Positivity (2011) 15:571–594
DOI 10.1007/s11117-011-0127-7

Positivity

Properties (u) and (V*) of Pelczynski in symmetric spaces of τ-measurable operators

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Received: 6 January 2011 / Accepted: 4 May 2011 / Published online: 18 May 2011

Abstract It is shown that order continuity of the norm and weak sequential completeness in non-commutative strongly symmetric spaces of τ-measurable operators are respectively equivalent to properties (u) and (V^*) of Pelczynski. In addition, it is shown that each strongly symmetric space with separable (Banach) bidual is necessarily reflexive. These results are non-commutative analogues of well-known characterisations in the setting of Banach lattices.

Keywords Measurable operators · Property (u) · Property (V*)

Mathematics Subject Classification (2000) Primary 46L52;
Secondary 46E30 · 47A30

1 Introduction

An important tool in the study of subspaces of Banach lattices is the (so-called) property (u) introduced by Pelczynski [19] (for the precise definition, see Sect. 3 below). It was shown by Pelczynski (see [19]) that each Banach space with an absolute basis has property (u); if a Banach space X has property (u), then so does each closed subspace of X; and if the Banach space X has property (u), then X is weakly sequentially com-

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plete if and only if no subspace of X is isomorphic to c_0. It was subsequently shown [15,26] that any Banach lattice with order continuous norm has property (u). Indeed [1,12,16], for Dedekind σ-complete Banach lattices, order continuity of the norm is, in fact, equivalent to property (u) and this, in turn, is equivalent to the Banach lattice not containing any isomorphic copy of l_∞. A principal result of this paper (Theorem 3.7) shows that these characterisations continue to hold for non-commutative strongly symmetric spaces of τ-measurable operators affiliated with a semi-finite von Neumann algebra $(\mathcal{M}, \tau)$. In this setting, the elegant Banach lattice arguments of [1,16] are no longer valid. Our approach is via the theory of non-commutative symmetric spaces as developed in [6–9]. In particular, a crucial role is played by the Yosida–Hewitt decomposition of the dual space of a strongly symmetric space and the sequential weak*-continuity of the corresponding Yosida–Hewitt projection.

Property (V^*) (see Sect. 4 below) was introduced by Pelczynski [20] in connection with his study of Banach spaces on which every unconditionally converging operator is weakly compact. It was shown in [20] that every abstract L-space has property (V^*), and if X is a closed subspace of a Banach space with an unconditional basis, then a necessary and sufficient condition that X should have property (V^*) is that no subspace of X be isomorphic to the Banach space c_0. It was shown by Pelczynski [20] that any Banach space with property (V^*) is weakly sequentially complete. In fact, as noted in [22], a Banach space has property (V^*) if and only if X is weakly sequentially complete and every sequence in X equivalent to l_1 contains a subsequence which generates a complemented copy of l_1. On the other hand, it was shown by Saab and Saab [23] that a separable subspace X of a Banach lattice with order continuous norm has property (V^*) if and only if X contains no copy of c_0 (or equivalently, if X is weakly sequentially complete). Meyer-Nieberg [16, Proposition 5.3.4], noted this continues to hold for arbitrary closed subspaces of a Banach lattice with order continuous norm. Further, it was shown by Pfitzner [21] that L-summands in their biduals have property (V^*). In the setting of spaces of τ-measurable operators, it was shown by Randrianantoanina [22] that if E is a (strongly)symmetric Banach function space on $[0,\infty)$ then the non-commutative space $E(\tau)$ corresponding to E has property (V^*) if E contains no copy of c_0. His approach is based on the cited result of Pfitzner which implies that non-commutative L^1-spaces have property (V^*). In Sect. 4 below, we prove directly (see Proposition 4.8) the equivalence of weak sequential completeness and property (V^*) for [7, Proposition 2.3]. This yields a very short and direct proof of the main result of [22] in the somewhat more general setting of strongly symmetric spaces which are not necessarily constructed as the non-commutative counterpart of rearrangement invariant spaces on the positive semi-axis. In particular, the fact that the non-commutative spaces $L^1(\tau)$ have property (V^*) now follows as an immediate consequence.

Section 5 contains several extensions of the main results to subspaces of strongly symmetric spaces with order continuous norm. Finally, the last section contains several complements on the relation of separability to order continuity of the norm. It is shown (Proposition 6.8) that if the strongly symmetric space E has order continuous norm, then E is separable if and only if the set of finite trace projections in $\mathcal{M}$ is separable for the measure topology. In the setting of (commutative) Banach function spaces, this goes back to the thesis of Luxemburg [13]. Finally, we show that any non-commutative strongly symmetric space with separable (Banach) bidual is necessarily reflexive. While true in the setting of Banach lattices, this result is well-known to be false in general Banach spaces as shown by an example due to R.C. James of a non-reflexive Banach space with separable Banach bidual.

2 Preliminaries and notation

Throughout this paper $\mathcal{M}$ will denote a von Neumann algebra on some Hilbert space $\mathcal{H}$. Unless otherwise stated, it will be assumed throughout that $\mathcal{M}$ is equipped with a fixed semifinite faithful normal trace τ. For standard facts concerning von Neumann algebras, we refer to [5,24]. The identity in $\mathcal{M}$ is denoted by $\mathbf{1}$ and we denote by $\mathcal{P}(\mathcal{M})$ the complete lattice of all (self-adjoint) projections in $\mathcal{M}$. A linear operator $x : \mathcal{D}(x) \to \mathcal{H}$, with domain $\mathcal{D}(x) \subseteq \mathcal{H}$, is said to be *affiliated with* $\mathcal{M}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. For any self-adjoint operator x on $\mathcal{H}$, its spectral measure is denoted by e^x. A self-adjoint operator x is affiliated with $\mathcal{M}$ if and only if $e^x(B) \in \mathcal{P}(\mathcal{M})$ for any Borel set $B \subseteq \mathbb{R}$.

The closed and densely defined operator x, affiliated with $\mathcal{M}$, is called τ*-measurable* if and only if there exists a number $s \geq 0$ such that

$$\tau(e^{|x|}(s, \infty)) < \infty.$$

The collection of all τ-measurable operators is denoted by $S(\tau)$. With the sum and product defined as the respective closures of the algebraic sum and product, it is well-known that $S(\tau)$ is a *-algebra. For $\epsilon,\ \delta > 0$, we denote by $V(\epsilon, \delta)$ the set of all $x \in S(\tau)$ for which there exists an orthogonal projection $p \in P(\mathcal{M})$ such that $p(\mathcal{H}) \subseteq \mathcal{D}(x)$, $\|xp\|_{\mathcal{B}(\mathcal{H})} \leq \epsilon$ and $\tau(\mathbf{1} - p) \leq \delta$. The sets $\{V(\epsilon, \delta) : \epsilon,\ \delta > 0\}$ form a base at 0 for a metrizable Hausdorff topology on $S(\tau)$, which is called the *measure topology*. Equipped with this topology, $S(\tau)$ is a complete topological $*$-algebra. These facts and their proofs can be found in the papers [17,25].

For $x \in S(\tau)$, the *singular value function* $\mu(\cdot; x) = \mu(\cdot; |x|)$ is defined by

$$\mu(t; x) = \inf\{s \geq 0 : \tau(e^{|x|}(s, \infty)) \leq t\}, \quad t \geq 0.$$

It follows directly that the singular value function $\mu(x)$ is a decreasing, right-continuous function on the positive half-line $[0, \infty)$. Moreover, $\mu(uxv) \leq \|u\|\|v\|\mu(x)$ for all $u, v \in \mathcal{M}$ and $x \in S(\tau)$ and $\mu(f(x)) = f(\mu(x))$ whenever $0 \leq x \in S(\tau)$ and f is an increasing continuous function on $[0, \infty)$ which satisfies $f(0) = 0$.

It should be observed that a sequence $\{x_n\}_{n=1}^{\infty}$ in $S(\tau)$ converges to zero for the measure topology if and only if $\mu(t; x_n) \to 0$ as $n \to \infty$ for all $t > 0$.

If m denotes Lebesgue measure on the semiaxis $[0, \infty)$, and if we consider $L^\infty(m)$ as an Abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2(m)$, with the trace given by integration with respect to m, then $S(m)$ consists of all measurable functions on $[0, \infty)$ which are bounded except on a set of finite measure, and that for $f \in S(m)$, the generalized singular value function $\mu(f)$ is precisely the classical decreasing rearrangement of the function $|f|$, which is usually denoted by f^*. In this setting, convergence for the measure topology coincides with the usual notion of convergence in measure. If $\mathcal{M} = \mathcal{L}(\mathcal{H})$ and τ is the standard trace, then $S(\tau) = \mathcal{M}$, the measure topology coincides with the operator norm topology. If $x \in S(\tau)$, then x is compact if and only if $\lim_{t\to\infty} \mu(t; x) = 0$; in this case,

$$\mu_n(x) = \mu(t; x), \quad t \in [n, n+1), \quad n = 0, 1, 2, \ldots,$$

and the sequence $\{\mu_n(x)\}_{n=0}^{\infty}$ is just the sequence of eigenvalues of $|x|$ in non-increasing order and counted according to multiplicity. In the general setting, we denote by $S_0(\tau)$ the two-sided $*$-ideal in $S(\tau)$ of all τ-compact operators, that is, of all $x \in S(\tau)$ such that $\lim_{t\to\infty} \mu(t; x) = 0$ (equivalently, $\tau(e^{|x|}(s, \infty)) < \infty$ for all $t > 0$).

