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Moonshine in fivebrane spacetimes

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ABSTRACT: We consider type II superstring theory on $K3 \times S^1 \times \mathbb{R}^{1,4}$ and study perturbative BPS states in the near-horizon background of two Neveu-Schwarz fivebranes whose world-volume wraps the $K3 \times S^1$ factor. These states are counted by the spacetime helicity supertrace $\chi_2(\tau)$ which we evaluate. We find a simple expression for $\chi_2(\tau)$ in terms of the completion of the mock modular form $H^{(2)}(\tau)$ that has appeared recently in studies of the decomposition of the elliptic genus of $K3$ surfaces into characters of the $N = 4$ superconformal algebra and which manifests a moonshine connection to the Mathieu group M_{24} .

KEYWORDS: Discrete and Finite Symmetries, Conformal Field Models in String Theory

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1 Introduction and motivation

Mock modular forms have appeared recently in a variety of physical and mathematical contexts. On the physical side, they play a central role in the counting of black hole states in string theory [1] and in computations of the elliptic genus of sigma models with non-compact target spaces [2–4]. In a more mathematical direction, a particular mock modular form with q expansion

$$H^{(2)}(\tau) = \sum_{n=0}^{\infty} c^{(2)}(8n-1) q^{n-1/8} = 2q^{-1/8}(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \dots) \quad (1.1)$$

appears in the decomposition of the elliptic genus of $K3$ surfaces into characters of the $N = 4$ superconformal algebra and reveals a mysterious moonshine property: the coefficients 45, 231, 770, 2277 are dimensions of irreducible representations of the largest sporadic Mathieu group M_{24} [5]. This “Mathieu Moonshine” has been further developed and tested through computation of the analogs of the McKay-Thompson series of Monstrous Moonshine [6], $H_g^{(2)}$ for $g \in M_{24}$ [7–10], and there is now a proof [12] of the existence of an infinite-dimensional M_{24} -module

$$K^{(2)} = \bigoplus_{n=0}^{\infty} K_{8n-1}^{(2)} \quad (1.2)$$

with $\dim K_{8n-1}^{(2)} = c^{(2)}(8n - 1)$ for $n \geq 1$, although so far no explicit construction of such a module is known.

There are many reasons to expect a construction based on Conformal Field Theory (CFT). These include the fact that there is such a construction [13, 14] that explains the similarly remarkable connection between the coefficients of the modular function $j(\tau)$ and dimensions of representations of the Monster group known as Monstrous Moonshine [15], and also the properties of the $H^{(2)}(\tau)$ constructed by twisting by elements of M_{24} . Monstrous Moonshine appears to have a generalization dubbed *generalized Moonshine* by Norton [16] which involves the existence of modular functions $Z_{g,h}(\tau)$ for congruence subgroups of $SL(2, \mathbb{R})$ for each pair of commuting elements (g, h) in the Monster group. These were given a conformal field theory interpretation in [17] in terms of the partition function twisted by h of an orbifold by g of the Monster CFT. A construction of many of these orbifold theories and their McKay-Thompson series can be found in [18] and references cited therein. Evidence for a similar generalization of Mathieu Moonshine has been presented in [19] and this can be regarded as further evidence that CFT is the correct framework in which to understand Mathieu Moonshine.

However, it is known that no classical $K3$ surface can exhibit the full M_{24} symmetry [20, 21]. Furthermore, there is also good evidence that the superconformal field theory (SCFT) describing any $K3$ surface also cannot exhibit the full M_{24} symmetry [22]. Thus it seems likely that one must look beyond the SCFT associated to $K3$ surfaces in the search for the origin of Mathieu Moonshine and an explicit construction of the infinite dimensional M_{24} module $K^{(2)}$ (see however [23] for an alternate point of view based on combining symmetry groups of distinct Kummer surfaces).

Another clue is provided by the existence of generalizations of the $(H^{(2)}, M_{24})$ moonshine to an umbral moonshine for vector-valued mock modular forms $H^{(X)}(\tau)$ and groups $G^{(X)}$ associated to the 23 Niemeier lattices [24, 25]. While some of the examples of umbral moonshine can also be related to weight zero Jacobi forms, the Jacobi forms are not the elliptic genera of any compact Calabi-Yau manifold, and for other instances of umbral moonshine it is weight one meromorphic Jacobi forms rather than weight zero Jacobi forms that are the primary objects leading to vector-valued mock modular forms. The existence of this large class of mock modular forms exhibiting Moonshine for finite groups but with no obvious connection to compact Calabi-Yau manifolds also points towards the need for an extended notion of SCFT if there is to be a universal construction for the infinite dimensional modules suggested by these constructions.

Yet another clue for the origin of the M_{24} -module and its generalizations may lie in the following detail of the original observation of M_{24} moonshine [5]. In order to obtain the mock modular form (1.1), the term proportional to the massless character of the $N = 4$ superconformal algebra had to be subtracted from the decomposition of the $K3$ elliptic genus into $N = 4$ characters. From the point of view of quantum field theory, removing part of the spectrum of the theory generically violates some fundamental property like locality or a defining symmetry of the theory. From this point of view, one may say that it is not too surprising that one does not find the M_{24} -module in a simple direct manner in the $K3$ SCFT.

Such a module is more likely to be present in a theory whose full spectrum of BPS states is counted directly by the function $H^{(2)}$. Given the recent appearance of mock modular forms as the elliptic genera of non-compact CFTs, it would be particularly natural if the target space of the CFT that we are looking for involved both $K3$ and a non-compact space.

The need to discard massless states is also reminiscent of the Frenkel-Lepowsky-Meurman construction of the Monster module denoted by V^{\natural} in [13, 14]. In physics terminology the construction starts with the holomorphic part of the bosonic string on the torus $\mathbb{R}^{24}/\Lambda_L$ where Λ_L is the Leech lattice. Since there are no points of length squared 2 in Λ_L , this theory has 24 massless states and a partition function that starts as

$$Z(\tau) = q^{-1} + 24 + 196884q + \dots \tag{1.3}$$

There is no 24-dimensional irreducible representation of the Monster, and the construction of FLM proceeds by the construction of an asymmetric $\mathbb{Z}/2$ orbifold which acts by $X^I \rightarrow -X^I$. This orbifold construction removes the 24 massless states and does not produce any new massless states in the twisted sector because the twist field has dimension 3/2. The orbifold construction also preserves modular invariance and so leads to a partition function which differs from eq. (1.3) only by the lack of a constant term.

Since mock modular forms appear in counting of supersymmetric, BPS black holes whose near horizon involves an Anti de Sitter space (AdS) component, it is also natural to wonder whether there might exist a BPS configuration of branes in string theory and an associated black hole counting problem where $H^{(2)}$ and its generalizations might appear. In light of the AdS/CFT correspondence this could provide a dual description of the CFT's associated to Mathieu and Umbral Moonshine. This idea is also supported by connections between semi-classical expansions in AdS and the Rademacher summability of [26–30].

As further motivation for the work presented here, we note that the appearance of the mock modular form $H^{(2)}(\tau)$ in the decomposition of the elliptic genus of $K3$ into characters of the $N = 4$ superconformal characters is a worldsheet phenomenon. It is often useful to find a translation of such worldsheet results into a spacetime computation involving BPS states since in that context one can apply the powerful ideas of string duality. This translation between worldsheet and spacetime points of view has been exploited heavily in the exact counting of BPS black hole states (see for example [31–33]). In the context of Type II string theory on $K3 \times S^1$ or $K3 \times T^2$ one might naively expect the elliptic genus of $K3$ to count perturbative 1/4 BPS states in intermediate representations of the $N = 4$ spacetime supersymmetry algebra since one can construct such states by combining purely left-moving excitations of the $K3$ SCFT with momentum and winding states on the S^1 or T^2 . However it is known that these states in fact combine into long representations of the $N = 4$ supersymmetry algebra and so do not contribute to the spacetime helicity index that counts BPS states [34].¹ Thus to find some spacetime, BPS image of the worldsheet decomposition it is natural to look at systems with the equivalent of $N = 2$ spacetime supersymmetry rather than $N = 4$ supersymmetry since in that case it is known that there are BPS states which are counted by the elliptic genus of $K3$. For example, this can be seen in the computation of threshold corrections in $N = 2$ heterotic string compactifications in [35] which depend on the new supersymmetric index which in turn can be seen

¹For a review of helicity supertraces see appendix B.

to count the difference between BPS vector and hypermultiplets. Connections between Mathieu Moonshine and threshold corrections in $N = 2$ heterotic string compactifications and their type II duals were recently explored in [36].

In this paper we take a first step in this direction through the computation of the second helicity index (often called the BPS index) $\chi_2(\tau)$ in the near horizon geometry of a background of two Neveu-Schwarz fivebranes in type II string theory on $K3 \times S^1$. This background has a spacetime supersymmetry algebra which has the same number of supersymmetries as an $N = 2$ theory in $\mathbb{R}^{1,3}$ and has perturbative BPS states which are counted by the index $\chi_2(\tau)$. We find that $\chi_2(\tau) = -(1/2)\eta(\tau)^3 \widehat{H}^{(2)}(\tau)$ where $\eta(\tau)$ is the Dedekind eta function and $\widehat{H}^{(2)}(\tau)$ is the completion of the mock modular form $H^{(2)}(\tau)$ determined by its shadow $g(\tau) = 24\eta(\tau)^3$. The outline of this paper is as follows. In the second section we discuss the fivebrane background we utilize and some details of the underlying conformal field theory. The third section goes through the calculation and interpretation of the BPS index while the fourth section discusses some properties of mock modular forms and the modification to this computation of the BPS index when we twist the theory by symplectic automorphisms of the $K3$ surface. The final section offers conclusions and a discussion of interesting directions suggested by our results. Some details of the analysis of an integral first analyzed by Gaiotto and Zagier are presented in appendix A while appendix B summarizes our conventions for theta functions as well as some Riemann theta relations that are used in our computations.

2 Wrapped fivebranes and the $K3 \times \text{SL}(2, \mathbb{R})/\text{U}(1)$ SCFT

Consider type II string theory in the background of k NS5-branes in ten-dimensional flat space. In the RNS formalism, fundamental string propagation in the near-horizon region of the branes is described by a two-dimensional superconformal field theory [37], which we denote as:

$$\mathbb{R}^{1,5} \times \rho \times \text{SU}(2)_k. \tag{2.1}$$

Here the first factor corresponds to the space-time which the 5-branes span, and represents six free bosons as well as their $N = 1$ superpartners. The second factor corresponds to an $N = 1$ linear dilaton theory with slope² $Q = \sqrt{\frac{2}{k}}$ and central charge $c = \frac{3}{2} + 3Q^2$, and represents the radial direction in the \mathbb{R}^4 transverse to the branes. The third factor is an $N = 1$ $\text{SU}(2)$ WZW model at level k with central charge $c = \frac{9}{2} - \frac{6}{k}$, and represents the S^3 of the transverse space.

