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Coagulation–Fragmentation Model for Animal Group-Size Statistics

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Abstract We study coagulation–fragmentation equations inspired by a simple model proposed in fisheries science to explain data for the size distribution of schools of pelagic fish. Although the equations lack detailed balance and admit no H-theorem, we are able to develop a rather complete description of equilibrium profiles and large-time behavior, based on recent developments in complex function theory for Bernstein and Pick functions. In the large-population continuum limit, a scaling-invariant regime is reached in which all equilibria are determined by a single scaling profile. This universal profile exhibits power-law behavior crossing over from exponent $-\frac{2}{3}$ for small size to $-\frac{3}{2}$ for large size, with an exponential cutoff.

Keywords Detailed balance · Fish schools · Bernstein functions · Complete monotonicity · Fuss–Catalan sequences · Convergence to equilibrium

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1 Introduction

A variety of methods have been used to account for the observed statistics of animal group size in population ecology. We refer to the works of Okubo (1986), Okubo and Levin (2001), and Couzin and Krause (2003) for extensive discussions of dynamical aspects of how animal groups form and evolve, including quantitative approaches. The present work focuses on coagulation–fragmentation equations, which model rates of merging and splitting of groups. These kinds of models seek to account for the observed frequency distribution of group sizes by simple rules involving the fission and fusion rates, which subsume all details of individual behavior and internal group structure.

The models that we study are particularly motivated by several works aimed at explaining observations in fisheries science that concern the size distribution of schools of pelagic fish, which roam in the mid-ocean (Bonabeau and Dagorn 1995; Bonabeau et al. 1998; Niwa 1996, 1998, 2003, 2004; Ma et al. 2011). For such creatures, it is plausible that group-size dynamics may be modeled as spatially homogeneous and dominated by random encounters. These and related works have also discussed data for other kinds of animal groups, including mammalian herds and flocks of geese (Hayakawa and Furuhashi 2012) and sparrows (Griesser et al. 2011).

Niwa (2003, 2004) reached the striking conclusion that a large amount of observational data indicates that the fish school-size distribution $(f_i)_{i\ge 1}$ is well described by a scaling relation of the form

$$f_i \propto \frac{1}{i_{\mathrm{av}}}\Phi\left(\frac{i}{i_{\mathrm{av}}}\right), \qquad i_{\mathrm{av}} = \frac{\sum_i i^2 f_i}{\sum_i i f_i}. \tag{1.1}$$

The scaling factor i_{av} is the expected group size averaged over individuals, and the universal profile Φ is highly non-Gaussian—instead it is a power law with an exponential cutoff at large size. Specifically Niwa proposed that

$$\Phi(x) = x^{-1}\exp\left(-x + \frac{1}{2}x\mathrm{e}^{-x}\right). \tag{1.2}$$

Earlier work had shown that group-size data for a variety of animals could be fit by distributions that crossover from power-law behavior at small sizes to exponential or Gaussian tails (Bonabeau et al. 1999; Sjöberg et al. 2000). In this respect, it is noteworthy that Niwa's distribution profile (1.2) has no fitting parameters. Nevertheless, (1.2) fits observed data rather well, as seen in Fig. 1.

Niwa discussed how his description (1.2) might be justified for pelagic fish in a couple of different ways. Of particular interest for us here is the fact that he performed kinetic Monte Carlo simulations of a coagulation–fragmentation or merging–splitting process with the following features:

- The ocean is modeled as a discrete set of sites that fish schools may occupy.
- Schools jump to a randomly chosen site at discrete time steps.

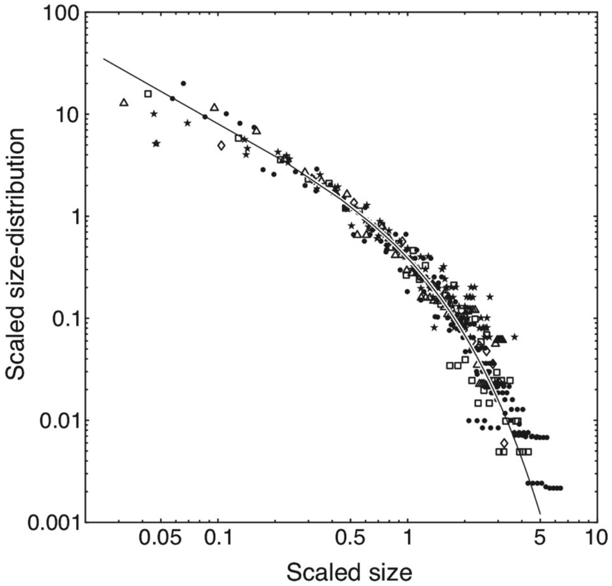

Fig. 1 Empirical school-size distribution of six types of pelagic fish. Data types and sources are listed in Niwa (2003, Table 1). Data are scaled by empirical average as in (1.1). *Solid line* corresponds to (1.2). Figure reprinted from *J. Theor. Biol.* 224, H. S. Niwa, Power-law versus exponential distributions of animal group sizes, 451–457, Copyright (2003), with permission from Elsevier

- Two schools arriving at the same site merge.
- Any school may split in two with fixed probability per time step,
  - independent of the school size i,
  - with uniform likelihood among the $i - 1$ splitting outcomes

$$(1, i - 1), (2, i - 2), \dots, (i - 1, 1).$$

Niwa's model is simple and compelling. It corresponds to mean-field coagulation–fragmentation equations with a constant rate kernel for coagulation and constant overall fragmentation rate; see Eqs. (2.1)–(2.3) and (2.7) in Sect. 2 below. Such equations were explicitly written and studied in a time-discrete form by Ma et al. in Ma et al. (2011). Yet the existing mathematical theory of coagulation–fragmentation equations reveals little about the nature of their equilibria and the dynamical behavior of solutions.

The reason for this dearth of theory is that the existing results that concern equilibria and long-time behavior almost all deal with systems that admit equilibria in detailed balance, with equal rates of merging and splitting for each reaction taking clusters of sizes i, j to one of size $i + j$. For modeling animal group sizes in particular, Gueron and Levin (1995) discussed several coagulation–fragmentation models for continuous-size distributions with explicit formulae for equilibria having detailed balance. However, Niwa argued explicitly in Niwa (2003) that the observational data for pelagic fish are inconsistent with these models.

Our purpose in the present paper is to carry out a mathematically rigorous investigation of coagulation–fragmentation equations motivated by Niwa's model and the work (Ma et al. 2011). For a continuous-size model, and for a discrete-size model that differs only slightly from that of Ma et al. (2011), we will describe the equilibria in detail, show that these solutions globally attract all solutions with finite total population, and establish convergence in the discrete-to-continuum limit.

Scale-invariant equilibria in the continuum limit. One of the principal contributions of the present paper is a demonstration that the coagulation–fragmentation system indeed achieves a scaling-invariant regime in the large-population, continuum limit. In this limit, all equilibrium size distributions $f_{\mathrm{eq}}(x)$, $x \in (0, \infty)$, are described rigorously in terms of a single scaling profile $\Phi_\star$, with

$$f_{\mathrm{eq}}(x) \propto \frac{1}{x_{\mathrm{av}}} \Phi_\star\left(\frac{x}{x_{\mathrm{av}}}\right), \qquad x_{\mathrm{av}} = \frac{\int_0^\infty x^2 f_{\mathrm{eq}}(x)\,\mathrm{d}x}{\int_0^\infty x\, f_{\mathrm{eq}}(x)\,\mathrm{d}x}. \tag{1.3}$$

The profile $\Phi_\star$ differs from (1.2) and appears to lack a simple expression. But it does admit an explicit infinite series representation (see Sect. 5.4), namely

$$\Phi_\star(x) = 2(6x)^{-2/3} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\frac{4}{3} - \frac{2}{3}n)} (6x)^{n/3}. \tag{1.4}$$

Further, we provide considerable qualitative information regarding the profile shape. In particular, we show that

$$\Phi_\star(x) = g(x)\,\mathrm{e}^{-\frac{8}{9}x}, \tag{1.5}$$

where g is a *completely monotone* function (infinitely differentiable with derivatives that alternate in sign), having the following asymptotic behavior:

$$g(x) \sim 6^{1/3} \frac{x^{-2/3}}{\Gamma(1/3)}, \qquad \text{when} \quad x \to 0, \tag{1.6}$$

$$g(x) \sim \frac{9}{8\sqrt{6}} \frac{x^{-3/2}}{\Gamma(1/2)}, \qquad \text{when} \quad x \to \infty. \tag{1.7}$$

Moreover, $\Phi_\star$ is a proper probability density, satisfying

$$1 = \int_0^\infty \Phi_\star(x)\,\mathrm{d}x = 6 \int_0^\infty x\Phi_\star(x)\,\mathrm{d}x = 6 \int_0^\infty x^2 \Phi_\star(x)\,\mathrm{d}x. \tag{1.8}$$

This characterization of equilibria in terms of the profile $\Phi_\star$ confirms Niwa's finding of scale invariance, as a rigorous consequence of coagulation–fragmentation modeling in the continuum limit. The power-law behavior exhibited by the exponential prefactor $g(x)$ changes as one goes from small to large group sizes, however, from $x^{-2/3}$ for small x to $x^{-3/2}$ for large x. This suggests that it could be difficult in practice to distinguish the profile in (1.5) from the expression in (1.2) with prefactor x^{-1}.

Indeed, in Fig. 2 we compare Niwa's profile in (1.2) to the new profile $\Phi_\star$ in (1.5) as computed using 45 terms from the desingularized power series in Sect. 5.4, and (5.7).

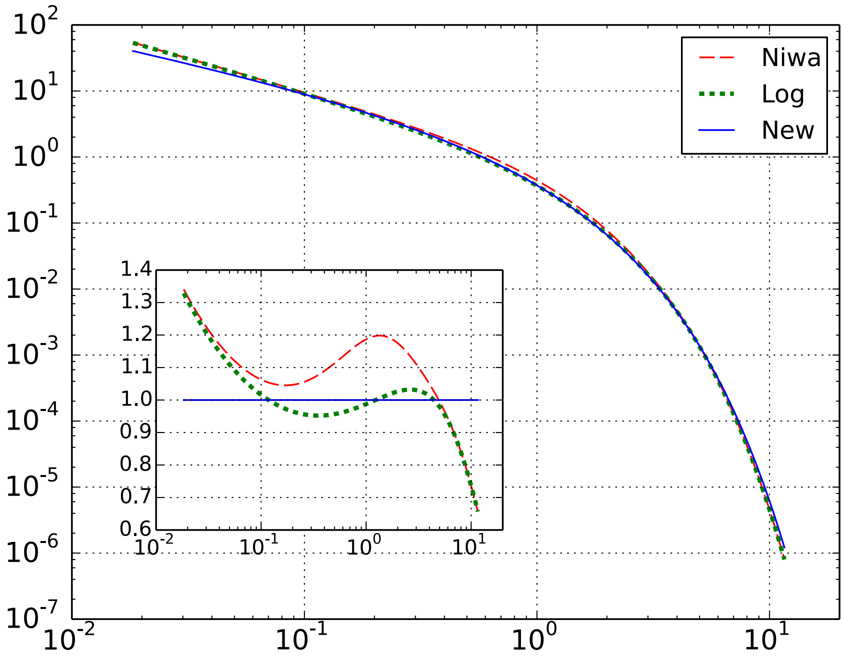

Fig. 2 Log–log plots of $\Phi(x)$ versus x for Niwa's, logarithmic, and new distribution profiles. *Inset* Ratios $\Phi/\Phi_\star$ for all three cases

The scaling of the log–log plot facilitates comparison with the scaled empirical data in Fig. 1 above (Niwa 2003, Figure 5), in which the solid line plots the profile (1.2). It is evident from this comparison that the new profile $\Phi_\star$ fits this data essentially as well as (1.2).

As reported in Ma et al. (2011), Niwa noted that as far as the quality of data fitting is concerned, the simple logarithmic distribution profile

$$\Phi(x) = x^{-1} \exp(-x) \tag{1.9}$$

serves about as well as (1.2)—the nested exponential term makes little difference. Thus, we also include this profile in Fig. 2 for comparison. The bulk of the empirical data in Fig. 1 corresponds to the "shoulder" region of the log–log plot, where the new profile differs by only a few percent from the logarithmic profile and by less than 20 percent from the profile in (1.2). This is so despite the difference in power-law exponents for small x and the slightly slower exponential decay rate in the tail of $\Phi_\star$ as compared to the other cases ($e^{-8x/9}$ vs. e^{-x}).

Dynamics without detailed balance. In addition to this description of equilibrium, we develop a rather complete theory of dynamics in the continuum limit, for weak solutions whose initial data are finite measures on $(0, \infty)$. We establish convergence to equilibrium for all solutions that correspond to finite total population (finite first moment). Furthermore, for initial data with infinite first moment, solutions converge to zero in a weak sense, meaning that the population concentrates in clusters whose size grows without bound as $t \to \infty$.

Previous mathematical studies of equilibria and dynamical behavior in coagulation–fragmentation models include work of Aizenman and Bak (1979), Carr (1992), Carr and Costa (1994), Laurençot and Mischler (2003), and Cañizo (2007), as well as a substantial literature related to Becker–Doering equations (which take into account only clustering events that involve the gain or loss of a single individual). These works all concern models having detailed balance and rely on some form of H theorem, or entropy/entropy-dissipation arguments. For models without detailed balance, there is work of Fournier and Mischler (2004), concerning initial data near equilibrium in discrete systems, and a recent study by Laurençot and Roessel (2015) of a model with a multiplicative coagulation rate kernel in critical balance with a fragmentation mechanism that produces an infinite number of fragments.

Arguments involving entropy are not available for the models that we need to treat here. Instead, it turns out to be possible to use methods from complex function theory related to the Laplace transform. Such methods have been used to great advantage to analyze the dynamics of pure coagulation equations with special rate kernels (Menon and Pego 2004, 2008; Laurençot and Roessel 2010, 2015).

Specifically what is relevant for the present work is the theory of Bernstein functions, as developed in the book of Schilling et al. (2010). In terms of the "Bernstein transform," the coagulation–fragmentation equation in the continuum limit transforms to a nonlocal integro-differential equation which turns out to permit a detailed analysis of equilibria and long-time dynamics.

A tractable size-discrete model. Unfortunately, the Bernstein transform does not appear to produce a tractable form for the discrete-size coagulation–fragmentation equations coming from Ma et al. (2011) that correspond to Niwa's merging–splitting process with the features described above (see Remark 10.1). We have found, however, that a simple change in the fragmentation mechanism allows one to reduce the Bernstein transform *exactly* to the same equation as obtained for the continuum limit. One only needs to change the splitting rule to assign uniform likelihood among the $i+1$ splitting outcomes

$$(0, i), (1, i-1), \dots, (i, 0).$$

In the extreme cases, of course, no splitting actually happens. This change effectively slows the fragmentation rate for small groups. However, the analysis becomes remarkably simpler, as we will see below.

In particular, for this discrete-size model, we can characterize its equilibrium distributions (which depend now on total population) in terms of completely monotone sequences with exponential cutoff. Whenever the total population is initially finite, every solution converges to equilibrium strongly with respect to a size-weighted norm. And for infinite total population, again the population concentrates in ever-larger clusters—the size distribution converges to zero pointwise while the zeroth moment goes to a nonzero constant.