The real vector space $S_h(\tau) = \{x \in S(\tau) : x = x^*\}$ is a partially ordered vector space with the ordering defined by setting $x \geq 0$ if and only if $\langle x\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{D}(x)$. The positive cone in $S_h(\tau)$ will be denoted by $S(\tau)^+$. If $0 \leq x_\alpha \uparrow_\alpha \leq x$ holds in $S(\tau)^+$, then $\sup_\alpha x_\alpha$ exists in $S(\tau)^+$. The trace τ extends to $S(\tau)^+$ as a non-negative extended real-valued functional which is positively homogeneous, additive, unitarily invariant and normal. This extension is given by

$$\tau(x) = \int_0^\infty \mu(t; x)\, dt, \quad x \in S(\tau)^+,$$

and satisfies $\tau(x^*x) = \tau(xx^*)$ for all $x \in S(\tau)$. It should be observed that if f is an increasing continuous function on $[0, \infty)$ satisfying $f(0) = 0$, then

$$\tau(f(|x|)) = \int_0^\infty \mu(t; f(|x|))\, dt = \int_0^\infty f(\mu(t; x))\, dt \tag{2.1}$$

for all $x \in S(\tau)$.

If $1 \leq p < \infty$, we set $L^p(\tau) = \{x \in S(\tau) : \tau(|x|^p) < \infty\}$. Note that it follows from (2.1) that $L^p(\tau)$ is also given by

$$L^p(\tau) = \{x \in S(\tau) : \mu(x) \in L^p(m)\},$$

where m denotes Lebesgue measure on $[0, \infty)$. The space $L^p(\tau)$ is a linear subspace of $S(\tau)$ and the functional $x \longmapsto \|x\|_{L^p(\tau)} = \tau(|x|^p)^{1/p}$, $x \in L^p(\tau)$, is a norm. It should be observed that $\|x\|_{L^p(\tau)} = \|\mu(x)\|_{L^p(m)}$ for all $x \in L^p(\tau)$. Equipped with this norm, $L^p(\tau)$ is a Banach space. In this setting, we also have that $L^\infty(\tau) = \mathcal{M}$.

In the commutative setting, the spaces $L^p(\tau)$ are the familiar Lebesgue spaces. In the special case that $\mathcal{M}$ is $\mathcal{B}(\mathcal{H})$ equipped with standard trace, the corresponding L^p-spaces are the Schatten classes $\mathfrak{S}_p$. As is well-known, the space $L^1(\tau)$ may be identified with the von Neumann algebra predual of $\mathcal{M}$ with respect to trace duality. If $x \in S(\tau)$, then the projection onto the closure of the range of $|x|$ is called the *support* of x and is denoted by $s(x)$. We set $\mathcal{F}(\tau) = \{x \in \mathcal{M} : \tau(s(x)) < \infty\}$.

If $(\mathcal{N}, \sigma)$ is a semifinite von Neumann algebra, if $x \in S(\tau)$ and $y \in S(\sigma)$ then x is said to be *submajorised* by y (in the sense of Hardy, Littlewood and Polya) if and only if

$$\int_0^t \mu(s; x)\, ds \leq \int_0^t \mu(s; y)\, ds$$

for all $t \geq 0$. We write $x \prec\prec y$, or equivalently, $\mu(x) \prec\prec \mu(y)$.

For further details and proofs, we refer the reader to [6,8,10].

It will be convenient to adopt the following terminology. A linear subspace $E \subseteq S(\tau)$, equipped with a norm $\|\cdot\|_E$ will be called

(i) *symmetrically normed* if $x \in E$, $y \in S(\tau)$ and $\mu(y) \leq \mu(x)$ imply that $y \in E$ and $\|y\|_E \leq \|x\|_E$;
(ii) *strongly symmetrically normed* if E is symmetrically normed and its norm has the additional property that $\|y\|_E \leq \|x\|_E$ whenever $x, y \in E$ satisfy $y \prec\prec x$.

In the present paper we shall only consider strongly symmetrically normed spaces. It should be pointed out that all results are also valid for symmetrically normed spaces if one assumes in addition that the von Neumann algebra $\mathcal{M}$ is either *non-atomic* (that is, does not contain any minimal projections) or *atomic* and all minimal projections have equal trace.

If a (strongly) symmetrically normed space is Banach, then it will be simply called a *(strongly) symmetric space*.

It may be shown that any strongly symmetrically normed space E for which $\bigvee_{x \in E} s(x) = \mathbf{1}$ (which will be always assumed) satisfies

$$\mathcal{F}(\tau) \subseteq E \subseteq L^1(\tau) + \mathcal{M},$$

with continuous inclusions (where $\mathcal{F}(\tau)$ is equipped with the $L^1 \cap L^\infty$-norm). If, in addition, E is a Banach space, then $L^1(\tau) \cap \mathcal{M} \subseteq E$, with continuous embedding.

Remark 2.1 We point out that the terminology introduced above differs from that which has been used elsewhere in the literature. We point out explicitly that the terms "symmetrically normed" and "strongly symmetrically normed" as defined above are used in the present paper instead of the earlier terminology of "rearrangement-invariant" and "symmetrically normed", respectively, used in the papers [7,8].

If $\mathcal{M}$ is $L^\infty(m)$, with m Lebesgue measure on the semiaxis $[0, \infty)$, then a symmetrically normed space $E \subseteq S(m)$ will be called, for simplicity, a symmetrically normed space on $[0, \infty)$.

If $E \subseteq S(\tau)$ is a strongly symmetrically normed space, then the embedding of E into $S(\tau)$ is continuous from the norm topology of E to the measure topology on $S(\tau)$.

A wide class of strongly symmetrically normed spaces may be constructed as follows. If $E \subseteq S(m)$ is a strongly symmetrically normed space on $[0, \infty)$, set

$$E(\tau) = \{x \in S(\tau) : \mu(x) \in E\}, \quad \|x\|_{E(\tau)} := \|\mu(x)\|_E$$

It may be shown as in [6] that $(E(\tau), \|\cdot\|_{E(\tau)})$ is a strongly symmetrically normed and is a Banach space if E is a Banach space.

Now suppose that $E \subseteq S(\tau)$ is a strongly symmetrically normed space and let

$$E^{\times} = \{y \in S(\tau) : \sup\{\tau(|xy|) : x \in E, \|x\|_E \leq 1\} < \infty\}$$

and

$$\|y\|_{E^{\times}} = \sup\{\tau(|xy|) : x \in E, \|x\|_E \leq 1\}.$$

If $y \in S(\tau)$, then $y \in E^{\times}$ if and only if

$$\sup\left\{\int_{[0,\infty)} \mu(x)\mu(y)\, dm : x \in E, \|x\|_E \leq 1\right\} < \infty,$$

in which case, the latter quantity is equal to $\|y\|_{E^{\times}}$. The space $(E^{\times}, \|\cdot\|_{E^{\times}})$ is a strongly symmetrically normed space, and is called the Köthe dual of E. The symmetrically normed space E is said to have the Fatou property if, whenever $0 \leq x_\alpha \uparrow_\alpha \subseteq E$ is an upwards directed system with $\sup \|x_\alpha\|_E < \infty$, it follows that $x = \sup_\alpha x_\alpha$ exists in E and $\|x\|_E = \sup_\alpha \|x_\alpha\|_E$. The Köthe dual $E^{\times}$ of any strongly symmetrically normed space $E \subseteq S(\tau)$ always has the Fatou property. If $y \in E^{\times}$, set $\phi_y : E \to \mathbb{C}$, $\phi_y(x) = \tau(xy)$, $x \in E$. The mapping $y \to \phi_y$, $y \in E^{\times}$ is an isometry of $E^{\times}$ into the Banach dual E^*. The linear functional ϕ on the symmetrically normed space $E \subseteq S(\tau)$ is said to be *normal* if whenever $x_\alpha \downarrow_\alpha \subseteq E$, it follows that $\phi(x_\alpha) \to_\alpha 0$ and *singular* if ϕ vanishes on some order dense ideal in E. See [9]. The space of all normal (respectively, singular) linear functionals on E is denoted by E_n^* (respectively, E_s^*). It is shown in [9] that, if $E \subseteq S(\tau)$ is strongly symmetric, then the Banach dual E^* admits the unique decomposition $E^* = E_n^* \oplus E_s^*$, called the Yosida–Hewitt decomposition, and that E_n^* may be identified with the Köthe dual $E^{\times}$ by trace duality. See also [8]. It is shown in [9] that the natural projections $P_n : E^* \to E_n^*$, $P_s : E^* \to E_s^*$ are positive, bounded projections which are sequentially continuous for the weak *-topology $\sigma(E^*, E)$. The norm on E is said to be *order-continuous* if $\|x_\alpha\|_E \downarrow_\alpha 0$ whenever $x_\alpha \downarrow_\alpha 0 \subseteq E$. If the norm on E is order-continuous, then every continuous linear functional on E is normal, and in this case, the Banach dual E^* may be identified with the Köthe dual $E^{\times}$.

