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Fuzzy quasi-triangular spaces, fuzzy sets of Pompeiu-Hausdorff type, and another extensions of Banach and Nadler theorems

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Abstract

Let $\mathcal{A}$ be an index set, and $C = \{C_\alpha\}_{\alpha\in\mathcal{A}} \in [1;\infty)^{\mathcal{A}}$. Fuzzy quasi-triangular space is defined to be $(X, \mathcal{M}_{C;\mathcal{A}}, *)$, where X is a nonempty set, a fuzzy family $\mathcal{M}_{C;\mathcal{A}} = \{M_\alpha : X \times X \times (0;\infty) \to (0;1], \alpha \in \mathcal{A}\}$ satisfies $\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\forall_{t,s\in(0;\infty)}\{M_\alpha(x,y,t) * M_\alpha(y,z,s) \le M_\alpha(x,z,C_\alpha(t+s))\}$, and $*$ is the continuous t-norm $* : [0;1] \times [0;1] \to [0;1]$. In $(X, \mathcal{M}_{C;\mathcal{A}}, *)$, left (right) $\mathcal{G}$-families and $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$ ($\mathcal{K}_{C;\mathcal{A}}$ generalize $\mathcal{M}_{C;\mathcal{A}}$) are defined and described. Using families $\mathcal{K}_{C;\mathcal{A}}$, three kinds of left (right) fuzzy sets of Pompeiu-Hausdorff type on $2^X \times 2^X \times (0;\infty)$ are introduced. Using these fuzzy sets, three kinds of left (right) set-valued fuzzy contractions $T : X \to 2^X$ are constructed, and for such fuzzy contractions, conditions guaranteeing the existence of periodic points and left (right) $\mathcal{M}_{C;\mathcal{A}}$-convergence to these periodic points of dynamic processes $(w^m : m \in \{0\} \cup \mathbb{N})$, $w^m \in T(w^{m-1})$ for $m \in \mathbb{N}$, starting at $w^0 \in X$, are established. Moreover, in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$, using left (right) $\mathcal{G}$-families and $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$, two kinds of left (right) single-valued fuzzy contractions $T : X \to X$ are constructed, and for such fuzzy contractions, the convergence, existence, approximation, uniqueness, periodic point, and fixed point result is also obtained. Examples are provided.

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1 Introduction

Let X be a (nonempty) set. A map $M : X \times X \times [0;\infty) \to [0;1]$ or $M : X \times X \times (0;\infty) \to [0;1]$ is called a *fuzzy set* (Zadeh [1]). The set X, together with a fuzzy set M and with a continuous t-norm $*$, is called a *fuzzy space* and is denoted by $(X, M, *)$. We recall the definition of Schweizer and Sklar [2].

Definition 1.1 A binary operation $* : [0;1] \times [0;1] \to [0;1]$ is called a continuous t-norm if $*$ satisfies the following conditions: (i) $a * b = b * a$; (ii) $a * b \le c * d$ for $a \le c$, $b \le d$; (iii) $(a * b) * c = a * (b * c)$; (iv) $a * 1 = a$; (v) $*$ is continuous, that is, for all convergent sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in $[0;1]$, $\lim_{m\to\infty}(x_m * y_m) = \lim_{m\to\infty} x_m * \lim_{m\to\infty} y_m$. Here $a, b, c, d \in [0;1]$.

The following fuzzy spaces are well known: fuzzy metric spaces (Kramosil and Michalek [3], George and Veeramani [4]), fuzzy quasi-metric and fuzzy quasi-pseudometric spaces (Gregori and Romaguera [5]), intuitionistic fuzzy metric spaces (Park [6]), fuzzy uniform spaces (Lowen and Wuyts [7]), fuzzy quasi-uniform spaces (Hutton [8]), and fuzzy topological spaces (Hutton [8]).

The following fuzzy metric space is a fuzzy version of probabilistic metric space.

Definition 1.2 A fuzzy metric space (in the sense of Kramosil and Michalek [3]) is an ordered triple $(X,M,*)$ where $M : X \times X \times [0;\infty) \to [0;1]$ satisfies (i) $\forall_{x,y\in X}\{M(x,y,0) = 0\}$, (ii) $\forall_{x\in X}\forall_{t\in(0;\infty)}\{M(x,x,t) = 1\}$, (iii) $\forall_{x,y\in X}\{\exists_{t\in(0;\infty)}\{M(x,y,t) = 1\}$ implies $x = y\}$, (iv) $\forall_{x,y\in X}\forall_{t\in(0;\infty)}\{M(x,y,t) = M(y,x,t)\}$, (v) $\forall_{x,y,z\in X}\forall_{t,s\in(0;\infty)}\{M(x,y,t) * M(y,z,s) \le M(x,z,t+s)\}$, and (vi) $\forall_{x,y\in X}\{M(x,y,\cdot) : [0;\infty) \to [0;1]$ is left continuous$\}$.

When (i)-(vi) hold, we will say that $(X,M,*)$ is a *KM-fuzzy metric space*.

New fuzzy metric spaces and a study of a Hausdorff topology in these spaces appeared in [4].

Definition 1.3 A fuzzy metric space (in the sense of George and Veeramani [4]) is an ordered triple $(X,M,*)$ where $M : X \times X \times (0;\infty) \to (0;1]$ satisfies (i) $\forall_{x,y\in X}\forall_{t\in(0;\infty)}\{M(x,y,t) > 0\}$, (ii) $\forall_{x\in X}\forall_{t\in(0;\infty)}\{M(x,x,t) = 1\}$, (iii) $\forall_{x,y\in X}\{\exists_{t\in(0;\infty)}\{M(x,y,t) = 1\}$ implies $x = y\}$, (iv) $\forall_{x,y\in X}\forall_{t\in(0;\infty)}\{M(x,y,t) = M(y,x,t)\}$, (v) $\forall_{x,y,z\in X}\forall_{t,s\in(0;\infty)}\{M(x,y,t) * M(y,z,s) \le M(x,z,t+s)\}$, and (vi) $\forall_{x,y\in X}\{M(x,y,\cdot) : (0;\infty) \to (0;1]$ is continuous$\}$.

When (i)-(vi) hold, we will say that $(X,M,*)$ is a *GV-fuzzy metric space*.

Note that these two concepts of fuzziness of metric spaces and the following two kinds of completeness in these spaces are important in the rich literature concerning fuzzy fixed point theory (see [3–5] and [9–13]).

Definition 1.4 (Grabiec [9]) Let $(X,M,*)$ be a KM-fuzzy metric space. A sequence $(x_m : m \in \mathbb{N}) \subset X$ is called *G-Cauchy* if

$$\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} M(x_m, x_{m+p}, t) = 1\right\}. \tag{1.1}$$

A sequence $(x_m : m \in \mathbb{N}) \subset X$ *converges* to $x \in X$ if $\forall_{t\in(0;\infty)}\{\lim_{m\to\infty} M(x_m, x, t) = 1\}$. $(X,M,*)$ is called *G-complete* if every G-Cauchy sequence in X is convergent in X.

Definition 1.5 (George and Veeramani [4]) Let $(X,M,*)$ be a GV-fuzzy metric space. A sequence $(x_m : m \in \mathbb{N}) \subset X$ is called *M-Cauchy* if

$$\forall_{t\in(0;\infty)}\left\{\lim_{n,m\to\infty} M(x_n, x_m, t) = 1\right\}. \tag{1.2}$$

A sequence $(x_m : m \in \mathbb{N}) \subset X$ *converges* to $x \in X$ if $\forall_{t\in(0;\infty)}\{\lim_{m\to\infty} M(x_m, x, t) = 1\}$. $(X,M,*)$ is called *M-complete* if every M-Cauchy sequence in X is convergent in X.

Note that an M-Cauchy sequence (see (1.2)) is a G-Cauchy sequence (see (1.1)), and the converse is not always true.

It is widely recognized that the existence, uniqueness, convergence, approximation, and fixed point result concerning single-valued contractions in complete metric spaces of Banach [14] (see also Caccioppoli [15]) deeply influenced the direction of fixed point theory.

Theorem 1.1 ([14]) *Let* (X,d) *be a complete metric space. If* $T: X \to X$ *and*

$$\exists_{\lambda\in[0;1)}\forall_{x,y\in X}\bigl\{d\bigl(T(x),T(y)\bigr)\leq\lambda d(x,y)\bigr\},\tag{1.3}$$

then the following are true: (i) T *has a unique fixed point* w *in* X (*i.e., there exists* $w\in X$ *such that* $w=T(w)$ *and* $\mathrm{Fix}(T)=\{w\}$); *and* (ii) *for each* $w^0\in X$, *the sequence* $(T^{[m]}(w^0):m\in\mathbb{N})$ *converges to* w.

The version of Theorem 1.1 for single-valued maps $T: X\to X$ satisfying the contractive condition

$$\exists_{\lambda\in(0;1)}\forall_{x,y\in X}\forall_{t\in(0;\infty)}\bigl\{M\bigl(T(x),T(y),\lambda t\bigr)\geq M(x,y,t)\bigr\}\tag{1.4}$$

in G-complete KM-fuzzy metric spaces was proved by Grabiec [9].

The methods and ideas introduced by Banach and Caccioppoli were generalized in various ways to analyze and solve an astonishing variety of convergence, existence, and approximation problems. Nadler's work (see [16, 17]) on the existence of fixed points was another major advance in this topic since it applies to the set-valued dynamic systems.

Theorem 1.2 ([16, 17]) *Let* (X,d) *be a complete metric space. If* $T: X\to\mathcal{CB}(X)$ *and*

$$\exists_{\lambda\in[0;1)}\forall_{x,y\in X}\bigl\{H^d\bigl(T(x),T(y)\bigr)\leq\lambda d(x,y)\bigr\},\tag{1.5}$$

then $\mathrm{Fix}(T)\neq\varnothing$ (*i.e., there exists* $w\in X$ *such that* $w\in T(w)$).

Here $\mathcal{CB}(X)$ is the class of all nonempty closed and bounded subsets of the metric space (X,d), and H^d is the *Pompeiu-Hausdorff metric* on $\mathcal{CB}(X)\times\mathcal{CB}(X)$ of the form

$$H^d(U,W)=\max\Bigl\{\sup_{u\in U}d(u,W),\sup_{w\in W}d(w,U)\Bigr\},\quad U,W\in\mathcal{CB}(X),\tag{1.6}$$

where $d(x,V)=\inf_{v\in V}d(x,v)$ for $x\in X$ and $V\in\mathcal{CB}(X)$.

There have been further new ideas, results, and perspectives on fuzzy fixed point theory in recent years concerning fuzzy extensions of Theorem 1.2 in the case of set-valued contractions with nonempty compact values.

New and important fixed point and endpoint results for set-valued fuzzy contractions $T: X\to\mathcal{K}(X)$ in M-complete GV-fuzzy metric spaces $(X,M,*)$ satisfying the contractive conditions

$$\exists_{\lambda\in(0;1)}\forall_{x,y\in X}\forall_{t\in(0;\infty)}\bigl\{H_M\bigl(T(x),T(y),\lambda t\bigr)\geq M(x,y,t)\bigr\}\tag{1.7}$$

(and their generalizations) are proved by Kiany and Amini-Harandi [11]; here the *Pompeiu-Hausdorff fuzzy metrics* H_M on $\mathcal{K}(X)\times\mathcal{K}(X)\times(0;\infty)$ is of the form

$$H_M(U,W,t)=\min\Bigl\{\inf_{u\in U}M(u,W,t),\inf_{w\in W}M(w,U,t)\Bigr\},\quad U,W\in\mathcal{K}(X),\tag{1.8}$$

where $\mathcal{K}(X)$ is the space of nonempty compact subsets of X, and $M(x,V,t) = \sup_{v\in V} M(x,v,t)$ for $x \in X$, $V \in \mathcal{K}(X)$, and $t \in (0;\infty)$. Further new results in this direction were proposed by Phiangsungnoen *et al.* [12].