To make a consistent string theory one must introduce the $N = 1$ ghost system (b, c, β, γ) with central charge $c = -15$. Spacetime supersymmetry can be introduced by the usual method of identifying an $N = 2$ structure in the above SCFT, and by imposing the GSO projection. This gives us a theory with 8 left-moving and 8 right-moving supercharges which transform non-trivially under the $\text{SU}(2)_L \times \text{SU}(2)_R = \text{SO}(4)$ rotations of the transverse \mathbb{R}^4 .

²We will set $\alpha' = 2$ throughout this paper.

The string coupling is given in terms of the radial coordinate by $g_s = g_s^{(0)} e^{-\rho}$ so that fundamental strings are weakly coupled in the asymptotic region $\rho \rightarrow \infty$, and they become arbitrarily strongly coupled deep inside the throat of the branes at $\rho \rightarrow -\infty$. In order to study string perturbation theory we would like to cap off the strong-coupling singularity. A way of doing so was suggested in [38], by spreading out the 5-branes on a ring in the transverse \mathbb{R}^4 thus breaking the $SO(4)$ R-symmetry to $U(1) \times \mathbb{Z}/k$. The authors of [38] proposed that the SCFT corresponding to this configuration is:

$$\mathbb{R}^{1,5} \times \left(\frac{SL(2, \mathbb{R})_k}{U(1)} \times \frac{SU(2)_k}{U(1)} \right) / (\mathbb{Z}/k), \tag{2.2}$$

where the \mathbb{Z}/k orbifold is required to implement the integrality of charges on which a $\mathbb{Z}/2$ GSO projection [39] can act. The level indicated in both the WZW models is the supersymmetric level, and the levels of the two bosonic algebras are related to k as

$$k_B^{sl(2)} = k + 2, \quad k_B^{su(2)} = k - 2. \tag{2.3}$$

The $\frac{SU(2)_k}{U(1)}$ factor in (2.2) is the well-understood compact $N = 2$ coset of central charge $c = 3 - \frac{6}{k}$. The $\frac{SL(2, \mathbb{R})_k}{U(1)}$ factor in (2.2) denotes the non-compact coset theory called the cigar theory or the Euclidean black hole [40], with $c = 3 + \frac{6}{k}$. In the large k limit, the coset has a geometric picture as a sigma model on the cigar geometry with curvature proportional to $1/k$. The algebraic approach, on the other hand, is exact in k . For the purposes of computing Euclidean path-integrals, we follow the treatment of [41–44]), in which the cigar theory is defined as the Euclidean coset $H_3^+/U(1)$ with $H_3^+ = SL(2, \mathbb{C})/SU(2)$.

Asymptotically, the cigar model consists of a linear dilaton direction ρ with slope $Q = \sqrt{\frac{2}{k}}$, and a $U(1)$ direction θ with $\theta \sim \theta + \frac{4\pi}{Q}$, and two fermions (ψ_ρ, ψ_θ) . Together, they make up an $N = 2$ SCFT with the following holomorphic currents (see e.g. [45]):

$$\begin{aligned} T_{\text{cig}} &= -\frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\theta)^2 - \frac{1}{2}(\psi_\rho\partial\psi_\rho + \psi_\theta\partial\psi_\theta) - \frac{1}{2}Q\partial^2\rho, \\ G_{\text{cig}}^\pm &= \frac{i}{2}(\psi_\rho \pm i\psi_\theta)\partial(\rho \mp i\theta) + \frac{i}{2}Q\partial(\psi_\rho \pm i\psi_\theta), \\ J_{\text{cig}} &= -i\psi_\rho\psi_\theta + iQ\partial\theta, \end{aligned} \tag{2.4}$$

as well as their anti-holomorphic counterparts. In combination with the $SU(2)/U(1)$ coset and the flat directions, one recovers the theory (2.1) in the asymptotic region. The strong coupling region, however, has now been capped off by the geometry of the cigar, and the string coupling has a maximum at the tip of the cigar, the value of which is a modulus of the string theory.

The full $N = 2$ worldsheet currents of the theory include the currents coming from the flat space and $SU(2)/U(1)$ factors in (2.2). Using this $N = 2$ structure, we can now construct spin fields and spacetime supersymmetry. We have 8 left-moving and 8 right-moving spacetime supercharges $\mathcal{S}_\alpha, \tilde{\mathcal{S}}_\alpha$, that obey the algebra

$$\{\mathcal{S}_\alpha, \bar{\mathcal{S}}_\beta\} = 2\gamma_{\alpha\beta}^\mu P_\mu, \quad \{\tilde{\mathcal{S}}_\alpha, \bar{\tilde{\mathcal{S}}}_\beta\} = 2\gamma_{\alpha\beta}^\mu P_\mu, \quad \mu = 0, 1 \dots 5. \tag{2.5}$$

The spinors S_α are minimal Weyl spinors of $\text{Spin}(1, 5)$, and the bar denotes charge conjugation. In the IIA theory, the chirality of the left-movers and the right-movers are the same, while in the IIB theory they are opposite.

We also have a global $U(1)$ symmetry coming from the momentum around the circle θ :

$$J_{\text{sp}} = P_L^\theta + P_R^\theta \equiv \frac{i}{Q} \oint \partial\theta dz + \frac{i}{Q} \oint \bar{\partial}\tilde{\theta} d\bar{z}, \quad (2.6)$$

under which all the spacetime supercharges are charged:

$$[J_{\text{sp}}, \mathcal{S}_\alpha] = \frac{1}{2}\mathcal{S}_\alpha, \quad [J_{\text{sp}}, \bar{\mathcal{S}}_\alpha] = -\frac{1}{2}\bar{\mathcal{S}}_\alpha. \quad (2.7)$$

There is a similar expression for the right-moving supercharges. The $U(1)$ momentum symmetry is thus a spacetime R-symmetry and the spacetime fermion number is $(-1)^{F_s} = e^{2\pi i J_{\text{sp}}}$.

It is clear from the above worldsheet construction that in order to study NS5-branes wrapped on a $K3$ surface, one simply replaces the $\mathbb{R}^{1,5}$ by $\mathbb{R}^{1,1} \times K3$. In this case the $K3$ breaks a further half of the supersymmetry, and we get a superstring theory with 4 left-moving and 4 right-moving supercharges. Translation invariance along the $K3$ directions is now broken, and the supercharges anti-commute to translations along the $\mathbb{R}^{1,1}$ directions.

At level $k = 2$, when the model represents the theory with two NS5-branes, something special happens³ [45]. The compact coset $SU(2)_k/U(1)$ (with central charge $c = 3 - 6/k$) disappears, and the free boson θ is equivalent to two free-fermions. These two fermions combined with the fermion ψ_θ obey an $SU(2)$ algebra, and these enhanced symmetries give rise to the expected $SU(2)_L \times SU(2)_R$ symmetries of the CHS model (2.1). On separating the two five-branes in the transverse \mathbb{R}^4 this is broken to an $SU(2) \times (\mathbb{Z}/2)$ global symmetry (instead of $U(1) \times (\mathbb{Z}/k)$ for $k > 2$), as expected from the spacetime picture of two 5-branes.

Finally we can, without any further issues, consider the single flat spatial direction to be a large circle to get type II superstring theory on

$$\mathbb{R}_t \times S^1 \times K3 \times \left(\frac{SL_2(\mathbb{R})_{k=2}}{U(1)} \right) / (\mathbb{Z}/2), \quad (2.8)$$

which is the model we shall study in this paper.

2.1 The generating function of perturbative BPS states

We would like to study the degeneracies of perturbative BPS states in the string theory (2.8). We consider a fundamental type II string propagating in time and wrapping the circle in (2.8). The covariant RNS description of the string has oscillators associated with the $\mathbb{R}_t \times S^1$ directions which are cancelled in all physical computations by the oscillators of the (b, c, β, γ) superghost system that gauge the $N = 1$ supergravity on the string

³The theory (2.2) at $k = 2$ is also the end-point $d = 6$ of another family of interesting superstring theories called non-critical superstrings [46], defined as an $N = 2$ generalization of Liouville theory combined with d flat spacetime dimensions. It was shown in [47] that the $N = 2$ Liouville theory is indeed mirror symmetric to the cigar supercoset.

world-sheet. One can also directly choose a gauge condition on the string world-sheet that eliminates the unphysical oscillators in the $\mathbb{R}_t \times S^1$ directions. To this end one can make a small modification to the usual light-cone gauge condition in $\mathbb{R}^{1,1}$ so as to keep only the transverse oscillators on the string world-sheet [48]. This leaves us with an $N = (4, 4)$ 2d SCFT with central charge $c = \tilde{c} = 12$ described by

$$K3 \times \left(\frac{SL_2(\mathbb{R})_{k=2}}{U(1)} \right) / (\mathbb{Z}/2). \tag{2.9}$$

If the string has momentum and winding labelled by integers n, w respectively, and we choose $n \geq 0, w \geq 0$, then in this compact light-cone gauge we have

$$M^2 = \frac{q_R^2}{2} + \tilde{h} + a = \frac{q_L^2}{2} + h + a, \tag{2.10}$$

where $M \equiv |p_0|$ denotes the energy of a state corresponding to an excitation of the SCFT (2.9) with left and right-moving conformal weights h, \tilde{h} and with

$$q_{R,L} = \frac{n}{R} \pm \frac{wR}{2} \tag{2.11}$$

where R is the radius of the S^1 . The constant in eq. (2.10) arises from the zero point energy and is equal to $a = -\frac{1}{2}$.

From the asymptotic supersymmetry algebra (2.5) compactified on $K3$, it follows that states annihilated by the right moving supercharges have $M = |q_R|$ which implies that $\tilde{h} + a_R = 0$. For such states, the level-matching condition (2.10) implies that the product of the winding and momenta

$$nw = h + a_L. \tag{2.12}$$

Perturbative BPS states in string theory in flat space can be summarised in a succinct way in terms of spacetime helicity supertraces [34]. We would like to compute similar BPS indices for our string theory. In particular, we are interested in generating functions of the form

$$\chi_n(\tau) = \text{Tr} (J_{\text{sp}})^n (-1)^{F_s} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \quad q = e^{2\pi i \tau}, \tag{2.13}$$

where τ is the modular parameter of the world-sheet torus and Tr indicates a sum over all the states in the theory (2.9). In the RNS formalism it represents a sum over Ramond and Neveu-Schwarz (NS) sectors with chiral GSO projections.