3 Property (u) of Pelczyński

If $\{x_n\}_{n=1}^{\infty}$ is a sequence in the Banach space X, then the series $\sum_{n=1}^{\infty} x_n$ is said to be *weakly unconditionally Cauchy* (or, *WUC*), whenever

$$\sum_{n=1}^{\infty} |\varphi(x_n)| < \infty, \quad \varphi \in X^*.$$

Observe that the series $\sum_{n=1}^{\infty} x_n$ is WUC if and only if the set

$$\sup\left\{\left\|\sum_{n\in F} x_n\right\| : F \subseteq \mathbb{N} \text{ finite}\right\} < \infty. \tag{3.1}$$

This is well-known for real spaces (see, for example, the monograph [4]), but the changes needed for the complex case are straightforward.

A weak Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ in the Banach space X is said to have *property* (u) if and only if there exists a WUC series $\sum_{n=1}^{\infty} u_n$ in X such that $x_n - \sum_{k=1}^{n} u_k \to 0$ weakly as $n \to \infty$. The Banach space X is said to have property (u) if and only if every weak Cauchy sequence in X has property (u).

Property (u) was introduced by Pelczyński [19], where it is shown that it is hereditary, that is, if the space X has property (u), then so does every closed subspace of X.

Before proving the main result of the present section, some technical preparations are necessary. Throughout, it is assumed that $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra on a Hilbert space $\mathcal{H}$.

Lemma 3.1 *If $E \subseteq S(\tau)$ is a strongly symmetric space and if $\{x_n\}_{n=1}^{\infty}$ is an increasing weak Cauchy sequence in E^+, then $\{x_n\}_{n=1}^{\infty}$ has property (u).*

Proof Define the sequence $\{u_n\}_{n=1}^{\infty} \subseteq E^+$ by setting $u_1 = x_1$ and $u_n = x_n - x_{n-1}$ for $n \geq 2$. Since each element of E^* is a linear combination of at most four positive linear functionals on E, it will clearly suffice to show that $\sum_{k=1}^{\infty} \varphi(u_k) < \infty$ for all $0 \leq \varphi \in E^*$. Observe that, if $0 \leq \varphi \in (E^*)^+$, then

$$\sum_{k=1}^{n} \varphi(u_k) = \varphi(x_n) \leq \|\varphi\|_{E^*}\|x_n\|_E \leq \|\varphi\|_{E^*} \sup_j \|x_j\|_E < \infty$$

for all n, using the fact that the sequence $\{x_n\}_{n=1}^{\infty}$ is norm bounded. □

Recall that a projection $e \in P(\mathcal{M})$ is called *σ-finite* if there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of finite trace projections in $P(\mathcal{M})$ such that $p_n \uparrow_n e$.

Lemma 3.2 *For a projection $e \in P(\mathcal{M})$ the following statements are equivalent.*

- (i) *e is σ-finite.*
- (ii) *There exists a sequence $\{p_n\}_{n=1}^{\infty}$ of finite trace projections in $P(\mathcal{M})$ such that $e \leq \bigvee_{n=1}^{\infty} p_n$.*
- (iii) *Every mutually orthogonal system $\{q_\alpha\}$ of non-zero projections in $P(\mathcal{M})$ satisfying $q_\alpha \leq e$ for all α, is at most countable.*
- (iv) *There exists a mutually orthogonal sequence $\{p_n\}_{n=1}^{\infty}$ of finite trace projections in $P(\mathcal{M})$ such that $e = \sum_{n=1}^{\infty} p_n$.*

Proof The implications (i)⇒(ii) and (iv)⇒(i) are evident. That (iii) implies (iv) follows immediately from the semi-finiteness of the trace τ.

(ii)⇒(iii). Suppose that $\{q_\alpha\}_{\alpha\in\mathbb{A}}$ is a mutually orthogonal system in $P(\mathcal{M})$ such that $0 < q_\alpha \leq e$ for all α. If $n \in \mathbb{N}$, then for each finite subset $\mathbb{F} \subseteq \mathbb{A}$,

$$\sum_{\alpha\in\mathbb{F}} p_n q_\alpha p_n = p_n \left(\sum_{\alpha\in\mathbb{F}} q_\alpha\right) p_n \leq p_n,$$

so that

$$\sum_{\alpha\in\mathbb{F}} \tau(p_n q_\alpha p_n) \leq \tau(p_n) < \infty.$$

This implies that the set $\mathbb{A}_n = \{\alpha \in \mathbb{A} : p_n q_\alpha p_n \neq 0\}$ is at most countable. Now observe that $\mathbb{A} = \bigcup_{n=1}^\infty \mathbb{A}_n$. Indeed, suppose that $\alpha \in \mathbb{A}$ satisfies $p_n q_\alpha p_n = 0$ for all $n \in \mathbb{N}$. It follows that $(q_\alpha p_n)^* q_\alpha p_n = 0$ and so, also $q_\alpha p_n = 0$ for all $n \in \mathbb{N}$. Hence, $p_n \leq \mathbf{1} - q_\alpha$ for all n and so, $e \leq \bigvee_{n=1}^\infty p_n \leq \mathbf{1} - q_\alpha$. Since $0 < q_\alpha \leq e$, this is a contradiction. Consequently, $\mathbb{A}$ is at most countable. □

From the above lemma, it is clear that if $f \leq e$ in $P(\mathcal{M})$ and e is σ-finite, then f is also σ-finite. Similarly, if $\{e_n\}_{n=1}^\infty$ is a sequence in $P(\mathcal{M})$ of σ-finite projections, then the projection $\bigvee_{n=1}^\infty e_n$ is also σ-finite.

Lemma 3.3 *Let $E \subseteq S(\tau)$ be a strongly symmetric space with order continuous norm. If $\{x_n\}_{n=1}^\infty$ is a sequence in E, then there exists a σ-finite projection $p \in P(\mathcal{M})$ such that $x_n = x_n p$ for all n.*

Proof First observe that for every $x \in E$ the support projection $s(x)$ is σ-finite. Indeed, since E has order continuous norm, it follows that $E \subseteq S_0(\tau)$. Hence, $\tau(e^{|x|}(1/n, \infty)) < \infty$ for all $n \in \mathbb{N}$. Observing that $e^{|x|}(1/n, \infty) \uparrow_n e^{|x|}(0, \infty) = s(|x|) = s(x)$, it is now clear that $s(x)$ is σ-finite. Therefore, if $\{x_n\}_{n=1}$ is a sequence in E, then the projection $p = \bigvee_{n=1}^\infty s(x_n)$ is σ-finite and satisfies $x_n = x_n p$ for all n. □

Lemma 3.4 *If $0 \leq x \in S(\tau)$ and if $s(x)$ is σ-finite, then there exists a sequence $\{x_n\}_{n=1}^\infty$ in $\mathcal{F}(\tau)$ such that $0 \leq x_n \uparrow_n x$.*

Proof It follows from Lemma 3.2 that there exists a mutually orthogonal sequence $\{q_n\}_{n=1}^\infty$ in $P(\mathcal{M})$ of finite trace projections such that $s(x) = \sum_{n=1}^\infty q_n$. It follows from the spectral theorem that, for each $n \in \mathbb{N}$, there exists a sequence $\{a_{n,k}\}_{k=1}^\infty$ in $\mathcal{M}^+$ such that $0 \leq a_{n,k} \uparrow_k x^{1/2} q_n x^{1/2}$. Observing that

$$\begin{aligned}\mu(x^{1/2} q_n x^{1/2}) &= \mu((q_n x^{1/2})^*(q_n x^{1/2})) = \mu((q_n x^{1/2})(q_n x^{1/2})^*) \\ &= \mu(q_n x q_n) \leq \mu(x)\chi_{[0,\tau(q_n))},\end{aligned}$$

it follows that $\tau(s(x^{1/2} q_n x^{1/2})) \leq \tau(q_n)$. Since $0 \leq a_{n,k} \leq x^{1/2} q_n x^{1/2}$ it follows that $s(a_{n,k}) \leq s(x^{1/2} q_n x^{1/2})$. Consequently, $\tau(s(a_{n,k})) \leq \tau(q_n) < \infty$ and so, $0 \leq a_{n,k} \in \mathcal{F}(\tau)$ for all $k, n \in \mathbb{N}$. Define $0 \leq x_n \in \mathcal{F}(\tau)$ by setting

$$x_n = a_{1,n} + a_{2,n} + \cdots + a_{n,n}, \quad n \in \mathbb{N}.$$

It is clear that $0 \leq x_n \leq x_{n+1}$ for all n. Further,

$$\begin{aligned} x_n &\leq x^{1/2} q_1 x^{1/2} + x^{1/2} q_2 x^{1/2} + \cdots + x^{1/2} q_n x^{1/2} \\ &= x^{1/2}(q_1 + q_2 + \cdots + q_n) x^{1/2} \leq x \end{aligned}$$

for all $n \in \mathbb{N}$. To show that $0 \leq x_n \uparrow_n x$ holds in $S(\tau)$, suppose that $0 \leq b \in S(\tau)$ satisfies $x_n \leq b$ for all $n \in \mathbb{N}$. It follows that

$$0 \leq a_{1,m} + \cdots + a_{n,m} \leq x_m \leq b, \quad m \geq n,$$

and this implies that,

$$x^{1/2} q_1 x^{1/2} + x^{1/2} q_2 x^{1/2} + \cdots + x^{1/2} q_n x^{1/2} \leq b, \quad n \in \mathbb{N}.$$

Since

$$\sum_{j=1}^{n} x^{1/2} q_j x^{1/2} = x^{1/2} \left(\sum_{j=1}^{n} q_j \right) x^{1/2} \uparrow_n x^{1/2} s(x) x^{1/2} = x,$$

it follows that $x \leq b$, and this suffices to complete the proof. □

We are now in a position to prove one of the main results of the present section.