A totally new idea was required in fuzzy spaces with asymmetric structures, for example, in fuzzy quasi-metric spaces. This new idea concerning convergence and completeness was proposed by Gregori and Romaguera [5] and Gregori *et al.* [10].

Definition 1.6 A fuzzy quasi-metric space (in the sense of George and Romaguera [5]) is an ordered triple $(X,M,*)$ where $M : X\times X\times(0;\infty) \to (0;1]$ satisfies (i) $\forall_{x,y\in X}\forall_{t\in(0;\infty)}\{M(x,y,t) > 0\}$, (ii) $\forall_{x\in X}\forall_{t\in(0;\infty)}\{M(x,x,t) = 1\}$, (iii) $\forall_{x,y\in X}\{\exists_{t\in(0;\infty)}\{M(x,y,t) = 1\}$ implies $x = y\}$, (iv) $\forall_{x,y,z\in X}\forall_{t,s\in(0;\infty)}\{M(x,y,t) * M(y,z,s) \le M(x,z,t+s)\}$, and (v) $\forall_{x,y\in X}\{M(x,y,\cdot) : (0;\infty) \to (0;1]$ is continuous$\}$.

When (i)-(v) hold, we will say that $(X,M,*)$ is a *GR-fuzzy quasi-metric space*.

Note that given important references are not exhaustive.

After the years 1922 and 1969, also with very strong assumptions (determining essential tools of investigation and playing an essential role in all known proofs) and with conclusions analogous to those mentioned before, many mathematicians constructed similar contractions and exhibited generalizations of Theorems 1.1 and 1.2 in various sequentially complete spaces.

However, we see that, without required restrictive assumptions and with conclusions more profound than in Theorems 1.1 and 1.2 or in the papers cited, it is not clear how one could construct new spaces, deliver new contractions in these spaces, and prove new analogous theorems for such contractions. This is one of the most fundamental and natural questions concerning theory of spaces and fixed point theory of set-valued and single-valued dynamic systems.

A *set-valued dynamic system* is defined as a pair (X,T), where X is a certain space, and T is a set-valued map $T : X \to 2^X$ (2^X denotes the family of all nonempty subsets of a space X). A *dynamic process* or a *trajectory starting at* $w_0 \in X$ or a *motion* of the system (X,T) at w^0 is a sequence $(w^m : m \in \{0\}\cup\mathbb{N})$ defined by $w^m \in T(w^{m-1})$ for $m \in \mathbb{N}$ (see Aubin and Siegel [18], Aubin and Ekeland [19], Aubin and Frankowska [20], and Yuan [21]). By $\mathrm{Fix}(T)$ and $\mathrm{Per}(T)$ we denote the sets of all *fixed points* and *periodic points* of T, respectively, that is, $\mathrm{Fix}(T) = \{w \in X : w \in T(w)\}$ and $\mathrm{Per}(T) = \{w \in X : w \in T^{[k]}(w)$ for some $k \in \mathbb{N}\}$.

Recall that a *single-valued dynamic system* is defined as a pair (X,T), where X is a certain space, and T is a single-valued map $T : X \to X$, that is, $\forall_{x\in X}\{T(x) \in X\}$. For each $w^0 \in X$, a sequence $(w^m = T^{[m]}(w^0) : m \in \{0\}\cup\mathbb{N})$ is called a *Picard iteration starting at* w^0 of the system (X,T); here, for $m \in \{0\}\cup\mathbb{N}$, we define $T^{[m]} = T\circ T\circ\cdots\circ T$ (m-times) and $T^{[0]} = I_X$ (the identity map on X). By $\mathrm{Fix}(T)$ and $\mathrm{Per}(T)$ we denote the sets of all *fixed points* and *periodic points* of T, respectively, that is, $\mathrm{Fix}(T) = \{w \in X : w = T(w)\}$ and $\mathrm{Per}(T) = \{w \in X : w = T^{[k]}(w)$ for some $k \in \mathbb{N}\}$.

In this paper our aim is twofold. First, we want to introduce and describe fuzzy quasi-triangular spaces. Second, we want to show how the fuzzy quasi-triangular spaces combined with some new ideas, methods, techniques, and tools of studying can be used to construct the set-valued and single-valued fuzzy contractions and next to study the problems concerning convergence, periodic points, and fixed points for such contractions. Then, in this more general setting, we formulate and prove fuzzy extensions of Theorems 1.1 and 1.2.

More precisely, this paper is divided into 15 sections. In Section 2, we define very general fuzzy spaces $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ called fuzzy quasi-triangular spaces. In Section 3, we introduce the notions of the left (right) $\mathcal{M}_{C;\mathcal{A}}$-convergence, $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequentially completeness, and $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequential completeness in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$. In Sections 4 and 5, we define left (right) $\mathcal{G}$-families and $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$, which generalize $\mathcal{M}_{C;\mathcal{A}}$, and next we define left (right) $\mathcal{K}_{C;\mathcal{A}}$-convergence, $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequential completeness, and $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequential completeness in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$; in particular, if $\mathcal{K}_{C;\mathcal{A}} = \mathcal{M}_{C;\mathcal{A}}$, then Section 5 reduces to Section 3. In Section 6, using families $\mathcal{K}_{C;\mathcal{A}}$, we introduce the notions of left (right) fuzzy sets of Pompeiu-Hausdorff type on $2^X \times 2^X \times (0;\infty)$, and using these fuzzy sets, we construct left (right) set-valued $T : X \to 2^X$ and single-valued $T : X \to X$ fuzzy contractions. In Section 7, we define $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-admissible and $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-admissible set-valued $T : X \to 2^X$ and single-valued $T : X \to X$ dynamic systems; this notions generalize in some sense the notions of $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequential completeness and $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequential completeness in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ presented in Section 5. Section 8 of the paper is devoted to the left (right) $\mathcal{M}_{C;\mathcal{A}}$-closed set-valued $T : X \to 2^X$ and single-valued $T : X \to X$ dynamic systems. Section 9 includes a convergence, approximation, and periodic point of Nadler-type result with its proof. This result concerns left (right) dynamic systems $T : X \to 2^X$ in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ with left (right) $\mathcal{G}$-families and left (right) $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$. The convergence, existence, periodic point, fixed point, and uniqueness of Banach-type result and its proof in the case of left (right) single-valued fuzzy contractions $T : X \to X$ in fuzzy quasi-triangular spaces $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ with left (right) $\mathcal{G}$-families $\mathcal{K}_{C;\mathcal{A}}$ and left (right) $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ (generated by $\mathcal{M}_{C;\mathcal{A}}$) are given in Section 10. Important relations between fuzzy quasi-triangular spaces and quasi-triangular spaces are to be found in Section 11. In Section 12, some fuzzy quasi-triangular spaces are constructed. The examples given in Sections 13 and 14 illustrate the fact that the results obtained here are different from those well known in the literature. Section 15 concerns some conclusions.

Results obtained here are new even in fuzzy metric spaces. This paper is a continuation of [22].

2 Fuzzy quasi-triangular spaces $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ and $\mathcal{M}_{C;\mathcal{A}}$-separability

It is worth noticing that the fuzzy quasi-triangular spaces $(X, \mathcal{M}_{C;\mathcal{A}}, *)$, introduced in this section, are not necessarily topological or Hausdorff or sequentially complete and are, in particular, substantial generalizations of fuzzy metric spaces [3, 4] and fuzzy quasi-metric spaces [5].

Definition 2.1 Let X be a (nonempty) set, $\mathcal{A}$ be an index set, $C = \{C_\alpha\}_{\alpha\in\mathcal{A}} \in [1;\infty)^{\mathcal{A}}$, and $* : [0;1] \times [0;1] \to [0;1]$ be a continuous t-norm.

(A) We say that a family $\mathcal{M}_{C;\mathcal{A}} = \{M_\alpha : X \times X \times (0;\infty) \to (0;1], \alpha \in \mathcal{A}\}$ of *fuzzy sets* M_α, $\alpha \in \mathcal{A}$, is a *fuzzy quasi-triangular family* on X if

$$\begin{aligned} &\forall_{\alpha\in\mathcal{A}} \forall_{x,y,z\in X} \forall_{t,s\in(0;\infty)} \bigl\{ M_\alpha(x,y,t) * M_\alpha(y,z,s) \\ &\quad \le M_\alpha\bigl(x,z,C_\alpha(t+s)\bigr) \bigr\}. \end{aligned} \tag{2.1}$$

A *fuzzy quasi-triangular space* $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ is a set X together with a fuzzy quasi-triangular family $\mathcal{M}_{C;\mathcal{A}} = \{M_\alpha : X \times X \times (0;\infty) \to (0;1], \alpha \in \mathcal{A}\}$ and with a continuous t-norm $*$.

(B) We say that $\mathcal{M}_{\mathcal{A}} = \{M_\alpha : X \times X \times (0;\infty) \to (0;1], \alpha \in \mathcal{A}\}$ is a *fuzzy triangular family* if *fuzzy triangular sets* M_α, $\alpha \in \mathcal{A}$, satisfy the condition

$$\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\forall_{t,s\in(0;\infty)}\bigl\{M_\alpha(x,y,t) * M_\alpha(y,z,s) \le M_\alpha(x,z,t+s)\bigr\}.$$

A *fuzzy triangular space* $(X, \mathcal{M}_{\mathcal{A}}, *)$ is a set X together with a fuzzy triangular family $\mathcal{M}_{\mathcal{A}} = \{M_\alpha : X \times X \times (0;\infty) \to (0;1], \alpha \in \mathcal{A}\}$ and a continuous t-norm $*$.

(C) Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space. We say that $\mathcal{M}_{C;\mathcal{A}}$ is *separating on X* if

$$\forall_{u,w\in X}\bigl\{u \ne w \Rightarrow \exists_{\alpha_0\in\mathcal{A}}\exists_{t_0\in(0;\infty)}\bigl\{M_{\alpha_0}(u,w,t_0) < 1 \vee M_{\alpha_0}(w,u,t_0) < 1\bigr\}\bigr\}. \tag{2.2}$$

(D) If $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ is a fuzzy quasi-triangular space and $\mathcal{M}^{-1}_{C;\mathcal{A}} = \{M^{-1}_\alpha, \alpha \in \mathcal{A}\}$, where $\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\forall_{t\in(0;\infty)}\{M^{-1}_\alpha(x,y,t) = M_\alpha(y,x,t)\}$, then $\mathcal{M}^{-1}_{C;\mathcal{A}}$ is a fuzzy quasi-triangular family on X, and we say that the fuzzy quasi-triangular space $(X, \mathcal{M}^{-1}_{C;\mathcal{A}}, *)$ is the *conjugation* of $(X, \mathcal{M}_{C;\mathcal{A}}, *)$.

Remark 2.1 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space. In general, the following properties do not necessarily hold:

(A) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\{\forall_{t\in(0;\infty)}\{M_\alpha(x,y,t) = 1\}$ iff $x = y\}$, or, equivalently, $\forall_{\alpha\in\mathcal{A}}\forall_{x\in X}\forall_{t\in(0;\infty)}\{M_\alpha(x,x,t) = 1\}$ and $\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\forall_{t\in(0;\infty)}\{x \ne y$ implies $M_\alpha(x,y,t) < 1\}$.
(B) $\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{x,y\in X}\{M_\alpha(x,y,t) = M_\alpha(y,x,t)\}$.
(C) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\{M_\alpha(x,y,\cdot) : (0;\infty) \to (0;1]$ is nondecreasing$\}$.
(D) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\{M_\alpha(x,y,\cdot) : (0;\infty) \to (0;1]$ is continuous$\}$.

3 Left (right) $\mathcal{M}_{C;\mathcal{A}}$-convergence in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$, Hausdorff property, and $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequential completeness, and $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequential completeness

The notion of G-completeness in KM-fuzzy metric spaces was introduced by Grabiec [9] (see Definition 1.4). The M-completeness in GV-fuzzy metric spaces was introduced in George and Veeramani [4] (see Definition 1.5). In fuzzy quasi-metric spaces, using ideas of Reilly *et al.* [23], the extensions of the notion of completeness were obtained by Gregori *et al.* [10] in a more complicated presentation, which is a consequence of asymmetric structures of these fuzzy spaces.