Plan. The plan of the paper is as follows. In Sect. 2, we describe the coagulation–fragmentation models under study in both the discrete-size setting (Model D) and continuous-size setting (Model C). A summary of results from the theory of Bernstein functions appears in Sect. 3. Our analysis of Model C is carried out in sections 4–9 (Part I), and Model D is treated in sections 10–13 (Part II). Lastly, in Part III (sections 14–16) we relate Model D to a discretization of Model C and prove a discrete-to-continuum limit theorem.

2 Coagulation–Fragmentation Models D and C

In this section, we describe the coagulation–fragmentation mean-field rate equations that model Niwa's merging–splitting simulations, and the corresponding equations for both discrete-size and continuous-size distributions that we focus upon in this paper.

2.1 Discrete-Size Distributions

We begin with a general description of coagulation–fragmentation equations for a system consisting of clusters (i) having discrete sizes $i \in \mathbb{N}^* = \{1, 2, \ldots\}$. Clusters can merge or split according to the following reactions:

$$\begin{aligned} (i) + (j) &\xrightarrow{a_{i,j}} (i + j) \quad \text{(binary coagulation)}, \\ (i) + (j) &\xleftarrow{b_{i,j}} (i + j) \quad \text{(binary fragmentation)}. \end{aligned}$$

Here, $a_{i,j}$ is the coagulation rate (i.e., the probability that clusters (i) and (j) with (unordered) respective sizes i and j merge into the cluster $(i + j)$ of size $i + j$ per unit of time) and $b_{i,j}$ is the fragmentation rate (i.e., the probability that a cluster $(i + j)$ splits into two clusters (i) and (j) per unit of time). Both $a_{i,j}$ and $b_{i,j}$ are assumed nonnegative, and symmetric in i, j.

We focus on a statistical description of this system in terms of the number density $f_i(t)$ of clusters of size i at time t. The size distribution $f(t) = (f_i(t))_{i\in\mathbb{N}^*}$ evolves according to the discrete coagulation–fragmentation equations, written in strong form as follows:

$$\frac{\partial f_i}{\partial t}(t) = Q_a(f)_i(t) + Q_b(f)_i(t), \tag{2.1}$$

$$Q_a(f)_i(t) = \frac{1}{2}\sum_{j=1}^{i-1} a_{j,i-j}\, f_j(t)\, f_{i-j}(t) - \sum_{j=1}^{\infty} a_{i,j}\, f_i(t)\, f_j(t), \tag{2.2}$$

$$Q_b(f)_i(t) = \sum_{j=1}^{\infty} b_{i,j}\, f_{i+j}(t) - \frac{1}{2}\sum_{j=1}^{i-1} b_{j,i-j}\, f_i(t). \tag{2.3}$$

Here, the terms in $Q_a(f)_i(t)$ (resp. in $Q_b(f)_i(t)$) account for gain and loss of clusters of size i due to aggregation/coagulation (resp. breakup/fragmentation). It is often useful to write this system in weak form, requiring that for any suitable test function φ_i (in a class to be specified later),

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\varphi_i\, f_i(t)=\frac{1}{2}\sum_{i,j=1}^{\infty}\left(\varphi_{i+j}-\varphi_i-\varphi_j\right)\left(a_{i,j}\, f_i(t)\, f_j(t)-b_{i,j}\, f_{i+j}(t)\right). \quad (2.4)$$

This equation can be recast as follows, in a form more suitable for describing the continuous-size analog:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\varphi_i\, f_i(t) &= \frac{1}{2}\sum_{i,j=1}^{\infty}\left(\varphi_{i+j}-\varphi_i-\varphi_j\right)a_{i,j}\, f_i(t)\, f_j(t)\\ &\quad -\frac{1}{2}\sum_{i=2}^{\infty}\left(\sum_{j=1}^{i-1}\left(\varphi_i-\varphi_j-\varphi_{i-j}\right)b_{j,i-j}\right)f_i(t). \end{aligned} \quad (2.5)$$

Taking $\varphi_i = i$, we obtain the formal conservation of total population (corresponding to mass in physical systems):

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty} i\, f_i(t) = 0. \quad (2.6)$$

However, depending on the rates, it can happen that population decreases due to a flux to infinite size. This phenomenon is known as gelation, associated with formation of infinite-size clusters).

The model written in Ma et al. (2011) essentially corresponds to a specific choice of rate coefficients which we take in the form

$$a_{i,j} = \alpha, \qquad b_{i,j} = \frac{\beta}{i+j-1}. \quad (2.7)$$

With these coefficients, the coagulation rate α is independent of cluster sizes. Moreover, the overall fragmentation rate for the breakdown of clusters of size i into clusters of any smaller size is given by

$$\frac{1}{2}\sum_{j=1}^{i-1} b_{j,i-j} = \frac{1}{2}\beta.$$

This rate is constant, independent of i, and these clusters break into pairs with sizes $(1, i-1), (2, i-2), \ldots (i-1, 1)$ with equal probability.

As mentioned earlier, we have found that a variant of this model is far more accessible to analysis by the transform methods which we will employ. Namely, we can imagine that clusters of size i now split into pairs $(0, i), (1, i-1), \ldots (i, 0)$ with equal probability, and take the rate coefficients in the form

$$a_{i,j} = \alpha, \qquad b_{i,j} = \frac{\beta}{i+j+1}. \quad (2.8)$$

We refer to the coagulation–fragmentation equations (2.1)–(2.3) with the coefficients in (2.8) as **Model D** (D for discrete size). This model also arises in a natural way as a discrete approximation of Model C; see Sect. 14. The overall effective fragmentation rate for clusters of size i becomes $\frac{1}{2}\beta\frac{i-1}{i+1}$, because we do not actually have clusters of size zero.

In general, an equilibrium solution $f = (f_i)$ of Eqs. (2.1)–(2.3) is in *detailed balance* if

$$a_{i,j} f_i f_j = b_{i,j} f_{i+j} \tag{2.9}$$

for all $i, j \in \mathbb{N}^*$. It is easy to see that for neither choice of coefficients in (2.7) nor (2.8) do the equations admit an equilibrium in detailed balance.

2.2 Continuous-Size Distributions

To study systems with large populations and typical cluster sizes, some simplicity is gained by passing to a continuum model in which clusters (x) may assume a continuous range of sizes $x \in \mathbb{R}_+ = (0, \infty)$. The corresponding reactions are written schematically now as

$$(x) + (y) \xrightarrow{a(x,y)} (x + y) \quad \text{(binary coagulation)},$$
$$(x) + (y) \xleftarrow{b(x,y)} (x + y) \quad \text{(binary fragmentation)},$$

where $a(x, y)$ and $b(x, y)$ are the coagulation and fragmentation rates, respectively. Again, both a and b are nonnegative and symmetric.

The distribution of cluster sizes is now described in terms of a (cumulative) distribution function $F_t(x)$, which denotes the number density of clusters with size in $(0, x]$ at time t. According to probabilistic convention, we use the same notation F_t to denote the measure on $(0, \infty)$ with this distribution function; thus, we write

$$F_t(x) = \int_{(0,x]} F_t(\mathrm{d}x).$$

The measure F_t evolves according to the following size-continuous coagulation–fragmentation equation, which we write in weak form. One requires that for any suitable test function $\varphi(x)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_+} \varphi(x)\, F_t(\mathrm{d}x) &= \frac{1}{2}\int_{\mathbb{R}_+^2} \big(\varphi(x+y) - \varphi(x) - \varphi(y)\big) a(x, y)\, F_t(\mathrm{d}x)\, F_t(\mathrm{d}y) \\ &\quad - \frac{1}{2}\int_{\mathbb{R}_+} \left(\int_0^x \big(\varphi(x) - \varphi(y) - \varphi(x-y)\big)\, b(y, x-y)\,\mathrm{d}y\right) F_t(\mathrm{d}x).\end{aligned} \tag{2.10}$$

Taking $\varphi(x) = x$, we obtain the formal conservation of total population:

$$\frac{\mathrm{d}}{\mathrm{d}t} m_1(F_t) = 0, \tag{2.11}$$

where in general, we denote the kth moment of F_t by

$$m_k(F_t) := \int_{\mathbb{R}_+} x^k \, F_t(\mathrm{d}x).$$

However, again there might be loss of mass due to gelation, and now also mass flux to zero may be possible (shattering, associated with the breakup of clusters to zero-size dust).

The specific rate coefficients that we will study correspond to constant coagulation rates and constant overall binary fragmentation rates with uniform distribution of fragments are

$$a(x, y) = A, \qquad b(x, y) = \frac{B}{x + y}, \tag{2.12}$$

We refer to the coagulation–fragmentation equations (2.10) with these coefficients as **Model C** (C for continuous size).

For size distributions with density, written as $F_t(\mathrm{d}x) = f(x, t)\,\mathrm{d}x$, Model C is written formally in strong form as follows:

$$\frac{\partial f}{\partial t}(x, t) = A\, Q_a(f)(x, t) + B\, Q_b(f)(x, t), \tag{2.13}$$

$$Q_a(f)(x, t) = \frac{1}{2} \int_0^x f(y, t)\, f(x - y, t)\,\mathrm{d}y - f(x, t) \int_0^\infty f(y, t)\,\mathrm{d}y, \tag{2.14}$$

$$Q_b(f)(x, t) = -\frac{1}{2} f(x, t) + \int_x^\infty \frac{f(y, t)}{y}\,\mathrm{d}y. \tag{2.15}$$

To argue that there are no equilibrium solutions of Model C in detailed balance, we need a suitable definition of detailed balance for the equations (2.10) in weak form. In Sect. 9 below, we will propose such a definition, and verify that no finite measures on $(0, \infty)$ satisfy it.

2.3 Scaling Relations for Models D and C

By simple scalings, we can relate the solutions of Models D and C to solutions of the same models with conveniently chosen coefficients.

Let us begin with Model D. Suppose $\hat{f}(t) = (\hat{f}_i(t))_{i \in \mathbb{N}^*}$ is a solution of Model D for the particular coefficients $\alpha = \beta = 2$. Then, a solution of Model D for general coefficients $\alpha, \beta > 0$ is given by

$$f_i(t) := \frac{\beta}{\alpha}\, \hat{f}_i(\beta t/2). \tag{2.16}$$

For the purposes of analysis, then it suffices to treat the case $\alpha = \beta = 2$, as we assume below.

For Model C, we have a similar expression. If $\hat{F}_t(x)$ is a weak solution of Model C for particular coefficients $A = B = 2$, then a solution for general $A, B > 0$ is given by

$$F_t(x) := \frac{B}{A}\,\hat{F}_{Bt/2}(x). \tag{2.17}$$

Thus, it suffices to deal with the case $A = B = 2$ as below.

Importantly, Model C has an additional scaling invariance involving dilation of size. If $\hat{F}_t(x)$ is any solution, for any A and B, then

$$F_t(x) := \hat{F}_t(Lx) \tag{2.18}$$

is also a solution, for the same A and B. Moreover, if we suppose that $\hat{F}_t(x)$ has first moment $\hat{m}_1 = \int_0^\infty x\,\hat{F}_t(\mathrm{d}x) = 1$, then we can obtain a solution $F_t(x)$ with arbitrary finite first moment

$$m_1 = \int_0^\infty x\,F_t(\mathrm{d}x) \tag{2.19}$$

by taking $L = 1/m_1$, so that

$$F_t(x) = \hat{F}_t(x/m_1). \tag{2.20}$$

For solutions with (fixed) total population, it therefore suffices to analyze the case that $m_1 = 1$.

3 Bernstein Functions and Transforms

Throughout this paper, we make use of various results from the theory of Bernstein functions, as laid out in the book (Schilling et al. 2010). We summarize here a number of key properties of Bernstein functions that we need in the sequel.

Recall that a function $g\colon (0,\infty) \to \mathbb{R}$ is *completely monotone* if it is infinitely differentiable and its derivatives satisfy $(-1)^n g^{(n)}(x) \ge 0$ for all real $x > 0$ and integer $n \ge 0$. By Bernstein's theorem, g is completely monotone if and only if it is the Laplace transform of some (Radon) measure on $[0,\infty)$.

Definition 3.1 A function $U\colon (0,\infty) \to \mathbb{R}$ is a *Bernstein function* if it is infinitely differentiable, nonnegative, and its derivative U' is completely monotone.

The main representation theorem for these functions (Schilling et al. 2010, Thm. 3.2) associates to each Bernstein function U a unique *Lévy triple* (a_0, a_∞, F) as follows. (Below, the notation $a \wedge b$ means the minimum of a and b.)

Theorem 3.2 *A function $U\colon (0,\infty) \to \mathbb{R}$ is a Bernstein function if and only if it has the representation*

$$U(s) = a_0 s + a_\infty + \int_{(0,\infty)} (1 - e^{-sx})\,F(\mathrm{d}x), \quad s \in (0,\infty), \tag{3.1}$$

where a_0, $a_\infty \ge 0$ and F is a measure satisfying

$$\int_{(0,\infty)} (x \wedge 1) F(\mathrm{d}x) < \infty. \tag{3.2}$$

In particular, the triple (a_0, a_∞, F) uniquely determines U and vice versa.

We point out that U determines a_0 and a_∞ via the relations

$$a_0 = \lim_{s\to\infty} \frac{U(s)}{s}, \qquad a_\infty = U(0^+) = \lim_{s\to 0} U(s). \tag{3.3}$$

Definition 3.3 Whenever (3.1) holds, we call U the **Bernstein transform** of the Lévy triple (a_0, a_∞, F). If $a_0 = a_\infty = 0$, we call U the Bernstein transform of the Lévy measure F, and write $U = \breve{F}$.

Many basic properties of Bernstein functions follow from Laplace transform theory. Yet Bernstein functions have beautiful and distinctive properties that are worth delineating separately. The second statement in the following proposition is proved in Lemma 2.3 of Iyer et al. (2015).

Proposition 3.4 *The composition of any two Bernstein functions is Bernstein. Moreover, if $V\colon (0,\infty) \to (0,\infty)$ is bijective and V' is Bernstein, then the inverse function V^{-1} is Bernstein.*

Topologies. Any pointwise limit of a sequence of Bernstein functions is Bernstein. The topology of this pointwise convergence corresponds to a notion of weak convergence related to the associated Lévy triples in a way that is not fully characterized in Schilling et al. (2010), but may be described as follows. Let $\mathcal{M}_+[0,\infty]$ denote the space of nonnegative finite (Radon) measures on the compactified half-line $[0,\infty]$.

To each Lévy triple (a_0, a_∞, F) we associate a finite measure $\kappa \in \mathcal{M}_+[0,\infty]$ by appending atoms of magnitude a_0 and a_∞ respectively at 0 and ∞ to the finite measure $(x \wedge 1)F(\mathrm{d}x)$, writing

$$\kappa(\mathrm{d}x) = a_0\delta_0 + a_\infty\delta_\infty + (x \wedge 1)F(\mathrm{d}x). \tag{3.4}$$

The correspondence $(a_0, a_\infty, F) \mapsto \kappa$ is bijective, as is the correspondence $(a_0, a_\infty, F) \to U$. We say that a sequence of finite measures κ_n *converges weakly* on $[0,\infty]$ to κ and write $\kappa_n \xrightarrow{w} \kappa$ if for each continuous $g\colon [0,\infty] \to \mathbb{R}$,

$$\int_{[0,\infty]} g(x)\,\kappa_n(\mathrm{d}x) \to \int_{[0,\infty]} g(x)\,\kappa(\mathrm{d}x) \quad \text{as } n \to \infty.$$

Theorem 3.5 *Let $(a_0^{(n)}, a_\infty^{(n)}, F^{(n)})$, be a sequence of Lévy triples, with corresponding Bernstein transforms U_n and κ-measures κ_n on $[0,\infty]$. Then, the following are equivalent:*

(i) $U(s) := \lim_{n\to\infty} U_n(s)$ exists for each $s \in (0,\infty)$.
(ii) $\kappa_n \xrightarrow{w} \kappa$ as $n \to \infty$, where κ is a finite measure on $[0,\infty]$.