Our general strategy to obtain $\chi_n(\tau)$ is to first compute

$$\chi(\tau, z) = \text{Tr} (-1)^{F_s} q^{L_0 - c/24} \bar{q}^{\tilde{L}_0 - \tilde{c}/24} \zeta^{P_L^\theta} \bar{\zeta}^{P_R^\theta}, \quad \zeta = e^{2\pi i z}, \tag{2.14}$$

and then act on it by the operator $\left(\frac{1}{2\pi i} (\partial_z - \partial_{\bar{z}}) \right)^n \Big|_{z=\bar{z}=0}$.

In a theory with $N = 2$ spacetime supersymmetry in four dimensions, the quantity $\chi_0(\tau)$ receives a vanishing contribution from long as well as short multiplets in the theory [34], this turns out to be true for our situation as well. We shall focus on the first non-vanishing helicity supertrace $\chi_2(\tau)$ here.

Our computation has both a space-time and a world-sheet interpretation. In the space-time without NS5-branes the partition functions (2.13) (after adding in the partition function of the winding and momentum modes around the S^1) would be precisely the Euclidean version of the helicity supertraces in four dimensional string theory on $K3 \times T^2$, as computed say in [34]. Indeed, one can check that the operator J_0 is the charge of the $U(1)$ that rotates two directions in the \mathbb{R}^4 transverse to the 5-branes [38].

We generalize this counting by working in a background sourced by two heavy defects, the NS5-branes. The first non-zero BPS index is then $\chi_2(\tau)$ and from (2.12), we see that the coefficients of the generating function are the degeneracies of such states in terms of the T-duality charge invariant nw [49, 50]. More precisely, we should sum over the partition function associated to the momentum and winding states in computing the full BPS index of the theory leading to

$$\sum_{n,w \in \mathbb{Z}} q^{q_L^2/2} \bar{q}^{q_R^2/2} \chi_2(\tau) \tag{2.15}$$

with $q_{L,R}$ given in (2.11). We will see that $\chi_2(\tau)$ is not holomorphic, but has a holomorphic part given by

$$\chi_2(\tau)|_{hol} = -\frac{1}{2} \eta(\tau)^3 H^{(2)}(\tau) = \sum_{N=0}^{\infty} c(N) q^N \tag{2.16}$$

which we will interpret as counting 1/4 BPS states that are localized near the tip of the cigar. The physical states satisfying level-matching are then those with equal powers of q and \bar{q} in (2.15), that is those states with

$$N = \frac{1}{2}(q_R^2 - q_L^2) = nw. \tag{2.17}$$

We can thus interpret the coefficients $c(N)$ as counting the contribution of 1/4 BPS states to the BPS index in the near horizon geometry of two NS5-branes with mass squared $M^2 = q_R^2/2$ and with T-duality invariant nw equal to N .

We expect to find a relation between the coefficients $c(N)$ in (2.16) and the degeneracy of small BPS black holes with charges (n, w) in the background of two NS5-branes. These black holes have vanishing horizon area in the two-derivative gravitational theory, but in a similar situation in flat space they can gain a finite string-scale size upon introducing higher-derivative corrections [51]. Since the function $\chi_2(\tau)|_{hol}$ does not have a polar term in its q -expansion, the coefficients $c(N)$ do not grow exponentially in \sqrt{N} as $N \rightarrow \infty$ as one might expect from the black hole picture. Perhaps the details of the relation between the gravitational index and degeneracy [32] plays a role in resolving this puzzle.

3 Computation of the BPS index

In this section we enter into the details of the computation of the BPS index. The reader who is only interested in the final answer can skip ahead to (3.24). Before getting started we note two general features of the analysis. First, in the RNS formulation, the two factors in the SCFT (2.9) are essentially decoupled except that the sum over the different fermion periodicities ties together the various free field pieces in the partition function. We shall

use the description of $K3$ as a $T^4/(\mathbb{Z}/2)$ orbifold, but as we shall see, the final answer depends only on the elliptic genus of $K3$ which is invariant across the $K3$ moduli space. Second, the partition function of the $SL(2, \mathbb{R})/U(1)$ coset involves an integral over a gauge field zero mode which is the source of the integral over the variable u in (3.24).

The analysis involves a number of Jacobi theta functions. Our conventions for these as well as some useful identities they obey are given in appendix B.

We now describe the relevant partition functions of the various pieces that make up the SCFT (2.9). In the fermionic sector we present the NS sector partition functions explicitly. The partition functions in the other sectors $NS(-1)^F, R, R(-1)^F$ follow easily from the free fermion analysis, one can also write them using worldsheet $N = 2$ spectral flow applied to the NS partition function.

3.1 The cigar piece

The functional integral for the indexed partition function of the $SL(2, \mathbb{R})_k/U(1)$ (cigar) SCFT has recently been explicitly computed in [2–4]⁴ based on the work of [41, 43, 44]. We shall follow this treatment in what follows. The main idea is to express the G/H WZW coset as $G \times H^{\mathbb{C}}/H$ where $H^{\mathbb{C}}$ is a complexification of the subgroup H that is gauged. To this one adds a (b, c) ghost system of central charge $c = -\dim(H^{\mathbb{C}})$. The three pieces are coupled only via zero modes.

Our case of interest here is the supersymmetric $SL(2, \mathbb{R})/U(1)$ WZW coset. The theory has a bosonic H_3^+ WZW model at level $k + 2$ of which a $U(1)$ subgroup is gauged, and two free fermions ψ^{\pm} (and their right-moving counterparts). The coset $H^{\mathbb{C}}/H$ is represented by the compact boson Y . The zero mode in question is the holonomy of the gauge field around the two cycles of the torus which is represented by a complex parameter⁵ $u = a\tau + b$. The $(b, c)^{\text{cig}}$ ghost system has central charge $c = -2$. The bosonic $SL(2, \mathbb{R})$, the two fermions, the Y boson, and the (b, c) ghosts are all solvable theories and are coupled by the holonomy u that has to be integrated over the elliptic curve $E(\tau) = \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$.

The various pieces have the following contributions. The bosonic $H_3^+ = SL(2, \mathbb{C})/SU(2)$ model contributes:

$$Z_{H_3^+}(\tau, u) = \frac{(k + 2)\sqrt{k}}{\tau_2^{1/2}} e^{2\pi u_2^2/\tau_2} \frac{1}{|\vartheta_{11}(\tau, u)|^2}. \tag{3.1}$$

The $(b, c)^{\text{cig}}$ ghosts have the contribution:

$$Z_{\text{gh}}(\tau) = \tau_2 |\eta(\tau)|^2. \tag{3.2}$$

The two left-moving fermions ψ^{\pm} have a contribution in the NS sector⁶ [53]:

$$Z_{\psi^{\pm}}^{\text{NS}}(\tau) = \frac{1}{\sqrt{k + 2}} e^{-\pi u_2^2/\tau_2} e^{2\pi i u_1 u_2/\tau_2} \frac{\vartheta_{00}(\tau, u)}{\eta(\tau)}, \tag{3.3}$$

⁴The holomorphic part of this partition function had been presented earlier in [52].

⁵Throughout this paper, we will use the subscripts 1 and 2 on a complex variable to denote its real and imaginary parts, i.e. $\tau = \tau_1 + i\tau_2$, $u = u_1 + iu_2$ etc.

⁶The prefactor in front of the usual expression for free fermions arises because of a factor of $k + 2$ in the action of these fermions. This prefactor cancels an equivalent one in the numerator of the bosons in (3.1).

and their right-moving counterparts have a similar contribution:

$$Z_{\psi^\pm}^{\text{NS}}(\tau) = \frac{1}{\sqrt{k+2}} e^{-\pi u_2^2/\tau_2} e^{-2\pi i u_1 u_2/\tau_2} \frac{\overline{\vartheta_{00}(\tau, u)}}{\eta(\tau)}. \tag{3.4}$$

Now we come to the boson $U(1)_Y$. The matching to the asymptotic fields (2.4) shows that $\psi^\pm = \psi^\rho \pm i\psi^\theta$, and the boson $Y^u \equiv Y + \Phi[u]$ with $\Phi[u] = \frac{i}{\tau_2}(u\bar{u} - \bar{u}u)$ should be identified with the boson θ . (The notations are those of [4].) For the case $k = 2$, we know that the boson θ is equivalent to two free fermions χ^\pm , so that in the asymptotic region the variables are the fields⁷ $(\rho, \psi_\rho, \psi^\theta, \chi^\pm)$. These four fermions along with the four fermions of $K3$ and the two fermions of $\mathbb{R}_t \times S^1$ are the analog of the ten free fermions of type II string theory in flat space. These considerations suggest that the boson Y^u should really be treated as a pair of fermions with their corresponding spin structure. The same conclusion can also be reached by looking at the worldsheet $N = 2$ algebra (2.4) which is used to build spacetime supercharges.

The boson Y^u is translationally charged under the potential u (see eq. (2.21) of [3]), and this means that the fermions χ^\pm have charges ± 1 under the corresponding $U(1)$ current. The contribution of these fermions is:

$$Z_{\chi^\pm}^{\text{NS}}(\tau) = e^{-\pi(u_2+z_2)^2/\tau_2} e^{2\pi i(u_1+z_1)(u_2+z_2)/\tau_2} \frac{\vartheta_{00}(\tau, z+u)}{\eta(\tau)}, \tag{3.5}$$

and their right moving counterparts contribute:

$$Z_{\bar{\chi}^\pm}^{\text{NS}}(\tau) = e^{-\pi(u_2-z_2)^2/\tau_2} e^{-2\pi i(z_1-u_1)(z_2-u_2)/\tau_2} \frac{\overline{\vartheta_{00}(\tau, z-u)}}{\eta(\tau)}. \tag{3.6}$$

We see here that the left- and right-movers are charged oppositely under the $U(1)$ gauge field — this can be traced to the fact that the coset is an axial gauging of the H_3^+ WZW model.⁸

3.2 The $K3$ piece

We evaluate the $K3$ partition function at an orbifold point $T^4/(\mathbb{Z}/2)$. The T^4 SCFT consists of four bosons X^i and four fermions ξ^i , $i = 1, \dots, 4$. The $\mathbb{Z}/2$ orbifold acts by reflection through the origin on the four bosons (i.e. as $X^i \rightarrow -X^i$). Supersymmetry requires that the orbifold acts in exactly the same way on the four fermions (i.e. as $\xi^i \rightarrow -\xi^i$).

Following standard procedure for orbifold theories, we need to sum over the twisted sectors and project to $\mathbb{Z}/2$ invariant states. Denoting the $\mathbb{Z}/2$ valued twist by $r \in \{0, 1\}$, this sum is equivalent to summing over all possible periodicities in both the directions of the worldsheet torus, i.e. over the sectors (r, s) , $r, s = 0, 1$.