The notions of $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequential completeness and $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequential completeness in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ are defined in this section.

A natural starting point is to define the notions of *left* (*right*) $\mathcal{M}_{C;\mathcal{A}}$-*convergence* of sequences in the fuzzy quasi-triangular spaces $(X, \mathcal{M}_{C;\mathcal{A}}, *)$.

Definition 3.1 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space.

(A) We say that the sequence $(x_m : m \in \mathbb{N})$ in X is *left* (*right*) $\mathcal{M}_{C;\mathcal{A}}$*-convergent to* $x \in X$ if $x \in LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ ($x \in LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$) where

$$LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} = \Big\{u \in X : \forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{\lim_{m\to\infty} M_\alpha(u, x_m, t) = 1\Big\}\Big\}$$

$$\Big(LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} = \Big\{u \in X : \forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{\lim_{m\to\infty} M_\alpha(x_m, u, t) = 1\Big\}\Big\}\Big).$$

(B) We say that a sequence $(x_m : m \in \mathbb{N})$ in X is *left* (*right*) $\mathcal{M}_{C;\mathcal{A}}$*-convergent in* X if $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$).

(C) We say that $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ is *left* (*right*) *Hausdorff* if for each left (right) $\mathcal{M}_{C;\mathcal{A}}$-convergent in X sequence $(x_m : m \in \mathbb{N})$, the set $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ ($LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$) is a singleton.

Remark 3.1 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space.

(A) It is clear that if $(x_m : m \in \mathbb{N})$ is left (right) $\mathcal{M}_{C;\mathcal{A}}$-convergent in X, then $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \subset LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$ ($LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \subset LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$) for each subsequence $(y_m : m \in \mathbb{N})$ of $(x_m : m \in \mathbb{N})$.

(B) The limit of left (right) $\mathcal{M}_{C;\mathcal{A}}$-convergent sequence in X need not be a singleton; see Examples 12.1-12.5.

Now we define $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequentially completeness and $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequentially completeness in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$.

Definition 3.2 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space.

(A) We say that a sequence $(x_m : m \in \mathbb{N})$ in X is an $\mathcal{M}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{G}$*-sequence in* X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Big\{\lim_{m\to\infty} M_\alpha(x_m, x_{m+p}, t) = 1\Big\} \tag{3.1}$$

$$\Big(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Big\{\lim_{m\to\infty} M_\alpha(x_{m+p}, x_m, t) = 1\Big\}\Big). \tag{3.2}$$

If every $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequence $(x_m : m \in \mathbb{N})$ in X is left (right) $\mathcal{M}_{C;\mathcal{A}}$-convergent in X (*i.e.*, $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$)), then $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ is called $\mathcal{M}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{G}$*-sequentially complete*.

(B) We say that a sequence $(x_m : m \in \mathbb{N})$ in X is an $\mathcal{M}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{W}$*-sequence in* X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{\lim_{m\to\infty} \inf_{n>m} M_\alpha(x_m, x_n, t) = 1\Big\} \tag{3.3}$$

$$\Big(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{\lim_{m\to\infty} \inf_{n>m} M_\alpha(x_n, x_m, t) = 1\Big\}\Big). \tag{3.4}$$

If every $\mathcal{M}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequence $(x_m : m \in \mathbb{N})$ in X is left (right) $\mathcal{M}_{C;\mathcal{A}}$-convergent in X (*i.e.*, $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$)), then $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ is called $\mathcal{M}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{W}$*-sequentially complete*.

Remark 3.2 Note that every $\mathcal{M}_{C;\mathcal{A}}$ left (respectively, right) $\mathcal{W}$-sequence is also an $\mathcal{M}_{C;\mathcal{A}}$ left (respectively, right) $\mathcal{G}$-sequence. Indeed, (3.3) and (3.4) imply

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{0<\varepsilon<1}\exists_{m_0\in\mathbb{N}}\forall_{m>m_0}\Big\{\inf_{n>m} M_\alpha(x_m, x_n, t) > 1-\varepsilon\Big\}$$

$$\Big(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{0<\varepsilon<1}\exists_{m_0\in\mathbb{N}}\forall_{m>m_0}\Big\{\inf_{n>m} M_\alpha(x_n, x_m, t) > 1-\varepsilon\Big\}\Big).$$

Hence,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{0<\varepsilon<1}\exists_{m_0\in\mathbb{N}}\forall_{n>m>m_0}\{M_\alpha(x_m,x_n,t)>1-\varepsilon\}$$
$$\bigl(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{0<\varepsilon<1}\exists_{m_0\in\mathbb{N}}\forall_{n>m>m_0}\{M_\alpha(x_n,x_m,t)>1-\varepsilon\}\bigr),$$

and this yields

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\forall_{0<\varepsilon<1}\exists_{m_0\in\mathbb{N}}\forall_{m>m_0}\{M_\alpha(x_m,x_{m+p},t)>1-\varepsilon\}$$
$$\bigl(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\forall_{0<\varepsilon<1}\exists_{m_0\in\mathbb{N}}\forall_{m>m_0}\{M_\alpha(x_{m+p},x_m,t)>1-\varepsilon\}\bigr).$$

Hence, we get (3.1) and (3.2).

4 Fuzzy quasi-triangular spaces $(X,\mathcal{M}_{C;\mathcal{A}},*)$ with left (right) $\mathcal{G}$-families $\mathcal{K}_{C;\mathcal{A}}$ and left (right) $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$ and relation between $\mathcal{K}_{C;\mathcal{A}}$-separability and $\mathcal{M}_{C;\mathcal{A}}$-separability

For given fuzzy quasi-triangular spaces $(X,\mathcal{M}_{C;\mathcal{A}},*)$, it is natural to define the notions of left (right) $\mathcal{G}$-families $\mathcal{K}_{C;\mathcal{A}}$ and left (right) $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$ (see Definitions 4.1 and 4.2), which provide new structures on X.

Definition 4.1 Let $(X,\mathcal{M}_{C;\mathcal{A}},*)$ be a fuzzy quasi-triangular space.

(A) The family $\mathcal{K}_{C;\mathcal{A}}=\{K_\alpha:\alpha\in\mathcal{A}\}$ of fuzzy sets $K_\alpha:X\times X\times(0,\infty)\to(0;1]$, $\alpha\in\mathcal{A}$, is said to be a *left* (*right*) $\mathcal{G}$*-family generated by* $\mathcal{M}_{C;\mathcal{A}}$ if:

($\mathcal{KG}1$) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\forall_{t,s\in(0;\infty)}\{K_\alpha(x,y,t)*K_\alpha(y,z,s)\le K_\alpha(x,z,C_\alpha(t+s))\}$.

($\mathcal{KG}2$) For any sequences $(x_m:m\in\mathbb{N})$ and $(y_m:m\in\mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Bigl\{\lim_{m\to\infty}K_\alpha(x_m,x_{m+p},t)=1\Bigr\}\tag{4.1}$$
$$\Bigl(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Bigl\{\lim_{m\to\infty}K_\alpha(x_{m+p},x_m,t)=1\Bigr\}\Bigr)\tag{4.2}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty}K_\alpha(y_m,x_m,t)=1\Bigr\}\tag{4.3}$$
$$\Bigl(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty}K_\alpha(x_m,y_m,t)=1\Bigr\}\Bigr),\tag{4.4}$$

we have

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty}M_\alpha(y_m,x_m,t)=1\Bigr\}\tag{4.5}$$
$$\Bigl(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty}M_\alpha(x_m,y_m,t)=1\Bigr\}\Bigr).\tag{4.6}$$

(B) $\mathbb{K}^{L;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$ ($\mathbb{K}^{R;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$) is the set of all left (right) $\mathcal{G}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$.

Definition 4.2 Let $(X,\mathcal{M}_{C;\mathcal{A}},*)$ be a fuzzy quasi-triangular space.

(A) The family $\mathcal{K}_{C;\mathcal{A}} = \{K_\alpha : \alpha \in \mathcal{A}\}$ of fuzzy sets $K_\alpha : X \times X \times (0,\infty) \to (0;1]$, $\alpha \in \mathcal{A}$, is said to be the *left* (*right*) $\mathcal{W}$*-family generated by* $\mathcal{M}_{C;\mathcal{A}}$ if:

($\mathcal{KW}1$) $\forall_{\alpha\in\mathcal{A}}\forall_{x,y,z\in X}\forall_{t,s\in(0;\infty)}\{K_\alpha(x,y,t) * K_\alpha(y,z,s) \le K_\alpha(x,z,C_\alpha(t+s))\}$.

($\mathcal{KW}2$) For any sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty}\inf_{n>m} K_\alpha(x_m,x_n,t) = 1\Bigr\} \tag{4.7}$$

$$\Bigl(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty}\inf_{n>m} K_\alpha(x_n,x_m,t) = 1\Bigr\}\Bigr) \tag{4.8}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty} K_\alpha(y_m,x_m,t) = 1\Bigr\} \tag{4.9}$$

$$\Bigl(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty} K_\alpha(x_m,y_m,t) = 1\Bigr\}\Bigr), \tag{4.10}$$

we have

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty} M_\alpha(y_m,x_m,t) = 1\Bigr\} \tag{4.11}$$

$$\Bigl(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty} M_\alpha(x_m,y_m,t) = 1\Bigr\}\Bigr). \tag{4.12}$$

(B) $\mathbb{K}^{L;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$ ($\mathbb{K}^{R;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$) is the set of all left (right) $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$.

A natural step is to understand the remarkable relations in $(X,\mathcal{M}_{C;\mathcal{A}},*)$ between $\mathcal{M}_{C;\mathcal{A}}$ and left (right) $\mathcal{G}$-families and $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$.

Remark 4.1 The left (right) $\mathcal{G}$-families $\mathcal{K}_{C;\mathcal{A}}$ and $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$ are substantial generalizations of $\mathcal{M}_{C;\mathcal{A}}$. Indeed, we have:

(A) Definitions 4.1 and 4.2 imply that $\mathcal{M}_{C;\mathcal{A}} \in \mathbb{K}^{L;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)} \cap \mathbb{K}^{R;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$ and $\mathcal{M}_{C;\mathcal{A}} \in \mathbb{K}^{L;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)} \cap \mathbb{K}^{R;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$.

(B) The arguments in Section 11 show that $\mathbb{K}^{L;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}\setminus\{\mathcal{M}_{C;\mathcal{A}}\} \neq \varnothing$, $\mathbb{K}^{R;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}\setminus\{\mathcal{M}_{C;\mathcal{A}}\} \neq \varnothing$, $\mathbb{K}^{L;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}\setminus\{\mathcal{M}_{C;\mathcal{A}}\} \neq \varnothing$, and $\mathbb{K}^{R;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}\setminus\{\mathcal{M}_{C;\mathcal{A}}\} \neq \varnothing$.

(C) $(X,\mathcal{K}_{C;\mathcal{A}},*)$ are fuzzy quasi-triangular spaces.

We introduce the notion of $\mathcal{K}_{C;\mathcal{A}}$-separability in $(X,\mathcal{M}_{C;\mathcal{A}},*)$ as follows.

Definition 4.3 Let $(X,\mathcal{M}_{C;\mathcal{A}},*)$ be a fuzzy quasi-triangular space, and $\mathcal{K}_{C;\mathcal{A}}$ be the left (right) $\mathcal{G}$-family or $\mathcal{W}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$. We say that $\mathcal{K}_{C;\mathcal{A}}$ is separating on X if

$$\begin{aligned}&\forall_{u,w\in X}\bigl\{u \neq w \Rightarrow \exists_{\alpha_0\in\mathcal{A}}\exists_{t_0\in(0;\infty)}\bigl\{K_{\alpha_0}(u,w,t_0) < 1\\ &\quad\vee K_{\alpha_0}(w,u,t_0) < 1\bigr\}\bigr\}.\end{aligned} \tag{4.13}$$

The notion of $\mathcal{K}_{C;\mathcal{A}}$-separability is used for the following interesting theorem.

