gluon scattering*, *Nucl. Phys.* **B 605** (2001) 467 [hep-ph/0102201] [INSPIRE].

[16] C. Anastasiou, E.W.N. Glover, C. Oleari and M.E. Tejeda-Yeomans, *Two loop QCD corrections to massless quark gluon scattering*, *Nucl. Phys.* **B 605** (2001) 486 [hep-ph/0101304] [INSPIRE].

[17] S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, *Int. J. Mod. Phys.* **A 15** (2000) 5087 [hep-ph/0102033] [INSPIRE].

[18] Z. Bern, A. De Freitas and L.J. Dixon, *Two loop helicity amplitudes for gluon-gluon scattering in QCD and supersymmetric Yang-Mills theory*, *JHEP* **03** (2002) 018 [hep-ph/0201161] [INSPIRE].

[19] O.V. Tarasov, *Computation of Grobner bases for two loop propagator type integrals*, *Nucl. Instrum. Meth.* **A 534** (2004) 293 [hep-ph/0403253] [INSPIRE].

[20] J. Gluza, K. Kajda and D.A. Kosower, *Towards a Basis for Planar Two-Loop Integrals*, *Phys. Rev.* **D 83** (2011) 045012 [arXiv:1009.0472] [INSPIRE].

[21] M.Y. Kalmykov and B.A. Kniehl, *Counting master integrals: Integration by parts versus differential reduction*, *Phys. Lett.* **B 702** (2011) 268 [arXiv:1105.5319] [INSPIRE].

[22] R.M. Schabinger, *A New Algorithm For The Generation Of Unitarity-Compatible Integration By Parts Relations*, *JHEP* **01** (2012) 077 [arXiv:1111.4220] [INSPIRE].

[23] A.V. Kotikov, *Differential equations method: New technique for massive Feynman diagrams calculation*, *Phys. Lett.* **B 254** (1991) 158 [INSPIRE].

JHEP06(2014)166

[24] E. Remiddi, *Differential equations for Feynman graph amplitudes*, *Nuovo Cim.* **A 110** (1997) 1435 [hep-th/9711188] [INSPIRE].

[25] T. Gehrmann and E. Remiddi, *Differential equations for two loop four point functions*, *Nucl. Phys.* **B 580** (2000) 485 [hep-ph/9912329] [INSPIRE].

[26] M. Argeri and P. Mastrolia, *Feynman Diagrams and Differential Equations*, *Int. J. Mod. Phys.* **A 22** (2007) 4375 [arXiv:0707.4037] [INSPIRE].

[27] J.M. Henn, *Multiloop integrals in dimensional regularization made simple*, *Phys. Rev. Lett.* **110** (2013) 251601 [arXiv:1304.1806] [INSPIRE].

[28] J.M. Henn and V.A. Smirnov, *Analytic results for two-loop master integrals for Bhabha scattering I*, *JHEP* **11** (2013) 041 [arXiv:1307.4083] [INSPIRE].

[29] J.M. Henn, A.V. Smirnov and V.A. Smirnov, *Evaluating single-scale and/or non-planar diagrams by differential equations*, *JHEP* **03** (2014) 088 [arXiv:1312.2588] [INSPIRE].

[30] M. Argeri et al., *Magnus and Dyson Series for Master Integrals*, *JHEP* **03** (2014) 082 [arXiv:1401.2979] [INSPIRE].

[31] M.C. Bergere and Y.-M.P. Lam, *Asymptotic Expansion Of Feynman Amplitudes. Part 1: The Convergent Case*, *Commun. Math. Phys.* **39** (1974) 1 [INSPIRE].

[32] N.I. Usyukina, *On a Representation for Three Point Function*, *Teor. Mat. Fiz.* **22** (1975) 300 [INSPIRE].

[33] V.A. Smirnov, *Analytical result for dimensionally regularized massless on shell double box*, *Phys. Lett.* **B 460** (1999) 397 [hep-ph/9905323] [INSPIRE].

[34] J.B. Tausk, *Nonplanar massless two loop Feynman diagrams with four on-shell legs*, *Phys. Lett.* **B 469** (1999) 225 [hep-ph/9909506] [INSPIRE].

[35] G. Passarino and M.J.G. Veltman, *One Loop Corrections for e^+e^- Annihilation Into $\mu^+\mu^-$ in the Weinberg Model*, *Nucl. Phys.* **B 160** (1979) 151 [INSPIRE].

[36] W.L. van Neerven and J.A.M. Vermaseren, *Large loop integrals*, *Phys. Lett.* **B 137** (1984) 241 [INSPIRE].