If either (i) or (ii) hold, then the respective limit U, κ is associated with a unique Lévy triple (a_0, a_∞, F).

This result is essentially a restatement of Theorem 3.1 in Menon and Pego (2008), where different terminology is used. A simpler, direct proof will appear in Iyer et al. (in preparation), however.

For finite measures on $[0,\infty)$, the topology of pointwise convergence of Bernstein functions is related to more usual notions of weak convergence as follows. Let $\mathcal{M}_+[0,\infty)$ be the space of nonnegative finite (Radon) measures on $[0,\infty)$. Given F, $F_n \in \mathcal{M}_+[0,\infty)$ for $n \in \mathbb{N}^*$, we say F_n converges to F *vaguely* on $[0,\infty)$ and write $F_n \xrightarrow{v} F$ provided

$$\int_{[0,\infty)} g(x)\, F_n(\mathrm{d}x) \to \int_{[0,\infty)} g(x)\, F(\mathrm{d}x) \tag{3.5}$$

for all functions $g \in C_0[0,\infty)$, the space of continuous functions on $[0,\infty)$ with limit zero at infinity. We say F_n converges to F *narrowly* and write $F_n \xrightarrow{n} F$ if (3.5) holds for all $g \in C_b[0,\infty)$, the space of bounded continuous functions on $[0,\infty)$. As is well known (Klenke 2014, p. 264, Thm. 13.35), $F_n \xrightarrow{n} F$ if and only if both $F_n \xrightarrow{v} F$ and $m_0(F_n) \to m_0(F)$.

Denote the Laplace transform of any $F \in \mathcal{M}_+[0,\infty)$ by

$$\mathcal{L}F(s) = \int_{[0,\infty)} \mathrm{e}^{-sx} F(\mathrm{d}x). \tag{3.6}$$

By the usual Laplace continuity theorem, $F_n \xrightarrow{v} F$ if and only if $\mathcal{L}F_n(s) \to \mathcal{L}F(s)$ for all $s \in (0,\infty)$. Regarding narrow convergence, we will make use of the following result for the space $\mathcal{M}_+(0,\infty)$ consisting of finite nonnegative measures on $(0,\infty)$. We regard elements of this space as elements of $\mathcal{M}_+[0,\infty)$ having no atom at 0.

Proposition 3.6 *Assume F, $F_n \in \mathcal{M}_+(0,\infty)$ for $n \in \mathbb{N}^*$. Then, the following are equivalent as $n \to \infty$.*

(i) F_n converges narrowly to F, i.e., $F_n \xrightarrow{n} F$.
(ii) The Laplace transforms $\mathcal{L}F_n(s) \to \mathcal{L}F(s)$, for each $s \in [0,\infty)$.
(iii) The Bernstein transforms $\breve{F}_n(s) \to \breve{F}(s)$, for each $s \in [0,\infty]$.
(iv) The Bernstein transforms $\breve{F}_n(s) \to \breve{F}(s)$, uniformly for $s \in (0,\infty)$.

Proof The equivalence of (i) and (ii) follows from the discussion above. Because

$$\breve{F}_n(s) = \int_{(0,\infty)} (1-\mathrm{e}^{-sx}) F_n(\mathrm{d}x), \qquad \breve{F}_n(\infty) = m_0(F_n) = \mathcal{L}F_n(0),$$

the equivalence of (ii) and (iii) is immediate. Now, the equivalence of (iii) and (iv) follows from an extension of Dini's theorem due to Pólya and Szegő (1998, Part II, Problem 127), because $\breve{F}_n$ and $\breve{F}$ are continuous and monotone on the compact interval $[0,\infty]$. □

Complete monotonicity. Our analysis of the equilibria of Model C relies on a striking result from the theory of the so-called *complete* Bernstein functions, as developed in Schilling et al. (2010, Chap. 6):

Theorem 3.7 *The following are equivalent.*

(i) The Lévy measure F in (3.1) has a completely monotone density g, so that

$$U(s) = a_0 s + a_\infty + \int_{(0,\infty)} (1-e^{-sx}) g(x)\, dx, \quad s \in (0,\infty). \tag{3.7}$$

(ii) U is a Bernstein function that admits a holomorphic extension to the cut plane $\mathbb{C}\backslash(-\infty, 0]$ satisfying $(\mathrm{Im}s)\mathrm{Im}U(s) \geq 0$.

In complex function theory, a function holomorphic on the upper half of the complex plane that leaves it invariant is called a *Pick function* or *Nevalinna-Pick function*. Condition (ii) of the theorem above says simply that U is a Pick function analytic and nonnegative on $(0, \infty)$. Such functions are called *complete Bernstein functions* in Schilling et al. (2010).

For our analysis of Model D, we will make use of an analogous criterion recently developed regarding sequences $(c_i)_{i\geq 0}$. Such a sequence is called completely monotone if all its backward differences are nonnegative:

$$(I - S)^k c_j \geq 0 \quad \text{for all} \quad j, k \geq 0, \tag{3.8}$$

where S denotes the backshift operator, $Sc_j = c_{j+1}$. The following result follows immediately from Corollary 1 in Liu and Pego (2016).

Theorem 3.8 *Let $c = (c_j)_{j\geq 0}$ be a real sequence with generating function*

$$G(z) = \sum_{j=0}^{\infty} c_j z^j. \tag{3.9}$$

Let $\lambda > 0$. Then, the following are equivalent.

(i) The sequence $(c_j \lambda^j)$ is completely monotone.
(ii) G is a Pick function that is analytic and nonnegative on $(-\infty, \lambda)$.

(Part I) Analysis of Model C

4 Equations for the Continuous-Size Model

In this part of the paper, we analyze the dynamics of Model C, requiring $A = B = 2$ as discussed above. Weak solutions of the model are then required to satisfy (the time-integrated version of) the equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} \varphi(x)\, F_t(\mathrm{d}x) &= \int_{\mathbb{R}_+^2} \big(\varphi(x+y) - \varphi(x) - \varphi(y)\big) F_t(\mathrm{d}x)\, F_t(\mathrm{d}y) \\ &\quad - \int_{\mathbb{R}_+} \frac{1}{x} \int_0^x \big(\varphi(x) - 2\varphi(y)\big)\, \mathrm{d}y\, F_t(\mathrm{d}x), \end{aligned} \tag{4.1}$$

for all continuous functions φ on $[0, \infty]$.

In successive sections to follow, we study the existence and uniqueness of weak solutions, characterize the equilibrium solutions, and study the long-time behavior of solutions for both finite and infinite total population size.

Bernstein transform. The results will be derived through study of the equation corresponding to (4.1) for the Bernstein transform

$$U(s,t) = \breve{F}_t(s) = \int_{\mathbb{R}_+} (1 - \mathrm{e}^{-sx})\, F_t(\mathrm{d}x). \tag{4.2}$$

We obtain the evolution equation for $\breve{F}_t$ by taking $\varphi_s(x) = 1 - \mathrm{e}^{-sx} = \varphi_1(sx)$ as test function in (4.1) and using the identities

$$\begin{aligned} -\varphi_s(x)\varphi_s(y) &= \varphi_s(x+y) - \varphi_s(x) - \varphi_s(y), \\ \frac{1}{x}\int_0^x \varphi_1(sy)\,\mathrm{d}y &= 1 - \frac{1}{sx}\varphi_1(sx) = \frac{1}{s}\int_0^s \varphi_1(rx)\,\mathrm{d}r. \end{aligned} \tag{4.3}$$

Then, for any weak solution F_t, the Bernstein transform must satisfy

$$\boxed{\frac{\partial U}{\partial t}(s,t) = -U^2 - U + \frac{2}{s}\int_0^s U(r,t)\,\mathrm{d}r.} \tag{4.4}$$

This equation is not as simple as in the case of pure coagulation studied in Menon and Pego (2004), without linear terms, but it turns out to be quite amenable to analysis.

Eq. (4.4) has a dilational symmetry corresponding to the symmetry that appears in (2.18). Namely, if $\hat{U}(s,t)$ is any solution of (4.4), then

$$U(s,t) = \hat{U}(s/L, t) \tag{4.5}$$

is another solution, for any constant $L > 0$.

Furthermore, the zeroth moment of any solution satisfies a logistic equation: By taking $\varphi \equiv 1$ in (4.1), we find that

$$m_0(F_t) = \int_{(0,\infty)} F_t(\mathrm{d}x) \tag{4.6}$$

satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} m_0(F_t) = -m_0(F_t)^2 + m_0(F_t). \tag{4.7}$$

5 Equilibrium Profiles for Model C

We begin the analysis of Model C by characterizing its equilibrium solutions. Due to (4.7), any nonzero steady-state solution $F_{\mathrm{eq}}(\mathrm{d}x)$ of Model C must have zeroth moment

$$m_0(F_{\mathrm{eq}}) = 1. \tag{5.1}$$

In this section, we prove the following theorem.

Theorem 5.1 *There is universal smooth profile $f_\star\colon \mathbb{R}_+ \to \mathbb{R}_+$, such that every nonzero steady-state solution F_{eq} of Model C has a smooth density, taking the form $F_{\text{eq}}(\mathrm{d}x) = f_{\text{eq}}(x)\,\mathrm{d}x$, where*

$$f_{\text{eq}}(x) = \frac{1}{\mu} f_\star\left(\frac{x}{\mu}\right), \tag{5.2}$$

with first moment

$$\mu = m_1(F_{\text{eq}}) = \int_{\mathbb{R}_+} x\, f_{\text{eq}}(x)\,\mathrm{d}x \in (0, \infty).$$

The universal profile $f_\star$ can be written as

$$f_\star(x) = \gamma_\star(x)\,\mathrm{e}^{-\frac{4}{27}x}, \tag{5.3}$$

where $\gamma_\star$ is a completely monotone function. Furthermore, $\gamma_\star$ has the following asymptotic behavior:

$$\gamma_\star(x) \sim \frac{x^{-2/3}}{\Gamma(1/3)}, \qquad \textit{when} \quad x \to 0, \tag{5.4}$$

$$\gamma_\star(x) \sim \frac{9}{8}\frac{x^{-3/2}}{\Gamma(1/2)}, \qquad \textit{when} \quad x \to \infty, \tag{5.5}$$

and the profile satisfies

$$1 = \int_0^\infty f_\star(x)\,\mathrm{d}x = \int_0^\infty x f_\star(x)\,dx = \frac{1}{6}\int_0^\infty x^2 f_\star(x)\,dx. \tag{5.6}$$

Remark 5.1 The profile $f_\star$ is related to the profile $\Phi_\star$ that appeared in (1.3) as follows. Given any equilibrium density $f_{\text{eq}}(x) = f_\star(x/m_1)/m_1$, with $m_2 = m_2(f_{\text{eq}})$ we compute $x_{\text{av}} = m_2/m_1 = 6m_1$, and therefore

$$f_{\text{eq}}(x) = \frac{6}{x_{\text{av}}} f_\star\left(\frac{6x}{x_{\text{av}}}\right).$$

Thus, we recover (1.3) and (1.5) with

$$\Phi_\star(x) = 6 f_\star(6x), \qquad g(x) = 6\gamma_\star(6x). \tag{5.7}$$

Remark 5.2 We mention that the same profiles as in (5.2) determine equilibrium solutions for coagulation–fragmentation equations with coagulation and fragmentation rates having the power-law form

$$a(x, y) = x^\alpha y^\alpha, \qquad b(x, y) = (x + y)^{\alpha - 1}. \tag{5.8}$$

Namely, for these rates, equilibria exist with the form

$$f_{\mathrm{eq}}(x) = \frac{x^{-\alpha}}{\mu} f_\star\left(\frac{x}{\mu}\right). \tag{5.9}$$

5.1 The Bernstein Transform at Equilibrium

The proof of Theorem 5.1 starts by determining the possible steady solutions of equation (4.4) for the Bernstein transform. A time-independent solution $U = \breve{F}_{\mathrm{eq}}$ of (4.4) must satisfy

$$U(s)^2 + U(s) = \frac{2}{s}\int_0^s U(r)\,dr. \tag{5.10}$$

Differentiating with respect to s and eliminating the integral yields

$$\frac{1}{s} = \frac{(2U+1)U'}{U(1-U)} = \frac{U'}{U} + \frac{3U'}{1-U}. \tag{5.11}$$

Because we must have $0 < U(s) < 1 = m_0(F)$ for all s, every relevant solution of this differential equation must satisfy $U(s) = U_\star(Cs)$ for some positive constant C, where

$$\boxed{\frac{U_\star(s)}{(1-U_\star(s))^3} = s.} \tag{5.12}$$

Because $U(0) = 0$ it follows necessarily that

$$m_1(F_{\mathrm{eq}}) = \int_{\mathbb{R}_+} x\, F_{\mathrm{eq}}(\mathrm{d}x) = U'(0^+) = C. \tag{5.13}$$

Thus, in order to prove the theorem, it is necessary to show that the solution $U_\star$ of (5.12) is a Bernstein function taking the form

$$U_\star(s) = \breve{f}_\star(s) = \int_0^\infty (1-\mathrm{e}^{-sx})\gamma_\star(x)\mathrm{e}^{-\frac{4}{27}x}\,\mathrm{d}x \tag{5.14}$$

where $\gamma_\star$ has the properties as stated. (We note that $-\frac{4}{27}$ is the minimum value of the function $u \mapsto u/(1-u)^3$, at $u = -\frac{1}{2}$.)

Remark An explicit expression for the solution of (5.12) is given by

$$U_\star(s) = \frac{1}{\sqrt{s}}\left(\left(\frac{\sqrt{s+s_0}+\sqrt{s}}{2}\right)^{1/3} - \left(\frac{\sqrt{s+s_0}-\sqrt{s}}{2}\right)^{1/3}\right)^3, \tag{5.15}$$

with $s_0 = \frac{4}{27}$. But we have not managed to derive any significant consequences from this.

5.2 Relation to Fuss–Catalan numbers

The Bernstein transform $U_\star(s) = \breve{f}_\star(s)$ of the universal equilibrium profile for Model C turns out to be very closely related to the generating function of the *Fuss–Catalan numbers* with single parameter p, given by

$$B_p(z) = \sum_{n=0}^{\infty} \binom{pn+1}{n} \frac{z^n}{pn+1}. \tag{5.16}$$

This function is well known (Graham et al. 1994; Młotkowski 2010) to satisfy

$$B_p(z) = 1 + zB_p(z)^p. \tag{5.17}$$

Consequently, as there is a unique real solution to $X = 1 - sX^3$ for $s > 0$, we find from (5.12) that

$$B_3(-s) = 1 - U_\star(s) = \int_0^\infty e^{-sx} f_\star(x)\,dx, \tag{5.18}$$

and therefore

$$U_\star(s) = -\sum_{n=1}^{\infty} \binom{3n+1}{n} \frac{(-s)^n}{3n+1}. \tag{5.19}$$

The function B_3 has the following properties, established in Liu and Pego (2016, Lemma 3).