Theorem 4.1 *Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space, and let $\mathcal{K}_{C;\mathcal{A}}$ be the left (right) $\mathcal{G}$-family or $\mathcal{W}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$. If $\mathcal{M}_{C;\mathcal{A}}$ is separating on X, then $\mathcal{K}_{C;\mathcal{A}}$ is separating on X.*

Proof We begin by supposing that $u_0, w_0 \in X$, $u_0 \neq w_0$, and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{K_\alpha(u_0, w_0, t) = 1 \wedge K_\alpha(w_0, u_0, t) = 1\bigr\}.$$

Then ($\mathcal{KG}1$) and ($\mathcal{KW}1$) imply

$$\begin{aligned}\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{&K_\alpha(u_0, u_0, t)\\ &\geq K_\alpha(u_0, w_0, t/2) * K_\alpha(w_0, u_0, t/2) = 1 * 1 = 1\bigr\}\end{aligned}$$

or, equivalently,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{K_\alpha(u_0, u_0, t) = K_\alpha(w_0, u_0, t) = 1\bigr\}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{K_\alpha(u_0, u_0, t) = K_\alpha(u_0, w_0, t) = 1\bigr\}.$$

Assuming that $x_m = u_0$ and $y_m = w_0$, $m \in \mathbb{N}$, we conclude that

$$\begin{aligned}&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Bigl\{\lim_{m\to\infty} K_\alpha(x_m, x_{m+p}, t) = \lim_{m\to\infty} K_\alpha(y_m, x_m, t) = 1\Bigr\},\\ &\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Bigl\{\lim_{m\to\infty} K_\alpha(x_{m+p}, x_m, t) = \lim_{m\to\infty} K_\alpha(x_m, y_m, t) = 1\Bigr\},\\ &\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty}\inf_{n>m} K_\alpha(x_m, x_n, t) = \lim_{m\to\infty} K_\alpha(y_m, x_m, t) = 1\Bigr\},\\ &\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty}\inf_{n>m} K_\alpha(x_n, x_m, t) = \lim_{m\to\infty} K_\alpha(x_m, y_m, t) = 1\Bigr\}.\end{aligned}$$

Therefore, it is not hard to see that (4.1)-(4.4) and (4.7)-(4.10) hold, and, by ($\mathcal{KG}2$) and ($\mathcal{KW}2$), the above considerations lead to the following conclusion:

$$u_0 \neq w_0 \wedge \forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty} M_\alpha(y_m, x_m, t) = \lim_{m\to\infty} M_\alpha(x_m, y_m, t) = 1\Bigr\}$$

or, equivalently,

$$u_0 \neq w_0 \wedge \forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{M_\alpha(w_0, u_0, t) = M_\alpha(u_0, w_0, t) = 1\bigr\}.$$

However, $\mathcal{M}_{C;\mathcal{A}}$ is separating (see (2.2)), a contradiction. Therefore, $\mathcal{K}_{C;\mathcal{A}}$ is separating. □

5 Left (right) $\mathcal{K}_{C;\mathcal{A}}$-convergence, $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequential completeness, and $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequential completeness in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$

Definition 5.1 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space, and $\mathcal{K}_{C;\mathcal{A}} = \{K_\alpha : X \times X \times (0,\infty) \to (0;1], \alpha \in \mathcal{A}\}$ be the left (right) $\mathcal{G}$-family or left (right) $\mathcal{W}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$.

We say that a sequence $(x_m : m \in \mathbb{N})$ in X is *left* (*right*) $\mathcal{K}_{C;\mathcal{A}}$*-convergent to* $x \in X$ if $x \in LIM^{L-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ ($x \in LIM^{R-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$) where

$$LIM^{L-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} = \left\{u \in X : \forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha(u, x_m, t) = 1\right\}\right\}$$
$$\left(LIM^{R-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} = \left\{u \in X : \forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha(x_m, u, t) = 1\right\}\right\}\right).$$

We say that a sequence $(x_m : m \in \mathbb{N})$ in X is *left* (*right*) $\mathcal{K}_{C;\mathcal{A}}$*-convergent in* X if $LIM^{L-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$).

Definition 5.2 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space, and $\mathcal{K}_{C;\mathcal{A}} = \{K_\alpha : X \times X \times (0,\infty) \to (0;1], \alpha \in \mathcal{A}\}$ be the left (right) $\mathcal{G}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$.

(A) We say that a sequence $(x_m : m \in \mathbb{N})$ in X is a $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{G}$*-sequence in* X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} K_\alpha(x_m, x_{m+p}, t) = 1\right\} \tag{5.1}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} K_\alpha(x_{m+p}, x_m, t) = 1\right\}\right). \tag{5.2}$$

(B) If every $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequence $(x_m : m \in \mathbb{N})$ in X is left (right) $\mathcal{K}_{C;\mathcal{A}}$-convergent in X (*i.e.*, $LIM^{L-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$)), then $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ is called $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{G}$*-sequentially complete.*

Definition 5.3 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space, and let $\mathcal{K}_{C;\mathcal{A}} = \{K_\alpha : X \times X \times (0,\infty) \to (0;1], \alpha \in \mathcal{A}\}$ be the left (right) $\mathcal{W}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$.

(A) We say that a sequence $(x_m : m \in \mathbb{N})$ in X is a $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{W}$*-sequence in* X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty}\inf_{n>m} K_\alpha(x_m, x_n, t) = 1\right\} \tag{5.3}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty}\inf_{n>m} K_\alpha(x_n, x_m, t) = 1\right\}\right). \tag{5.4}$$

(B) If every $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequence $(x_m : m \in \mathbb{N})$ in X is left (right) $\mathcal{K}_{C;\mathcal{A}}$-convergent in X (*i.e.*, $LIM^{L-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{K}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$)), then $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ is called $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{W}$*-sequentially complete.*

Remark 5.1

(A) Every $\mathcal{K}_{C;\mathcal{A}}$ left (respectively, right) $\mathcal{W}$*-sequence in* X is $\mathcal{K}_{C;\mathcal{A}}$ left (respectively, right) $\mathcal{G}$*-sequence in* X. This can be proved by using (5.1)-(5.4) and by adopting the arguments in Remark 3.2.

(B) Note that every $\mathcal{K}_{C;\mathcal{A}}$ left (respectively, right) $\mathcal{G}$-sequentially complete fuzzy quasi-triangular space $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ is also $\mathcal{K}_{C;\mathcal{A}}$ left (respectively, right) $\mathcal{W}$-sequentially complete. Indeed, assume that $(x_m : m \in \mathbb{N})$ in X is $\mathcal{K}_{C;\mathcal{A}}$ left (respectively, right) $\mathcal{W}$*-sequence in* X. Then, by (A) it is also $\mathcal{K}_{C;\mathcal{A}}$ left (respectively, right) $\mathcal{G}$*-sequence in* X, and this means that it is left (respectively, right) $\mathcal{K}_{C;\mathcal{A}}$-convergent in X.

6 Left (right) fuzzy sets of Pompeiu-Hausdorff type on $2^X \times 2^X \times (0;\infty)$ and set-valued $T: X \to 2^X$ and single-valued $T: X \to X$ left (right) fuzzy contractions in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$

The Pompeiu-Hausdorff metric (1.6) on closed or compact sets in metric spaces (X,d) plays an essential role in mathematics and its applications. See Berinde and Păcurar [24] for a comprehensive treatment of the foundations of this metric.

In this section, in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ with left (right) $\mathcal{G}$-families $\mathcal{K}_{C;\mathcal{A}}$ and left (right) $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$, we define three kinds of left (right) fuzzy sets of Pompeiu-Hausdorff type on $2^X \times 2^X \times (0;\infty)$, and, using these fuzzy sets, we construct three kinds of set-valued left (right) fuzzy contractions $T: X \to 2^X$ and two kinds of single-valued left (right) fuzzy contractions $T: X \to X$.

Definition 6.1 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space, $\mathcal{K}_{C;\mathcal{A}}$ be the left (right) $\mathcal{G}$-family or $\mathcal{W}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$, and

$$\begin{aligned}&\forall_{\alpha\in\mathcal{A},x\in X,V\in 2^X,t\in(0;\infty)}\bigl\{K_\alpha(x,V,t)=\sup\bigl\{K_\alpha(x,v,t):v\in V\bigr\}\\&\quad\wedge K_\alpha(V,x,t)=\sup\bigl\{K_\alpha(v,x,t):v\in V\bigr\}\bigr\}.\end{aligned}\tag{6.1}$$

(A) If $\mathcal{K}_{C;\mathcal{A}} \in \mathbb{K}^{L;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)} \cup \mathbb{K}^{L;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$, $\eta \in \{1,2,3\}$, and

$$\forall_{\alpha\in\mathcal{A}}\forall_{U,W\in 2^X}\forall_{t\in(0;\infty)}\left\{F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,2^X}(U,W,t)\right.$$
$$=\left\{\begin{array}{ll}\min\{\inf_{u\in U}K_\alpha(u,W,t),\inf_{w\in W}K_\alpha(U,w,t)\} & \text{if } \eta=1,\\ \min\{\inf_{u\in U}K_\alpha(u,W,t),\inf_{w\in W}K_\alpha(w,U,t)\} & \text{if } \eta=2,\\ \inf_{u\in U}K_\alpha(u,W,t) & \text{if } \eta=3\end{array}\right\},$$

then a family $\mathcal{F}^{L-\mathcal{K}_{C;\mathcal{A}}}_{\eta,2^X} = \{F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,2^X}, \alpha\in\mathcal{A}\}$ is said to be a *left fuzzy set of Pompeiu-Hausdorff type on* $2^X \times 2^X \times (0;\infty)$.

(B) If $\mathcal{K}_{C;\mathcal{A}} \in \mathbb{K}^{R;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)} \cup \mathbb{K}^{R;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$, $\eta \in \{1,2,3\}$, and

$$\forall_{\alpha\in\mathcal{A}}\forall_{U,W\in 2^X}\forall_{t\in(0;\infty)}\left\{F^{R-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,2^X}(U,W,t)\right.$$
$$=\left\{\begin{array}{ll}\min\{\inf_{u\in U}K_\alpha(u,W,t),\inf_{w\in W}K_\alpha(U,w,t)\} & \text{if } \eta=1,\\ \min\{\inf_{u\in U}K_\alpha(u,W,t),\inf_{w\in W}K_\alpha(w,U,t)\} & \text{if } \eta=2,\\ \inf_{u\in U}K_\alpha(u,W,t) & \text{if } \eta=3\end{array}\right\},$$

then a family $\mathcal{F}^{R-\mathcal{K}_{C;\mathcal{A}}}_{\eta,2^X} = \{F^{R-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,2^X}, \alpha\in\mathcal{A}\}$ is said to be a *right fuzzy set of Pompeiu-Hausdorff type on* $2^X \times 2^X \times (0;\infty)$.

Definition 6.2 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space, $\mathcal{K}_{C;\mathcal{A}}$ be the left (right) $\mathcal{G}$-family or left (right) $\mathcal{W}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$, $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathcal{A}} \in (0;1)^{\mathcal{A}}$, (X,T) be a set-valued dynamic system, $T: X \to 2^X$, and $\eta \in \{1,2,3\}$.

(A) If $\mathcal{K}_{C;\mathcal{A}} \in \mathbb{K}^{L;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)} \cup \mathbb{K}^{L;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$ and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{x,y\in X}\bigl\{F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,2^X}\bigl(T(x),T(y),\lambda_\alpha t\bigr) \\ \geq K_\alpha(x,y,C_\alpha t)\bigr\}, \tag{6.2}$$

then we say that (X,T) is a *fuzzy* $(\mathcal{F}^{L-\mathcal{K}_{C;\mathcal{A}}}_{\eta,2^X},\lambda)$*-left contraction.*

(B) If $\mathcal{K}_{C;\mathcal{A}} \in \mathbb{K}^{R;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)} \cup \mathbb{K}^{R;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$ and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{x,y\in X}\bigl\{F^{R-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,2^X}\bigl(T(x),T(y),\lambda_\alpha t\bigr) \\ \geq K_\alpha(x,y,C_\alpha t)\bigr\}, \tag{6.3}$$

then we say that (X,T) is a *fuzzy* $(\mathcal{F}^{R-\mathcal{K}_{C;\mathcal{A}}}_{\eta,2^X},\lambda)$*-right contraction.*

The following Definition 6.3 can be stated as a single-valued version of Definitions 6.1 and 6.2.