⁷The fields (ρ, ψ_ρ) form an $N = 1$ theory, and the three free fermions $(\psi^\theta, \psi^{1,2})$ form an $N = 1$ $SU(2)$ current algebra at level $k = 2$. This $SU(2)$ and the corresponding one from right-movers form the currents of the asymptotic $SO(4)$ theory of the theory of two 5-branes. This $SO(4)$ is then broken to $SU(2) \times \mathbb{Z}_2$ by the cigar interactions, see [45], section 3.4 for details.

⁸One can compare the relative charge assignments of the boson Y^u with respect to the momentum $U(1)$ (∂Y) and the gauged $U(1)$ (u). This is written down clearly in [4], equations (2.28)–(2.32). We see that, indeed, the charge assignments are consistent with the assignment of the potentials in (3.5), (3.6).

The partition function of the bosons in the untwisted sector is given by

$$Z_{K3(0,0)}^{\text{bos}}(\tau) = \frac{\Theta^{4,4}(\tau, \bar{\tau})}{|\eta(\tau)^4|^2}, \quad (3.7)$$

where the $\Theta^{4,4}$ indicates the sum over the $\Gamma^{4,4}$ Narain lattice of the T^4 . The left moving fermionic oscillator modes (with NS boundary conditions) is:

$$Z_{K3(0,0)}^{\text{ferNS}}(\tau) = \frac{\vartheta_{00}(\tau, 0)^2}{\eta(\tau)^2}, \quad (3.8)$$

and there is a corresponding factor from the right movers. Note that the fields of the $K3$ are not charged under the chemical potentials u (from the gauging of the coset), nor are they charged under the spacetime $U(1)$ R-symmetry.

In the sectors $(r, s) \neq (0, 0)$, there is no lattice sum. The bosonic partition function of the oscillator modes is:

$$Z_{K3(r,s)}^{\text{bos}}(\tau) = 16 \left| \frac{\eta(\tau)^2}{\vartheta_{11}(\tau, (s+r\tau)/2)^2} \right|^2. \quad (3.9)$$

The left-moving NS sector fermionic partition function is:

$$Z_{(r,s)}^{K3\text{NS}}(\tau) = \frac{\vartheta_{00}(\tau, (s+r\tau)/2)^2}{\eta(\tau)^2}, \quad (3.10)$$

and there is a corresponding partition function for the right-movers.

3.3 Putting the pieces together

The full partition function is obtained by multiplying the various bosonic and fermionic pieces of the cigar and the $K3$ SCFT, summing over NS, $\text{NS}(-1)^F$, R and $\text{R}(-1)^F$ fermion periodicities in each (r, s) twisted sector, and then summing over the twists. We include a factor of $1/2$ for each projection in the sum.

The untwisted sector. In the untwisted sector, we obtain:

$$\begin{aligned} Z_{(0,0)}(\tau, \bar{\tau}, u, \bar{u}, z, \bar{z}) &= \frac{1}{\sqrt{2}} \tau_2^{1/2} e^{-2\pi u_2^2/\tau_2 - 2\pi z_2^2/\tau_2} e^{4\pi i(u_1 z_2 + z_2 u_1)/\tau_2} \times \\ &\times \frac{|\eta(\tau)^2|^2}{|\vartheta_1(\tau, u)|^2} \frac{\Theta^{4,4}(\tau, \bar{\tau})}{|\eta(\tau)^4|^2} Z_{(0,0)}^{\text{fer,sum}}(\tau, u, z) \bar{Z}_{(0,0)}^{\text{fer,sum}}(\bar{\tau}, \bar{u}, \bar{z}), \end{aligned} \quad (3.11)$$

where $Z_{(0,0)}^{\text{fer,sum}}(\tau, u, z)$ denotes the sum over all the left-moving fermionic pieces of the theory, and is given by:

$$\begin{aligned} Z_{(0,0)}^{\text{fer,sum}}(\tau, u, z) &= \frac{1}{2} \frac{1}{\eta(\tau)^4} \left(\vartheta_{00}(\tau, u) \vartheta_{00}(\tau, z+u) \vartheta_{00}(\tau)^2 - \vartheta_{01}(\tau, u) \vartheta_{01}(\tau, z+u) \vartheta_{01}(\tau)^2 \right. \\ &\quad \left. - \vartheta_{10}(\tau, u) \vartheta_{10}(\tau, z+u) \vartheta_{10}(\tau)^2 - \vartheta_{11}(\tau, u) \vartheta_{11}(\tau, z+u) \vartheta_{11}(\tau)^2 \right), \\ &= \frac{1}{2} \frac{1}{\eta(\tau)^4} \vartheta_{11}(\tau, z/2)^2 \vartheta_{11}(\tau, z/2+u)^2. \end{aligned} \quad (3.12)$$

In going to the second line, we have used the Riemann identity R5 of [54]. Similarly, the right-movers evaluate to

$$\overline{Z}_{(0,0)}^{\text{fer,sum}}(\overline{\tau}, \overline{u}, \overline{z}) = \frac{1}{\eta(\tau)^4} \overline{\vartheta}_{11}(\tau, z/2 - u)^2 \overline{\vartheta}_{11}(\tau, z/2)^2. \quad (3.13)$$

Note that

$$Z_{(0,0)}^{\text{fer,sum}} \overline{Z}_{(0,0)}^{\text{fer,sum}} \sim z^2 \overline{z}^2 \quad \text{as } z \rightarrow 0. \quad (3.14)$$

The twisted sectors. In the twisted sector $(r, s) \neq (0, 0)$, we obtain:

$$\begin{aligned} Z_{(r,s)}(\tau, \overline{\tau}, u, \overline{u}, z, \overline{z}) &= 8\sqrt{2} \tau_2^{1/2} e^{-2\pi u_2^2/\tau_2 - 2\pi z_2^2/\tau_2} e^{4\pi i(u_1 z_2 + z_2 u_1)/\tau_2} \frac{|\eta(\tau)|^2}{|\vartheta_1(\tau, u)|^2} \times \\ &\times \left| \frac{\eta(\tau)^2}{\vartheta_{11}(\tau, (s+r\tau)/2)^2} \right|^2 Z_{(r,s)}^{\text{fer,sum}}(\tau, u, z) \overline{Z}_{(r,s)}^{\text{fer,sum}}(\overline{\tau}, \overline{u}, \overline{z}). \end{aligned} \quad (3.15)$$

The left-moving fermion partition functions involve a sum over the various fermion periodicities and in each case, a Riemann theta identity (see appendix B) allows us to sum them up into a product form. They are given by:

$$\begin{aligned} Z_{(0,1)}^{\text{fer,sum}}(\tau, u, z) &= \frac{1}{\eta(\tau)^4} \vartheta_{11}(\tau, z/2)^2 \vartheta_{10}(\tau, z/2 + u)^2, \\ Z_{(1,0)}^{\text{fer,sum}}(\tau, u, z) &= \frac{1}{\eta(\tau)^4} \vartheta_{11}(\tau, z/2)^2 \vartheta_{01}(\tau, z/2 + u)^2, \\ Z_{(1,1)}^{\text{fer,sum}}(\tau, u, z) &= \frac{1}{\eta(\tau)^4} \vartheta_{11}(\tau, z/2)^2 \vartheta_{00}(\tau, z/2 + u)^2. \end{aligned} \quad (3.16)$$

On the right-moving side, we get:

$$\begin{aligned} \overline{Z}_{(0,1)}^{\text{fer,sum}}(\overline{\tau}, \overline{u}, \overline{z}) &= \frac{1}{\eta(\tau)^4} \overline{\vartheta}_{11}(\tau, z/2 - u)^2 \overline{\vartheta}_{10}(\tau, z/2)^2, \\ \overline{Z}_{(1,0)}^{\text{fer,sum}}(\overline{\tau}, \overline{u}, \overline{z}) &= \frac{1}{\eta(\tau)^4} \overline{\vartheta}_{11}(\tau, z/2 - u)^2 \overline{\vartheta}_{01}(\tau, z/2)^2, \\ \overline{Z}_{(1,1)}^{\text{fer,sum}}(\overline{\tau}, \overline{u}, \overline{z}) &= \frac{1}{\eta(\tau)^4} \overline{\vartheta}_{11}(\tau, z/2 - u)^2 \overline{\vartheta}_{00}(\tau, z/2)^2. \end{aligned} \quad (3.17)$$

Note that

$$Z_{(r,s)}^{\text{fer,sum}} \overline{Z}_{(r,s)}^{\text{fer,sum}} \sim z^2 \quad \text{as } z \rightarrow 0. \quad (3.18)$$

3.4 Helicity supertrace

Now we are in a position to compute the helicity supertraces:

$$\chi_n(\tau) = \text{Tr } J_{sp}^n (-1)^{F_s} q^{L_0} = \int_{E(\tau)} \frac{du_1 du_2}{\tau_2} \left(\frac{1}{2\pi i} (\partial_z - \partial_{\overline{z}}) \right)^n Z(\tau, u, \overline{u}, z, \overline{z}) \Big|_{z=\overline{z}=0}, \quad (3.19)$$

with

$$Z(\tau, u, \overline{u}, z, \overline{z}) = \sum_{r,s=0,1} Z_{(r,s)}(\tau, \overline{\tau}, u, \overline{u}, z, \overline{z}). \quad (3.20)$$

From (3.14), it is clear that the untwisted partition function $Z_{(0,0)}$ (3.11)–(3.13) does not contribute to χ_0 and χ_2 , and the first non-vanishing result to which it contributes is χ_4 . This is consistent with the fact that the untwisted sector has the same number of fermion zero modes as the theory on T^4 . Similarly, it is clear from (3.18) that the twisted sector partition functions $Z_{(r,s)}$ (3.15)–(3.17) do not contribute to χ_0 but they do contribute to χ_2 .