Lemma 5.2 *The function B_3 is a Pick function analytic on $\mathbb{C}\setminus\left[\frac{4}{27},\infty\right)$ and nonnegative on the real interval $\left(-\infty,\frac{4}{27}\right)$.*

5.3 Proof of Theorem 5.1

1. We infer from Lemma 5.2 that $U_\star$ is a Pick function analytic on $\left(-\frac{4}{27},\infty\right)$. Because $B_3\left(\frac{4}{27}\right) = \frac{3}{2}$ due to (5.17), we have

$$U_\star\left(-\frac{4}{27}\right) = -\frac{1}{2}. \tag{5.20}$$

We make the change of variables

$$V = U_\star + \frac{1}{2}, \qquad z = s + \frac{4}{27}. \tag{5.21}$$

Then, V is a Pick function analytic and nonnegative on $(0,\infty)$. Invoking Theorem 3.7, we deduce that V is a Bernstein function whose Lévy measure has a completely monotone density $\gamma_\star$. Note that $V(0^+) = 0$, and from (5.12) we have $U_\star(s) \to 1$ as $s \to \infty$, hence

$$\lim_{z\to\infty} \frac{V(z)}{z} = \lim_{s\to\infty} \frac{U_\star(s)}{s} = (1 - U_\star(\infty))^3 = 0.$$

Therefore, by Theorem 3.7 and (3.3) it follows that

$$V(z) = \int_0^\infty (1 - \mathrm{e}^{-zx})\gamma_\star(x)\,\mathrm{d}x, \tag{5.22}$$

2. We will analyze the asymptotic behavior of $\gamma_\star(x)$ using Tauberian arguments, starting with the range $x \sim 0$. Because $1 - U_\star \sim s^{-1/3}$ as $s \to \infty$ due to (5.12), we can write

$$\frac{3}{2} - V(z) = \int_0^\infty \mathrm{e}^{-zx}\gamma_\star(x)\,\mathrm{d}x \sim z^{-1/3} \qquad \text{as} \quad z \to \infty. \tag{5.23}$$

By a result that follows from Karamata's Tauberian theorem (Feller 1971, Thm. XIII.5.4) it follows

$$\gamma_\star(x) \sim \frac{x^{-2/3}}{\Gamma(1/3)} \qquad \text{as } x \to 0.$$

3. Next, we deal with limiting behavior as $x \to \infty$. Transformation of (5.12) yields

$$G(V(z)) = z, \qquad G(V) = \frac{16\,V^2\,(9 - 2V)}{27\,(3 - 2V)^3}. \tag{5.24}$$

We notice that

$$G(0) = G'(0) = 0 \,<\, G''(0) = 2\left(\frac{4}{9}\right)^2, \tag{5.25}$$

and that G is monotone increasing from $\left[0, \frac{3}{2}\right)$ onto $[0, +\infty)$. Denote by $G^{-1}(z)$ the inverse function of G, which is monotone increasing from $[0, +\infty)$ onto $\left[0, \frac{3}{2}\right)$. Eq. (5.24) is equivalent to saying $V(z) = G^{-1}(z)$.

Thanks to (5.12), we can use Taylor's theorem to write

$$G(V) = \left(\frac{4}{9}V\right)^2 (1 + h(V))^2, \qquad \text{as} \quad V \to 0,$$

where $h(V)$ is analytic in the neighborhood of $V = 0$ and is such that $h(0) = 0$. Now, introducing $\zeta = \sqrt{z}$ where the complex square root is taken with branch cut along the interval $(-\infty, 0)$, Eq. (5.24) is written

$$\frac{4}{9}\tilde{V}\,(1 + h(\tilde{V})) = \zeta,$$

where $\tilde{V}(\zeta) = V(z)$. This equation shows that $\tilde{V}$ is an analytic function of ζ in the neighborhood of $\zeta = 0$, such that

$$\tilde{V}(\zeta) = \frac{9}{4}\zeta + O(\zeta^2), \qquad \tilde{V}'(\zeta) = \frac{9}{4} + O(\zeta).$$

Going back to $V(z)$, we find that as $z \to 0$, $V(z) \sim \frac{9}{4}z^{1/2}$ and

$$\int_0^\infty \mathrm{e}^{-zx} x\gamma_\star(x)\,\mathrm{d}x = V'(z) \sim \frac{9}{8}z^{-1/2}. \tag{5.26}$$

Although the Tauberian theorem cited previously does not apply directly (because we do not know $x \mapsto x\gamma_\star(x)$ is monotone), the selection argument used in the proof of Theorem XIII.5.4 from Feller (1971) works without change. It follows

$$x\gamma_\star(x) \sim \frac{9}{8}\frac{x^{-1/2}}{\Gamma(1/2)} \qquad \text{as } x \to \infty.$$

This proves (5.5) and completes the characterization of $\gamma_\star$.

4. Finally, by using (5.22) we may express $U_\star$ in the form

$$U_\star(s) = \int_0^\infty \left(1 - \mathrm{e}^{-\left(s+\frac{4}{27}\right)x}\right)\gamma_\star(x)\,\mathrm{d}x - \frac{1}{2}. \tag{5.27}$$

Note that thanks to (5.22) we have

$$\int_0^\infty (1 - \mathrm{e}^{-\frac{4}{27}x})\,\gamma_\star(x)\,\mathrm{d}x = V(\frac{4}{27}) = U_\star(0) + \frac{1}{2} = \frac{1}{2}.$$

Therefore, we can recast (5.27) into:

$$\begin{aligned} U_\star(s) &= \int_0^\infty \left(1 - \mathrm{e}^{-\left(s+\frac{4}{27}\right)x}\right)\gamma_\star(x)\,\mathrm{d}x - \int_0^\infty (1 - \mathrm{e}^{-\frac{4}{27}x})\,\gamma_\star(x)\,\mathrm{d}x \\ &= \int_0^\infty \left(1 - \mathrm{e}^{-sx}\right)\mathrm{e}^{-\frac{4}{27}x}\,\gamma_\star(x)\,\mathrm{d}x. \end{aligned} \tag{5.28}$$

By the uniqueness property of Bernstein transforms, this proves that $F_{\mathrm{eq}}(\mathrm{d}x) = f_\star(x)\,\mathrm{d}x$ where $f_\star$ is given by (5.3) as claimed. It remains only to discuss the moment relations in (5.6). We already know

$$m_0(f_\star) = U_\star(\infty) = 1, \qquad m_1(f_\star) = U_\star'(0^+) = 1.$$

By multiplying (5.12) by $(1 - U_\star)^3$ and differentiating twice, one finds

$$m_2(f_\star) = -U_\star''(0^+) = 6.$$

This finishes the proof of Theorem 5.1. □

5.4 Series Expansion for the Equilibrium Profile

Start with the equation (5.12) and change variables via

$$s = z^{-3}, \qquad W(z) = 1 - U_\star(s) = \int_0^\infty \mathrm{e}^{-sx} f_\star(x)\,\mathrm{d}x, \tag{5.29}$$

to obtain

$$\frac{W}{\phi(W)} = z, \qquad \phi(W) = (1 - W)^{1/3}.$$

By the Lagrange inversion formula (see Whittaker and Watson 1996, pp. 128–133; Henrici 1964), we find that

$$W = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n s^{-n/3}, \tag{5.30}$$

where na_n is the coefficient of w^{n-1} in the (binomial) series expansion of $\phi(w)^n$. Thus, we find

$$a_n = \frac{(-1)^{n-1}}{n}\binom{n/3}{n-1} = \frac{(-1)^{n-1}}{3(n-1)!}\frac{\Gamma(n/3)}{\Gamma(2-2n/3)}. \tag{5.31}$$

Term-by-term Laplace inversion using $\int_0^\infty \mathrm{e}^{-sx} x^{p-1}\,\mathrm{d}x = \Gamma(p)s^{-p}$ with $p = n/3$ yields

$$\boxed{f_\star(x) = \frac{x^{-2/3}}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma\left(\frac{4}{3}-\frac{2}{3}n\right)}\frac{x^{n/3}}{n!}.} \tag{5.32}$$

Every third term of this series vanishes because of the poles of the Gamma function at negative integers.

Replacing n by $3k$, $3k+1$, $3k+2$, respectively, yields zero in the last case, and the series reduces to

$$f_\star(x) = \frac{x^{-2/3}}{3}\sum_{k=0}^{\infty}\left(\frac{(-1)^k}{\Gamma(\frac{4}{3}-2k)}\frac{x^k}{(3k)!} + \frac{(-1)^{k+1}}{\Gamma(\frac{2}{3}-2k)}\frac{x^{k+1/3}}{(3k+1)!}\right). \tag{5.33}$$

6 Global Existence and Mass Conservation

Next, we deal with the initial-value problem for Model C. Although this can be studied using rather standard techniques from kinetic theory, we find it convenient to study this problem using Bernstein function theory.

We require solutions take values in the space $\mathcal{M}_+(0,\infty)$, which we recall to be the space of nonnegative finite measures on $(0,\infty)$. The main result of this section is the following.

Theorem 6.1 *Given any $F_{\text{in}} \in \mathcal{M}_+(0,\infty)$, there is a unique narrowly continuous map $t \mapsto F_t \in \mathcal{M}_+(0,\infty)$, $t \in (0,\infty)$ that satisfies (4.1). Furthermore,*

(i) If the first moment $m_1(F_{\text{in}}) < \infty$, then

$$m_1(F_t) = m_1(F_{\text{in}}) \tag{6.1}$$

for all $t \in [0,\infty)$, and $t \mapsto x\,F_t(dx) \in \mathcal{M}_+(0,\infty)$ is narrowly continuous.

(ii) If $F_{\text{in}}(x) = f_{\text{in}}(x)\,dx$ where f_{in} is a completely monotone function, then for any $t \geq 0$, $F_t(dx) = f_t(x)\,dx$ where f_t is also a completely monotone function.

Proof We shall construct the solution F_t from an unconditionally stable approximation scheme restricted to a discrete set of times $t_n = n\Delta t$. We approximate F_{t_n} by measures F_n, discretizing the gain term from fragmentation explicitly, and the rest of the terms implicitly. We require that for every test function φ continuous on $[0, \infty]$,

$$\begin{aligned} &\frac{1}{\Delta t}\int_{\mathbb{R}_+} \varphi(x)\,(F_{n+1}(\mathrm{d}x) - F_n(\mathrm{d}x)) \\ &\quad = \int_{\mathbb{R}_+^2} \big(\varphi(x+y) - \varphi(x) - \varphi(y)\big)\, F_{n+1}(\mathrm{d}x)\, F_{n+1}(\mathrm{d}y) \\ &\qquad - \int_{\mathbb{R}_+} \varphi(x)\, F_{n+1}(\mathrm{d}x) + \int_{\mathbb{R}_+} \frac{2}{x}\int_0^x \varphi(y)\,\mathrm{d}y\, F_n(\mathrm{d}x). \end{aligned} \tag{6.2}$$

Taking $\varphi(x) = 1 - \mathrm{e}^{-sx}$ in particular, the above scheme reduces to the following implicit–explicit scheme for Eq. (4.4) for the Bernstein transform $U_n = \breve{F}_n$: Given U_n, first compute

$$\hat{U}_n(s) = U_n(s) + 2\Delta t\int_0^1 U_n(s\tau)\,\mathrm{d}\tau, \tag{6.3}$$

then determine $U_{n+1}(s)$ by solving

$$(1+\Delta t)U_{n+1}(s) + \Delta t\, U_{n+1}(s)^2 = \hat{U}_n(s). \tag{6.4}$$

In order to construct a solution to the difference scheme (6.2), we first show by induction that for the solution of this scheme in (6.3)–(6.4), U_n is a Bernstein function for all $n \geq 0$. Naturally, $U_0 = \breve{F}_{\text{in}}$ is Bernstein, so suppose U_n is Bernstein for some $n \geq 0$. Because the set of Bernstein functions is a convex cone that is closed under the topology of pointwise convergence, it is clear that $\hat{U}_n$ is Bernstein. Now, the function

$$u \mapsto v = G_{\Delta t}(u) := (1+\Delta t)u + \Delta t\, u^2$$

is bijective on $(0, \infty)$ and its inverse is given by

$$u = G_{\Delta t}^{-1}(v) = -\alpha + \sqrt{\alpha^2 + v/\Delta t}, \qquad \alpha = \frac{1+\Delta t}{2\Delta t}. \tag{6.5}$$

This function is Bernstein, and the composition $U_{n+1} = G_{\Delta t}^{-1} \circ \hat{U}_n$ is Bernstein, by Proposition 3.4.

Because $U_n(s)$ is increasing in s, it is clear that $\hat{U}_n(s) \leq (1+2\Delta t)U_n(s)$, and therefore that for all $n > 0$ and $s \in (0, \infty]$,

$$U_n(s) \leq \frac{1+2\Delta t}{1+\Delta t}U_{n-1}(s) \leq \mathrm{e}^{\Delta t}U_{n-1}(s) \leq \mathrm{e}^{n\Delta t}U_0(s). \tag{6.6}$$

Therefore, for each $n \geq 0$, we have

$$U_n(0) = 0, \qquad \lim_{s\to\infty} \frac{U_n(s)}{s} = 0,$$
$$U_n(\infty) \leq \mathrm{e}^{n\Delta t} U_0(\infty) = \mathrm{e}^{n\Delta t} m_0(F_{\mathrm{in}}).$$

By (3.3), it follows $U_n = \breve{F}_n$ for some finite measure $F_n \in \mathcal{M}_+(0, \infty)$.

We note that by the concavity of U_n and the bound (6.6), we have the following uniform bound on derivatives of U_n: for any positive $\sigma, \tau > 0$, as long as $(n+1)\Delta t \leq \tau$ and $s \in [\sigma, \infty)$,

$$U_n'(s) \leq \frac{U_n(s)}{s} \leq \mathrm{e}^{\tau} \frac{U_0(s)}{s} \leq \mathrm{e}^{\tau} \frac{U_0(\sigma)}{\sigma}, \tag{6.7}$$

and

$$\frac{1}{\Delta t} |U_{n+1}(s) - U_n(s)| \leq U_{n+1}(s) + U_{n+1}(s)^2 + 2U_n(s) \leq C(\tau). \tag{6.8}$$

Due to these bounds, the piecewise-linear interpolant in time is Lipschitz continuous, uniformly for (s, t) in any compact subset of $(0, \infty) \times [0, \infty)$. By passing to a subsequence, using the Arzela–Ascoli theorem and a diagonal extraction argument, we obtain a limit $U(s, t)$ continuous on $(0, \infty) \times [0, \infty)$ which is Bernstein for each fixed $t \geq 0$ and satisfies the time-integrated form of Eq. (4.4):

$$U(s,t) = U_0(s) + \int_0^t \left(-U^2 - U + \frac{2}{s} \int_0^s U(r,\tau)\,\mathrm{d}r \right) \mathrm{d}\tau. \tag{6.9}$$

By consequence, (4.4) holds. From (6.6), the function $U(s, t)$ inherits the bounds

$$U(s,t) \leq \mathrm{e}^t U_0(s), \tag{6.10}$$

and we conclude

$$U(0,t) = 0, \qquad \lim_{s\to\infty} \frac{U(s,t)}{s} = 0,$$
$$U(\infty,t) \leq \mathrm{e}^t m_0(F_{\mathrm{in}}).$$

As previously, by (3.3) we infer that for each $t \geq 0$, $U(s,t) = \breve{F}_t$ for some finite measure $F_t \in \mathcal{M}_+(0, \infty)$, and $U(\infty, t) = m_0(F_t)$.