[37] Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated one loop integrals*, *Phys. Lett.* **B 302** (1993) 299 [*Erratum ibid.* **B 318** (1993) 649] [hep-ph/9212308] [INSPIRE].

[38] Z. Bern, L.J. Dixon and D.A. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B 412** (1994) 751 [hep-ph/9306240] [INSPIRE].

[39] R.K. Ellis and G. Zanderighi, *Scalar one-loop integrals for QCD*, *JHEP* **02** (2008) 002 [arXiv:0712.1851] [INSPIRE].

[40] A. Denner and S. Dittmaier, *Reduction schemes for one-loop tensor integrals*, *Nucl. Phys.* **B 734** (2006) 62 [hep-ph/0509141] [INSPIRE].

[41] G. Duplancic and B. Nizic, *Reduction method for dimensionally regulated one loop N-point Feynman integrals*, *Eur. Phys. J.* **C 35** (2004) 105 [hep-ph/0303184] [INSPIRE].

[42] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226] [INSPIRE].

JHEP06(2014)166

[43] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *Fusing gauge theory tree amplitudes into loop amplitudes*, Nucl. Phys. **B 435** (1995) 59 [hep-ph/9409265] [INSPIRE].

[44] Z. Bern, L.J. Dixon and D.A. Kosower, *Progress in one loop QCD computations*, Ann. Rev. Nucl. Part. Sci. **46** (1996) 109 [hep-ph/9602280] [INSPIRE].

[45] Z. Bern and A.G. Morgan, *Massive loop amplitudes from unitarity*, Nucl. Phys. **B 467** (1996) 479 [hep-ph/9511336] [INSPIRE].

[46] Z. Bern, L.J. Dixon and D.A. Kosower, *One loop amplitudes for e^+e^- to four partons*, Nucl. Phys. **B 513** (1998) 3 [hep-ph/9708239] [INSPIRE].

[47] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop selfdual and $N = 4$ super Yang-Mills*, Phys. Lett. **B 394** (1997) 105 [hep-th/9611127] [INSPIRE].

[48] R. Britto, F. Cachazo and B. Feng, *Generalized unitarity and one-loop amplitudes in $N = 4$ super-Yang-Mills*, Nucl. Phys. **B 725** (2005) 275 [hep-th/0412103] [INSPIRE].

[49] R. Britto, F. Cachazo and B. Feng, *Computing one-loop amplitudes from the holomorphic anomaly of unitarity cuts*, Phys. Rev. **D 71** (2005) 025012 [hep-th/0410179] [INSPIRE].

[50] S.J. Bidder, N.E.J. Bjerrum-Bohr, L.J. Dixon and D.C. Dunbar, *$N = 1$ supersymmetric one-loop amplitudes and the holomorphic anomaly of unitarity cuts*, Phys. Lett. **B 606** (2005) 189 [hep-th/0410296] [INSPIRE].

[51] S.J. Bidder, N.E.J. Bjerrum-Bohr, D.C. Dunbar and W.B. Perkins, *One-loop gluon scattering amplitudes in theories with $N < 4$ supersymmetries*, Phys. Lett. **B 612** (2005) 75 [hep-th/0502028] [INSPIRE].

[52] S.J. Bidder, D.C. Dunbar and W.B. Perkins, *Supersymmetric Ward identities and NMHV amplitudes involving gluinos*, JHEP **08** (2005) 055 [hep-th/0505249] [INSPIRE].

[53] Z. Bern, N.E.J. Bjerrum-Bohr, D.C. Dunbar and H. Ita, *Recursive calculation of one-loop QCD integral coefficients*, JHEP **11** (2005) 027 [hep-ph/0507019] [INSPIRE].

[54] Z. Bern, L.J. Dixon and D.A. Kosower, *Bootstrapping multi-parton loop amplitudes in QCD*, Phys. Rev. **D 73** (2006) 065013 [hep-ph/0507005] [INSPIRE].

[55] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, *One-loop amplitudes of gluons in SQCD*, Phys. Rev. **D 72** (2005) 065012 [hep-ph/0503132] [INSPIRE].

[56] R. Britto, B. Feng and P. Mastrolia, *The Cut-constructible part of QCD amplitudes*, Phys. Rev. **D 73** (2006) 105004 [hep-ph/0602178] [INSPIRE].

[57] P. Mastrolia, *On Triple-cut of scattering amplitudes*, Phys. Lett. **B 644** (2007) 272 [hep-th/0611091] [INSPIRE].

[58] A. Brandhuber, S. McNamara, B.J. Spence and G. Travaglini, *Loop amplitudes in pure Yang-Mills from generalised unitarity*, JHEP **10** (2005) 011 [hep-th/0506068] [INSPIRE].