The first non-vanishing result is thus χ_2 , and this receives contributions only from the sectors with $(r, s) \neq (0, 0)$:

$$\sum_{(r,s) \neq (0,0)} Z_{(r,s)}(\tau, \bar{\tau}, u, \bar{u}, z, \bar{z}) = 8\sqrt{2} \tau_2^{1/2} e^{-2\pi u_2^2/\tau_2 - 2\pi z_2^2/\tau_2} e^{4\pi i(u_1 z_2 + u_2 z_1)/\tau_2} \frac{1}{|\vartheta_1(\tau, u)|^2} \times \\ \times \vartheta_1(\tau, z/2)^2 \overline{\vartheta_1(\tau, z/2 - u)^2} \sum_{i=2,3,4} \frac{\vartheta_i(\tau, z/2 + u)^2}{\vartheta_i(\tau, 0)^2} \frac{\overline{\vartheta_i(\tau, z/2)^2}}{\vartheta_i(\tau, 0)^2}. \quad (3.21)$$

We have:

$$\left(\frac{1}{2\pi i}(\partial_z - \partial_{\bar{z}})\right)^2 Z(\tau, u, \bar{u}, z, \bar{z}) \Big|_{z=\bar{z}=0} = \left(\frac{1}{2\pi i}\partial_z\right)^2 Z(\tau, u, \bar{u}, z, \bar{z}) \Big|_{z=\bar{z}=0} \\ = 4\sqrt{2} \tau_2^{1/2} e^{-2\pi u_2^2/\tau_2} \frac{1}{|\vartheta_1(\tau, u)|^2} \eta(\tau)^6 \overline{\vartheta_1(\tau, u)^2} \sum_{i=2,3,4} \frac{\vartheta_i(\tau, z/2 + u)^2}{\vartheta_i(\tau, 0)^2}, \\ = 4\sqrt{2} \tau_2^{1/2} e^{-2\pi u_2^2/\tau_2} \frac{\eta(\tau)^6}{\vartheta_1(\tau, u)} \overline{\vartheta_1(\tau, u)} \sum_{i=2,3,4} \frac{\vartheta_i(\tau, u)^2}{\vartheta_i(\tau, 0)^2}. \quad (3.22)$$

Note that although we started with a full string theory with all the fermion periodicities, the spacetime computation is such that after summing over all the twisted sectors, the final answer only depends on⁹ the elliptic genus of $K3$

$$\mathcal{Z}^{\text{ell}}(K3; \tau, u) = 8 \sum_{i=2,3,4} \frac{\vartheta_i(\tau, u)^2}{\vartheta_i(\tau, 0)^2}. \quad (3.23)$$

We thus obtain our main result for the second helicity supertrace:

$$\chi_2(\tau) = \frac{1}{2} \int_{E(\tau)} \frac{du_1 du_2}{\tau_2} (2\tau_2)^{1/2} e^{-2\pi u_2^2/\tau_2} \frac{\eta(\tau)^6}{\vartheta_1(\tau, u)} \overline{\vartheta_1(\tau, u)} \mathcal{Z}^{\text{ell}}(K3; \tau, u). \quad (3.24)$$

It is useful to rewrite the integral (3.24) in the language of Jacobi forms. We first write down some notation and standard facts [55] that will be useful. A Jacobi form is a holomorphic function $\varphi(\tau, u)$ from $\mathbb{H} \times \mathbb{C}$ to \mathbb{C} which is “modular in τ and elliptic in u ” in the sense that it transforms under the modular group as

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{u}{c\tau + d}\right) = (c\tau + d)^k e^{\frac{2\pi i m c u^2}{c\tau + d}} \varphi(\tau, u) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2; \mathbb{Z}) \quad (3.25)$$

and under the translations of u by $\mathbb{Z}\tau + \mathbb{Z}$ as

$$\varphi(\tau, u + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda u)} \varphi(\tau, u) \quad \forall \lambda, \mu \in \mathbb{Z}, \quad (3.26)$$

⁹A similar phenomenon happens in the computation of helicity supertraces in flat space [34].

where k is an integer and m is a positive integer. We denote Jacobi forms of weight k and index m by $\varphi_{k,m}$. The ring of Jacobi forms of even weight is generated by the two Jacobi forms (our conventions for theta functions are given in appendix B).

$$\varphi_{-2,1}(\tau, u) = \frac{\vartheta_1(\tau, u)^2}{\eta(\tau)^6}, \quad \varphi_{0,1}(\tau, u) = 4 \sum_{i=2,3,4} \frac{\vartheta_i(\tau, u)^2}{\vartheta_i(\tau, 0)^2}. \quad (3.27)$$

The function

$$P(\tau, u) = \frac{\varphi_{0,1}(\tau, u)}{\varphi_{-2,1}(\tau, u)} = -\frac{3}{\pi^2} \wp(\tau, u) \quad (3.28)$$

with $\wp(\tau, u)$ the usual Weierstrass function is a Jacobi form of weight 2 and index 0, which implies that it is invariant under the elliptic transformations (3.26) of the Jacobi group. It has double poles of residue $-3/\pi^2$ at $z = 0$ and its translates by the lattice $\mathbb{Z}\tau + \mathbb{Z}$. We have chosen this normalization to streamline the notation here and in the manipulations of the integrals in appendix A. We also define the non-holomorphic function:

$$H(\tau, u) = (2\tau_2)^{1/2} e^{-2\pi u^2/\tau_2} |\vartheta_1(\tau, u)|^2, \quad (3.29)$$

which is invariant under the full Jacobi group as can be easily checked. We then have

$$\chi_2(\tau) = \int_{E(\tau)} \frac{du_1 du_2}{\tau_2} P(\tau, u) H(\tau, u). \quad (3.30)$$

This integral has been evaluated by Gaiotto and Zagier [56]. We present a brief analysis and a slightly different method of evaluation in appendix A. The result is:

$$\chi_2(\tau) = -\frac{1}{2} \eta(\tau)^3 \widehat{H}^{(2)}(\tau). \quad (3.31)$$

where $\widehat{H}^{(2)}(\tau)$ is the modular completion of a mock modular form discussed in the introduction. We discuss the notion of the modular completion in the following section.

4 The mock theta function $H^{(2)}(\tau)$ and the twisted BPS index

In this section we give a quick summary of the definition of mock modular forms and of the mock theta function which appeared in the previous section in the computation of the BPS index $\chi_2(\tau)$. We then consider the effects of twisting.

A holomorphic function $h(\tau)$ on the upper half plane \mathbb{H} is called a weakly holomorphic mock modular form of weight k for $\Gamma_1 = SL_2(\mathbb{Z})$ if it has at most exponential growth as $\tau \rightarrow i\infty$ and if there exists a modular form $g(\tau)$ of weight $k - 2$ on Γ_1 such that the completion of $h(\tau)$ given by

$$\widehat{h}(\tau) = h(\tau) + (4i)^{k-1} \int_{-\bar{\tau}}^{\infty} (z + \tau)^{-k} \overline{g(-\bar{z})} dz \quad (4.1)$$

transforms like a holomorphic modular form of weight k on Γ_1 with some multiplier system ν . The modular form $g(\tau)$ is called the shadow of the mock modular form $h(\tau)$. The completion \widehat{h} obeys

$$4i (\tau_2/2)^k \frac{\partial \widehat{h}(\tau)}{\partial \bar{\tau}} = \overline{g(\tau)}. \quad (4.2)$$

When the shadow g is a unary theta series of weight $1/2$ or $3/2$, then the mock modular form h is called a mock theta function of weight $3/2$ or $1/2$, respectively.

The example appearing in this paper is the mock theta function $H^{(2)}(\tau)$ which appeared in the physics literature in the decomposition of the elliptic genus of $K3$ in terms of characters of the $N = 4$ superconformal algebra [5]. It can also be defined as follows [1]. Let

$$F_2^{(2)}(\tau) = \sum_{\substack{r>s>0 \\ r-s \text{ odd}}} (-1)^r s q^{rs/2} = q + q^2 - q^3 + q^4 - q^5 + \dots \quad (4.3)$$

Then the function $H^{(2)}$ and its Fourier coefficients $c^{(2)}$ are defined by:

$$H^{(2)}(\tau) = \frac{48F_2^{(2)}(\tau) - 2E_2(\tau)}{\eta(\tau)^3} = \sum_{n=0}^{\infty} c^{(2)}(8n-1) q^{n-1/8} \quad (4.4)$$

$$= 2q^{-1/8} \left(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \dots \right) \quad (4.5)$$

where $E_2(\tau)$ is the usual Eisenstein series and $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ is the Dedekind eta function. $H^{(2)}(\tau)$ is a weight $1/2$ mock modular form with shadow $24\eta(\tau)^3$ and a multiplier system conjugate to that of $\eta(\tau)^3$. Evaluating the integral in (4.1) gives an explicit formula for the completion

$$\widehat{H}^{(2)}(\tau) = H^{(2)}(\tau) + 24 \sum_{k \in \mathbb{Z}} \text{sgn}(4k+1) q^{-(4k+1)^2/8} \left(-1 + \text{Erf} \left[\frac{4k+1}{2} \sqrt{2\pi\tau_2} \right] \right) \quad (4.6)$$

with τ_2 the imaginary part of τ and $\text{Erf}[x]$ the error function.

The first few coefficients $c^{(2)}(8n-1)$ in the q -expansion of $H^{(2)}(\tau)$ are dimensions of irreducible representations of the Mathieu group M_{24} [5]. It is natural to think that each coefficient $c^{(2)}(8n-1)$ should be identified with the dimension of an M_{24} module K_n so that $c^{(2)}(8n-1) = \dim K_n = \text{Tr}_{K_n} 1$. This idea by itself is ambiguous because there are many possible decompositions of the coefficients into dimensions of irreducible representations (irreps) of M_{24} . To test the idea one follows the same logic as in the computation of the McKay-Thompson series of Monstrous Moonshine [6, 15] and studies the series $H_g^{(2)}(\tau) = \sum_n \text{Tr}_{K_n} g q^{n-1/8}$ for $g \in M_{24}$. These McKay-Thompson series depend only on the conjugacy class of g , and if for each conjugacy class the $H_g^{(2)}(\tau)$ are also mock modular forms this is interpreted as positive evidence for a correct choice of decomposition into irreps as well as for a moonshine connection between the mock modular form $H^{(2)}(\tau)$ and the finite simple group M_{24} . This strategy has been used in [7–10] to compute the mock modular forms $H_g^{(2)}(\tau)$ for all conjugacy classes of M_{24} and thus determine the decomposition of the coefficients $c^{(2)}(8n-1)$ into irreps of M_{24} . Using the notation of the review [11] the resulting mock modular forms can be written in the form

$$H_g^{(2)}(\tau) = \frac{\chi(g)}{24} H^{(2)}(\tau) - \frac{\widetilde{T}_g(\tau)}{\eta(\tau)^3} \quad (4.7)$$

where $\chi(g)$ is the character of g in the 24-dimensional permutation representation of M_{24} with a decomposition $24 = 23 \oplus 1$ in terms of irreps. Here the $\widetilde{T}_g(\tau)$ are a set of weight

two modular forms for congruence subgroups which can be found tabulated in [11] and the $H_g^{(2)}(\tau)$ are weight 1/2 mock modular forms for $\Gamma_0(N_g)$ with shadow $\chi(g)\eta(\tau)^3$. The number N_g is an integer known as the level of g and determined by the cycle shape of g in the 24-dimensional permutation representation of M_{24} . See the review [11] for details.