We may now deduce (4.7) by taking $s \to \infty$ in (6.9). (Indeed, because $U(s, t)$ increases as $s \to \infty$ toward $m_0(F_t)$, one finds $s^{-1} \int_0^s U(r,\tau)\,\mathrm{d}r \to m_0(F_\tau)$ for each $\tau > 0$.) Because $t \mapsto \breve{F}_t(s)$ is continuous for each $s \in [0, \infty]$, we may now invoke Proposition 3.6 to conclude that $t \mapsto F_t$ is narrowly continuous.

At this point, we know that (4.1) holds for each test function of the form $\varphi(x) = 1 - \mathrm{e}^{-sx}$, $s \in (0, \infty]$. Linear combinations of these functions are dense in the space of continuous functions on $[0, \infty]$. (This follows by using the homeomorphism $[0, \infty] \to [0, 1]$ given by $x \mapsto \mathrm{e}^{-x}$ together with the Weierstrass approximation theorem.) Then, because $m_0(F_t)$ is uniformly bounded, it is clear that one obtains (4.1) for arbitrary continuous φ on $[0, \infty]$ by approximation.

It remains to prove the further statements in (i) and (ii) of the Theorem. Suppose $m_1(F_{\text{in}}) < \infty$. Then, we claim that the first moments of the sequence (F_n) remain constant:

$$m_1(F_n) = U_n'(0) = U_0'(0) = m_1(F_{\text{in}}) \tag{6.11}$$

for all $n \geq 0$. The proof by induction is as follows. Differentiating (6.3)–(6.4) we have

$$\hat{U}_n'(s) = U_n'(s) + 2\Delta t \int_0^1 U_n'(s\tau)\tau \,\mathrm{d}\tau \to (1 + \Delta t)U_n'(0) \tag{6.12}$$

as $s \to 0$, and

$$(1 + \Delta t + 2\Delta t\, U_{n+1}(s))U_{n+1}'(s) = \hat{U}_n'(s) \to (1 + \Delta t)U_{n+1}'(0). \tag{6.13}$$

The claim follows.

Next, we claim that the first moment of the limit F_t is also constant in time. To show this, first note that $U_n(s)/s \leq U_n'(0) = m_1(F_{\text{in}})$. Taking $\Delta t \to 0$ we infer $U(s, t)/s \leq m_1(F_{\text{in}})$, and taking $s \to 0$ we get

$$m_1(F_t) = \partial_s U(0, t) \leq m_1(F_{\text{in}}).$$

Next note that because $U_n'(s)$ is decreasing in s, (6.12)–(6.13) imply

$$(1 + \Delta t + 2\Delta t\, U_{n+1}(s))U_{n+1}'(s) \geq (1 + \Delta t)U_n'(s). \tag{6.14}$$

Then, by the bound (6.6), for $(n + 1)\Delta t \leq \tau$ we have

$$\exp(2\Delta t \mathrm{e}^\tau U_0(s))\, U_{n+1}'(s) \geq (1 + 2\Delta t\, U_{n+1}(s))\, U_{n+1}'(s) \geq U_n'(s),$$

where the last inequality follows by dividing (6.14) by $1 + \Delta t$. Therefore,

$$\exp(2\tau \mathrm{e}^\tau U_0(s))\, U_n'(s) \geq U_0'(s). \tag{6.15}$$

This inequality persists in the limit, whence we infer by taking $s \to 0$ that $\partial_s U(0, t) \geq \partial_s U_0(0)$, meaning $m_1(F_t) \geq m_1(F_{\text{in}})$. This proves (6.1) holds.

Because $t \mapsto (x \wedge 1)F_t$ is weakly continuous on $[0, \infty]$ by the continuity theorem 3.5, and $m_1(F_t)$ is bounded, the finite measure $x\, F_t(\mathrm{d}x)$ is vaguely continuous, hence it is narrowly continuous because $m_1(F_t)$ remains constant in time.

Finally we prove (ii). The hypothesis implies that U_0 is a complete Bernstein function by Theorem 3.7. Complete Bernstein functions form a convex cone that is closed with respect to pointwise limits and composition, according to Schilling et al. (2010, Cor. 7.6).

We prove by induction that U_n is a complete Bernstein function for every $n \geq 0$. For if U_n has this property, then clearly $\hat{U}_n$ does. From the formula (6.5), $G_{\Delta t}^{-1}$ is a complete Bernstein function, as it is positive on $(0, \infty)$, analytic on $\mathbb{C}\setminus(-\infty, 0]$ and leaves the upper half plane invariant. Therefore, U_{n+1} is completely Bernstein.

Passing to the limit as $\Delta t \to 0$, we deduce that $U(\cdot, t)$ is completely Bernstein for each $t \geq 0$. By the representation theorem 3.7, we deduce that the measure F_t has a completely monotone density as stated in (ii).

This completes the proof of Theorem 6.1, except for uniqueness. Uniqueness is proved by a Gronwall argument very similar to that used below in Sect. 16 to study the discrete-to-continuum limit. We refer to Eqs. (16.1)–(16.5) and omit further details. □

7 Convergence to Equilibrium with Finite First Moment

In this section, we prove that any solution with finite first moment converges to equilibrium in a weak sense. Our main result is the following.

Theorem 7.1 *Suppose $F_t(dx)$ is any solution of Model C with initial data F_0 having finite first moment $m_1 = m_1(F_0)$. Then, we have*

$$F_t(dx) \xrightarrow{n} \frac{1}{m_1} f_\star(x/m_1)\, dx \tag{7.1}$$

and

$$x F_t(dx) \xrightarrow{n} \frac{1}{m_1} x f_\star(x/m_1)\, dx \tag{7.2}$$

narrowly, as $t \to \infty$.

Due to the scaling invariance of Model C, we may assume $m_1 = 1$ without loss of generality. To prove the theorem, the main step is to study Eq. (4.4) satisfied by the Bernstein transform $U(s, t) = \breve{F}_t(s)$, and prove

$$\breve{F}_t(s) \to \breve{f}_\star(s) \quad \text{as } t \to \infty, \ \text{for all } s \in [0, \infty). \tag{7.3}$$

We make use of the following proposition which will also facilitate the treatment of the discrete-size Model D later.

Proposition 7.2 *Let $\bar{s} > 0$, and suppose $U(s, t)$ is a C^1 solution of (4.4) for $0 < s < \bar{s}$, $0 \leq t < \infty$, such that for every $t \geq 0$, $s \mapsto U(s, t)$ is increasing and concave, with $U(0^+, t) = 0$ and $\partial_s U(0^+, t) = 1$. Then, for all $s \in (0, \bar{s})$,*

$$U(s, t) \to U_\star(s) = \breve{f}_\star(s) \quad \textit{as } t \to \infty.$$

Proof *Step 1: Change variables.* We write $U_0(s) = U(s, 0)$ and

$$U(s, t) = s(1 - v(s, t)), \quad U_\star(s) = s(1 - v_\star(s)), \quad U_0(s) = s(1 - v_0(s)). \tag{7.4}$$

Note $s \mapsto U(s, t)/s$ decreases, so $s \mapsto v(s, t)$ increases and $0 < v(s, t) < 1$. Writing

$$v(s, t) = v_\star(s) + w(s, t), \qquad v_0(s) = v_\star(s) + w_0(s), \tag{7.5}$$

we have

$$-v_\star(s) \le w(s,t) \le 1 - v_\star(s), \qquad w_0(s) = v_0(s) - v_\star(s) < v_0(s), \tag{7.6}$$

for all $s, t > 0$. Because $\partial_t U_\star = 0$, we find w satisfies

$$\partial_t w(s,t) = s(w + 2v_\star - 2)w - w + \frac{2}{s^2}\int_0^s w(r,t)\, r \,\mathrm{d}r. \tag{7.7}$$

□

Step 2: Upper barrier. Our proof of convergence is based on comparison principles, which will be established with the aid of two lemmas.

Lemma 7.3 *There exists a decreasing function $\bar{b}(t) \to 0$ as $t \to \infty$ such that $\bar{b}(0) = 1$ and that for all $s \in (0, \bar{s})$ and $t > 0$,*

$$\partial_t \bar{w}(s,t) \ge -U_\star(s)\bar{w}(s,t), \quad \textit{where} \quad \bar{w}(s,t) := v_0(s) \wedge \bar{b}(t).$$

Proof of Lemma 7.3. It suffices to choose $\bar{b}(t)$ to solve an ODE of the form

$$\partial_t b(t) = -\bar{J}(b(t)), \qquad \bar{b}(0) = 1, \tag{7.8}$$

provided the function $\bar{J}$ is chosen positive so that for all $s > 0$,

$$b < v_0(s) \quad \text{implies} \quad \bar{J}(b) \le U_\star(s)b. \tag{7.9}$$

Let v_0^{-1} denote the inverse function. Then, v_0^{-1} is increasing on its domain $[0, 1)$, with the property

$$b < v_0(s) \quad \text{implies} \quad v_0^{-1}(b) < s \quad \text{implies} \quad U_\star(v_0^{-1}(b)) < U_\star(s).$$

We may then let

$$\bar{J}(b) = U_\star \circ \left(v_0^{-1}(b \wedge \frac{1}{2}) \right) b. \tag{7.10}$$

This expression is well defined for all $b > 0$ and works as desired. □

Lemma 7.4 *For all $s \in (0, \bar{s})$ and $t > 0$, $w(s,t) \le \bar{w}(s,t)$.*

Proof of Lemma 7.4. Let $\epsilon > 0$, $\bar{s} > 0$. We claim $w(s,t) < \bar{w}(s,t) + \epsilon$ for all $t \ge 0$, $0 \le s \le \bar{s}$. This is true at $t = 0$ with $\bar{b}(0) = 1$. Note $w(0,t) = 0$ for all t. Suppose that for some minimal $t > 0$,

$$w(s,t) = \bar{w}(s,t) + \epsilon \quad \text{with } 0 < s \le \bar{s}.$$

Then, $\partial_t w(s,t) \geq \partial_t \bar{w}(s,t)$. However, because $s \mapsto \bar{w}(s,t)$ is increasing, we have

$$\frac{2}{s^2}\int_0^s w(r,t)\, r\, \mathrm{d}r < \frac{2}{s^2}\int_0^s (\bar{w}(r,t)+\epsilon)\, r\, \mathrm{d}r < \bar{w}(s,t)+\epsilon = w(s,t),$$

Then

$$-w + \frac{2}{s^2}\int_0^s w(r,t)\, r\, \mathrm{d}r < 0,$$

so, because $0 < w + v_\star \leq 1$ and $s(v_\star(s)-1) = -U_\star(s) < 0$, we deduce from (7.7) that

$$\partial_t w(s,t) < s(v_\star(s)-1)\bar{w} = -U_\star(s)\bar{w}(s,t) \leq \partial_t \bar{w}(s,t) \leq \partial_t w(s,t).$$

This contradiction proves the claim. Because $\epsilon > 0$ and $\bar{s} > 0$ are arbitrary, the result of the Lemma follows. □

Step 3: Lower barrier. In a similar way (we omit details) we can find $\underline{b}(t)$ increasing with $\underline{b}(t) \to 0$ as $t \to \infty$ such that for all $s \in (0,\bar{s})$ and $t > 0$,

$$\partial_t \underline{w}(s,t) \leq -U_\star(s)\underline{w}(s,t), \quad \text{where} \quad \underline{w}(s,t) := (-v_\star(s)) \vee \underline{b}(t). \tag{7.11}$$

(Here $a \vee b$ means $\max(a,b)$.) Then, it follows $w(s,t) \geq \underline{w}(s,t)$ for all $s, t > 0$.

Now, because $\underline{w}(s,t) \leq w(s,t) \leq \bar{w}(s,t)$ and $\underline{w}(s,t), \bar{w}(s,t) \to 0$ as $t \to \infty$ for each $s > 0$, the convergence result for $U(s,t)$ in the Proposition follows. This finishes the proof of the Proposition. □

Proof of Theorem 7.1 The result of the Proposition applies to the Bernstein transform $U(s,t) = \breve{F}_t(s)$ for arbitrary $\bar{s} > 0$, and this yields (7.3). This finishes the main step of the proof, and it remains to deduce (7.1) and (7.2). We know that $m_0(F_t) = U(\infty,t) \to 1 = U_\star(\infty)$ as $t \to \infty$, hence by Proposition 3.6 it follows $F_t(\mathrm{d}x) \to f_\star(x)\,\mathrm{d}x$ narrowly as $t \to \infty$.

By consequence, because $m_1(F_t) \equiv 1$ is bounded, the measures $x F_t(\mathrm{d}x)$ converge to $x f_\star(x)\,\mathrm{d}x$ vaguely, and because $m_1(F_t) \equiv m_1(f_\star)$, this convergence also holds narrowly. This finishes the proof of the Theorem. □

8 Weak Convergence to Zero with Infinite First Moment

Next, we study the case with infinite first moment. If the initial data have infinite first moment, the solution converges to zero in a weak sense, with all clusters growing asymptotically to infinite size, loosely speaking.

Theorem 8.1 *Assume $m_1(F_{\mathrm{in}}) = \int_0^\infty x F_{\mathrm{in}}(dx) = \infty$. Then as $t \to \infty$,*

$$F_t(dx) \xrightarrow{v} 0$$

vaguely on $[0,\infty)$, while $m_0(F_t) \to 1$.

Proof The main step of the proof involves showing that

$$U(s,t) \to 1 \quad \text{as } t \to \infty, \qquad \text{for all} \quad s \in (0,\infty). \tag{8.1}$$

The assumption $m_1 = \infty$ implies $\partial_s U(0^+, 0) = \infty$. Then, for any positive constant k, we compare with a solution corresponding to $F_0(\mathrm{d}x) = ka^2 \mathrm{e}^{-ax}\,\mathrm{d}x$, whose Bernstein transform is

$$u_0(s) = \int_0^\infty (1 - \mathrm{e}^{-sx}) ka^2 \mathrm{e}^{-ax}\,\mathrm{d}x = \frac{kas}{a+s}.$$

Note that

$$u_0(0) = 0, \quad \partial_s u_0(0) = k, \tag{8.2}$$

and for a sufficiently small we have

$$u_0(s) < U_0(s) \quad \text{for all} \quad s > 0. \tag{8.3}$$

This is so because $u_0(s) \le k(s \wedge a)$, while U_0 is increasing with $U_0(s) \ge ks$ for all s sufficiently small.

Let $u(s,t)$ denote the solution of Eq. (4.4) with initial data $u(s,0) = u_0(s)$ for $s > 0$. By a comparison argument similar to that above, we have

$$u(s,t) \le U(s,t) \quad \text{for all } s, t > 0.$$

Because $u(s/k, t)$ is also a solution of (4.4), with first moment equal to 1, it follows from Theorem 7.1 that for every $s > 0$, $u(s/k,t) \to \breve{f}_\star(s)$ as $t \to \infty$, hence due to the dilational invariance of (4.4) we have

$$u(s,t) \to \breve{f}_\star(ks) \quad \text{as } t \to \infty, \ \text{for every } s > 0. \tag{8.4}$$

It follows that for every $s > 0$,

$$\liminf_{t\to\infty} U(s,t) \ge \breve{f}_\star(ks). \tag{8.5}$$

This is true for every $k > 0$, hence we infer

$$\liminf_{t\to\infty} U(s,t) \ge 1. \tag{8.6}$$

On the other hand, we know $U(s,t) \le m_0(t) \to 1$ from (4.7). Therefore, we conclude $U(s,t) \to 1$ as $t \to \infty$.