Definition 6.3 Let $(X,\mathcal{M}_{C;\mathcal{A}},*)$ be a fuzzy quasi-triangular space, let $\mathcal{K}_{C;\mathcal{A}}$ be the left (right) $\mathcal{G}$-family or left (right) $\mathcal{W}$-family on X generated by $\mathcal{M}_{C;\mathcal{A}}$, and let $\eta \in \{1,2\}$. Let $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathcal{A}} \in (0;1)^{\mathcal{A}}$, and let (X,T) be a single-valued dynamic system, $T: X \to X$.

(A) If $\mathcal{K}_{C;\mathcal{A}} \in \mathbb{K}^{L;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)} \cup \mathbb{K}^{L;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$, then we define a *left fuzzy set* $\mathcal{F}^{L-\mathcal{K}_{C;\mathcal{A}}}_{\eta,X}$ *on* $X\times X\times(0;\infty)$ by $\mathcal{F}^{L-\mathcal{K}_{C;\mathcal{A}}}_{\eta,X} = \{F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,X}, \alpha\in\mathcal{A}\}$ where

$$\forall_{\alpha\in\mathcal{A}}\forall_{u,w\in X}\forall_{t\in(0;\infty)}F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,X}(u,w,t) \\ = \begin{cases} \min\{K_\alpha(u,w,t),K_\alpha(w,u,t)\} & \text{if } \eta=1, \\ K_\alpha(u,w,t) & \text{if } \eta=2. \end{cases}$$

We say that (X,T) is a *fuzzy* $(\mathcal{F}^{L-\mathcal{K}_{C;\mathcal{A}}}_{\eta,X},\lambda)$*-left contraction* if

$$\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\forall_{t\in(0;\infty)}\bigl\{F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,X}\bigl(T(x),T(y),\lambda_\alpha t\bigr) \\ \geq K_\alpha(x,y,C_\alpha t)\bigr\}. \tag{6.4}$$

(B) If $\mathcal{K}_{C;\mathcal{A}} \in \mathbb{K}^{R;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)} \cup \mathbb{K}^{R;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}},*)}$, then we define a *right fuzzy set* $\mathcal{F}^{R-\mathcal{K}_{C;\mathcal{A}}}_{\eta,X}$ *on* $X\times X\times(0;\infty)$ by $\mathcal{F}^{R-\mathcal{K}_{C;\mathcal{A}}}_{\eta,X} = \{F^{R-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,X}, \alpha\in\mathcal{A}\}$ where

$$\forall_{\alpha\in\mathcal{A}}\forall_{u,w\in X}\forall_{t\in(0;\infty)}(PH)^{R-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,X}(u,w,t) \\ = \begin{cases} \min\{K_\alpha(u,w,t),K_\alpha(w,u,t)\} & \text{if } \eta=1, \\ K_\alpha(u,w,t) & \text{if } \eta=2. \end{cases}$$

We say that (X,T) is a *fuzzy* $(\mathcal{F}^{R-\mathcal{K}_{C;\mathcal{A}}}_{\eta,X},\lambda)$*-right contraction* if

$$\forall_{\alpha\in\mathcal{A}}\forall_{x,y\in X}\forall_{t\in(0;\infty)}\bigl\{F^{R-\mathcal{K}_{C;\mathcal{A}}}_{\alpha;\eta,X}\bigl(T(x),T(y),\lambda_\alpha t\bigr) \\ \geq K_\alpha(x,y,C_\alpha t)\bigr\}. \tag{6.5}$$

Remark 6.1 By Definition 2.1, Remark 4.1, and the property $\mathcal{K}(X) \subset \mathcal{CB}(X) \subset 2^X$ it follows that, even when $\eta = 2$, Definitions 6.1 and 6.2 extend (1.8) and (1.4).

7 $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-admissible and $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-admissible set-valued $T : X \to 2^X$ and single-valued $T : X \to X$ dynamic systems in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$

The following terminology will be often used in the sequel.

Definition 7.1 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space, and (X, T) be a set-valued dynamic system, $T : X \to 2^X$.

(A) Let $\mathcal{K}_{C;\mathcal{A}}$ be the left (right) $\mathcal{G}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$.

We say that (X, T) is $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{G}$-*admissible in a point* $w^0 \in X$ if each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ starting at w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1} \in T(w^m)\}$, and satisfying the property

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} K_\alpha\left(w^m, w^{m+p}, t\right) = 1\right\}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} K_\alpha\left(w^{m+p}, w^m, t\right) = 1\right\}\right)$$

is left (right) $\mathcal{K}_{C;\mathcal{A}}$-convergent, that is, there exists $w \in X$ such that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha\left(w, w^m, t\right) = 1\right\}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha\left(w^m, w, t\right) = 1\right\}\right).$$

We say that (X, T) is $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{G}$-*admissible on* X if (X, T) is $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-admissible in each point $w^0 \in X$.

(B) Let $\mathcal{K}_{C;\mathcal{A}}$ be the left (right) $\mathcal{W}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$.

We say that (X, T) is $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{W}$-*admissible in a point* $w^0 \in X$ if each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ starting at w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1} \in T(w^m)\}$, and satisfying the property

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} \inf_{n>m} K_\alpha\left(w^m, w^n, t\right) = 1\right\}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} \inf_{n>m} K_\alpha\left(w^n, w^m, t\right) = 1\right\}\right)$$

is left (right) $\mathcal{K}_{C;\mathcal{A}}$-convergent, that is, there exists $w \in X$ such that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha\left(w, w^m, t\right) = 1\right\}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha\left(w^m, w, t\right) = 1\right\}\right).$$

We say that (X, T) is $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{W}$-*admissible on* X if (X, T) is $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-admissible in each point $w^0 \in X$.

Definition 7.2 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space, and (X, T) be a single-valued dynamic system, $T : X \to X$.

(A) Let $\mathcal{K}_{C;\mathcal{A}}$ be the left (right) $\mathcal{G}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$.

We say that (X,T) is $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{G}$*-admissible in a point* $w^0 \in X$ if a sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfying the property

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} K_\alpha\left(w^m, w^{m+p}, t\right) = 1\right\}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} K_\alpha\left(w^{m+p}, w^m, t\right) = 1\right\}\right)$$

is left (right) $\mathcal{K}_{C;\mathcal{A}}$-convergent, that is, there exists $w \in X$ such that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha\left(w, w^m, t\right) = 1\right\}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha\left(w^m, w, t\right) = 1\right\}\right).$$

We say that (X,T) is $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{G}$*-admissible on* X if (X,T) is $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-admissible in each point $w^0 \in X$.

(B) Let $\mathcal{K}_{C;\mathcal{A}}$ be the left (right) $\mathcal{W}$-family generated by $\mathcal{M}_{C;\mathcal{A}}$.

We say that (X,T) is $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{W}$*-admissible in a point* $w^0 \in X$ if a sequence $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ satisfying the property

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty}\inf_{n>m} K_\alpha\left(w^m, w^n, t\right) = 1\right\}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty}\inf_{n>m} K_\alpha\left(w^n, w^m, t\right) = 1\right\}\right)$$

is left (right) $\mathcal{K}_{C;\mathcal{A}}$-convergent, that is, there exists $w \in X$ such that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha\left(w, w^m, t\right) = 1\right\}$$
$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha\left(w^m, w, t\right) = 1\right\}\right).$$

We say that (X,T) is $\mathcal{K}_{C;\mathcal{A}}$ *left* (*right*) $\mathcal{W}$*-admissible on* X if (X,T) is $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-admissible in each point $w^0 \in X$.

Remark 7.1 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space with left (right) $\mathcal{G}$-family $\mathcal{K}_{C;\mathcal{A}}$ or with left (right) $\mathcal{W}$-family $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$. Let (X,T) be a set-valued dynamic system $T : X \to 2^X$ or a single-valued dynamic system $T : X \to 2^X$. If $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ is a $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-sequentially complete or a $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-sequentially complete fuzzy quasi-triangular space, then (X,T) is $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{G}$-admissible or $\mathcal{K}_{C;\mathcal{A}}$ left (right) $\mathcal{W}$-admissible on X, respectively, but the converse does not necessarily hold.

8 Left (right) $\mathcal{M}_{C;\mathcal{A}}$-closed set-valued $T : X \to 2^X$ and single-valued $T : X \to X$ dynamic systems in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$

The continuity has been extended in several directions and has been applied to problems in different fields.

In this section we define the following generalization of continuity of set-valued dynamic systems in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$.

Definition 8.1 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space, $k \in \mathbb{N}$, and (X,T) be a set-valued dynamic system, $T : X \to 2^X$. A set-valued dynamic system $(X, T^{[k]})$ is said to be *left* (*right*) $\mathcal{M}_{C;\mathcal{A}}$-*closed* on X if for every sequence $(x_m : m \in \mathbb{N})$ in $T^{[k]}(X)$, left (right) $\mathcal{M}_{C;\mathcal{A}}$-converging in X (thus $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$)) and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m \in T^{[k]}(u_m)\}$, the following property holds: there exists $x \in LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ ($x \in LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$) such that $x \in T^{[k]}(x)$ ($x \in T^{[k]}(x)$).

We further state the generalization of continuity of single-valued dynamic systems in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$.

Definition 8.2 Let $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ be a fuzzy quasi-triangular space. Let (X, T) be a single-valued dynamic system, $T : X \to X$, and let $k \in \mathbb{N}$. The single-valued dynamic system $(X, T^{[k]})$ is said to be *left* (*right*) $\mathcal{M}_{C;\mathcal{A}}$-*closed on* X if for each sequence $(x_m : m \in \mathbb{N})$ in $T^{[k]}(X)$, left (right) $\mathcal{M}_{C;\mathcal{A}}$-converging in X (thus $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$ ($LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})} \neq \varnothing$)) and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T^{[k]}(u_m)\}$, the following property holds: there exists $x \in LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$ ($x \in LIM^{R-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}$) such that $x = T^{[k]}(x)$ ($x = T^{[k]}(x)$).

9 Convergence, existence, approximation, and periodic point theorem in $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ for set-valued left (right) fuzzy contractions $T : X \to 2^X$

In this section, in fuzzy quasi-triangular spaces $(X, \mathcal{M}_{C;\mathcal{A}}, *)$ with left (right) $\mathcal{G}$-families $\mathcal{K}_{C;\mathcal{A}}$ and with left (right) $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ (generated by $\mathcal{M}_{C;\mathcal{A}}$), the convergence, approximation, and periodic point theorem concerning set-valued fuzzy left (right) contractions $T : X \to 2^X$ is proved.

We use the notation

$$\forall_{\alpha\in\mathcal{A}}\forall_{m\in\mathbb{N}}\left\{b^{(\alpha)}_m = \prod_{l=1}^{m}\left(1 + \beta^{(l)}_\alpha\right)\right\},$$

where, for each $m \in \mathbb{N}$, $\{\beta^{(m)}_\alpha\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$.

Theorem 9.1 *Assume that* $(X, \mathcal{M}_{C;\mathcal{A}}, *)$, *where* $C = \{C_\alpha\}_{\alpha\in\mathcal{A}} \in [1;\infty)^{\mathcal{A}}$, *is a fuzzy quasi-triangular space*, $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathcal{A}} \in (0;1)^{\mathcal{A}}$, $\eta \in \{1, 2, 3\}$, *and* (X, T) *is a set-valued dynamic system*, $T : X \to 2^X$.