Theorem 3.5 *If $E \subseteq S(\tau)$ is strongly symmetric and if E has order continuous norm, then E has property (u).*

Proof If $\phi \in E^*$, then the functional $\phi^* \in E^*$ is defined by setting $\phi^*(x) = \overline{\phi(x^*)}$, $x \in E$. It follows that, if $\{x_n\}_{n=1}^{\infty}$ is a weak Cauchy sequence in E, then so is the sequence $\{x_n^*\}_{n=1}^{\infty}$. Using that linear combinations of sequences with property (u) also have property (u), it will suffice to show that each weak Cauchy sequence consisting of self-adjoint elements of E has property (u).

Suppose then that $\{x_n\}_{n=1}^{\infty} \subseteq E$ is a weak Cauchy sequence in E such that $x_n^* = x_n$ for all $n \in \mathbb{N}$. Since the norm on E is order continuous, the Banach dual E^* may be identified with the Köthe dual $E^{\times}$ via trace duality. Since the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded, there exists $\Phi \in (E^{\times})^*$ such that

$$\Phi(y) = \lim_{n \to \infty} \tau(x_n y), \quad y \in E^{\times},$$

that is, $x_n \to \Phi$ with respect to the $\sigma((E^{\times})^*, E^{\times})$-topology. Let $P = P_n$ be the projection in $(E^{\times})^*$ onto $(E^{\times})_n^*$ corresponding to the Yosida–Hewitt decomposition $(E^{\times})^* = (E^{\times})_n^* \oplus (E^{\times})_s^*$. It is shown in [9] that the projection P is sequentially continuous with respect to the $\sigma((E^{\times})^*, E^{\times})$-topology and hence, $P x_n \to P\Phi$ in the $\sigma((E^{\times})^*, E^{\times})$-topology. Since $P x_n = x_n$ for all $n \in \mathbb{N}$, it follows that $P\Phi = \Phi$, that is, $\Phi \in (E^{\times})^*_n$. Identifying $(E^{\times})^*_n$ with $E^{\times\times}$ via trace duality, it follows that there exists $z \in E^{\times\times}$ such that

$$\tau(zy) = \lim_{n\to\infty} \tau(x_n y), \quad y \in E^{\times}.$$

Since

$$\begin{aligned}\tau(zy) &= \lim_{n\to\infty} \tau(x_n y) = \lim_{n\to\infty} \tau(x_n^* y) \\ &= \lim_{n\to\infty} \overline{\tau(x_n y^*)} = \overline{\tau(zy^*)} = \tau(z^* y)\end{aligned}$$

for all $y \in E^{\times}$, it is clear that $z^* = z$.

By Lemma 3.3, there exists a σ-finite projection $p \in P(\mathcal{M})$ such that $x_n = x_n p$ for all $n \in \mathbb{N}$. Observing that

$$\tau(zy) = \lim_{n\to\infty} \tau(x_n y) = \lim_{n\to\infty} \tau(x_n p y) = \tau(zpy), \quad y \in E^{\times},$$

it follows that $z = zp$ and so, $s(z) \leq p$. Since $s(z^+) \leq s(z) \leq p$, it follows that $s(z^+)$ is σ-finite and so, by Lemma 3.4, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subseteq \mathcal{F}(\tau) \subseteq E$ such that $0 \leq u_n \uparrow z^+$. Similarly, there exists a $\{v_n\}_{n=1}^{\infty} \subseteq \mathcal{F}(\tau)$ such that $0 \leq v_n \uparrow z^-$. It is also clear that $u_n \to z^+$ and $v_n \to z^-$ and hence, $u_n - v_n \to z$ with respect to the weak topology $\sigma((E^{\times})^*, E^{\times})$. It follows that $x_n - (u_n - v_n) \to 0$ weakly in E. Furthermore, it follows from Lemma 3.1 that each of the sequences $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ has property (u) and so, the sequence $\{u_n - v_n\}_{n=1}^{\infty}$ has property (u). It is now clear that the sequence $\{x_n\}_{n=1}^{\infty}$ has also property (u), and this completes the proof of the theorem. □

It will be shown next that property (u) is actually equivalent to order continuity of the norm. The proof is based on the following result.

Proposition 3.6 *If $E \subseteq S(\tau)$ is strongly symmetric and if E does not have order continuous norm, then E contains an isomorphic copy of l_∞.*

Proof If E does not have order continuous norm, then it follows from [9] Theorem 6.13 that there exists $\epsilon > 0$, an element $0 \leq x \in E$ and a mutually orthogonal sequence $\{e_n\}_{n=1}^{\infty} \subseteq P(\mathcal{M})$ of projections such that $\|e_n x\|_E \geq \epsilon$ for all $n \in \mathbb{N}$. If $(\alpha_n) \in l_\infty$, then the series $\sum_{n=1}^{\infty} \alpha_n e_n$ is strongly convergent in $\mathcal{M}$ and $\|\sum_{n=1}^{\infty} \alpha_n e_n\|_{B(H)} = \|(\alpha_n)\|_\infty$. Define the linear mapping $T : l_\infty \to E$ by setting

$$T(\alpha_n) = \left(\sum_{n=1}^{\infty} \alpha_n e_n\right) x, \quad (\alpha_n) \in l_\infty.$$

It is clear that

$$\|T(\alpha_n)\|_E \leq \left\|\sum_{n=1}^{\infty} \alpha_n e_n\right\|_{B(H)} \|x\|_E = \|(\alpha_n)\|_\infty \|x\|_E, \quad (\alpha_n) \in l_\infty,$$

and so T is bounded with $\|T\| \leq \|x\|_E$. For each $m \in \mathbb{N}$, it is clear that

$$e_m T(\alpha_n) = \alpha_m e_m x$$

and so

$$\epsilon|\alpha_m| \leq \|\alpha_m e_m x\|_E = \|e_m T(\alpha_n)\|_E \leq \|T(\alpha_n)\|_E.$$

This implies that

$$\epsilon\|(\alpha_n)\|_\infty \leq \|T(\alpha_n)\|_E, \quad (\alpha_n) \in l_\infty,$$

and hence T is an isomorphism onto its range. This suffices to complete the proof. □

Theorem 3.7 *If $E \subseteq S(\tau)$ is strongly symmetric, then the following statements are equivalent.*

(i) *E has order continuous norm.*
(ii) *E has property (u).*
(iii) *E does not contain a copy of l_∞.*

Proof The implication (i)$\Longrightarrow$(ii) is just the statement of Theorem 3.5 and the implication (iii)$\Longrightarrow$(i) follows from Proposition 3.6. It will suffice to show that (ii)$\Longrightarrow$(iii). If E has property (u), then E cannot contain an isomorphic copy of l_∞, since it is well-known that l_∞ does not have property (u), and each closed subspace of a space with property (u) again has property (u). This suffices to complete the proof. □

An immediate consequence of the preceding theorem is that order continuity of the norm is a Banach space property within the class of strongly symmetric spaces, that is, if $E \subseteq S(\tau)$, $F \subseteq S(\sigma)$ are isomorphic as Banach spaces, then the norm on E is order continuous if and only if the norm on F is order continuous.

In the setting of Banach lattices, the preceding Theorem 3.7 is due to Lozanovskiĭ [12].

Corollary 3.8 *Suppose that $E \subseteq S(m)$ is strongly symmetric. If E has property (u), then so does $E(\tau)$ for all semifinite $(\mathcal{M}, \tau)$.*

Proof Suppose that $E \subseteq S(m)$ has property (u). By the commutative specialisation of Theorem 3.7, it follows that E has order continuous norm. By [8] Proposition 3.6, it follows that $E(\tau)$ has order continuous norm for all semifinite $(\mathcal{M}, \tau)$. The assertion of the corollary now follows from the implication (i)$\Longrightarrow$(ii) of Theorem 3.7. □

4 Property (V*) of Pelczynski

The following definition was introduced in [20].

Definition 4.1 A subset K of a Banach space X is called a (V^*) *-set* if

$$\sup\{|\varphi_n(x)| : x \in K\} \to 0, \quad n \to \infty,$$

for every WUC series $\sum_{n=1}^{\infty} \varphi_n$ in X^*.

It should be noted that every (V^*)-set is bounded. Further, it is easily seen that if K is a V^*-set in X and if $T : X \to Y$ is a continuous linear mapping from X to the Banach space Y, then $T(K)$ is a (V^*)-set in Y.

It will be convenient to recall the following alternative characterisation of (V^*)-sets. See [2], Proposition 1.1.

Proposition 4.2 *If X is a Banach space and $K \subseteq X$, then following statements are equivalent.*

(i) *K is a (V^*)-set.*
(ii) *For every WUC series $\sum_{n=1}^{\infty} \varphi_n$ in X^* we have*

$$\sup\left\{\sum_{n=m}^{\infty} |\varphi_n(x)| : x \in K\right\} \to 0, \quad m \to \infty. \tag{4.1}$$

(iii) *Every bounded linear operator $T : X \to l_1$ maps K onto a relatively compact subset of l_1.*
(iv) *K contains no sequence $\{x_n\}_{n=1}^{\infty}$ equivalent to the unit vector basis of l_1 for which the closed linear span $[x_n]$ is complemented in X.*

Definition 4.3 A subset K of a Banach space X is called *sequentially $\sigma(X, X^*)$-precompact* if every sequence in K has a $\sigma(X, X^*)$-Cauchy subsequence.