Because $m_0(F_t) = U(\infty, t) \to 1$ as $t \to \infty$, we have

$$m_0(F_t) - U(s,t) = \int_0^\infty \mathrm{e}^{-sx} F_t(\mathrm{d}x) \to 0 \quad \text{for all} \quad s \in (0,\infty).$$

Hence, $F_t(\mathrm{d}x) \xrightarrow{v} 0$ by the standard continuity theorem for Laplace transforms. □

9 No Detailed Balance for Model C

Here, we verify that the size-continuous Model C admits no equilibrium finite measure solution $F_*(\mathrm{d}x)$ supported on $\mathbb{R}_+$ that satisfies a natural weak form of the detailed balance condition. First, note that the detailed balance condition (2.9) for the discrete coagulation–fragmentation equations can be written in a weak form by requiring that for any bounded sequence $(\psi_{i,j})_{i,j\in\mathbb{N}^*}$

$$\sum_{i,j=1}^{\infty} \psi_{i,j} a_{i,j} f_i f_j = \sum_{i=2}^{\infty} \left(\sum_{j=1}^{i} \psi_{i-j,j} b_{i-j,j} \right) f_i. \tag{9.1}$$

For a general coagulation–fragmentation equation in the form (2.10), we will say that F_* *satisfies detailed balance in weak form* if for any smooth bounded test function $\psi(x, y)$,

$$\int_{\mathbb{R}_+^2} \psi(x, y) a(x, y)\, F_*(\mathrm{d}x)\, F_*(\mathrm{d}y) = \int_{\mathbb{R}_+} \left(\int_0^x \psi(x-y, y) b(x-y, y)\, \mathrm{d}y \right) F_*(\mathrm{d}x). \tag{9.2}$$

Note that if some F_* exists satisfying this condition, then it is an equilibrium solution of (2.10), as one can check by taking

$$\psi(x, y) = \varphi(x+y) - \varphi(x) - \varphi(y).$$

Theorem 9.1 *For Model C, no finite measure F_* on $(0, \infty)$ exists that satisfies condition (9.2) for detailed balance in weak form.*

Proof Recall that $a(x, y) = 2, b(x, y) = 2/(x+y)$ for Model C. By taking $\psi(x, y) = 1$ in (9.2) we find $m_0(F_*)^2 = m_0(F_*)$, so a nontrivial solution requires $m_0(F_*) = 1$. Next, with $\psi(x, y) = 1 - \mathrm{e}^{-sy}$ we find

$$\breve{F}_*(s) = \int_{\mathbb{R}_+} \frac{\mathrm{e}^{-sx} - 1 + sx}{sx}\, F_*(\mathrm{d}x) =: \frac{G(s)}{s},$$

for all $s > 0$. We compute that $G'(s) = \breve{F}_*(s) = G(s)/s$, whence $G(s) = Cs$ for some constant $C > 0$. But this gives $\breve{F}_*(s) \equiv C$, which is not possible for a finite measure F_* supported in $(0, \infty)$. □

Remark 9.1 If detailed balance holds in the general weak form (9.2), there is a formal H theorem in the following sense: Assume the measure $F_t(\mathrm{d}x)$ is absolutely continuous with respect to $F_*(\mathrm{d}x)$ for all t, with Radon–Nikodym derivative $f_t(x)$ so that $F_t(\mathrm{d}x) = f_t(x) F_*(\mathrm{d}x)$. Then, the weak-form coagulation–fragmentation equation (2.10) takes the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}_+} \varphi(x)\, f_t(x)\, F_*(\mathrm{d}x) \\ &\quad = \frac{1}{2} \int_{\mathbb{R}_+^2} \big(\varphi(x+y) - \varphi(x) - \varphi(y)\big)\big(f_t(x) f_t(y) - f_t(x+y)\big) K_*(\mathrm{d}x, \mathrm{d}y), \end{aligned} \tag{9.3}$$

where $K_*(\mathrm{d}x,\mathrm{d}y) = a(x,y)F_*(\mathrm{d}x)F_*(\mathrm{d}y)$. If we now put

$$\mathcal{H} = \int_{\mathbb{R}_+} f_t(x)(\ln f_t(x) - 1)\, F_*(\mathrm{d}x), \tag{9.4}$$

$$\mathcal{I} = \frac{1}{2}\int_{\mathbb{R}_+^2} \left(\frac{f_t(x+y)}{f_t(x)\, f_t(y)} - 1\right) \ln \frac{f_t(x+y)}{f_t(x)\, f_t(y)}\, K_t(\mathrm{d}x,\mathrm{d}y), \tag{9.5}$$

where $K_t(\mathrm{d}x,\mathrm{d}y) = f_t(x) f_t(y) K_*(\mathrm{d}x,\mathrm{d}y)$, then $\mathcal{I} \geq 0$ and formally

$$\frac{d}{\mathrm{d}t}\mathcal{H} + \mathcal{I} = 0. \tag{9.6}$$

(Part II) Analysis of Model D

10 Equations for the Discrete-Size Model

As discussed in Sect. 2.3, we take $\alpha = \beta = 2$ in the expression (2.8) for the coagulation and fragmentation rates to get the equations for Model D that will be studied below. In weak form, we require that for any bounded test sequence (φ_i),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty} \varphi_i\, f_i(t) &= \sum_{i,j=1}^{\infty} \left(\varphi_{i+j} - \varphi_i - \varphi_j\right) f_i(t)\, f_j(t) \\ &\quad + \sum_{i=1}^{\infty}\Big(-\varphi_i + \frac{2}{i+1}\sum_{j=1}^{i}\varphi_j\Big) f_i(t).\end{aligned} \tag{10.1}$$

In strong form, the system is written as follows:

$$\frac{\partial f_i}{\partial t}(t) = Q_a(f)_i(t) + Q_b(f)_i(t), \tag{10.2}$$

$$Q_a(f)_i(t) = \sum_{j=1}^{i-1} f_j(t)\, f_{i-j}(t) - 2\sum_{j=1}^{\infty} f_i(t)\, f_j(t), \tag{10.3}$$

$$\begin{aligned}Q_b(f)_i(x,t) &= -f_i(t) + 2\sum_{j=i}^{\infty} \frac{1}{j+1}\, f_j(t) \\ &= -\left(\frac{i-1}{i+1}\right) f_i(t) + 2\sum_{j=i+1}^{\infty} \frac{1}{j+1}\, f_j(t).\end{aligned} \tag{10.4}$$

Bernstein transform. It is useful to describe the long-time dynamics of the discrete model in terms of the Bernstein transform of the discrete measure $\sum_{j=1}^{\infty} f_j(t)\,\delta_j(\mathrm{d}x)$. We define

$$\breve{f}(s,t) = \sum_{j=1}^{\infty}(1 - \mathrm{e}^{-js}) f_j(t). \tag{10.5}$$

Taking the test function $\varphi_j = 1 - \mathrm{e}^{-js}$ in (10.1) and using the fact that

$$\varphi_{i+j} - \varphi_i - \varphi_j = -\varphi_i \varphi_j,$$

we find that $\breve{f}(s,t)$ satisfies the equation

$$\boxed{\partial_t \breve{f}(s,t) = -\breve{f}^2 - \breve{f} + 2A_1(\breve{f}),} \tag{10.6}$$

for any $s, t > 0$, where

$$\begin{aligned} A_1(\breve{f})(s,t) &= \sum_{i=1}^{\infty} \frac{f_i(t)}{i+1} \sum_{j=0}^{i} (1 - \mathrm{e}^{-sj}) \qquad (10.7)\\ &= m_0(f) - \sum_{i=1}^{\infty} \frac{f_i(t)}{i+1} \frac{1 - \mathrm{e}^{-s(i+1)}}{1 - \mathrm{e}^{-s}} \\ &= \sum_{i=1}^{\infty} f_i(t) \left(1 - \frac{1}{1 - \mathrm{e}^{-s}} \int_0^s \mathrm{e}^{-r(i+1)}\, \mathrm{d}r \right) \\ &= \sum_{i=1}^{\infty} \frac{f_i(t)}{1 - \mathrm{e}^{-s}} \int_0^s (1 - \mathrm{e}^{-ri}) \mathrm{e}^{-r}\, \mathrm{d}r \\ &= \frac{1}{1 - \mathrm{e}^{-s}} \int_0^s \breve{f}(r,t) \mathrm{e}^{-r}\, \mathrm{d}r. \qquad (10.8) \end{aligned}$$

Eq. (10.6) is a nonlocal analog of the logistic equation that one would obtain if the averaging term $2A_1(\breve{f})$ were replaced by $2\breve{f}$.

It is very remarkable that Eq. (10.6) for the Bernstein transform of Model D transforms *exactly* into Eq. (4.4) for the Bernstein transform of Model C, by a simple change of variables. With

$$\sigma = 1 - \mathrm{e}^{-s}, \qquad u(\sigma, t) = \breve{f}(s,t), \tag{10.9}$$

one finds that (10.6) for $s \in (0, \infty)$, $t > 0$, is equivalent to

$$\partial_t u(\sigma, t) = -u^2 - u + \frac{2}{\sigma} \int_0^\sigma u(r,t)\, \mathrm{d}r, \tag{10.10}$$

for $\sigma \in (0, 1)$, $t > 0$. This equation will be used below in sections 11 and 13 below to study the equilibria and long-time behavior of solutions to Model D.

Remark 10.1 As we have mentioned in the introduction, the coagulation–fragmentation rates for Model D in (2.8) lead to a much more tractable transform than the rates in (2.7) which correspond most directly to Niwa's size-discrete model and which appear in time-discrete form in the work of Ma et al. (2011). With the rates in (2.7), the weak form of the discrete-size coagulation–fragmentation equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{\infty}\varphi_i\, f_i(t) = \sum_{i,j=1}^{\infty}\left(\varphi_{i+j}-\varphi_i-\varphi_j\right) f_i(t)\, f_j(t) + \sum_{i=2}^{\infty}\Big(-\varphi_i+\frac{2}{i-1}\sum_{j=1}^{i-1}\varphi_j\Big) f_i(t). \tag{10.11}$$

In terms of the Bernstein transform defined as in (10.5), by calculations similar to those above, we find this equation yields the evolution equation

$$\partial_t \breve{f}(s,t) = -\breve{f}^2 - \breve{f} + \frac{2}{\mathrm{e}^s-1}\int_0^s \breve{f}(r,t)\mathrm{e}^r\,\mathrm{d}r + f_1(t)\left(\frac{2s}{\mathrm{e}^s-1}-1-\mathrm{e}^{-s}\right).$$

Due to the explicit presence of $f_1(t)$ in this equation, its analysis appears to be much more complicated than that of (10.6), and we leave its study to future work.

11 Equilibrium Profiles for Model D

As will become clear from the well-posedness theory to come, any equilibrium solution $f = (f_i)$ of Model D has a finite zeroth moment

$$m_0(f) = \sum_{i=1}^{\infty} f_i.$$

Theorem 11.1 *For every $\mu \in [0,\infty)$, there is a unique equilibrium solution f_μ of Model D such that*

$$m_1(f_\mu) = \sum_{i=1}^{\infty} i f_{\mu i} = \mu.$$

The solution has the form

$$f_{\mu i} = \gamma_{\mu i}\,\lambda_\mu^{-i}, \qquad \lambda_\mu = 1 + \frac{4}{27\mu}, \tag{11.1}$$

where γ_μ is a completely monotone sequence with the asymptotic behavior

$$\gamma_{\mu i} \sim \frac{9}{8}\left(\frac{\mu\lambda_\mu}{\pi}\right)^{1/2} i^{-3/2} \quad \text{as } i \to \infty. \tag{11.2}$$

Every equilibrium solution has the form f_μ for some μ.

Proof To start the proof of the theorem, let $f = (f_i)$ be a nonzero equilibrium solution of Model D with Bernstein transform $\breve{f}(s)$. Change variables to $\sigma = 1 - \mathrm{e}^{-s} \in (0,1)$

as in (10.9) and introduce $u(\sigma) = \breve{f}(s)$, $\sigma \in (0,1)$. Then, u is a stationary solution of (10.6) and satisfies (5.10), and it follows that

$$u(\sigma) = U_\star(\mu\sigma), \quad \text{or} \quad \breve{f}(s) = U_\star(\mu - \mu \mathrm{e}^{-s}), \tag{11.3}$$

for some $\mu \in (0,\infty)$. Then, by differentiation it follows that

$$\mu = u'(0) = \breve{f}'(0) = \sum_{i=1}^{\infty} i f_i = m_1(f). \tag{11.4}$$

Note that by (5.12) the zeroth moment of f satisfies

$$m_0(f) = \breve{f}(\infty) = U_\star(\mu), \qquad \frac{m_0(f)}{(1-m_0(f))^3} = \mu. \tag{11.5}$$

Generating function. The properties of the equilibrium sequence f shall be derived from the behavior of the *generating function*

$$G(z) := \sum_{i=0}^{\infty} f_i\, z^i, \tag{11.6}$$

where we find it convenient (see Sect. 14) to define

$$f_0 = 1 - m_0(f) = 1 - U_\star(\mu). \tag{11.7}$$

Complete monotonicity. Due to (11.3) and (11.7), we have

$$U_\star(\mu\sigma) = \breve{f}(s) = 1 - G(\mathrm{e}^{-s}) = 1 - G(1-\sigma).$$

Hence by (5.18), we obtain

$$\boxed{G(z) = B_3(\mu(z-1)).} \tag{11.8}$$

By Lemma 5.2 (from Liu and Pego 2016), G is a Pick function analytic and nonnegative on the interval $(-\infty, \lambda_\mu)$ where $\lambda_\mu = 1 + \frac{4}{27\mu}$. Nonnegativity, and indeed complete monotonicity, of the sequence γ_μ given by

$$\gamma_{\mu i} = f_i\, \lambda_\mu^i, \tag{11.9}$$

now follows immediately from Theorem 3.8.

Asymptotics. The decay rate of the sequence f shall be deduced from the derivative of $G(z)$ using Tauberian arguments, as developed in the book of Flajolet and Sedgewick (2009).

Recall that $U_\star(s)$ has a branch point at $s = -\frac{4}{27}$, with $U_\star'\left(s - \frac{4}{27}\right) \sim \frac{9}{8} s^{-1/2}$ due to (5.26). The generating function $G(z)$ has a corresponding branch point at $z = \lambda_\mu$.