At special points in the moduli space of $K3$, one has a SCFT description of the $K3$ surface. At such points, all the discrete symmetries of $K3$ that preserve supersymmetry can be classified [22]. This list includes and extends the symplectic automorphisms of the $K3$ surface that were classified by Mukai [20] and by Kondo [21], but does not include all elements of M_{24} . For elements $g \in M_{24}$ that are within this class, one has a somewhat better understanding of the McKay-Thompson series (4.7). Using the SCFT description, one can compute a twisted version of the elliptic genus:

$$\mathcal{Z}_g^{\text{ell}}(K3; \tau, u) = \text{Tr}_{RR} (-1)^F g q^{L_0 - c/24} \tilde{q}^{\tilde{L}_0 - \tilde{c}/24} \zeta^{J_0}, \tag{4.8}$$

where the trace is over the RR sector of the Hilbert space of the $K3$ SCFT. The twisted elliptic genus $\mathcal{Z}_g^{\text{ell}}(K3; \tau, u)$ is also a Jacobi form over a subgroup of the full Jacobi group. Using this fact, one can decompose it into the basis elements (3.27) with coefficients being modular forms on a subgroup of $\text{SL}(2, \mathbb{Z})$. For all elements g for which the twisted elliptic genus has been computed, one finds [11]:

$$\mathcal{Z}_g^{\text{ell}}(K3; \tau, u) = \frac{\chi(g)}{24} \mathcal{Z}^{\text{ell}}(K3; \tau, u) + \tilde{T}_g(\tau) \varphi_{-2,1}(\tau, u). \tag{4.9}$$

On decomposing the twisted elliptic genus into characters of the $N = 4$ superconformal algebra and throwing out the massless representation as before, one obtains the McKay-Thompson series $H_g^{(2)}(\tau)$.

From the point of view of this paper, the NS5-brane system naturally produces the McKay-Thompson series $H_g^{(2)}(\tau)$. The integral in eq. (3.24) that gives us a map from the elliptic genus of $K3$ to the completion of the weight two mixed mock modular form $-(1/2)\eta(\tau)^3 H^{(2)}(\tau)$ that can be obviously generalized to a map from the twisted form of the elliptic genus given in eq. (4.9) to a twisted version of the completion. We can check that this correctly leads to the twisted mock modular form $H_g^{(2)}(\tau)$ as follows. We define

$$\chi_{2,g}(\tau) = \int_{E(\tau)} P_g(\tau, u) H(\tau, u) \frac{du_1 du_2}{\tau_2} \tag{4.10}$$

with

$$P_g(\tau, u) = \frac{1}{2} \frac{\mathcal{Z}_g^{\text{ell}}(K3; \tau, u)}{\varphi_{-2,1}(\tau, u)}. \tag{4.11}$$

Then using eq. (4.9) and eq. (4.7) as well as the integral in eq. (A.15) of appendix A we find

$$\chi_{2,g} = -\frac{1}{2} \eta(\tau)^3 \widehat{H}_g^{(2)}(\tau) \tag{4.12}$$

where $\widehat{H}_g^{(2)}(\tau)$ is the completion of $H_g^{(2)}(\tau)$. Thus the map from twisted elliptic genera to twisted mock modular forms provided by the integral in eq. (3.24) agrees with the map

given by the decomposition of the twisted elliptic genus into characters of the $N = 4$ superconformal algebra.

Further, the superstring computation in section 3 that led to the integral in eq. (3.24) can itself be generalised to include the twist g . If g is a symmetry of the $K3$ SCFT that preserves the worldsheet supersymmetry, then it can be lifted to a corresponding symmetry of the superstring theory discussed in section 2, and we can compute

$$\chi_{2,g}(\tau) = \text{Tr} (J_{\text{sp}})^2 (-1)^{F_s} g q^{L_0 - c/24} \tilde{q}^{\tilde{L}_0 - \tilde{c}/24} \tag{4.13}$$

in this superstring theory. The sum over NS and R sectors with the insertion of the GSO projection for the twisted superstring index collapses as before in such a way that the final answer only depends on the twisted SCFT elliptic genus (4.8). The main technical point here is that the sum over NS and R sectors with the GSO projection involves eight free worldsheet fermions, and the Riemann identities used in section 3.4 to sum the various expressions are a manifestation of spacetime supersymmetry, as is the case for superstring theory in 10 flat dimensions. We can identify the spinorial charges that the spacetime supercharges have under the various rotational symmetries of the theory, but we have not explicitly constructed the Green-Schwarz superstring for the cigar theory (see [45] for some more discussion of this subject).

More generally, since we are working at the level of superconformal field theory, we can consider automorphisms of the full superconformal field theory (2.8) which preserve spacetime supersymmetry. These transformations certainly include such symmetries of the $K3$ component of our superconformal field theory as were analyzed in [22]. The full extension of this classification to the superconformal field theories considered here is a very interesting problem that we hope to return to in the future.

5 Discussion and conclusions

As mentioned in the introduction, our goal in this paper was to find a BPS state counting problem in string theory that leads to the mock modular form $H^{(2)}(\tau)$ (or its modular completion) and we suggested that the required construction would remove the massless string states from the spectrum. The two NS5-brane system on $K3 \times S^1$ achieves what we want in a natural manner, but the connection to our earlier discussion may not be completely clear so here we make some further remarks on our interpretation of the calculation performed in this paper.

The $K3$ elliptic genus can be written in terms of Jacobi forms as (see e.g [58] or eq. (7.39) of [1])

$$\mathcal{Z}^{ell}(K3, \tau, u) = 2\varphi_{0,1}(\tau, u) = -24\mu(\tau, u)\eta(\tau)^3\varphi_{-2,1}(\tau, u) - \eta(\tau)^3H^{(2)}(\tau)\varphi_{-2,1}(\tau, u) \tag{5.1}$$

with

$$\mu(\tau, u) = \frac{e^{\pi i u}}{\vartheta_1(\tau, u)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{(n^2+n)/2} e^{2\pi i n u}}{1 - q^n e^{2\pi i u}}. \tag{5.2}$$

The first term on the right hand side of (5.1) is related to a massless character of the world sheet $N = 4$ superconformal algebra and from a spacetime point of view encodes the massless graviton degree of freedom and its descendants. These modes have wave functions that are delocalized along the length of the cigar. In comparison, the second term corresponds to massive modes that are localized near the tip of the cigar. The holomorphic mock modular form $H^{(2)}$ counts the localized modes (up to a factor of $-(1/2)\eta(\tau)^3$), while the delocalized modes contribute to the non-holomorphic part of the full BPS index $-(1/2)\eta(\tau)^3\hat{H}^{(2)}$. The 5-brane background and the process of taking the near horizon limit has in a sense removed some of the massless modes associated to the first term in (5.1) which give a holomorphic term such that the sum in (5.1) is modular, and replaced them by a set of delocalized modes which give a non-holomorphic contribution which also leads to a modular answer.

Naively we would expect the BPS state counting formula to be holomorphic based on the argument of pairing of bosonic and fermionic modes while the answer we obtain is clearly not holomorphic. The resolution of this puzzle arises from recent studies of non-compact SCFTs in which such a phenomenon has been unravelled [2–4]. The point is that the non-compactness requires us to specify normalizability conditions for all the modes in the spectrum, and supersymmetry does not commute with these conditions. Note that the form of the spacetime supercharges that we write down in eq. (2.5) are only valid in the asymptotic region of the cigar, and their exact form is more complicated. From a technical point of view, the non-compactness produces a continuum and an associated density of states of bosons and fermions that are not equal. The difference in the density of states is proportional to the reflection coefficient of a wave sent down the throat of the cigar [2].

A notion of holography exists for the theory of NS fivebranes in string theory [59, 60]. From this point of view, we expect that the BPS states studied in this paper are related to the BPS states of the non-gravitational low-energy theory of the fluctuations of the fivebranes wrapped on $K3$. Theories of fivebranes in M-theory wrapping various two and four dimensional surfaces have generated great interest in the last few years following the work of [61] and it would be very interesting to make this relation precise.

If we had not taken the near-horizon limit of the NS5-brane, but instead looked for bound states of NS5-branes with fundamental strings carrying momentum, we would have obtained a BPS three charge black hole with a macroscopic horizon size in five dimensional asymptotically flat space. It would be very interesting to understand the relation of these “big” black holes to the counting problem we have analyzed and thus possibly to moonshine. Three charge BPS black holes in five dimensions are also closely related to four-charge black holes in four dimensions that exhibit the wall-crossing phenomenon. Mathematically they are described by a family of mock modular forms [1] that are a priori unrelated to the mock modular form that we study in this paper. It would be interesting to find relations between the mock modular forms appearing in these two counting problems.

Finally, there are several obvious generalizations of the present work that we hope to return to in the near future. One of these is the extension of our analysis to an arbitrary number of fivebranes. It would be particularly interesting to see if there is any connection between the ADE classification of fivebranes and the ADE classification which appears in the analysis of umbral moonshine [25]. Another promising direction involves the computa-

tion of the BPS index for CHL models constructed as $(K3 \times S^1)/(\mathbb{Z}/n)$ where \mathbb{Z}/n acts as an order n shift on the S^1 and as an order n symplectic automorphism of $K3$. Finally, it would be interesting to analyze the full group of supersymmetry preserving automorphisms for the BPS configuration analyzed here and its generalization to CHL models and arbitrary numbers of fivebranes.

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A Analysis of the integral for the second helicity supertrace χ_2

In this appendix, we analyze and evaluate the integral (3.30) that gives the second helicity supertrace. In terms of the functions

$$P(\tau, u) = \frac{\varphi_{0,1}(\tau, u)}{\varphi_{-2,1}(\tau, u)} = -\frac{3}{\pi^2} \wp(\tau, u) \tag{A.1}$$

and

$$H(\tau, u) = (2\tau_2)^{1/2} e^{-2\pi u^2/\tau_2} |\vartheta_1(\tau, u)|^2, \tag{A.2}$$

the integral is written as:

$$\chi_2(\tau) = \int_{E(\tau)} P(\tau, u) H(\tau, u) \frac{du_1 du_2}{\tau_2}. \tag{A.3}$$

where $E(\tau)$ is the elliptic curve $\mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z})$. We use the notation $q = e^{2\pi i\tau}$, $y = e^{2\pi iu}$.

On the right-hand side of this equation, the integrand, the integration region, and the measure are all manifestly invariant under the elliptic transformations. Further, the integrand is a (meromorphic) Jacobi form of weight 2. If the integral is well-defined, it is thus manifest that the function $\chi_2(\tau)$ transforms as a holomorphic modular form of weight $k = 2$. We say “transforms as”, and not “is” a holomorphic modular form because, as we shall see below, the function χ_2 is not holomorphic in τ , it is the non-holomorphic completion of a (mixed) mock modular form.