Note: In [2], sequentially $\sigma(X, X^*)$ -precompact sets are called *weakly conditionally compact* sets. As observed in [2, Corollary 1.3], Proposition 4.2 has the following consequence.

Corollary 4.4 *If X is a Banach space, then every sequentially $\sigma(X, X^*)$-precompact subset of X is a (V^*)-set.*

Proof Let $K \subseteq X$ be sequentially $\sigma(X, X^*)$-precompact. If $T : X \to l_1$ is a bounded linear map, then $T(K)$ is sequentially $\sigma(l_1, l_\infty)$-precompact in l_1, as the image under T of a weakly Cauchy sequence in X is a weakly Cauchy sequence in l_1. Hence, $T(K)$ is relatively weakly compact, since l_1 is weakly sequentially complete. Consequently, since weak and norm convergence of sequences coincide in l_1, $T(K)$ is relatively compact in l_1. The result now follows from Proposition 4.2. □

Definition 4.5 A Banach space X is said to have *property* (V^*) if every (V^*)-subset of X is relatively weakly compact.

It is easily seen that, if the Banach space X has property (V^*), then so does each closed subspace $Y \subseteq X$. Indeed, it suffices to observe that the restriction to Y of any WUC series in X^* is again a WUC series in Y^*. This is an immediate consequence of the characterisation of a WUC series in terms of bounded sums given in equation (3.1) which implies that any (V^*)-subset of Y is also a (V^*)-subset of X.

Further, the Banach space c_0 does not have property (V^*). Indeed, since $c_0^* = l_1$ is weakly sequentially complete and weak and norm convergence of sequences coincide in l_1, it follows that every bounded set in c_0 is a (V^*)-set. If c_0 had property (V^*), it would then follow that every norm-bounded subset of c_0 is relatively weakly compact, which is clearly not the case. Consequently, if the Banach space X has property (V^*) then X cannot contain any copy of c_0.

The following simple observation is worth noting. See [2], Remark 1.4 (a). The proof is included for the sake of completeness.

Lemma 4.6 *For a Banach space X the following two statements are equivalent:*

(i) *X has property (V^*);*
(ii) *X is weakly sequentially complete and every (V^*)-subset of X is sequentially $\sigma(X, X^*)$-precompact.*

Proof (i)⇒(ii). Any (V^*)-subset of X is relatively weakly compact and hence, sequentially $\sigma(X, X^*)$-precompact. Furthermore, if $\{x_n\}_{n=1}^{\infty}$ is a $\sigma(X, X^*)$-Cauchy sequence in X, then the set $K = \{x_n : n \in \mathbb{N}\}$ is a (V^*)-set (as follows from Corollary 4.4). Hence, K is relatively weakly compact. This implies that the sequence $\{x_n\}_{n=1}^{\infty}$ is weakly convergent in X. Consequently, X is weakly sequentially complete.

(ii)⇒(i). Let $K \subseteq X$ be a (V^*) -set and $\{x_n\}_{n=1}^{\infty}$ be a sequence in K. By hypothesis, K is sequentially $\sigma(X, X^*)$-precompact and so, $\{x_n\}_{n=1}^{\infty}$ has a weak Cauchy sequence $\{x_{n_k}\}_{k=1}^{\infty}$. Since X is weakly sequentially complete, $\{x_{n_k}\}_{k=1}^{\infty}$ has a weak limit in X. Hence, K is relatively weakly (sequentially) compact. The proof is complete. □

It will now be shown that, for strongly symmetric spaces $E \subseteq S(\tau)$, weak sequential completeness coincides with property (V^*).We begin with the following simple observation.

Lemma 4.7 *If $x \in E$ and if $\{e_n\}_{n=1}^{\infty} \subseteq P(\mathcal{M})$ is any sequence of mutually orthogonal projections then the series $\sum_{n=1}^{\infty} xe_n$ and $\sum_{n=1}^{\infty} e_n x$ are both WUC.*

Proof If $F \subseteq \mathbb{N}$ is finite, then

$$\left\| \sum_{n \in F} xe_n \right\| = \left\| x \left(\sum_{n \in F} e_n \right) \right\|_E \leq \left\| \sum_{n \in F} e_n \right\|_{B(H)} \|x\|_E \leq \|x\|_E$$

It now follows from the remark at the beginning of the previous section that $\sum_{n=1}^{\infty} xe_n$ is a WUC series. The second statement is proved similarly. □

We recall that a strongly symmetric space $E \subseteq S(\tau)$ is said to be a KB-space if and only if every norm-bounded increasing sequence in E is convergent in the norm of E, or equivalently, $E \subseteq S(\tau)$ is a KB-space if and only if the norm on E is order continuous and E has the Fatou property.

Proposition 4.8 *If $E \subseteq S(\tau)$ is a strongly symmetrically normed space then the following statements are equivalent.*

(i) *E is weakly sequentially complete.*
(ii) *E is a KB-space.*
(iii) *E contains no copy of c_0.*
(iv) *E has property (V^*).*

Proof The equivalences (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iii) are established in [7] Proposition 3.2. The implication (iv)$\Longrightarrow$(iii) follows from the above remarks. It will suffice to show that (i)$\Longrightarrow$(iv). Suppose then that E is weakly sequentially complete. In particular, E is a KB-space and so the norm on E is order continuous, E^* coincides with the Köthe dual $E^\times$ and the natural embedding of E into $E^{\times\times} \subseteq E^{**}$ is a surjective isometry. Now suppose that $K \subseteq (E^\times)^\times$ is bounded and that K is a (V^*)-set. Let $0 \leq y \in E^\times$ and suppose that $\{e_n\}_{n=1}^\infty \subseteq P(\mathcal{M})$ is a sequence of mutually orthogonal projections. By Lemma 4.7, the series $\sum_{n=1}^\infty xe_n$ and $\sum_{n=1}^\infty e_n x$ are both WUC. Since K is a (V^*)-set, it follows that

$$\sup\{\max(|\tau(yxe_n)|, |\tau(ye_n x)|) : y \in K\} \to_n 0.$$

By the weak compactness criterion given in [7] Proposition 2.3, it follows that K is relatively $\sigma((E^\times)^\times, E^\times) = \sigma(E, E^*)$-compact, and this completes the proof of the Proposition. □

Since the space $L_1(\tau)$ is weakly sequentially complete, the following consequence is immediate and was first proved for the more general class of spaces which are L-summands in their bidual by Pfitzner [21]

Corollary 4.9 *The space $L_1(\tau)$ has property (V^*).*

Corollary 4.10 *Suppose that $E \subseteq S(m)$ is a strongly symmetrically normed space. If E has property (V^*), then so does $E(\tau)$ for all $(\mathcal{M}, \tau)$.*

This follows from the fact that, if E is a KB-space, then so is $E(\tau)$, as follows from [8, Corollary 5.12].

In the Banach lattice setting, the equivalence of (iii), (iv) in the preceding Proposition 4.8 may be found in [23], where it is pointed out that the implication (iii)$\Longrightarrow$(iv) fails in general Banach spaces. In the non-commutative setting, the preceding Corollary 4.10 follows from the main result in [22], although it is tacitly assumed there that if $E \subseteq S(m)$ is a strongly symmetric space which contains no copy of c_0, then the corresponding non-commutative space $E(\tau)$ is weakly sequentially complete. As has been observed earlier, if $E \subseteq S(m)$ is a KB-space, then so is the non-commutative space $E(\tau)$ for all $(\mathcal{M}, \tau)$, and so this assertion is actually a consequence of Proposition 4.8. The proof of Corollary 4.10 given in the present paper is more direct and is a considerable simplification of the method of [22].

5 Subspaces of strongly symmetric spaces with order continuous norm

Before proceeding, some technical preparation is required.

Lemma 5.1 *Let $E \subseteq S(\tau)$ be a strongly symmetric space and let $K \subseteq E$ be a (V^*)-set.*

(i) *If $\{y_n\}_{n=1}^{\infty}$ is a sequence in $E^{\times}$ such that $y_n \downarrow 0$, then*

$$\sup\{|\tau(xy_n)| : x \in K\} \to 0, \quad n \to \infty.$$

(ii) *If $\{p_n\}_{n=1}^{\infty}$ is a sequence in $P(\mathcal{M})$ such that $p_n \downarrow 0$ and if $y \in E^{\times}$, then*

$$\sup\{|\tau(xp_n y)| : x \in K\} \to 0, \quad n \to \infty.$$

Proof (i) Note that $y_n \downarrow 0$ in $E^{\times}$ implies that $\tau(xy_n) \to 0$ for all $x \in E$. Defining $z_n = y_n - y_{n+1}$, it follows that

$$0 \le \sum_{n \in F} z_n \le y_1$$

and so, $\|\sum_{n\in F} z_n\|_{E^\times} \le \|y_1\|_{E^\times}$ for all finite subsets F of $\mathbb{N}$. Hence, the series $\sum_{n=1}^{\infty} z_n$ is WUC in E^* (identifying $E^{\times}$ with a closed linear subspace of E^* via trace duality). Therefore, it follows from Proposition 4.2 that

$$\sup\left\{\sum_{n=m}^{\infty} |\tau(xz_n)| : x \in K\right\} \to 0, \quad m \to \infty.$$

If $m < N$ in $\mathbb{N}$ and $x \in K$, then

$$\sum_{n=m}^{N} \tau(xz_n) = \sum_{n=m}^{N} \{\tau(xy_n) - \tau(xy_{n+1})\} = \tau(xy_m) - \tau(xy_{N+1})$$

and so,

$$\tau(xy_m) = \sum_{n=m}^{\infty} \tau(xz_n).$$

Hence,

$$\sup\{|\tau(xy_m)| : x \in K\} \le \sup\left\{\sum_{n=m}^{\infty} |\tau(xz_n)| : x \in K\right\} \to 0, \quad m \to \infty.$$

The proof of (i) is complete.