Assume, moreover, that one of the following (A) *or* (B) *holds*:

(A) *There exist a left* (*respectively, right*) $\mathcal{G}$-*family* $\mathcal{K}_{C;\mathcal{A}}$ *generated by* $\mathcal{M}_{C;\mathcal{A}}$ *and a point* $w^0 \in X$ *such that*:

(A1) (X, T) *is fuzzy* $(\mathcal{F}^{L-\mathcal{K}_{C;\mathcal{A}}}_{\eta,2^X}, \lambda)$-*left contraction* (*respectively, fuzzy* $(\mathcal{F}^{R-\mathcal{K}_{C;\mathcal{A}}}_{\eta,2^X}, \lambda)$-*right contraction*);

(A2) (X, T) *is* $\mathcal{K}_{C;\mathcal{A}}$ *left* (*respectively, right*) $\mathcal{G}$-*admissible in a point* w^0;

(A3) *for every* $x \in X$ *and for every* $\beta = \{\beta_\alpha\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$, *there exists* $y \in T(x)$ *such that*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{K_\alpha\left(x, T(x), t\right) \leq K_\alpha\left(x, y, t(1 + \beta_\alpha)\right)\right\} \tag{9.1}$$

(*respectively*,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{K_\alpha\left(T(x), x, t\right) \leq K_\alpha\left(y, x, t(1 + \beta_\alpha)\right)\right\}); \tag{9.2}$$

and either

(A4) *there exists* $w^1 \in T(w^0)$ *and, for each* $m \in \mathbb{N}$, *there exists* $\{\beta_\alpha^{(m)}\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$ *such that* $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} b_m^{(\alpha)} \in (0;\infty)\}$ *and*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} *_{i=m}^{m+p-1} K_\alpha\left(w^0, w^1, \frac{tC_\alpha^{m-1}}{p\lambda_\alpha^i b_i^{(\alpha)}}\right) = 1\right\} \tag{9.3}$$

(*respectively*,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} *_{i=m}^{m+p-1} K_\alpha\left(w^1, w^0, \frac{tC_\alpha^{m-1}}{p\lambda_\alpha^i b_i^{(\alpha)}}\right) = 1\right\}\Bigg); \tag{9.4}$$

or

(A5) *there exists* $w^1 \in T(w^0)$ *such that*

$$\forall_{\alpha\in\mathcal{A}}\left\{\lim_{t\to\infty} K_\alpha\left(w^0, w^1, t\right) = 1\right\} \tag{9.5}$$

(*respectively*,

$$\forall_{\alpha\in\mathcal{A}}\left\{\lim_{t\to\infty} K_\alpha\left(w^1, w^0, t\right) = 1\right\}\Bigg). \tag{9.6}$$

(B) *There exist a left* (*respectively, right*) $\mathcal{W}$*-family* $\mathcal{K}_{C;\mathcal{A}}$ *generated by* $\mathcal{M}_{C;\mathcal{A}}$ *and a point* $w^0 \in X$ *such that*:

(B1) (X, T) *is a fuzzy* $(\mathcal{F}_{\eta,2^X}^{L-\mathcal{K}_{C;\mathcal{A}}}, \lambda)$*-left contraction* (*respectively, fuzzy* $(\mathcal{F}_{\eta,2^X}^{R-\mathcal{K}_{C;\mathcal{A}}}, \lambda)$*-right contraction*);

(B2) (X, T) *is* $\mathcal{K}_{C;\mathcal{A}}$ *left* (*respectively, right*) $\mathcal{W}$*-admissible in a point* w^0;

(B3) *for every* $x \in X$ *and for every* $\beta = \{\beta_\alpha\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$, *there exists* $y \in T(x)$ *such that*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{K_\alpha\left(x, T(x), t\right) \le K_\alpha\left(x, y, t(1+\beta_\alpha)\right)\right\} \tag{9.7}$$

(*respectively*,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{K_\alpha\left(T(x), x, t\right) \le K_\alpha\left(y, x, t(1+\beta_\alpha)\right)\right\}\Big); \tag{9.8}$$

(B4) *there exists* $w^1 \in T(w^0)$, *for each* $\alpha \in \mathcal{A}$, *there exists a sequence* $(a_m^{(\alpha)} : m \in \mathbb{N}) \subset (0;1)$, *and, for each* $m \in \mathbb{N}$, *there exists* $\{\beta_\alpha^{(m)}\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$ *for which* $\forall_{\alpha\in\mathcal{A}}\{\sum_{m=1}^\infty a_m^{(\alpha)} = 1\}$, $\forall_{\alpha\in\mathcal{A}}\{\lim_{m\to\infty} b_m^{(\alpha)} \in (0;\infty)\}$,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} *_{i=m}^{\infty} K_\alpha\left(w^0, w^1, \frac{ta_i^{(\alpha)}C_\alpha^{m-1}}{\lambda_\alpha^i b_i^{(\alpha)}}\right) = 1\right\} \tag{9.9}$$

(*respectively*,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} *_{i=m}^{\infty} K_\alpha\left(w^1, w^0, \frac{ta_i^{(\alpha)}C_\alpha^{m-1}}{\lambda_\alpha^i b_i^{(\alpha)}}\right) = 1\right\}\Bigg), \tag{9.10}$$

and, in addition, either

$$\forall_{\alpha\in\mathcal{A}}\left\{K_\alpha\left(w^0, w^1, \cdot\right) : (0;\infty) \to (0;1] \text{ is nondecreasing}\right\} \tag{9.11}$$

(*respectively*,

$$\forall_{\alpha\in\mathcal{A}}\{K_\alpha(w^1,w^0,\cdot):(0;\infty)\to(0;1]\ \textit{is nondecreasing}\}) \tag{9.12}$$

or

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{x\in W^0=\bigcup_{m=1}^{\infty}T^{[m]}(w^0)}\{K_\alpha(x,x,t)=1\}. \tag{9.13}$$

Then the following statements hold:

(C) *There exist a dynamic process* $(w^m : m\in\{0\}\cup\mathbb{N})$ *of the system* (X,T) *starting at* w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1}\in T(w^m)\}$, *and a point* $w\in X$ *such that* $(w^m : m\in\{0\}\cup\mathbb{N})$ *is left* (*respectively, right*) $\mathcal{M}_{C;\mathcal{A}}$*-convergent to* w.

(D) *If a set-valued dynamic system* $(X,T^{[k]})$ *is left* (*respectively, right*) $\mathcal{M}_{C;\mathcal{A}}$*-closed on* X *for some* $k\in\mathbb{N}$, *then* $\mathrm{Fix}(T^{[k]})\neq\emptyset$, *and there exist a dynamic process* $(w^m : m\in\{0\}\cup\mathbb{N})$ *of the system* (X,T) *starting at* w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1}\in T(w^m)\}$, *and a point* $w\in\mathrm{Fix}(T^{[k]})$ *such that* $(w^m : m\in\{0\}\cup\mathbb{N})$ *is left* (*respectively, right*) $\mathcal{M}_{C;\mathcal{A}}$*-convergent to* w.

Proof We prove only the case where $\mathcal{K}_{C;\mathcal{A}}$ is a left $\mathcal{G}$-family or a left $\mathcal{W}$-family on X, (X,T) is fuzzy $(\mathcal{F}_{\eta,2^X}^{L-\mathcal{K}_{C;\mathcal{A}}},\lambda)$-left contraction, (X,T) is $\mathcal{K}_{C;\mathcal{A}}$ left $\mathcal{G}$-admissible or left $\mathcal{W}$-admissible in a point $w^0\in X$, and $(X,T^{[k]})$ is left $\mathcal{M}_{C;\mathcal{A}}$-closed on X. We omit the case of 'right' since the reasoning is based on an analogous technique.

PART A. Further, in Steps I-IV, we consider the situation where assumptions (A) hold.

Step I. *Assume that* (A1)-(A3) *hold and let*

$$w^1\in T(w^0) \tag{9.14}$$

be arbitrary and fixed. For arbitrary and fixed $\beta^{(m)}=\{\beta_\alpha^{(m)}\}_{\alpha\in\mathcal{A}}\in(0;\infty)^{\mathcal{A}}$, $m\in\mathbb{N}$, *there exists* $(w^m : m\geq 2)$ *such that*

$$w^{m+1}\in T(w^m),\quad m\in\mathbb{N} \tag{9.15}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\{K_\alpha(w^m,w^{m+1},t)\geq K_\alpha(w^0,w^1,t(C_\alpha/\lambda_\alpha)^m/b_m^{(\alpha)})\}, \tag{9.16}$$

where $\forall_{\alpha\in\mathcal{A}}\forall_{m\in\mathbb{N}}\{b_m^{(\alpha)}=\prod_{l=1}^m(1+\beta_\alpha^{(l)})\}$.

Indeed, we apply (9.1) for w^1 and for $\beta^{(1)}=\{\beta_\alpha^{(1)}\}_{\alpha\in\mathcal{A}}$ to find

$$w^2\in T(w^1) \tag{9.17}$$

such that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\{K_\alpha(w^1,T(w^1),t)\leq K_\alpha(w^1,w^2,t(1+\beta_\alpha^{(1)}))\}. \tag{9.18}$$

Observe that then

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\{K_\alpha(w^1,w^2,t)\geq K_\alpha(w^0,w^1,t(C_\alpha/\lambda_\alpha)/(1+\beta_\alpha^{(1)}))\}. \tag{9.19}$$

Indeed, from (9.18), Definitions 6.1 and 6.2, and using (9.14), for each case where $\eta = 1$ or $\eta = 2$ or $\eta = 3$, we get

$$\begin{aligned}&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{K_\alpha\bigl(w^1,w^2,t\lambda_\alpha\bigl(1+\beta_\alpha^{(1)}\bigr)\bigr)\\&\quad\geq K_\alpha\bigl(w^1,T\bigl(w^1\bigr),t\lambda_\alpha\bigr)\\&\quad\geq \inf\bigl\{K_\alpha\bigl(u,T\bigl(w^1\bigr),t\lambda_\alpha\bigr):u\in T\bigl(w^0\bigr)\bigr\}\\&\quad\geq F_{\alpha;\eta,2^X}^{L-\mathcal{K}_{C;\mathcal{A}}}\bigl(T\bigl(w^0\bigr),T\bigl(w^1\bigr),t\lambda_\alpha\bigr)\\&\quad\geq K_\alpha\bigl(w^0,w^1,tC_\alpha\bigr)\bigr\}.\end{aligned}$$

This yields (9.19).

Applying (9.1) for w^2 and for $\beta^{(2)} = \{\beta_\alpha^{(2)}\}_{\alpha\in\mathcal{A}}$, we conclude that there exists

$$w^3 \in T\bigl(w^2\bigr)$$

such that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{K_\alpha\bigl(w^2,T\bigl(w^2\bigr),t\bigr)\leq K_\alpha\bigl(w^2,w^3,t\bigl(1+\beta_\alpha^{(2)}\bigr)\bigr)\bigr\}. \tag{9.20}$$

We seek to show that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{K_\alpha\bigl(w^2,w^3,t\bigr)\geq K_\alpha\bigl(w^1,w^2,t(C_\alpha/\lambda_\alpha)/\bigl(1+\beta_\alpha^{(2)}\bigr)\bigr)\bigr\}. \tag{9.21}$$

By (9.20), Definitions 6.1 and 6.2, and using (9.17), for each case where $\eta = 1$ or $\eta = 2$ or $\eta = 3$, it follows that

$$\begin{aligned}&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{K_\alpha\bigl(w^2,w^3,t\lambda_\alpha\bigl(1+\beta_\alpha^{(2)}\bigr)\bigr)\\&\quad\geq K_\alpha\bigl(w^2,T\bigl(w^2\bigr),\lambda_\alpha t\bigr)\geq \inf\bigl\{K_\alpha\bigl(u,T\bigl(w^2\bigr),t\lambda_\alpha\bigr):u\in T\bigl(w^1\bigr)\bigr\}\\&\quad\geq F_{\alpha;\eta,2^X}^{L-\mathcal{K}_{C;\mathcal{A}}}\bigl(T\bigl(w^1\bigr),T\bigl(w^2\bigr),t\lambda_\alpha\bigr)\\&\quad\geq K_\alpha\bigl(w^1,w^2,tC_\alpha\bigr)\bigr\}.\end{aligned}$$

This implies (9.21).