[59] Z. Bern, L.J. Dixon and D.A. Kosower, *On-Shell Methods in Perturbative QCD*, Annals Phys. **322** (2007) 1587 [arXiv:0704.2798] [INSPIRE].

[60] D. Forde, *Direct extraction of one-loop integral coefficients*, Phys. Rev. **D 75** (2007) 125019 [arXiv:0704.1835] [INSPIRE].

[61] S.D. Badger, *Direct Extraction Of One Loop Rational Terms*, JHEP **01** (2009) 049 [arXiv:0806.4600] [INSPIRE].

[62] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *D-dimensional unitarity cut method*, *Phys. Lett.* **B 645** (2007) 213 [hep-ph/0609191] [INSPIRE].

[63] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *Unitarity cuts and Reduction to master integrals in d dimensions for one-loop amplitudes*, *JHEP* **03** (2007) 111 [hep-ph/0612277] [INSPIRE].

[64] W.T. Giele, Z. Kunszt and K. Melnikov, *Full one-loop amplitudes from tree amplitudes*, *JHEP* **04** (2008) 049 [arXiv:0801.2237] [INSPIRE].

[65] R. Britto and B. Feng, *Unitarity cuts with massive propagators and algebraic expressions for coefficients*, *Phys. Rev.* **D 75** (2007) 105006 [hep-ph/0612089] [INSPIRE].

[66] R. Britto and B. Feng, *Integral coefficients for one-loop amplitudes*, *JHEP* **02** (2008) 095 [arXiv:0711.4284] [INSPIRE].

[67] R. Britto, B. Feng and P. Mastrolia, *Closed-Form Decomposition of One-Loop Massive Amplitudes*, *Phys. Rev.* **D 78** (2008) 025031 [arXiv:0803.1989] [INSPIRE].

[68] R. Britto, B. Feng and G. Yang, *Polynomial Structures in One-Loop Amplitudes*, *JHEP* **09** (2008) 089 [arXiv:0803.3147] [INSPIRE].

[69] B. Feng and G. Yang, *Unitarity Method with Spurious Pole*, *Nucl. Phys.* **B 811** (2009) 305 [arXiv:0806.4016] [INSPIRE].

[70] R. Britto and B. Feng, *Solving for tadpole coefficients in one-loop amplitudes*, *Phys. Lett.* **B 681** (2009) 376 [arXiv:0904.2766] [INSPIRE].

[71] C.F. Berger and D. Forde, *Multi-Parton Scattering Amplitudes via On-Shell Methods*, *Ann. Rev. Nucl. Part. Sci.* **60** (2010) 181 [arXiv:0912.3534] [INSPIRE].

[72] Z. Bern, J.J. Carrasco, T. Dennen, Y.-t. Huang and H. Ita, *Generalized Unitarity and Six-Dimensional Helicity*, *Phys. Rev.* **D 83** (2011) 085022 [arXiv:1010.0494] [INSPIRE].

[73] R. Britto, *Loop Amplitudes in Gauge Theories: Modern Analytic Approaches*, *J. Phys.* **A 44** (2011) 454006 [arXiv:1012.4493] [INSPIRE].

[74] L.J. Dixon, *A brief introduction to modern amplitude methods*, arXiv:1310.5353 [INSPIRE].

[75] E.I. Buchbinder and F. Cachazo, *Two-loop amplitudes of gluons and octa-cuts in $N = 4$ super Yang-Mills*, *JHEP* **11** (2005) 036 [hep-th/0506126] [INSPIRE].

[76] F. Cachazo, *Sharpening The Leading Singularity*, arXiv:0803.1988 [INSPIRE].

[77] F. Cachazo, M. Spradlin and A. Volovich, *Leading Singularities of the Two-Loop Six-Particle MHV Amplitude*, *Phys. Rev.* **D 78** (2008) 105022 [arXiv:0805.4832] [INSPIRE].

[78] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *A Duality For The S Matrix*, *JHEP* **03** (2010) 020 [arXiv:0907.5418] [INSPIRE].

[79] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, A.B. Goncharov, A. Postnikov and J. Trnka, *Scattering Amplitudes and the Positive Grassmannian*, arXiv:1212.5605 [INSPIRE].

[80] D.A. Kosower and K.J. Larsen, *Maximal Unitarity at Two Loops*, *Phys. Rev.* **D 85** (2012) 045017 [arXiv:1108.1180] [INSPIRE].