(ii) Observe first that $\tau(xp_n y) \to 0$ as $n \to \infty$ for all $x \in E$. Indeed, $\tau(xp_n y) = \tau(p_n yx)$ for all n and $\|p_n yx\|_1 \to 0$ as $n \to \infty$ by the order continuity of the norm in $L_1(\tau)$. Define $q_n = p_n - p_{n+1}$ and note that

$$\sum_{k=n}^{N} \tau(xq_k y) = \sum_{k=n}^{N} \{\tau(xp_k y) - \tau(xp_{k+1} y)\}$$
$$= \tau(xp_n y) - \tau(xp_{N+1} y)$$

whenever $n < N$ in $\mathbb{N}$. Hence,

$$\tau(xp_n y) = \sum_{k=n}^{\infty} \tau(xq_k y).$$

If $F \subseteq \mathbb{N}$ is finite, then $\|\sum_{n\in F} q_n y\|_{E^\times} \leq \|\sum_{n\in F} q_n\|_\infty \|y\|_{E^\times} \leq \|y\|_{E^\times}$ and so, the series $\sum_{n=1}^{\infty} q_n y$ is WUC in E^*. Hence, it follows from Proposition 4.2 that

$$\sup\{|\tau(xp_n y)| : x \in K\} \leq \sup\left\{\sum_{k=n}^{\infty} |\tau(xq_k y)| : x \in K\right\} \to 0$$

as $n \to \infty$. The proof is complete. □

Lemma 5.2 *Let E be a strongly symmetric space with order continuous norm, let $K \subseteq E$ be a (V^*)-set and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K. If $p \in P(\mathcal{M})$ with $\tau(p) < \infty$, then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{n_k} p\}_{k=1}^{\infty}$ is a $\sigma(E, E^*)$-Cauchy sequence.*

Proof If $x \in E$, then $xp \in L^1(\tau)$ and $\|xp\|_1 \leq \|p\|_{E^\times} \|x\|_E$. Hence, $x \longmapsto xp$, $x \in E$, is a bounded linear map from E into $L^1(\tau)$. As observed earlier, it follows that $\{xp : x \in K\}$ is a (V^*)-set in $L_1(\tau)$. Since $L^1(\tau)$ has property (V^*), it follows that $\{xp : x \in K\}$ is relatively weakly compact in $L^1(\tau)$. Hence, there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{n_k} p\}_{k=1}^{\infty}$ is $\sigma(L^1(\tau), \mathcal{M})$-Cauchy.

It will now be shown that $\{x_{n_k} p\}_{k=1}^{\infty}$ is a $\sigma(E, E^*)$-Cauchy sequence. Since E has order continuous norm, it follows that $E^* = E^\times$ and so it needs to be shown that $\{\tau(x_{n_k} py)\}_{k=1}^{\infty}$ is a Cauchy sequence for all $y \in E^\times$. It will be sufficient to show this for all $0 \leq y \in E^\times$. So, let $0 \leq y \in E^\times$ be given. It follows from the spectral theorem that there exists a sequence $\{y_n\}_{n=1}^{\infty}$ in $E^\times \cap \mathcal{M}$ such that $0 \leq y_n \uparrow y$ in $E^\times$. It follows from Lemma 5.1, (i) that

$$\sup\{|\tau(x_{n_k} p(y - y_n))| : k \in \mathbb{N}\} \to 0, \quad n \to \infty.$$

Consequently, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|\tau(x_{n_k} p(y - y_N))| \leq \varepsilon/3, \quad \forall\, k \in \mathbb{N}.$$

The sequence $\{x_{n_k} p\}_{k=1}^{\infty}$ is $\sigma(L^1(\tau), \mathcal{M})$-Cauchy and so, there exists $J \in \mathbb{N}$ such that

$$|\tau(x_{n_k} p y_N) - \tau(x_{n_l} p y_N)| \leq \varepsilon/3, \quad k, l \geq J.$$

Hence, if $k, l \geq J$, then

$$\begin{aligned} &|\tau(x_{n_k} p y) - \tau(x_{n_l} p y)| \\ &\quad \leq |\tau(x_{n_k} p(y - y_N))| + |\tau(x_{n_k} p y_N) - \tau(x_{n_l} p y_N)| + |\tau(x_{n_l} p(y - y_N))| \\ &\quad \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

The proof is complete. □

Proposition 5.3 *If $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm, then every (V^*)-set is sequentially $\sigma(E, E^*)$-precompact.*

Proof Let $K \subseteq E$ be a (V^*)-set and $\{x_n\}_{n=1}^{\infty}$ be a sequence in K. By Lemma 3.3, there exists a σ-finite projection $p \in P(\mathcal{M})$ such that $x_n p = x_n$ for all $n \in \mathbb{N}$. Let $\{p_k\}_{k=1}^{\infty}$ be a sequence in $P(\mathcal{M})$ such that $p_k \uparrow p$ and $\tau(p_k) < \infty$ for all k. It follows from Lemma 5.2 combined with a diagonal argument that there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of the sequence $\{x_n\}_{n=1}^{\infty}$ with the property that $\{x_{n_j} p_k\}_{j=1}^{\infty}$ is a $\sigma(E, E^*)$-Cauchy sequence for all $k \in \mathbb{N}$.

For simplicity, denote x_{n_j} by x_n. It follows that $\{x_n\}_{n=1}^{\infty}$ is a $\sigma(E, E^*)$-Cauchy sequence. Indeed, fix $y \in E^{\times}$ $(= E^*)$ and let $\varepsilon > 0$ be given. It follows from Lemma 5.1 (ii) that

$$\sup\{|\tau(x_n(p - p_k)y)| : n \in \mathbb{N}\} \to 0, \quad k \to \infty.$$

Hence, there exists $J \in \mathbb{N}$ such that

$$|\tau(x_n(p - p_k)y)| \leq \varepsilon/3, \quad n \in \mathbb{N}, \quad k \geq J.$$

The sequence $\{x_n p_J\}_{n=1}^{\infty}$ is $\sigma(E, E^*)$-Cauchy and so, there exists $N \in \mathbb{N}$ such that

$$|\tau(x_n p_J) - \tau(x_m p_J)| \leq \varepsilon/3, \quad n, m \geq N.$$

Consequently, if $n, m \geq N$, then

$$\begin{aligned} &|\tau(x_n y) - \tau(x_m y)| = |\tau(x_n p y) - \tau(x_m p y)| \\ &\quad \leq |\tau(x_n(p - p_J)y)| + |\tau(x_n p_J y) - \tau(x_m p_J y)| + |\tau(x_m(p - p_J)y)| \\ &\quad \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

This completes the proof. □

Corollary 5.4 *Let $E \subseteq S(\tau)$ be a strongly symmetric space with order continuous norm. If $X \subseteq E$ is a closed linear subspace, then a subset $K \subseteq X$ is a (V^*)-set if and only if K is sequentially $\sigma(X, X^*)$ -precompact.*

Proof If $K \subseteq X$ is sequentially $\sigma(X, X^*)$ -precompact, then it follows from Corollary 4.4 that K is a (V^*)-set.

Conversely, if $K \subseteq X$ is a (V^*)-set, then K is a (V^*)-set in E and so, by Proposition 5.3, K is sequentially $\sigma(E, E^*)$-precompact. The Hahn–Banach theorem implies that K is sequentially $\sigma(X, X^*)$-precompact. □

The following result is shown in [2] Corollary 1.6. We include the proof for sake of completeness.

Proposition 5.5 *If X is a Banach space with the property that each (V^*)-set is sequentially $\sigma(X, X^*)$-precompact, and if X contains a copy of l_1, then X contains a complemented copy of l_1.*

Proof If X contains no complemented copy of l_1 then the implication (iv)$\Longrightarrow$(i) of Proposition 4.2 implies that every bounded set of X has property (V^*). Consequently, every bounded subset of X is weakly sequentially precompact. It follows that X contains no copy of l_1, and this is a contradiction. □

The following corollary is then an immediate consequence of Proposition 5.5 and Corollary 5.4.

Corollary 5.6 *Let $E \subseteq S(\tau)$ be a strongly symmetric space and suppose that $X \subseteq E$ is a closed linear subspace. If the norm on E is order continuous then the following statements are equivalent.*

(i) *X contains a copy of l_1.*
(ii) *X contains a complemented copy of l_1.*

In the setting of Banach lattices, the preceding corollary is due to Tzafriri [26].

Note that l_∞ does not have the property that each (V^*)-set is weakly sequentially precompact. Indeed, if this were the case, then it would follow from Proposition 5.5 that there exists a continuous projection $P : l_\infty \to Y$ from l_∞ to a closed linear subspace $Y \subseteq l_\infty$ which is isomorphic to l_1. Since l_1 is weakly sequentially complete, a well-known theorem of A. Grothendieck (see [3], Chapter VI.2, Theorem 15) shows that the projection P is weakly compact. From the open mapping theorem, it follows that the unit ball of Y is contained the image under P of a multiple of the unit ball in l_∞ and is therefore relatively weakly compact. This implies that Y is reflexive, and this is clearly a contradiction.