From (9.21) and (9.19) we get

$$\begin{aligned}&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\bigl\{K_\alpha\bigl(w^2,w^3,t\bigr)\\&\quad\geq K_\alpha\bigl(w^0,w^1,t(C_\alpha/\lambda_\alpha)^2/\bigl[\bigl(1+\beta_\alpha^{(1)}\bigr)\bigl(1+\beta_\alpha^{(2)}\bigr)\bigr]\bigr)\bigr\}.\end{aligned}$$

Proceeding as before, using Definitions 6.1 and 6.2, we get that there exists a sequence $(w^m : m \in \mathbb{N})$ in X satisfying (9.15) and, for calculational purposes, upon letting $\forall_{m\in\mathbb{N}}\{\beta^{(m)} = \{\beta_\alpha^{(m)}\}_{\alpha\in\mathcal{A}}\}$,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\bigl\{K_\alpha\bigl(w^m,T\bigl(w^m\bigr),t\bigr)\leq K_\alpha\bigl(w^m,w^{m+1},t\bigl(1+\beta_\alpha^{(m)}\bigr)\bigr)\bigr\}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\{K_\alpha(w^m, w^{m+1}, t\lambda_\alpha(1+\beta_\alpha^{(m)})) \\ \geq K_\alpha(w^{m-1}, w^m, tC_\alpha)\},$$

that is,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\{K_\alpha(w^m, w^{m+1}, t) \\ \geq K_\alpha(w^{m-1}, w^m, t(C_\alpha/\lambda_\alpha)/(1+\beta_\alpha^{(m)})) \\ \geq K_\alpha(w^0, w^1, t(C_\alpha/\lambda_\alpha)^m/b_m^{(\alpha)})\}.$$

Consequently, we proved that with arbitrary and fixed w^1 satisfying (9.14), the dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of the system (X, T) starting at w^0 constructed here satisfies (9.16).

Step II. *Assume that* (A1)-(A3) *hold and that* (A4) *or* (A5) *holds. Then there exists a dynamic process* $(w^m : m \in \{0\} \cup \mathbb{N})$ *of the system* (X, T) *starting at* w^0 *such that*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Big\{\lim_{m\to\infty} K_\alpha(w^m, w^{m+p}, t) = 1\Big\}, \tag{9.22}$$

and there exists $w \in X$ *such that*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{\lim_{m\to\infty} K_\alpha(w, w^m, t) = 1\Big\} \tag{9.23}$$

(*i.e.*, $(w^m : m \in \{0\} \cup \mathbb{N})$ *is left* $\mathcal{K}_{C;\mathcal{A}}$*-converging to* w) *and*

$$w \in LIM_{(w^m:m\in\mathbb{N})}^{L-\mathcal{M}_{C;\mathcal{A}}} = \Big\{x \in X : \lim_{m\to\infty}^{L-\mathcal{M}_{C;\mathcal{A}}} w^m = x\Big\} \tag{9.24}$$

(*i.e.*, $(w^m : m \in \{0\} \cup \mathbb{N})$ *is left* $\mathcal{M}_{C;\mathcal{A}}$*-converging to* w).

We consider two cases.

Case II.1. Let $m, p \in \mathbb{N}$, and let (A4) hold, that is, $w^1 \in T(w^0)$ and $\{\beta_\alpha^{(m)}\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$, $m \in \mathbb{N}$, are such that (9.3) holds. By Step I, for such $w^1 \in T(w^0)$ and $\{\beta_\alpha^{(m)}\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$, $m \in \mathbb{N}$, there exists $(w^m : m \geq 2)$ such that $w^{m+1} \in T(w^m)$, $m \in \mathbb{N}$, and a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of the system (X, T) starting at w^0 satisfies (9.16). Then, using Definition 4.1($\mathcal{KG}$1), property (9.16), and Definition 1.1(ii), we get

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m,p\in\mathbb{N}}\{K_\alpha(w^m, w^{m+p}, t) \\
&\quad = K_\alpha(w^m, w^{m+p}, t/p + (p-1)t/p) \\
&\quad \geq K_\alpha(w^m, w^{m+1}, t/(pC_\alpha)) * K_\alpha(w^m, w^{m+1}, (p-1)t/(pC_\alpha)) \\
&\quad \geq *_{i=m}^{m+p-1} K_\alpha(w^i, w^{i+1}, t/(pC_\alpha^{i-m+1})) \\
&\quad \geq *_{i=m}^{m+p-1} K_\alpha(w^0, w^1, t[(C_\alpha/\lambda_\alpha)^i/b_i^{(\alpha)}]/(pC_\alpha^{i-m+1})) \\
&\quad = *_{i=m}^{m+p-1} K_\alpha(w^0, w^1, tC_\alpha^{m-1}/(\lambda_\alpha^i p b_i^{(\alpha)}))\}.
\end{aligned} \tag{9.25}$$

Hence, by Definition 1.1(v) and property (9.3) we obtain

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Big\{\lim_{m\to\infty} K_\alpha\big(w^m, w^{m+p}, t\big) \\ \geq \lim_{m\to\infty} *_{i=m}^{m+p-1} K_\alpha\big(w^0, w^1, tC_\alpha^{m-1}/\big(\lambda_\alpha^i p b_i^{(\alpha)}\big)\big) = 1\Big\}.$$

This means that (9.22) holds.

Case II.2. Let $m, p \in \mathbb{N}$, and let (A5) hold, that is, let $w^1 \in T(w^0)$, and let (9.5) hold. Let now $\{\beta_\alpha^{(m)}\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$, $m \in \mathbb{N}$, be arbitrary and fixed and such that, for each $\alpha \in \mathcal{A}$, $(\prod_{i=1}^m (1 + \beta_m^{(i)}) : m \in \mathbb{N})$ converges to $\beta_\alpha \in (0;\infty)$. By Step I we may construct a sequence $(w^m : m \geq 2)$ satisfying $w^{m+1} \in T(w^m)$, $m \in \mathbb{N}$, and such that a dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of the system (X, T) starting at w^0 satisfies (9.16). Using next Definition 4.1($\mathcal{KG}$1), property (9.16), and Definition 1.1(ii), we get (see (9.25))

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m,p\in\mathbb{N}}\big\{K_\alpha\big(w^m, w^{m+p}, t\big) \\ \geq *_{i=m}^{m+p-1} K_\alpha\big(w^0, w^1, tC_\alpha^{m-1}/\big(p\lambda_\alpha^i b_i^{(\alpha)}\big)\big)\big\}.$$

However, since $\forall_{\alpha\in\mathcal{A}}\{\beta_\alpha \in (0;\infty)\}$, condition (9.5) implies

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\forall_{i\in\{m,\ldots,m+p-1\}}\Big\{\lim_{m\to\infty} K_\alpha\big(w^0, w^1, tC_\alpha^{m-1}/\big(\lambda_\alpha^i p b_i^{(\alpha)}\big)\big) = 1\Big\}.$$

Consequently,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Big\{\lim_{m\to\infty} K_\alpha\big(w^m, w^{m+p}, t\big) \\ \geq *_{i=m}^{m+p-1} \lim_{m\to\infty} K_\alpha\big(w^0, w^1, tC_\alpha^{m-1}/\big(\lambda_\alpha^i p b_i^{(\alpha)}\big)\big) = 1 * \cdots * 1 = 1\Big\}.$$

This means that (9.22) also holds in this case.

Now, since (X, T) is $\mathcal{K}_{C;\mathcal{A}}$ left $\mathcal{G}$-admissible in a point $w^0 \in X$, by Definition 7.1(A) property (9.22) implies that there exists $w \in X$ such that (9.23) holds.

Next, defining $x_m = w^m$ and $y_m = w$ for $m \in \mathbb{N}$, by (9.22) and (9.23) we see that conditions (4.1) and (4.3) hold for the sequences $(x_m : m \in \mathbb{N})$ and $(y_m : m \in \mathbb{N})$ in X. Consequently, by Definition 4.1($\mathcal{KG}$2) we get (4.5), which implies that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{\lim_{m\to\infty} M_\alpha\big(w, w^m, t\big) = \lim_{m\to\infty} M_\alpha(y_m, x_m, t) = 1\Big\},$$

and so, in particular, by Definition 3.1 we have $w \in LIM_{(w^m:m\in\mathbb{N})}^{L-\mathcal{M}_{C;\mathcal{A}}} = \{x \in X : \lim_{m\to\infty}^{L-\mathcal{M}_{C;\mathcal{A}}} w^m = x\}$, that is, (9.24) holds.

Step III. *Assume that* (A1)-(A3) *and* (A4) *or* (A5) *hold and that, for some* $k \in \mathbb{N}$, $(X, T^{[k]})$ *is left* $\mathcal{M}_{C;\mathcal{A}}$*-closed on* X. *Then*

$$\exists_{w\in LIM_{(w^m:m\in\{0\}\cup\mathbb{N})}^{L-\mathcal{M}_{C;\mathcal{A}}}}\big\{w \in T^{[k]}(w)\big\}.$$

Indeed, by Steps I and II, $LIM_{(w^m:m\in\{0\}\cup\mathbb{N})}^{L-\mathcal{M}_{C;\mathcal{A}}} \neq \varnothing$ and $w^{m+1} \in T(w^m)$ for $m \in \{0\} \cup \mathbb{N}$; thus, $w^{(m+1)k} \in T^{[k]}(w^{mk})$ for $m \in \{0\} \cup \mathbb{N}$, and defining $(x_m = w^{m-1+k} : m \in \mathbb{N})$, we see that $(x_m : m \in \mathbb{N}) \subset T^{[k]}(X)$, $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\{0\}\cup\mathbb{N})} = LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} \neq \emptyset$, the sequences $(y_m = w^{(m+1)k} : m \in \mathbb{N}) \subset T^{[k]}(X)$, and $(z_m = w^{mk} : m \in \mathbb{N}) \subset T^{[k]}(X)$ satisfy $\forall_{m\in\mathbb{N}}\{y_m \in T^{[k]}(z_m)\}$ and, as subsequences of $(x_m : m \in \{0\} \cup \mathbb{N})$, are left $\mathcal{M}_{C;\mathcal{A}}$-converging to each point of the set $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. Moreover, by Remark 3.1(A), $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(w^m:m\in\mathbb{N})} \subset LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(y_m:m\in\mathbb{N})}$ and $LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(w^m:m\in\mathbb{N})} \subset LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(z_m:m\in\mathbb{N})}$. From this by Definition 8.1, since $T^{[k]}$ is left $\mathcal{M}_{C;\mathcal{A}}$-closed, we conclude that

$$\exists_{w\in LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}=LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(x_m:m\in\mathbb{N})}}\left\{w \in T^{[k]}(w)\right\}.$$

Step IV. The results in the case where $\mathcal{K}_{C;\mathcal{A}}$ is a left $\mathcal{G}$-family, (X,T) is left $\mathcal{G}$-admissible in a point $w^0 \in X$, and $(X,T^{[k]})$ is left $\mathcal{M}_{C;\mathcal{A}}$-closed on X now follow at once from Steps I-III.

PART B. Further, in Steps V-VIII we consider situation where assumptions (B) hold.

Step V. *Assume that* (B1)-(B3) *hold and let* $w^1 \in T(w^0)$ *be arbitrary and fixed. For arbitrary and fixed* $\{\beta^{(m)}_\alpha\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$, $m \in \mathbb{N}$, *there exists* $(w^m : m \geq 2)$ *such that* $w^{m+1} \in T(w^m)$, $m \in \mathbb{N}$, *and*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\left\{K_\alpha\left(w^m, w^{m+1}, t\right) \geq K_\alpha\left(w^0, w^1, t(C_\alpha/\lambda_\alpha)^m/b^{(\alpha)}_m\right)\right\}, \tag{9.26}$$

where $\forall_{\alpha\in\mathcal{A}}\forall_{m\in\mathbb{N}}\{b^{(\alpha)}_m = \prod_{i=1}^m(1 + \beta^{(i)}_\alpha)\}$.