[81] K.J. Larsen, *Global Poles of the Two-Loop Six-Point $N = 4$ SYM integrand*, *Phys. Rev.* **D 86** (2012) 085032 [arXiv:1205.0297] [INSPIRE].

JHEP06(2014)166

[82] S. Caron-Huot and K.J. Larsen, *Uniqueness of two-loop master contours*, JHEP **10** (2012) 026 [arXiv:1205.0801] [INSPIRE].

[83] H. Johansson, D.A. Kosower and K.J. Larsen, *Two-Loop Maximal Unitarity with External Masses*, Phys. Rev. **D 87** (2013) 025030 [arXiv:1208.1754] [INSPIRE].

[84] H. Johansson, D.A. Kosower and K.J. Larsen, *An Overview of Maximal Unitarity at Two Loops*, PoS(LL2012)066 [arXiv:1212.2132] [INSPIRE].

[85] M. Søgaard, *Global Residues and Two-Loop Hepta-Cuts*, JHEP **09** (2013) 116 [arXiv:1306.1496] [INSPIRE].

[86] H. Johansson, D.A. Kosower and K.J. Larsen, *Maximal Unitarity for the Four-Mass Double Box*, Phys. Rev. **D 89** (2014) 125010 [arXiv:1308.4632] [INSPIRE].

[87] M. Søgaard and Y. Zhang, *Multivariate Residues and Maximal Unitarity*, JHEP **12** (2013) 008 [arXiv:1310.6006] [INSPIRE].

[88] G. Ossola, C.G. Papadopoulos and R. Pittau, *Reducing full one-loop amplitudes to scalar integrals at the integrand level*, Nucl. Phys. **B 763** (2007) 147 [hep-ph/0609007] [INSPIRE].

[89] P. Mastrolia and G. Ossola, *On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes*, JHEP **11** (2011) 014 [arXiv:1107.6041] [INSPIRE].

[90] S. Badger, H. Frellesvig and Y. Zhang, *Hepta-Cuts of Two-Loop Scattering Amplitudes*, JHEP **04** (2012) 055 [arXiv:1202.2019] [INSPIRE].

[91] P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, *Scattering Amplitudes from Multivariate Polynomial Division*, Phys. Lett. **B 718** (2012) 173 [arXiv:1205.7087] [INSPIRE].

[92] R.H.P. Kleiss, I. Malamos, C.G. Papadopoulos and R. Verheyen, *Counting to One: Reducibility of One- and Two-Loop Amplitudes at the Integrand Level*, JHEP **12** (2012) 038 [arXiv:1206.4180] [INSPIRE].

[93] S. Badger, H. Frellesvig and Y. Zhang, *An Integrand Reconstruction Method for Three-Loop Amplitudes*, JHEP **08** (2012) 065 [arXiv:1207.2976] [INSPIRE].

[94] P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, *Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division*, Phys. Rev. **D 87** (2013) 085026 [arXiv:1209.4319] [INSPIRE].

[95] R. Huang and Y. Zhang, *On Genera of Curves from High-loop Generalized Unitarity Cuts*, JHEP **04** (2013) 080 [arXiv:1302.1023] [INSPIRE].

[96] S. Badger, H. Frellesvig and Y. Zhang, *A Two-Loop Five-Gluon Helicity Amplitude in QCD*, JHEP **12** (2013) 045 [arXiv:1310.1051] [INSPIRE].

[97] Y. Zhang, *Integrand-Level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods*, JHEP **09** (2012) 042 [arXiv:1205.5707] [INSPIRE].

[98] B. Feng and R. Huang, *The classification of two-loop integrand basis in pure four-dimension*, JHEP **02** (2013) 117 [arXiv:1209.3747] [INSPIRE].

[99] P. Mastrolia, E. Mirabella, G. Ossola, T. Peraro and H. van Deurzen, *The Integrand Reduction of One- and Two-Loop Scattering Amplitudes*, PoS(LL2012)028 [arXiv:1209.5678] [INSPIRE].

[100] R.N. Lee and A.A. Pomeransky, *Critical points and number of master integrals*, JHEP **11** (2013) 165 [arXiv:1308.6676] [INSPIRE].

[101] F. Cachazo, P. Svrček and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [hep-th/0403047] [INSPIRE].

[102] G. Heinrich, G. Ossola, T. Reiter and F. Tramontano, *Tensorial Reconstruction at the Integrand Level*, *JHEP* **10** (2010) 105 [arXiv:1008.2441] [INSPIRE].

[103] S. Abreu, R. Britto, C. Duhr and E. Gardi, *From multiple unitarity cuts to the coproduct of Feynman integrals*, arXiv:1401.3546 [INSPIRE].