Corollary 5.7 *If $E \subseteq S(\tau)$, then the following statements are equivalent.*

(i) *Every (V^*)-set in E is sequentially $\sigma(E, E^*)$-precompact.*
(ii) *E contains no isomorphic copy of l_∞.*
(iii) *E has order continuous norm.*

Proof The equivalence (ii)$\Longleftrightarrow$(iii)is given in Theorem 3.7, while the implication (iii)$\Longrightarrow$(i) follows from Corollary 5.4. The implication (i)$\Longrightarrow$(ii), follows from the fact that l_∞ fails the assertion of (i), as is noted above □

In the case of Banach lattices, the preceding characterisation may be found in [2] Corollary 2.4.

It is now possible to state the main result of this section.

Theorem 5.8 *Suppose that $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm. If $X \subseteq E$ is a closed subspace, then the following statements are equivalent.*

(i) *X is weakly sequentially complete.*
(ii) *X contains no isomorphic copy of c_0.*
(iii) *X has property (V^*)*

Proof It is clear that if X is weakly sequentially complete, then X can contain no isomorphic copy of c_0 and this establishes the implication (i)$\Longrightarrow$(ii). To see the implication (ii)$\Longrightarrow$(i), assume that X contains no isomorphic copy of c_0. If X is not weakly sequentially complete, there exists a $\sigma(X, X^*)$- Cauchy sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ which is not convergent for the weak topology $\sigma(X, X^*)$. Since E has order continuous norm, it follows from Theorem 3.5 that E has property (u), and so also X has property (u). Consequently, there exists a sequence $\{u_n\}_{n=1}^{\infty} \subseteq X$ such that $\sum_{n=1}^{\infty} |\phi(u_n)| < \infty$ for every $\phi \in X^*$ with the property that $x_n - \sum_{k=1}^{n} u_k \to_n 0$ weakly. It follows that the series $\sum_{n=1}^{\infty} u_n$ is not weakly convergent in X. By a well-known characterisation of Bessaga–Pelczynski (see [5, Theorem V.8]), it follows that X contains an isomorphic copy of c_0.

The implication (iii)$\Rightarrow$(i) has already been observed in Lemma 4.6.

(i)$\Rightarrow$(iii). It follows from Proposition 5.3 that every (V^*)-set in X is sequentially $\sigma(X, X^*)$-precompact. Since X is weakly sequentially complete, this implies that X has property (V^*) (again by Lemma 4.6). □

For separable subspaces of a Banach lattice with order continuous norm, the equivalence (ii)$\Longleftrightarrow$(iii) is shown in [23]. That the restriction of separability could be removed was noted in [16] Proposition 5.3.4. In the non-commutative setting, the assertion of Theorem 5.8 is new.

Proposition 5.9 *Suppose that $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm. If $X \subseteq E$ is a closed subspace, then the following statements are equivalent.*

(i) *X is reflexive.*
(ii) *X contains no copy of c_0 or l_1.*

Proof It is clear that if X is reflexive, then X can contain no isomorphic copy of either c_0 or l_1. It will suffice, therefore, to prove only the implication (ii)$\Longrightarrow$(i). Suppose then that X contains no copy of c_0 or l_1. Since X contains no copy of c_0, it follows from the preceding theorem that X is weakly sequentially complete. Since X contains no copy of l_1, Rosenthal's l_1- theorem (see [4, Chapter XI]) implies that the unit ball of X is sequentially $\sigma(X, X^*)$-precompact. Consequently the unit ball of X is weakly compact and so X is reflexive. □

In the Banach lattice setting, the preceding proposition is due to Tzafriri [26].

6 Order continuity of the norm and separability

We begin this section with an observation which must be surely well-known. Its proof is based on the fact that every positive functional φ in the predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ may be represented as

$$\varphi(x) = \sum_{i=1}^{\infty} \langle x\xi_i, \xi_i \rangle, \quad x \in \mathcal{M},$$

for some sequence $\{\xi_i\}_{i=1}^{\infty}$ in $\mathcal{H}$ satisfying $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$ (see, for example, [24, Proposition 3.20]).

Proposition 6.1 *If $\mathcal{M}$ is a von Neumann algebra acting in a separable Hilbert space $\mathcal{H}$, then its predual $\mathcal{M}_*$ is a separable Banach space.*

We state some simple consequences.

Corollary 6.2 *If $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, then:*

(i) *the Banach space $L^1(\tau)$ is separable;*
(ii) *the space $L^1(\tau)$ is separable with respect to the measure topology;*
(iii) *the set $\mathcal{F}(\tau) \cap P(\mathcal{M})$ is separable with respect to the measure topology.*

Proof Via trace duality, the space $L_1(\tau)$ is isometrically isomorphic with $\mathcal{M}_*$ and so, (i) follows immediately from Proposition 6.1. Since the embedding of $L_1(\tau)$ into $S(\tau)$, equipped with the measure topology, is continuous, (ii) is also clear. Statement (iii) is now evident, as $\mathcal{F}(\tau) \cap P(\mathcal{M})$ is a subset of $L^1(\tau)$. □

Remark 6.3 Suppose that $(\mathcal{M}, \tau)$ is a semi-finite von Neumann algebra. If $\mathcal{F}(\tau) \cap P(\mathcal{M})$ is separable for the measure topology, then $\mathcal{F}(\tau)$ is also separable for the measure topology. Indeed, it clearly follows that $\operatorname{span} P(\mathcal{M})$ is separable. If $0 \le x \in \mathcal{F}(\tau)$, then it follows from the spectral theorem that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\operatorname{span} P(\mathcal{M})$ such that $0 \le x_n \uparrow_n x$. Since, in particular, $x \in S_0(\tau)$, this implies that $x_n \to x$ with respect to the measure topology. This suffices to show that $\mathcal{F}(\tau)$ is separable.

Lemma 6.4 *Let $E \subseteq S(\tau)$ be a strongly symmetric space with order continuous norm. If $\{p_n\}_{n=1}^{\infty}$ is a sequence of finite trace projections in $P(\mathcal{M})$ such that $\tau(p_n) \to 0$ as $n \to \infty$, then $\|p_n\|_E \to 0$.*

Proof Suppose that $\|p_n\|_E \not\to 0$ as $n \to \infty$. Passing to a subsequence, if necessary, we may assume that $\tau(p_n) \le 2^{-n}$ and $\|p_n\|_E \ge \varepsilon$ for some $\varepsilon > 0$ and all n. Defining $q_n \in P(\mathcal{M})$ by $q_n = \bigvee_{k=n}^{\infty} p_k$, it follows that $q_n \downarrow_n$ and $\tau(q_n) \le 2^{-n+1}$ for all n. Hence, $q_n \in E$ and $q_n \downarrow_n 0$. By the order continuity of the norm, this implies that $\|q_n\|_E \downarrow_n 0$. Since $p_n \le q_n$, it is also clear that $\|p_n\|_E \le \|q_n\|_E$ for all n, which yields a contradiction. □

In the proof of the lemma which follows, the following notation will be used. For $\alpha \geq 0$, define the real functions f_α, g_α by setting $f_\alpha(t) = t \wedge \alpha$ and $g_\alpha(t) = (t-\alpha)^+$, $t \in [0, \infty)$. If $0 \leq x \in S(\tau)$, set $x \wedge \alpha \mathbf{1} = f_\alpha(x)$ and $(x - \alpha\mathbf{1})^+ = g_\alpha(x)$. Note that $x = x \wedge \alpha\mathbf{1} + (x - \alpha\mathbf{1})^+$. Further, for all $\alpha > 0$

$$\mu(f_\alpha(x)) = f_\alpha(\mu(x)), \quad \mu(g_\alpha(x)) = g_\alpha(\mu(x))$$

as follows from [10, Lemma 2.5 (iv)]. It follows that $(x - \alpha_k\mathbf{1})^+ \downarrow_k 0$ holds in $S(\tau)$ whenever $\alpha_k \uparrow_k \infty$. In fact, if $0 \leq z \in S(\tau)$ satisfies $0 \leq z \leq (x - \alpha_k\mathbf{1})^+$ for all k, then

$$\mu(z) \leq \mu((x - \alpha_k\mathbf{1})^+) = (\mu(x) - \alpha_k)^+) \downarrow_k 0,$$

and this clearly implies that $z = 0$. Similarly, if $\alpha_k \downarrow_k 0$, then $x \wedge \alpha_k\mathbf{1} \downarrow_k 0$ in $S(\tau)$.