We rescale, replacing z by $\lambda_\mu z$ to write

$$G(\lambda_\mu z) = \sum_{i=0}^{\infty} \gamma_{\mu i} z^i = 1 - U_\star(\mu(1-\lambda_\mu z))$$

Then, differentiate, writing $\hat{\lambda} = \mu\lambda_\mu$, to find that

$$\begin{aligned}\sum_{i=1}^{\infty} i\gamma_{\mu i} z^{i-1} &= \hat{\lambda} U'_\star(\mu(1-\lambda_\mu z)) = \hat{\lambda} U'_\star\left(\hat{\lambda}(1-z) - \frac{4}{27}\right) \\ &\sim \frac{9\hat{\lambda}^{1/2}}{8}(1-z)^{-1/2} \qquad (11.10)\end{aligned}$$

By Corollary VI.1 from Flajolet and Sedgewick (2009) we deduce that as $i \to \infty$,

$$i\gamma_{\mu i} \sim \frac{9\hat{\lambda}^{1/2}}{8}\frac{(i-1)^{-1/2}}{\Gamma(1/2)} \sim \frac{9\hat{\lambda}^{1/2}}{8}\frac{i^{-1/2}}{\Gamma(1/2)},$$

This yields (11.2), since $\Gamma(1/2) = \sqrt{\pi}$. □

11.1 Recursive Computation of Equilibria for Model D

At equilibrium, the profile is required to satisfy

$$0 = \sum_{i,j=1}^{\infty} (\varphi_{i+j} - \varphi_i - \varphi_j)\, f_i\, f_j + \sum_{i=1}^{\infty}\left(-\varphi_i + \frac{2}{i+1}\sum_{j=1}^{i}\varphi_j\right) f_i. \qquad (11.11)$$

Define

$$\nu_0 = m_0(f) = \sum_{j=1}^{\infty} f_j, \qquad \beta_i = \sum_{j=i}^{\infty} \frac{1}{j+1}\, f_j. \qquad (11.12)$$

Taking $\varphi_j \equiv 1$ yields

$$0 = -\nu_0^2 - \nu_0 + 2\sum_{i=1}^{\infty}\frac{i}{i+1} f_i = -\nu_0^2 + \nu_0 - 2\beta_1. \qquad (11.13)$$

Next, taking $\varphi_k = 1$ if $k = i$ and 0 otherwise yields

$$0 = \sum_{j=1}^{i-1} f_j\, f_{i-j} - (2\nu_0 + 1) f_i + 2\beta_i, \qquad i \ge 1. \qquad (11.14)$$

Based on these equations, we can compute the f_j recursively, in a manner analogous to the computation of equilibria in the models studied by Ma et al. (2011). Starting from any given value of the parameter $\nu_0 \in (0, 1)$, set β_1 according to (11.13), defining

$$\beta_1 = \frac{1}{2}\left(\nu_0 - \nu_0^2\right). \tag{11.15}$$

Then, for $i = 1, 2, 3, \ldots$ compute

$$f_i = (1 + 2\nu_0)^{-1}\left(2\beta_i + \sum_{j=1}^{i-1} f_j f_{i-j}\right), \tag{11.16}$$

$$\beta_{i+1} = \beta_i - \frac{f_i}{i+1}. \tag{11.17}$$

These formulae are used to compute the discrete profile that is compared to the continuous profile $f_\star$ in Fig. 3 in Sect. 15 below.

12 Well Posedness for Model D

Here, we consider the discrete dynamics described by Eqs. (10.1)–(10.4), and establish well posedness of the initial-value problem by a simple strategy of proving local Lipschitz estimates on an appropriate Banach space.

We first introduce some preliminary notations. For $k \in \mathbb{R}$, and a real sequence $f = (f_i)_{i=1}^\infty$, we define the k-th moment $m_k(f)$ of f and associated norm $\|f\|_k$ by:

$$m_k(f) = \sum_{i=1}^\infty i^k f_i, \qquad \|f\|_k = \sum_{i=1}^\infty i^k |f_i|.$$

We introduce the Banach space $\ell_{1,k}$ as the vector space of sequences f with finite norm $\|f\|_k < \infty$ and denote the positive cone in this space by

$$\ell_{1,k}^+ = \left\{f \in \ell_{1,k} \text{ such that } f_i \geq 0 \text{ for all } i\right\}.$$

Theorem 12.1 *Let $k \geq 0$ and let $f_{\mathrm{in}} = (f_{\mathrm{in},i})_{i=1}^\infty$ be given in $\ell_{1,k}^+$. Then, there exists a unique global-in-time solution $f \in C^1([0,\infty), \ell_{1,k}^+)$ for system (10.1)–(10.4) with initial condition $f(0) = f_{\mathrm{in}}$. Moreover, f is C^∞, and for all $t \geq 0$ we have*

$$\partial_t m_0(f(t)) \leq -m_0(f(t))^2 + m_0(f(t)), \tag{12.1}$$

and

$$m_0(f(t)) \leq m_0(f_{\mathrm{in}}) + 1. \tag{12.2}$$

If $k \geq 1$, then
$m_1(f(t)) = m_1(f_{\mathrm{in}})$ for all $t \geq 0$.

Proof We first transform the problem to simplify the proof of positivity. Fix $\lambda = 2m_0(f_{\text{in}}) + 3$ and change variables using $\hat{f}_i(t) = \mathrm{e}^{\lambda t} f_i(t)$. Then, Eq. (10.1) is equivalent to

$$\frac{\partial \hat{f}_i}{\partial t} = Q_\lambda(\hat{f})_i := \lambda \hat{f}_i + Q_a(\hat{f})_i \mathrm{e}^{-\lambda t} + Q_b(\hat{f})_i. \tag{12.3}$$

Using the inequality $i^k \le 2^k(j^k + (i-j)^k)$, we find that the quadratic map $f \mapsto Q_a(f)$ is locally Lipschitz on $\ell_{1,k}$, satisfying

$$\begin{aligned} 2^{-k-1}\|Q_a(f) - Q_a(g)\|_k \le \|f - g\|_k(\|f\|_0 + \|g\|_0) \\ + \|f - g\|_0(\|f\|_k + \|g\|_k). \end{aligned} \tag{12.4}$$

Also, the linear map $f \mapsto Q_b(f)$ is bounded on $\ell_{1,k}$, due to the estimate

$$\sum_{i=1}^{\infty} i^k \sum_{j=i}^{\infty} \frac{1}{j+1}|f_j| = \sum_{j=1}^{\infty}\sum_{i=1}^{j} \frac{i^k}{j+1}|f_j| \le \sum_{j=1}^{\infty} j^k |f_j|.$$

Consequently, we have local existence of a unique smooth solution $\hat{f}$ to (12.3) with values in $\ell_{1,k}$, and a corresponding smooth solution f to (10.1). If $k \ge 1$, then because $f \mapsto m_1(f)$ is bounded on $\ell_{1,k}$, we find $m_1(f(t))$ is constant in time by taking $\varphi_i = i$ in (10.1).

It remains to prove positivity and global existence, for any $k \ge 0$. The solution $\hat{f}$ is the limit of Picard iterates $\hat{f}^{(n)}$ starting with $\hat{f}^{(0)}(t) \equiv f_{\text{in}}$. Note that the coefficient of $\hat{f}_i$ in $Q_\lambda(\hat{f})_i$ is, from (10.3)–(10.4),

$$\lambda - 2m_0(\hat{f})\mathrm{e}^{-\lambda t} - \frac{i-1}{i+1} > 2(\hat{M} - m_0(\hat{f})\mathrm{e}^{-\lambda t}), \qquad \hat{M} := \frac{\lambda - 1}{2}.$$

Let $\nu_n(t) = m_0(\hat{f}^{(n)}(t))\mathrm{e}^{-\lambda t}$. We show by induction the following statement: For all $n \in \mathbb{N}^*$, for all $t \ge 0$ in the interval of existence,

$$0 \le \nu_n(t) \le \hat{M} \quad \text{and} \quad \hat{f}_i^{(n)}(t) \ge 0 \quad \text{for all} \quad i. \tag{12.5}$$

For the induction step, note that by the induction hypothesis, $Q_\lambda(\hat{f}^{(n)})_i \ge 0$ for all i, therefore $\hat{f}_i^{(n+1)}(t) \ge 0$ for all i and $t \ge 0$. We also have

$$\sum_{i=1}^{\infty} Q_a(\hat{f}^{(n)}(t))_i \mathrm{e}^{-\lambda t} = -\nu_n(t)^2 \mathrm{e}^{\lambda t}, \qquad \sum_{i=1}^{\infty} Q_b(\hat{f}^{(n)}(t))_i \le \nu_n(t)\mathrm{e}^{\lambda t},$$

hence

$$\mathrm{e}^{-\lambda t}\partial_t(\mathrm{e}^{\lambda t}\nu_{n+1}) = \mathrm{e}^{-\lambda t}\partial_t m_0(\hat{f}^{(n+1)}) \le \lambda\nu_n - \nu_n^2 + \nu_n \le \lambda\hat{M}. \tag{12.6}$$

The last inequality holds because $x \mapsto \lambda x - x^2 + x$ increases on $[0, \hat{M}]$ and $\hat{M} > 1$. Upon integration, we deduce $0 \le \nu_{n+1}(t) \le \hat{M}$.

Taking $n \to \infty$, we find that $\hat{f}_i(t) \geq 0$ and $m_0(\hat{f}(t)) \leq \hat{M} e^{\lambda t}$, hence

$$f(t) = e^{-\lambda t} \hat{f}(t) \in \ell^+_{1,k} \quad \text{and} \quad m_0(f(t)) \leq \hat{M}$$

on the maximal interval of existence. Due to the local Lipschitz bounds established above, the solution can be continued to exist in $\ell^+_{1,k}$ for all $t \in [0, \infty)$.

To obtain (12.1), take $\varphi_i \equiv 1$ in (10.1), or take $s \to \infty$ in (10.6). □

13 Long-Time Behavior for Model D

By using much of the same analysis as in the continuous-size case, we obtain strong convergence to equilibrium for solutions with a finite first moment, and weak convergence to zero for solutions with infinite first moment.

Theorem 13.1 (Strong convergence with finite first moment) *Let $f(t) = (f_i(t))$ be any solution of Model D with initial data having finite first moment $\mu = m_1(f_{\text{in}})$, so $f_{\text{in}} \in \ell^+_{1,1}$. Then, the solution converges strongly to the equilibrium solution f_μ having the same first moment:*

$$\|f(t) - f_\mu\|_1 = \sum_{i=1} i |f_i(t) - f_{\mu i}| \to 0 \quad \textit{as } t \to \infty. \tag{13.1}$$

Proof Let $\breve{f}(s,t)$ from (10.5) be the Bernstein transform of the solution, and let $\mu = m_1(f)$ be the first moment (constant in time). Similarly to (10.9), we change variables according to

$$\sigma = \mu(1 - e^{-s}), \quad u(\sigma, t) = \breve{f}(s,t) = \sum_{i=1}^{\infty} \left(1 - \left(1 - \frac{\sigma}{\mu}\right)^i\right) f_i(t). \tag{13.2}$$

This function $u(\sigma, t)$ then is a solution to (10.10) for $0 < \sigma < \mu$ and has the properties that for all $t > 0$, $u(0,t) = 0$ and $\sigma \mapsto u(\sigma, t)$ is increasing and concave.

We now invoke Proposition 7.2 with $\bar{s} = \mu$, and conclude that as $t \to \infty$, $u(\sigma, t) \to U_*(\sigma)$ for all $\sigma \in (0, \mu)$. It follows that

$$\breve{f}(s,t) \to \breve{f}_\mu(s) \quad \text{for all} \quad s > 0.$$

By the continuity theorem 3.5, it follows that as $t \to \infty$, the discrete measures

$$F_t(dx) = \sum_i f_i(t) \delta_i(dx) \xrightarrow{w} F_\mu(dx) = \sum_i f_{\mu i} \delta_i(dx)$$

weakly on $[0, \infty]$. But this implies $f_i(t) \to f_{\mu i}$ for every $i \in \mathbb{N}^*$. Then, (13.1) follows, see Ball et al. (1986, Lemma 3.3), for example. □

Theorem 13.2 (Weak convergence to zero with infinite first moment) *Let $f(t)$ be any solution of Model D with initial data $f_{\mathrm{in}} \in \ell_{1,0}$ having infinite first moment. Then, as $t \to \infty$ we have $f_i(t) \to 0$ for all i, and*

$$m_0(f(t)) = \sum_{i=1}^{\infty} f_i(t) \to 1.$$

Remark 13.1 The conclusion means that the total number of groups $m_0(f(t)) \to 1$, while the number of groups of any fixed size i tends to zero. Thus as time increases, individuals cluster in larger and larger groups, leaving no groups of finite size in the large-time limit.

Proof We use the change of variables in (10.9) and obtain a solution $u(\sigma, t)$ to (10.10) that satisfies $\partial_\sigma u(0^+, 0) = \infty$. By the same arguments as in the proof of Theorem 8.1, we obtain the analog of (8.6), namely

$$\liminf_{t\to\infty} u(\sigma, t) \geq 1 \quad \text{for all} \quad \sigma \in (0, 1). \tag{13.3}$$

We know, though, that $u(\sigma, t) \leq m_0(f(t))$ and that

$$\limsup_{t\to\infty} m_0(f(t)) \leq 1$$

due to (12.1). It follows that $m_0(f(t)) \to 1$ and $\breve{f}(s, t) \to 1$ for all $s > 0$. Therefore, the Laplace transform $\sum_{i=1}^{\infty} \mathrm{e}^{-si} f_i(t) \to 0$, and the conclusions of the Theorem follow. □

(Part III) From Discrete to Continuous Size

14 Discretization of Model C

In this section, we discuss how a particular discretization of Model C is naturally related to Model D. We start from the weak form of Model C expressed in (4.1). Introduce a grid size $h > 0$, corresponding to a scaled size increment, and introduce the approximation

$$f_i^h \approx \int_{I_i^h} F_t(\mathrm{d}x), \qquad I_i^h := (ih, (i+1)h], \quad i = 0, 1, \ldots \tag{14.1}$$

for the number of clusters with size in the interval I_i^h. (We find it convenient to not scale this number by the width of I_i^h, for purposes of comparison.)

Then, by formal discretization of the integrals in (4.1) by the left-endpoint rule, we require

$$\sum_{i=0}^{\infty} \varphi(ih) \frac{df_i^h}{dt}(t) = \sum_{i,j=0}^{\infty} \Big(\varphi\big((i+j)h\big) - \varphi(ih) - \varphi(jh) \Big) f_i^h(t)\, f_j^h(t)$$
$$+ \sum_{i=0}^{\infty} \Big(-\varphi(ih) + \frac{2}{i+1} \sum_{j=0}^{i} \varphi(jh) \Big) f_i^h(t). \tag{14.2}$$

"Ghost" clusters. Because of the left-endpoint discretization, the sums in (14.2) start with $i, j = 0$. Note, however, that if we take $\varphi(0) = 0$ and $\varphi_i = \varphi(ih)$, all terms with $i = 0$ or $j = 0$ drop, and Eq. (14.2) becomes identical to the weak form of Model D in (10.1). Thus, the dynamics of the sequence $(f_i^h(t))_{i=1}^{\infty}$ is governed exactly by Model D, and is decoupled from the behavior of $f_0^h(t)$. The equation governing f_0^h corresponds to the coefficient of $\varphi(0)$ in Eq. (14.2) and takes the form

$$\partial_t f_0^h(t) = -(f_0^h)^2 - 2 f_0^h \sum_{i=1}^{\infty} f_i^h + f_0^h + 2 \sum_{i=1}^{\infty} \frac{1}{i+1} f_i^h. \tag{14.3}$$

We see the behavior of $f_0^h(t)$ is slaved to that of $(f_i^h(t))_{i=1}^{\infty}$.

In the discrete-size model, f_0^h has the interpretation as number density of clusters whose size is less than the bin width h. Merging and splitting interactions with such clusters have a negligible effect upon dynamics in the discrete approximation, so we can say these clusters become "ghosts." It is sometimes convenient, however, to include them in the tally of total cluster numbers, for the following reason: Taking $\varphi \equiv 1$ in (14.2) we find that the quantity

$$\hat{\nu}_0(t) := m_0 \Big((f_i^h(t))_{i=0}^{\infty} \Big) = \sum_{i=0}^{\infty} f_i^h(t)$$

is an exact solution of the logistic equation:

$$\partial_t \hat{\nu}_0(t) = -\hat{\nu}_0^2 + \hat{\nu}_0. \tag{14.4}$$

(Recall that without the $i = 0$ term, we have only the inequality (12.1).) Naturally $\hat{\nu}_0 = 1$ in equilibrium, which helps to explain the convenient choice of f_0 in (11.7): The function G in (11.6) is the generating function of $(f_i)_{i=0}^{\infty}$, an equilibrium for Model D extended to include ghost clusters.