We now show that the integral (A.3) is well defined. The only possible problems come from the behavior as u approaches $0, 1, \tau, \tau + 1$. To analyze the behavior near these points we cut out a pizza slice of radius $\varepsilon \ll 1$ around each of these points in $E(\tau)$ so that $E(\tau) = E^\varepsilon + D^\varepsilon$ and then study the limit $\varepsilon \rightarrow 0$. Here E^ε is the “ticket-shaped” region obtained by removing the pizza slices from $E(\tau)$ and D^ε is the disc of radius ε formed by assembling the four slices into a single disc of radius ε at the origin using the elliptic invariance of the integrand.

Now consider the integral over the disc D^ε . As $u \rightarrow 0$ we have $P(\tau, u) \sim u^{-2}$ and $\vartheta_1(\tau, u) \sim u$. Therefore the only potentially problematic part of the integrand is

$$\int_{D^\varepsilon} e^{-2\pi u_2^2/\tau_2} \frac{\bar{u}}{u} du_1 du_2. \tag{A.4}$$

Using polar coordinates $u = \rho e^{i\theta}$ this becomes

$$\int_0^{2\pi} d\theta \int_0^\varepsilon \rho d\rho e^{-2\pi\rho^2 \sin^2 \theta/\tau_2} e^{-2i\theta} \tag{A.5}$$

$$= -\frac{\tau_2}{4\pi} \int_0^{2\pi} d\theta \frac{e^{-2i\theta}}{\sin^2 \theta} (1 - e^{-2\pi\varepsilon^2 \sin^2 \theta/\tau_2}) = \frac{\pi^2}{4\tau_2} \varepsilon^4 + O(\varepsilon^6). \tag{A.6}$$

Since this vanishes as $\varepsilon \rightarrow 0$ we can simply define the integral as

$$\chi_2(\tau) = \lim_{\varepsilon \rightarrow 0} \int_{E^\varepsilon} H(\tau, u) P(\tau, u) \frac{du_1 du_2}{\tau_2}. \tag{A.7}$$

Since $P(\tau, u)$ is analytic in the region E^ε we can safely set $\partial_{\bar{u}} P(\tau, u) = 0$ inside the integral in the manipulations below.

We now compute the $\bar{\tau}$ derivative of the function χ_2 . By a change of variables $u = a\tau + b$, we have:

$$\chi_2(\tau) = \int_0^1 \int_0^1 H(\tau, a\tau + b) P(\tau, a\tau + b) db da. \tag{A.8}$$

Since P is meromorphic in τ , the only local $\bar{\tau}$ dependence comes from the function H . We have:

$$\partial_{\bar{\tau}} \chi_2(\tau) = \int_0^1 \int_0^1 (\partial_{\bar{\tau}} H(\tau, a\tau + b)) P(\tau, a\tau + b) db da. \tag{A.9}$$

One can check that:

$$\partial_{\bar{\tau}} H(\tau, a, b) \equiv \partial_{\bar{\tau}} H(\tau, a\tau + b) = \left(\frac{i}{4\pi} \partial_{\bar{u}}^2 H(\tau, u) \right)_{u=a\tau+b}. \tag{A.10}$$

Plugging (A.10) into (A.9), and changing variables to $u = u_1 + iu_2, \bar{u} = u_1 - iu_2$, we obtain:

$$\begin{aligned} \partial_{\bar{\tau}} \chi_2(\tau) &= \frac{1}{8\pi} \int_{E^\varepsilon} (\partial_{\bar{u}}^2 H(\tau, u)) P(\tau, u) \frac{d\bar{u} du}{\tau_2} \\ &= \frac{1}{8\pi} \int_{E^\varepsilon} \partial_{\bar{u}} \left(\partial_{\bar{u}} H(\tau, u) P(\tau, u) \frac{1}{\tau_2} \right) d\bar{u} du \\ &= \frac{1}{8\pi} \oint_{\partial E^\varepsilon} \partial_{\bar{u}} H(\tau, u) P(\tau, u) \frac{1}{\tau_2} du. \end{aligned} \tag{A.11}$$

The integral along the four straight edges of E^ε adds up to zero since we go around the opposite straight edges in opposite directions, and the integrand is equal by elliptic invariance. Therefore we have:

$$\partial_{\bar{\tau}} \chi_2(\tau) = -\frac{1}{8\pi} \oint_{\partial D^\varepsilon} \partial_{\bar{u}} H(\tau, u) P(\tau, u) \frac{du}{\tau_2}$$

$$\begin{aligned}
 &= -\sqrt{2} \frac{1}{8\pi\tau_2} (2\pi i) \operatorname{Res}_{u \rightarrow 0} \left(\partial_{\bar{u}} H(\tau, u) \frac{1}{u^2} \right) \\
 &= -i\sqrt{2} \frac{1}{4\tau_2} \left(\partial_u \partial_{\bar{u}} H(\tau, u) \right)_{u=0} = i\sqrt{2} \pi^2 \tau_2^{-1/2} \eta(\tau)^3 \overline{\eta(\tau)^3}. \quad (\text{A.12})
 \end{aligned}$$

The function χ_2/η^3 transforms as a holomorphic modular form of weight $k = 1/2$ and the above shows that it obeys the holomorphic anomaly equation:

$$\frac{1}{i\sqrt{2}\pi^2} \tau_2^{1/2} \partial_\tau \frac{\chi_2(\tau)}{\eta(\tau)^3} = \overline{\eta(\tau)^3}. \quad (\text{A.13})$$

In other words, χ_2/η^3 is a mock modular form of weight $k = 1/2$ and shadow $-12\eta^3$.

Following [56] we now evaluate this integral and find

$$\chi_2(\tau) = -\frac{1}{2} \eta(\tau)^3 \widehat{H}^{(2)}(\tau). \quad (\text{A.14})$$

As a first step towards this result we show that

$$I^{(2)}(\tau) = \int_{E(\tau)} H(\tau, u) \frac{du_1 du_2}{\tau_2} = 1. \quad (\text{A.15})$$

We use the expansion

$$|\vartheta_1(\tau, u)|^2 = \sum_{n, m \in \mathbb{Z}} q^{(n+1/2)^2/2} \bar{q}^{(m+1/2)^2/2} e^{2\pi i[(n+1/2)(u+1/2) - (m+1/2)(\bar{u}+1/2)]} \quad (\text{A.16})$$

and write τ, u in terms of real and imaginary parts $\tau = \tau_1 + i\tau_2$, $u = u_1 + iu_2$ to give

$$|\vartheta_1(\tau, u)|^2 = \sum_{n, m \in \mathbb{Z}} \operatorname{Exp} \left[2\pi i \left(\frac{\tau_1}{2} (n^2 - m^2 + n - m) + i \frac{\tau_2}{2} \left(n^2 + m^2 + n + m + \frac{1}{2} \right) \right) \right] \quad (\text{A.17})$$

$$+ u_1(n - m) + iu_2(n + m + 1) + \frac{n - m}{2} \Big]. \quad (\text{A.18})$$

Now change variables from (u_1, u_2) to (a, b) with $u = a\tau + b$. The Jacobian gives a factor of τ_2 and the only term involving b is

$$\int_0^1 db e^{2\pi i b(n-m)} = \delta_{n,m} \quad (\text{A.19})$$

so we are left with the integral

$$I^{(2)}(\tau) = \sqrt{2\tau_2} \int_0^1 da \sum_{n \in \mathbb{Z}} \operatorname{Exp}[-2\pi a^2 \tau_2 - \pi \tau_2 (2n^2 + 2n + 1/2) - 2\pi a \tau_2 (2n + 1)] \quad (\text{A.20})$$

$$= \sqrt{2\tau_2} \int_0^1 da \sum_{n \in \mathbb{Z}} \operatorname{Exp}[-2\pi \tau_2 (a + n + 1/2)^2] \quad (\text{A.21})$$

$$= \sqrt{2\tau_2} \sum_{n \in \mathbb{Z}} \int_n^{n+1} da_n \operatorname{Exp}[-2\pi \tau_2 (a_n + 1/2)^2] \quad (\text{A.22})$$

$$= \sqrt{2\tau_2} \int_{-\infty}^{+\infty} dx \text{Exp}[-2\pi\tau_2(x + 1/2)^2] = 1 \tag{A.23}$$

where we changed variables to $a_n = a + n$ to convert the integral of the sum to a sum of integrals over the interval $[n, n + 1]$.

We now move on to the evaluation of the integral (A.7). We first use the identity (see for example [58] or eq. (7.39) of [1])

$$P(\tau, u) = \frac{\varphi_{0,1}(\tau, u)}{\varphi_{-2,1}(\tau, u)} = -12\mu(\tau, u)\eta^3(\tau) - \frac{1}{2}\eta^3(\tau)H^{(2)}(\tau) \tag{A.24}$$

with

$$\mu(\tau, u) = \frac{e^{\pi i u}}{\vartheta_1(\tau, u)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{(n^2+n)/2} e^{2\pi i n u}}{1 - q^n e^{2\pi i u}}. \tag{A.25}$$

Substituting this into the integrand gives

$$\chi_2(\tau) = -12\eta^3(\tau) \int_{E(\tau)} H(\tau, u) \mu(\tau, u) \frac{du_1 du_2}{\tau_2} - \frac{1}{2}\eta^3(\tau)H^{(2)}(\tau) \int_{T(\tau)} H(\tau, u) \frac{du_1 du_2}{\tau_2} \tag{A.26}$$

which using the earlier result for $I^{(2)}$ gives

$$\chi_2(\tau) = -\frac{1}{2}\eta^3(\tau) \left(H^{(2)}(\tau) + 24 \int_{E(\tau)} H(\tau, u) \mu(\tau, u) \frac{du_1 du_2}{\tau_2} \right). \tag{A.27}$$

To evaluate the remaining integral first note that the factors of ϑ_1 cancel out so that

$$H(\tau, u) \mu(\tau, u) = (2\tau_2)^{1/2} e^{-2\pi u_2^2/\tau_2} \overline{\vartheta_1(\tau, u)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{(n^2+n)/2} e^{2\pi i(n+1/2)u}}{1 - q^n e^{2\pi i u}}. \tag{A.28}$$