Lemma 6.5 *Let $E \subseteq S(\tau)$ be a strongly symmetric space with order continuous norm. Let $\{x_n\}_{n=1}^\infty$ be a sequence in E and $y \in E$ be such that $\mu(x_n) \leq \mu(y)$ for all n. If $x_n \to 0$ for the measure topology, then $\|x_n\|_E \to 0$.*

Proof For the proof of the lemma it may be assumed, without loss of generality, that $y \geq 0$ and $x_n \geq 0$ for all n. We assume first, in addition, that $0 \leq x_n \leq N\mathbf{1}$ for all n and some $N \in \mathbb{N}$. Given $\delta > 0$, we have

$$0 \leq x_n = x_n e^{x_n}[0, \delta] + x_n e^{x_n}(\delta, \infty) \leq x_n e^{x_n}[0, \delta] + N e^{x_n}(\delta, \infty).$$

It follows from $x_n \to 0$ in measure that $\tau(e^{x_n}(\delta, \infty)) \to 0$ and so, by Lemma 6.4, $\|e^{x_n}(\delta, \infty)\|_E \to 0$ as $n \to \infty$. Hence,

$$\limsup_{n\to\infty} \|x_n\|_E \leq \limsup_{n\to\infty} \|x_n e^{x_n}[0, \delta]\|_E.$$

Since $\mu(x_n e^{x_n}[0, \delta]) \leq \delta$ and $\mu(x_n e^{x_n}[0, \delta]) \leq \mu(y)$, it follows that

$$\mu(x_n e^{x_n}[0, \delta]) \leq \mu(y) \wedge \delta\mathbf{1} = f_\delta(\mu(y)) = \mu(f_\delta(y)) = \mu(y \wedge \delta\mathbf{1})$$

and so, $\|x_n e^{x_n}[0, \delta]\|_E \leq \|y \wedge \delta\mathbf{1}\|_E$. Hence, $\limsup_{n\to\infty} \|x_n\|_E \leq \|y \wedge \delta\mathbf{1}\|_E$ for all $\delta > 0$. Since $y \wedge \delta\mathbf{1} \downarrow 0$ as $\delta \downarrow 0$, we have $\|y \wedge \delta\mathbf{1}\|_E \downarrow 0$ and so, $\limsup_{n\to\infty} \|x_n\|_E = 0$.

Assume now that $0 \leq x_n \in E$ and $0 \leq y \in E$ are such that $x_n \to 0$ in measure and $\mu(x_n) \leq \mu(y)$ for all n. Since $(y - N\mathbf{1})^+ \downarrow_N 0$ in E, it follows from the order continuity of the norm that $\|(y - N\mathbf{1})^+\|_E \downarrow_N 0$. Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $\|(y - N\mathbf{1})^+\|_E \leq \varepsilon/2$. For $n \in \mathbb{N}$ we have

$$\begin{aligned}\mu\left((x_n - N\mathbf{1})^+\right) &= \mu(g_N(x_n)) = g_N(\mu(x_n)) = (\mu(x_n) - N\mathbf{1})^+ \\ &\leq (\mu(y) - N\mathbf{1})^+ = \mu((y - N\mathbf{1})^+)\end{aligned}$$

and so, $\|(x_n - N\mathbf{1})^+\|_E \leq \|(y - N\mathbf{1})^+\|_E \leq \varepsilon/2$. Hence,

$$\begin{aligned}\|x_n\|_E &\leq \|x_n \wedge N\mathbf{1}\|_E + \|(x_n - N\mathbf{1})^+\|_E \\ &\leq \|x_n \wedge N\mathbf{1}\|_E + \varepsilon/2.\end{aligned}$$

for all n. Since $0 \leq x_n \wedge N\mathbf{1} \leq N\mathbf{1}$ for all n and $x_n \wedge N\mathbf{1} \to 0$ as $n \to \infty$ for the measure topology (as $\mu(x_n \wedge N\mathbf{1}) \leq \mu(x_n)$), it follows from the first part of the proof that $\|x_n \wedge N\mathbf{1}\|_E \to 0$ as $n \to \infty$. So, there is an $M \in \mathbb{N}$ such that $\|x_n \wedge N\mathbf{1}\|_E \leq \varepsilon/2$ for all $n \geq M$. Hence, $\|x_n\|_E \leq \varepsilon$ for all $n \geq M$. The proof is complete. □

The following is now a simple consequence of the above lemma.

Lemma 6.6 *Let $E \subseteq S(\tau)$ be a strongly symmetric space with order continuous norm. If $\{p_n\}_{n=1}^{\infty}$ is a sequence in $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ and $e \in P(\mathcal{M}) \cap \mathcal{F}(\tau)$ such that $p_n \to e$ with respect to the measure topology, then $\|e - p_n\|_E \to 0$ as $n \to \infty$.*

Proof It follows from $p_n \to e$ in measure, that

$$\chi_{[0,\tau(p_n))}(t) = \mu(t; p_n) \to \mu(t; e) = \chi_{[0,\tau(e))}(t), \quad n \to \infty,$$

for all $t > 0$ where $\mu(e)$ is continuous, that is, for all $t \neq \tau(e)$. Therefore, there exists $N \in \mathbb{N}$ such that $\tau(p_n) < \tau(e) + 1$ for all $n \geq N$. Hence, $\tau(e \vee p_n) \leq 2\tau(e) + 1$ for all $n \geq N$. Let $q \in P(\mathcal{M}) \cap \mathcal{F}(\tau)$ be such that $\tau(e \vee p_n) \leq \tau(q)$ for all $n \geq N$ (such a q always exists: if $\tau(\mathbf{1}) < \infty$, then we take $q = \mathbf{1}$; if $\tau(\mathbf{1}) = \infty$, then $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ contains projections of arbitrarily large trace). Since $s(e - p_n) \leq e \vee p_n$ and $\|e - p_n\|_\infty \leq 2$, it is now clear that $\mu(e - p_n) \leq \mu(2q)$ for all $n \geq N$. The statement of the lemma now follows from Lemma 6.5. □

Corollary 6.7 *Let $E \subseteq S(\tau)$ be a strongly symmetric space with order continuous norm. If $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ is separable with respect to the measure topology, then $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ is separable with respect to the norm topology in E.*

Proof If $\mathcal{C}$ is a countable subset of $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ which is dense in $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ for the measure topology, then it follows immediately from Lemma 6.6 that $\mathcal{C}$ is dense in $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ for the norm topology in E. □

Proposition 6.8 *If $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm, then the following two statements are equivalent:*

(i) *E is separable;*
(ii) *$P(\mathcal{M}) \cap \mathcal{F}(\tau)$ is separable for the measure topology.*

Proof (i)⇒(ii). Separability of E clearly implies that of $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ for the norm of E and hence, also for the measure topology, by the continuity of the embedding of E into $S(\tau)$.

(ii)⇒(i). By Corollary 6.7, the set $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ is norm separable. If $0 \leq x \in E$, then $0 \leq x \in S_0(\tau)$ (as the norm in E is order continuous) and hence, by the spectral theorem, there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\operatorname{span}(P(\mathcal{M}) \cap \mathcal{F}(\tau))$ such that $0 \leq x_n \uparrow x$. This implies that $\|x - x_n\|_E \to 0$ as $n \to \infty$. This shows that $\operatorname{span}(P(\mathcal{M}) \cap \mathcal{F}(\tau))$ is dense in E and we may conclude that E is separable. □

Proposition 6.9 *If $E \subseteq S(\tau)$ is a strongly symmetric space, then the following statements are equivalent:*

(i) *E has order continuous norm and $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ is separable for the measure topology;*
(ii) *E is separable.*

Proof The implication (i)⇒(ii) follows immediately from Proposition 6.8.

If E is separable, then E cannot contain a copy of ℓ_∞ and hence, E has order continuous norm (see [Theorem 3.8]). Furthermore, $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ is separable in E and hence with respect to the measure topology (as the embedding of E into $S(\tau)$ is continuous. This shows that (ii) implies (i). □

The following result is now an immediate consequence of Proposition 6.9 (and of Corollary 6.2).

Corollary 6.10 *Suppose that $P(\mathcal{M}) \cap \mathcal{F}(\tau)$ is separable for the measure topology (which is, in particular, the case if the underlying Hilbert space $\mathcal{H}$ is separable). If $E \subseteq S(\tau)$ is a strongly symmetric space, then the following statements are equivalent:*

(i) *E has order continuous norm;*
(ii) *E is separable.*

We conclude the paper with several results which relate separability to reflexivity.

Proposition 6.11 *Let $E \subseteq S(\tau)$ be a strongly symmetric space. If the Banach dual E^* is separable and if E has the Fatou property, then E is reflexive.*

Proof As is well-known, separability of E^* implies that of E. Theorem 3.7 now implies that the norm on E is order continuous. This implies that the Banach dual E^* coincides with the Köthe dual $E^\times$. Separability of E^* again implies that the norm on $E^\times = E^*$ is order continuous. Since E has the Fatou property and the norm on E is order continuous, it follows that E is a KB space. Since $E^\times$ has the Fatou property, order continuity of the norm on $E^\times$ implies that $E^\times$ is also a KB space and the reflexivity of E now follows from [8, Theorem 5.15]. □

Corollary 6.12 *Let $E \subseteq S(\tau)$ be a strongly symmetric space. If E^{**} is separable, then E is reflexive.*

Proof Since E^{**} is separable, it follows that each of E, E^* are separable. In particular, $E^* = E^\times \subseteq S(\tau)$ is a strongly symmetric space with the Fatou property and $(E^\times)^* = E^{**}$ is separable. It follows from Proposition 6.11 that E^*, and hence also E, is reflexive. □

Remarks Proposition 6.11 and Corollary 6.12 are valid for in the setting of Banach lattices and may be found in [14], where the Banach lattice counterpart of Corollary 6.12 is attributed to Ogasawara [18]. In the Banach function space setting, Proposition 6.11 goes back to the thesis of Luxemburg [13]. That Corollary 6.12 fails in the setting of general Banach spaces is a well-known result of R.C. James.

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