Bernstein transform. By taking $\varphi(x) = 1 - \mathrm{e}^{-sx}$, we obtain the h-scaled Bernstein transform

$$U^h(s,t) = \sum_{i=1}^{\infty} (1 - \mathrm{e}^{-shi}) f_i^h(t) = \breve{F}_t^h(s), \tag{14.5}$$

where F_t^h is the discrete measure on the grid $\{ih : i = 1, \dots\}$ formed from the solution $f(t) = (f_i^h(t))_{i=1}^{\infty}$ of Model D:

$$F_t^h(\mathrm{d}x) = \sum_{i=1}^{\infty} f_i^h(t)\,\delta_{ih}(\mathrm{d}x).$$

(Ghost clusters would have no effect on U^h, and we do not include them here, in order to focus on how Model D compares to Model C.) The function U^h satisfies the following scaled variant of (10.6):

$$\boxed{\partial_t U^h(s,t) = -(U^h)^2 - U^h + 2A_h(U^h),} \tag{14.6}$$

where the scaled averaging operator

$$A_h(U^h)(s,t) = \frac{h}{1-\mathrm{e}^{-sh}} \int_0^s U^h(r,t)\mathrm{e}^{-rh}\,\mathrm{d}r. \tag{14.7}$$

In the limit $h \to 0$, the operator A_h reduces formally to the running average operator A_0 as defined by

$$A_0(U)(s) := \frac{1}{s}\int_0^s U(r)\,\mathrm{d}r. \tag{14.8}$$

15 Limit Relations at Equilibrium

At equilibrium, $f^h = (f_i^h)_{i=1}^{\infty}$ is constant in time, and the zeroth and first moments satisfy

$$\nu_h = m_0(F^h) = \sum_{i=1}^{\infty} f_i^h, \qquad \mu_h = m_1(F^h) = \sum_{i=1}^{\infty} ihf_i^h. \tag{15.1}$$

Because f^h is an equilibrium solution of Model D, from relations (11.4)–(11.5) we find that these moments are related by

$$\frac{\nu_h}{(1-\nu_h)^3} = \frac{\mu_h}{h}. \tag{15.2}$$

If we consider the rescaled mass μ_h as fixed, the leading behavior of ν_h as $h \to 0$ is given by

$$\nu_h \sim 1 - (h/\mu_h)^{1/3}. \tag{15.3}$$

In these terms, the tail behavior of the Model D equilibrium from Theorem 11.1 can be recast in the form

$$\frac{1}{h} f_i^h \sim \frac{1}{\mu_h}\frac{9}{8\sqrt{\pi}}\left(\frac{ih}{\mu_h}\right)^{-3/2}\left(1+\frac{4}{27}\frac{h}{\mu_h}\right)^{-i+\frac{1}{2}}, \quad i \to \infty. \tag{15.4}$$

As $h \to 0$ and $i \to \infty$ with $ih \to x$ and $\mu_h \to \mu$, the right-hand side converges to

$$\frac{1}{\mu}\frac{9}{8\sqrt{\pi}}\left(\frac{x}{\mu}\right)^{-3/2}\exp\left(-\frac{4}{27}\frac{x}{\mu}\right). \tag{15.5}$$

This is consistent with the large-size asymptotic behavior of the solution of Model C with first moment μ from Theorem 5.1.

We have the following rigorous convergence theorem for the continuum limit of the discrete equilibria.

Theorem 15.1 *Let f^h be a family of equilibria of Model D and let $F^h(dx) = \sum_{i=1}^{\infty} f_i^h \delta_{ih}$. If $\mu_h = m_1(F^h) \to \mu$ as $h \to 0$, then F^h converges narrowly to $F_{\mathrm{eq}}(dx) = f_{\mathrm{eq}}(x)\,dx$ where*

$$f_{\mathrm{eq}}(x) = \frac{1}{\mu} f_\star\left(\frac{x}{\mu}\right).$$

Proof The Bernstein transform of the scaled discrete-size distribution $F^h(\mathrm{d}x) = \sum_{i=1}^{\infty} f_i^h \delta_{ih}$ has the following representation, due to (11.3):

$$\breve{F}^h(s) = \sum_{i=1}^{\infty}(1 - \mathrm{e}^{-sih}) f_i^h = U_\star\left(\mu_h \frac{1 - \mathrm{e}^{-sh}}{h}\right). \tag{15.6}$$

As $h \to 0$, from (15.6) we have $\breve{F}^h(s) \to \breve{F}_{\mathrm{eq}}(s) = U_\star(\mu s)$, for every $s \in [0, \infty]$. Now, the result follows from Proposition 3.6. □

See Fig. 3 for a comparison of the discrete and continuous profiles f_i^h/h and $f_\star(ih)$, for the parameters $\nu_h = 0.6$, $\mu_h = \mu = 1$, corresponding to $h = 0.1066\bar{6}$. In the inset, we compare the ratio of these profiles with the asymptotic expression coming from (15.4)–(15.5),

$$\frac{f_i^h/h}{f_\star(ih)} \sim \exp\left(\frac{4ih}{27}\right)\left(1 + \frac{4h}{27}\right)^{-i+\frac{1}{2}}, \tag{15.7}$$

which reflects the different exponential decay rates for the discrete and continuous profiles. The discrete profile also exhibits a transient behavior for small i, starting from the value

$$\frac{1}{h} f_1^h = \frac{\nu_h - \nu_h^2}{1 + 2\nu_h}\,\frac{\nu_h}{(1 - \nu_h)^3} \sim \frac{1}{3} h^{-2/3}, \tag{15.8}$$

coming from (11.16), (15.2) and (15.3). The asymptotic value of the ratio

$$(f_1^h/h)/f_\star(h) \sim \frac{1}{3}\Gamma\left(\frac{1}{3}\right) \tag{15.9}$$

(with $f_\star(h)$ approximated by (5.3)–(5.4) in Theorem 5.1) is plotted as a cross at the point $(h, \frac{1}{3}\Gamma(\frac{1}{3}))$ in the inset.

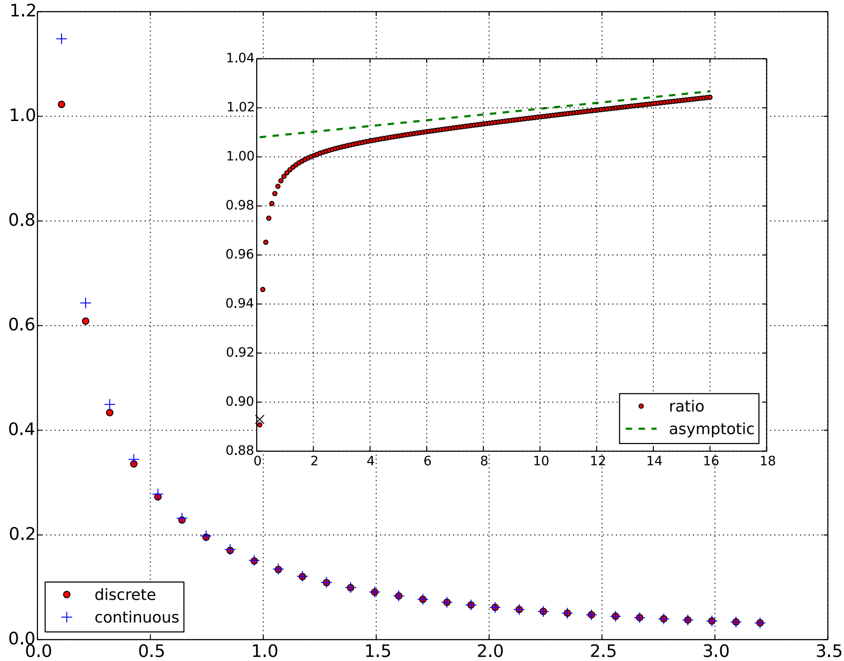

Fig. 3 Discrete and continuous profiles: f_i^h/h and $f_\star(ih)$ versus ih for $\nu_h = 0.6$, $\mu_h = 1$, $h = 0.10667$. *Inset* Ratio $(f_i^h/h)/f_\star(ih)$ and asymptotics in (15.7) versus ih. *Cross in inset* at $(h, \frac{1}{3}\Gamma(\frac{1}{3})) \approx (0.1, 0.893)$, see (15.9)

16 Discrete-to-Continuum Limit

We can rigorously prove a weak-convergence result for time-dependent solutions of Model D to solutions of Model C, as follows.

Theorem 16.1 *Let F be a solution of Model C with initial data F_0 a finite measure on $(0,\infty)$, and let f^h, $h \in I$ be solutions of Model D with initial data that satisfy $F_0^h \to F_0$ narrowly as $h \to 0$. Then, for each $t > 0$, we have $F_t^h \to F_t$ narrowly as $h \to 0$.*

Proof of the Theorem 1. The Bernstein functions $U^h = \breve{F}_t^h$ and $U = \breve{F}_t$ satisfy (14.6) and (4.4) respectively. For h small enough, these functions are uniformly bounded globally in time by $C_0 = m_0(F_0) + 2$, due to the fact that $m_0(F_0^h) \to m_0(F_0)$ and the bounds coming from (4.7) and (12.2).

2. Next we let

$$\omega(s,t) = (U - U^h)(s,t), \qquad \Omega(s,t) = \sup_{0<r\le s} |(U - U^h)(r,t)|,$$

and compute

$$\partial_t \omega(s,t) = -\omega(U + U^h + 1) + 2A_h(\omega) + 2(A_0(U) - A_h(U)). \tag{16.1}$$

Observe that $|A_h(\omega)(s,t)| \le \Omega(s,t)$, and that

$$A_0(U) - A_h(U) = \frac{1}{s}\int_0^s U(r,t)\left(1 - \frac{she^{-rh}}{1-e^{-sh}}\right) dr$$

satisfies the bound

$$|A_0(U) - A_h(U)| \le U(s,t)\eta_0(sh), \tag{16.2}$$

where

$$\eta_0(s) := \sup_{0<r<s}\left|1 - \frac{s}{1-e^{-s}}e^{-r}\right| \underset{s\to 0}{\to} 0. \tag{16.3}$$

Now, we multiply Eq. (16.1) by 2ω and integrate in time to obtain

$$\begin{aligned} \omega(s,t)^2 - \omega(s,0)^2 &\le \int_0^t 4\omega(s,\tau)(\Omega(s,\tau) + C_0\eta_0(sh))\, d\tau \\ &\le \int_0^t 8\Omega(s,\tau)^2\, d\tau + tC_0^2\eta(sh)^2. \end{aligned} \tag{16.4}$$

Taking the sup over $s \in [0,\hat{s}]$ and using a standard Gronwall argument, we infer that for any $s \in (0,\infty)$ and $t > 0$,

$$\sup_{0<r<s}|(U - U^h)(r,t)| = \Omega(s,t) \le e^{4t}\left(\Omega(s,0) + \sqrt{t}C_0\eta(sh)\right) \to 0 \tag{16.5}$$

as $h \to 0$.

3. We separately study the case $s = \infty$, writing

$$\hat{m}^h(t) = m_0(F_t^h) = U^h(\infty,t), \qquad \hat{m}(t) = m_0(F_t) = U(\infty,t).$$

Due to (14.6) and (4.7), these moments satisfy

$$\partial_t\hat{m}^h(t) \le -(\hat{m}^h)^2 + \hat{m}^h, \qquad \partial_t\hat{m}(t) = -(\hat{m})^2 + \hat{m}. \tag{16.6}$$

By the assumption that initial data converge narrowly, we have $\hat{m}^h(0) \to \hat{m}(0)$, and it is not difficult to deduce that for each $t > 0$,

$$\limsup_{h\to 0}\hat{m}^h(t) \le \hat{m}(t).$$

Now, however, by the monotonicity of $s \mapsto U^h(s,t)$, we have

$$U(s,t) = \lim_{h\to 0} U^h(s,t) \le \liminf_{h\to 0}\hat{m}^h(t).$$

Because we know from Theorem 6.1 that $\hat{m}(t) = \lim_{s\to\infty} U(s,t)$, we deduce $\hat{m}^h(t) \to \hat{m}(t)$ as $h \to 0$.

The narrow convergence $F_t^h \to F_t$ now follows from Proposition 3.6. □

Remark 16.1 Uniqueness for Model C is proved by a simple Gronwall estimate analogous to (16.5). We omit details.

17 Discussion and Conclusions

Niwa (2003) simulated random merging and splitting dynamics according to the rules we listed in the introduction, but he did so for an indirect purpose: The simulations were used as a tool in order to estimate noise terms in a stochastic differential equation that he formulated to model the group size experienced by an individual. He then solved this SDE to obtain self-consistently the equilibrium group size in the form of his profile (1.2).

The present paper is a study of mean-field, deterministic coagulation–fragmentation equations that closely correspond to Niwa's simulations. As stochastic effects can be expected to become nontrivial as populations decrease, it would be interesting to study stochastic models in a manner compatible with interaction rules and the deterministic limit. Niwa's individual-based point of view could lead to a kind of model different from the Markus–Lushnikov merging processes whose large-population limits are known rigorously to converge to deterministic coagulation equations (Norris 1999). Because one can expect size of the group containing a given individual to change discontinuously upon mergers, appropriate self-consistent individual-based models should involve jump processes rather than SDEs.

Consistent with Niwa's major finding, we have shown that in the continuum limit corresponding to large population size, the equilibrium group-size distribution does achieve a scaling-invariant form. Moreover, the exact equilibrium profile $\Phi_\star$ in the continuum limit is computable in terms of explicit series representations as described in Sect. 5.4. It is not as simple as Niwa's expression (1.2), but we have determined its asymptotic form as an exponential with a smooth prefactor having different power-law behavior at small *vs.* large group sizes.

Much of the empirical data exhibited by Niwa (2003) is in a range where the profile $\Phi_\star$ differs little from (1.2) or from the logarithmic distribution (1.9). In principle, it would be interesting to determine whether the profile $\Phi_\star$ is a better model for real data than other alternatives, but it may not be easy given the difficulties of dealing with noisy population data. In particular, the crossover that we found in the power-law behavior of the exponential prefactor suggests that extracting power-law exponents from data, as is done in a number of papers in the literature including Bonabeau et al. (1999) and Sjöberg et al. (2000), may be difficult to do accurately and reliably.

In broader terms, however, what our study does support is the idea that indeed this kind of widely dispersed, non-Gaussian group-size distribution can arise as an emergent property in a mathematically well-formulated model of random encounters between groups and having a simple kind of group instability.

In our investigation of dynamical behavior, we have proved that for a given finite total population, the unique equilibrium globally attracts all solutions, both for continuous-size and discrete-size variants. Because of the rigidity of the techniques that we use based on Bernstein transform, this finding does not extend further, even to models that have the same equilibria but differing rates as indicated in Remark 5.2.

More information about how solutions approach equilibrium would be interesting to obtain. Some numerical investigations related to equilibria and dynamical stability for variant discretizations of our continuous-size Model C have been performed in separate work (Degond and Engel 2016).

The systems that we study lack detailed balance, which means that their equilibria are maintained by a steady cycling in the network of reactions between groups of different sizes. Remarkably little is known about dynamics in such systems. We wonder, for example, whether any such coagulation–fragmentation systems may have dynamically cycling (time-periodic) solutions.

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