Now $|q^n y| = e^{-2\pi n \tau_2} e^{-2\pi u_2}$. Using modular invariance we can choose τ to be in the usual fundamental domain of $SL(2, \mathbb{Z})$ so that $\tau_2 \geq \sqrt{3}/2$ and since $u \in E(\tau)$ we have $0 \leq u_2 \leq \tau_2$. Now we have a wall-crossing type phenomenon. For $n \geq 0$ we expand¹⁰

$$\frac{1}{1 - q^n y} = \sum_{k=0}^{\infty} q^{nk} y^k, \tag{A.29}$$

while for $n \leq -1$ we can write

$$\frac{1}{1 - q^n y} = -\frac{q^{-n} y^{-1}}{1 - q^{-n} y^{-1}} = -\sum_{k=0}^{\infty} q^{-n(k+1)} y^{-(k+1)}. \tag{A.30}$$

This then gives us

$$H(\tau, u) \mu(\tau, u) = (2\tau_2)^{1/2} e^{-2\pi u_2^2/\tau_2} \overline{\vartheta_1(\tau, u)} \left(\mathcal{S}_< + \mathcal{S}_> \right) \tag{A.31}$$

¹⁰For $n \geq 1$ and $n \leq -2$ the expansions in eq. (A.29) and eq. (A.30) are correct since $|q^n y| < 1$ for all q, y . They can be extended to $n \geq 0$ and $n \leq -1$ because the contribution from the boundary where $|q^n y| = 1$ vanishes due to the prefactor in eq. (A.28).

where

$$\mathcal{S}_< = - \sum_{n=-\infty}^{-1} \sum_{k=0}^{\infty} (-1)^n q^{\frac{n^2-n-2nk}{2}} y^{n-k-1/2}, \quad (\text{A.32})$$

$$\mathcal{S}_> = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^n q^{\frac{n^2+n+2nk}{2}} y^{n+k+1/2}. \quad (\text{A.33})$$

Writing $\overline{\vartheta}_1$ as the sum

$$\overline{\vartheta}_1(\tau, u) = \sum_{m \in \mathbb{Z}} \overline{q}^{\frac{(m+1/2)^2}{2}} \overline{y}^{m+1/2} (-1)^m \quad (\text{A.34})$$

gives us an expression for the integral of $H\mu$ which has two terms:

$$I_< = - \int_{E(\tau)} \frac{du_1 du_2}{\tau_2} \left((2\tau_2)^{1/2} e^{-2\pi u_2^2/\tau_2} \sum_{m \in \mathbb{Z}} \sum_{n=-\infty}^{-1} \sum_{k=0}^{\infty} (-1)^{n+m} q^{\frac{n^2-n-2nk}{2}} \overline{q}^{\frac{m^2+m+1/4}{2}} y^{n-k-1/2} \overline{y}^{m+1/2} \right), \quad (\text{A.35})$$

$$I_> = \int_{E(\tau)} \frac{du_1 du_2}{\tau_2} \left((2\tau_2)^{1/2} e^{-2\pi u_2^2/\tau_2} \sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n+m} q^{\frac{n^2+n+2nk}{2}} \overline{q}^{\frac{m^2+m+1/4}{2}} y^{n+k+1/2} \overline{y}^{m+1/2} \right). \quad (\text{A.36})$$

Let's consider $I_>$ first. Changing variables via $u = a\tau + b$ the integral over b gives $\delta_{n+k,m}$ and we are left after some simplifications with

$$I_> = (2\tau_2)^{1/2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k (q\overline{q})^{\frac{n^2+n+2nk}{2}} \overline{q}^{\frac{(k+1/2)^2}{2}} \int_0^1 da e^{-2\pi a^2 \tau_2} e^{-4\pi a \tau_2 (n+k+1/2)}. \quad (\text{A.37})$$

Similarly we find

$$I_< = (2\tau_2)^{1/2} \sum_{n=-\infty}^{-1} \sum_{k=0}^{\infty} (-1)^k (q\overline{q})^{\frac{n^2-n-2nk}{2}} \overline{q}^{\frac{(k+1/2)^2}{2}} \int_0^1 da e^{-2\pi a^2 \tau_2} e^{-4\pi a \tau_2 (n-k-1/2)}. \quad (\text{A.38})$$

We can rewrite these expressions in the form

$$I_> = (2\tau_2)^{1/2} \sum_{k=0}^{\infty} (-1)^k q^{-\frac{(k+1/2)^2}{2}} \sum_{n=0}^{\infty} \int_0^1 da e^{-2\pi \tau_2 (a+n+k+1/2)^2}, \quad (\text{A.39})$$

$$I_< = (2\tau_2)^{1/2} \sum_{k=0}^{\infty} (-1)^k q^{-\frac{(k+1/2)^2}{2}} \sum_{n=-\infty}^{-1} \int_0^1 da e^{-2\pi \tau_2 (a+n-k-1/2)^2}. \quad (\text{A.40})$$

Changing variables as before leads to

$$I_< + I_> = (2\tau_2)^{1/2} \sum_{k=0}^{\infty} (-1)^k q^{-\frac{(2k+1)^2}{8}} \left(\int_{-\infty}^{+\infty} e^{-2\pi \tau_2 x^2} - \int_{-k-1/2}^{k+1/2} e^{-2\pi \tau_2 x^2} dx \right) \quad (\text{A.41})$$

$$= \sum_{k=0}^{\infty} (-1)^k q^{-\frac{(2k+1)^2}{8}} \left(1 - \text{Erf} \left[\frac{1+2k}{2} \sqrt{2\pi \tau_2} \right] \right). \quad (\text{A.42})$$

So finally we find after substitution into eq. (A.27)

$$\chi_2(\tau) = -\frac{1}{2} \eta^3(\tau) \left(H^{(2)}(\tau) + 24 \sum_{k=0}^{\infty} (-1)^k q^{-\frac{(2k+1)^2}{8}} \left(1 - \text{Erf} \left[\frac{1+2k}{2} \sqrt{2\pi \tau_2} \right] \right) \right) \quad (\text{A.43})$$

$$= -\frac{1}{2} \eta^3(\tau) \widehat{H}^{(2)}(\tau). \quad (\text{A.44})$$

B Theta function conventions and Riemann identities

The classical Jacobi theta functions are (with $q = e^{2\pi i\tau}$, $\zeta = e^{2\pi iz}$)

$$\begin{aligned} \vartheta_{00}(\tau, z) &= \vartheta_3(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 + \zeta q^{n-1/2})(1 + \zeta^{-1} q^{n-1/2}) \\ &= \sum_{m \in \mathbb{Z}} q^{m^2/2} \zeta^m, \end{aligned} \tag{B.1}$$

$$\begin{aligned} \vartheta_{01}(\tau, z) &= \vartheta_4(\tau, z) = \prod_{n=1}^{\infty} (1 - q^n)(1 - \zeta q^{n-1/2})(1 - \zeta^{-1} q^{n-1/2}) \\ &= \sum_{m \in \mathbb{Z}} e^{\pi i m} q^{m^2/2} \zeta^m, \end{aligned} \tag{B.2}$$

$$\begin{aligned} \vartheta_{10}(\tau, z) &= \vartheta_2(\tau, z) = q^{1/8} \zeta^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 + \zeta q^n)(1 + \zeta^{-1} q^{n-1}) \\ &= \sum_{m \in \mathbb{Z}} q^{(m+1/2)^2/2} \zeta^{m+\frac{1}{2}}, \end{aligned} \tag{B.3}$$

$$\begin{aligned} \vartheta_{11}(\tau, z) &= -i\vartheta_1(\tau, z) = -q^{1/8} \zeta^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 - \zeta q^n)(1 - \zeta^{-1} q^{n-1}) \\ &= \sum_{m \in \mathbb{Z}} e^{\pi i(m+\frac{1}{2})} q^{(m+1/2)^2/2} \zeta^{m+\frac{1}{2}}. \end{aligned} \tag{B.4}$$

The conventions for $\vartheta_{00}, \vartheta_{01}, \vartheta_{10}, \vartheta_{11}$ agree with [54] and the conventions for ϑ_i , $i = 1, 2, 3, 4$ agree with the appendix of [24] Also the above convention for ϑ_{11} differs from [57].

Write $\vartheta_{ab}(x) \equiv \vartheta_{ab}(\tau, x)$ and let

$$\begin{aligned} x_1 &= \frac{1}{2}(x + y + u + v), & y_1 &= \frac{1}{2}(x + y - u - v), \\ u_1 &= \frac{1}{2}(x - y + u - v), & v_1 &= \frac{1}{2}(x - y - u + v). \end{aligned} \tag{B.5}$$

Then we have the following Riemann theta relations, taken from [54].

$$\begin{aligned} (R5) &: \vartheta_{00}\vartheta_{00}\vartheta_{00}\vartheta_{00} - \vartheta_{01}\vartheta_{01}\vartheta_{01}\vartheta_{01} - \vartheta_{10}\vartheta_{10}\vartheta_{10}\vartheta_{10} + \vartheta_{11}\vartheta_{11}\vartheta_{11}\vartheta_{11} = 2\vartheta_{11}\vartheta_{11}\vartheta_{11}\vartheta_{11}, \\ (R9) &: \vartheta_{00}\vartheta_{00}\vartheta_{01}\vartheta_{01} - \vartheta_{01}\vartheta_{01}\vartheta_{00}\vartheta_{00} - \vartheta_{10}\vartheta_{10}\vartheta_{11}\vartheta_{11} + \vartheta_{11}\vartheta_{11}\vartheta_{10}\vartheta_{10} = -2\vartheta_{10}\vartheta_{10}\vartheta_{11}\vartheta_{11}, \\ (R11) &: \vartheta_{00}\vartheta_{00}\vartheta_{10}\vartheta_{10} + \vartheta_{01}\vartheta_{01}\vartheta_{11}\vartheta_{11} - \vartheta_{10}\vartheta_{10}\vartheta_{00}\vartheta_{00} - \vartheta_{11}\vartheta_{11}\vartheta_{01}\vartheta_{01} = 2\vartheta_{01}\vartheta_{01}\vartheta_{11}\vartheta_{11}, \\ (R15) &: \vartheta_{00}\vartheta_{00}\vartheta_{11}\vartheta_{11} + \vartheta_{01}\vartheta_{01}\vartheta_{10}\vartheta_{10} - \vartheta_{10}\vartheta_{10}\vartheta_{01}\vartheta_{01} - \vartheta_{11}\vartheta_{11}\vartheta_{00}\vartheta_{00} = 2\vartheta_{00}\vartheta_{00}\vartheta_{11}\vartheta_{11}. \end{aligned} \tag{B.6}$$

In the above the arguments of the theta functions on the left hand side are x, y, u, v , in that order, and on the right hand side the arguments are x_1, y_1, u_1, v_1 , in that order.

When $x = y = u = v = 0$, using the fact that $\vartheta_{11}(0) = 0$, (R5) reads

$$\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 = 0 \tag{B.7}$$

which is Jacobi's ‘‘abstruse identity’’ demonstrating equal numbers of space-time bosons and fermions in the GSO projected superstring.

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