The proof of this step is identical to the proof of Step I and is omitted.

Step VI. *Assume that* (B1)-(B4) *hold. Then there exists a dynamic process* $(w^m : m \in \{0\} \cup \mathbb{N})$ *of the system* (X,T) *starting at* w^0 *such that*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty}\inf_{n>m} K_\alpha\left(w^m, w^n, t\right) = 1\right\}, \tag{9.27}$$

and there exists $w \in X$ *such that*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} K_\alpha\left(w, w^m, t\right) = 1\right\} \tag{9.28}$$

(*i.e.*, $(w^m : m \in \{0\} \cup \mathbb{N})$ *is left* $\mathcal{K}_{C;\mathcal{A}}$*-converging to* w) *and*

$$w \in LIM^{L-\mathcal{M}_{C;\mathcal{A}}}_{(w^m:m\in\mathbb{N})} = \left\{x \in X : \overset{L-\mathcal{M}_{C;\mathcal{A}}}{\lim_{m\to\infty}} w^m = x\right\} \tag{9.29}$$

(*i.e.*, $(w^m : m \in \{0\} \cup \mathbb{N})$ *is left* $\mathcal{M}_{C;\mathcal{A}}$*-converging to* w).

Indeed, let $m,n \in \mathbb{N}$ and $n > m$. Let $\beta^{(m)} = \{\beta^{(m)}_\alpha\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$, $m \in \mathbb{N}$, be such that, for each $\alpha \in \mathcal{A}$, $\lim_{m\to\infty}\prod_{i=1}^m(1 + \beta^{(i)}_\alpha) \in (0;\infty)$. Denote

$$\forall_{\alpha\in\mathcal{A}}\forall_{m\in\mathbb{N}}\left\{S^{(\alpha)}_m = \sum_{i=m}^{\infty} a^{(\alpha)}_i \text{ and } s^{(\alpha)}_m = \sum_{i=1}^{m} a^{(\alpha)}_i\right\}.$$

Using then Definition 4.2($\mathcal{KW}$1), we get

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m,n\in\mathbb{N},n>m}\left\{K_\alpha\left(w^m, w^n, t\right)\right.\\
&\quad = K_\alpha\left(w^m, w^n, tS^{(\alpha)}_1\right) = K_\alpha\left(w^m, w^n, ta^{(\alpha)}_m + tS^{(\alpha)}_{m+1} + ts^{(\alpha)}_{m-1}\right)\\
&\quad \geq K_\alpha\left(w^m, w^{m+1}, ta^{(\alpha)}_m/C_\alpha\right) * K_\alpha\left(w^{m+1}, w^n, tS^{(\alpha)}_{m+1}/C_\alpha + ts^{(\alpha)}_{m-1}/C_\alpha\right)
\end{aligned}$$

$$\begin{aligned}
&\geq K_\alpha\left(w^m, w^{m+1}, ta_m^{(\alpha)}/C_\alpha\right) * K_\alpha\left(w^{m+1}, w^{m+2}, ta_{m+1}^{(\alpha)}/C_\alpha^2\right) \\
&\quad * K_\alpha\left(w^{m+2}, w^n, tS_{m+2}^{(\alpha)}/C_\alpha^2 + ts_{m-1}^{(\alpha)}/C_\alpha^2\right) \\
&\geq \cdots \geq *_{i=m}^{n-2} K_\alpha\left(w^i, w^{i+1}, ta_i^{(\alpha)}/C_\alpha^{i-m+1}\right) \\
&\quad * K_\alpha\left(w^{n-1}, w^n, tS_{n-1}^{(\alpha)}/C_\alpha^{n-m-1} + ts_{m-1}^{(\alpha)}/C_\alpha^{n-m-1}\right)\bigr\}.
\end{aligned} \tag{9.30}$$

Now let us prove that

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m,n\in\mathbb{N},n>m}\bigl\{K_\alpha\left(w^m, w^n, t\right) \\
&\quad \geq *_{i=m}^{n-1} K_\alpha\left(w^i, w^{i+1}, ta_i^{(\alpha)}/C_\alpha^{i-m+1}\right)\bigr\}.
\end{aligned} \tag{9.31}$$

With this aim, we consider two cases.

Case VI.1. *Let* (9.11) *hold*. We see that

$$S_{n-1}^{(\alpha)}/C_\alpha^{n-m-1} + s_{m-1}^{(\alpha)}/C_\alpha^{n-m-1} \geq a_{n-1}^{(\alpha)}/C_\alpha^{n-m}. \tag{9.32}$$

Therefore, using (9.32) in (9.30), we immediately obtain (9.31).

Case VI.2. *Let* (9.13) *hold*. By Definition 4.2($\mathcal{KW}$1) and (9.13) we see that

$$\begin{aligned}
&K_\alpha\left(w^{n-1}, w^n, tS_{n-1}^{(\alpha)}/C_\alpha^{n-m-1} + ts_{m-1}^{(\alpha)}/C_\alpha^{n-m-1}\right) \\
&\quad = K_\alpha\left(w^{n-1}, w^n, ta_{n-1}^{(\alpha)}/C_\alpha^{n-m-1} + tS_n^{(\alpha)}/C_\alpha^{n-m-1} + ts_{m-1}^{(\alpha)}/C_\alpha^{n-m-1}\right) \\
&\quad \geq K_\alpha\left(w^{n-1}, w^n, ta_{n-1}^{(\alpha)}/C_\alpha^{n-m}\right) * K_\alpha\left(w^n, w^n, tS_n^{(\alpha)}/C_\alpha^{n-m} + ts_{m-1}^{(\alpha)}/C_\alpha^{n-m}\right) \\
&\quad = K_\alpha\left(w^{n-1}, w^n, ta_{n-1}^{(\alpha)}/C_\alpha^{n-m}\right).
\end{aligned} \tag{9.33}$$

Now (9.31) is a consequence of (9.30) and (9.33).

Now (9.31) and (9.26) imply

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m,n\in\mathbb{N},n>m}\bigl\{K_\alpha\left(w^m, w^n, t\right) \\
&\quad \geq *_{i=m}^{n-1} K_\alpha\left(w^0, w^1, tC_\alpha^{m-1}a_i^{(\alpha)}/\left(\lambda_\alpha^i b_i^{(\alpha)}\right)\right)\bigr\}.
\end{aligned} \tag{9.34}$$

Further, in view of (ii) and (iv) of Definition 1.1 and property (9.34), for each $l \in \mathbb{N}$, we obtain that

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m,n\in\mathbb{N},n>m}\bigl\{K_\alpha\left(w^m, w^n, t\right) * 1 \\
&\quad \geq *_{i=m}^{n-1+l} K_\alpha\left(w^0, w^1, tC_\alpha^{m-1}a_i^{(\alpha)}/\left(\lambda_\alpha^i b_i^{(\alpha)}\right)\right)\bigr\}.
\end{aligned}$$

Using this, we conclude that (9.34) implies

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\Bigl\{\inf_{n>m} K_\alpha\left(w^m, w^n, t\right) \\
&\quad \geq \inf_{n>m} *_{i=m}^{n-1} K_\alpha\left(w^0, w^1, tC_\alpha^{m-1}a_i^{(\alpha)}/\left(\lambda_\alpha^i b_i^{(\alpha)}\right)\right) \\
&\quad \geq *_{i=m}^{\infty} K_\alpha\left(w^0, w^1, tC_\alpha^{m-1}a_i^{(\alpha)}/\left(\lambda_\alpha^i b_i^{(\alpha)}\right)\right)\Bigr\}.
\end{aligned} \tag{9.35}$$

In view of (9.9), from (9.35) it follows that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{\lim_{m\to\infty}\inf_{n>m}K_\alpha\big(w^m,w^n,t\big)$$
$$\geq\lim_{m\to\infty}*_{i=m}^{\infty}K_\alpha\big(w^0,w^1,tC_\alpha^{m-1}a_i^{(\alpha)}/\big(\lambda_\alpha^i b_i^{(\alpha)}\big)\big)=1\Big\},\tag{9.36}$$

which implies (9.27).

On the other hand, since (X,T) is $\mathcal{K}_{C;\mathcal{A}}$ left $\mathcal{W}$-admissible in $w^0\in X$, by Definition 7.1(B) property (9.27) implies that there exists $w\in X$ such that (9.28) holds.

Finally, defining $x_m=w^m$ and $y_m=w$ for $m\in\mathbb{N}$, by (9.27) and (9.28) we see that conditions (4.7) and (4.9) hold for the sequences $(x_m:m\in\mathbb{N})$ and $(y_m:m\in\mathbb{N})$ in X. Consequently, by $(\mathcal{K}\mathcal{W}2)$ of Definition 4.2 we get (4.11), which implies that

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{\lim_{m\to\infty}M_\alpha\big(w,w^m,t\big)=\lim_{m\to\infty}M_\alpha(y_m,x_m,t)=1\Big\},$$

and so, in particular, we see that $w\in LIM_{(w^m:m\in\mathbb{N})}^{L-\mathcal{M}_{C;\mathcal{A}}}=\{x\in X:\lim_{m\to\infty}^{L-\mathcal{M}_{C;\mathcal{A}}}w^m=x\}$.

Step VII. *Assume that* (B1)-(B4) *hold and that, for some* $k\in\mathbb{N}$, $(X,T^{[k]})$ *is left* $\mathcal{M}_{C;\mathcal{A}}$-*closed on* X. *Then*

$$\exists_{w\in LIM_{(w^m:m\in\{0\}\cup\mathbb{N})}^{L-\mathcal{M}_{C;\mathcal{A}}}=LIM_{(x_m:m\in\mathbb{N})}^{L-\mathcal{M}_{C;\mathcal{A}}}}\big\{w\in T^{[k]}(w)\big\}.$$

The proof of this step is identical to the proof of Step III and is omitted.

Step VIII. The results in the case where $\mathcal{K}_{C;\mathcal{A}}$ is a left $\mathcal{W}$-family, (X,T) is left $\mathcal{W}$-admissible in a point $w^0\in X$, and $(X,T^{[k]})$ is left $\mathcal{M}_{C;\mathcal{A}}$-closed on X now follow at once from Steps V-VII. □

10 Convergence, existence, approximation, periodic point, fixed point, and uniqueness theorem in $(X,\mathcal{M}_{C;\mathcal{A}},*)$ for single-valued fuzzy left (right) contractions $T:X\to X$

Using Theorem 9.1, we prove the following convergence, existence, periodic point, fixed point, and uniqueness theorem for two kinds of single-valued fuzzy left (right) contractions in $(X,\mathcal{M}_{C;\mathcal{A}},*)$ with left (right) $\mathcal{G}$-families $\mathcal{K}_{C;\mathcal{A}}$ and with left (right) $\mathcal{W}$-families $\mathcal{K}_{C;\mathcal{A}}$ (generated by $\mathcal{M}_{C;\mathcal{A}}$).

Theorem 10.1 *Assume that* $(X,\mathcal{M}_{C;\mathcal{A}},*)$, *where* $C=\{C_\alpha\}_{\alpha\in\mathcal{A}}\in[1;\infty)^{\mathcal{A}}$, *is a fuzzy quasi-triangular space*, $\lambda=\{\lambda_\alpha\}_{\alpha\in\mathcal{A}}\in(0;1)^{\mathcal{A}}$, $\eta\in\{1,2\}$, *and* (X,T) *is a single-valued dynamic system*, $T:X\to X$.

Assume, *moreover*, *that one of the following* (A) *or* (B) *holds*:

(A) *There exist a left* (*respectively*, *right*) $\mathcal{G}$-*family* $\mathcal{K}_{C;\mathcal{A}}$ *generated by* $\mathcal{M}_{C;\mathcal{A}}$ *and a point* $w^0\in X$ *such that*:

(A1) (X,T) *is fuzzy* $(\mathcal{F}_{\eta,X}^{L-\mathcal{K}_{C;\mathcal{A}}},\lambda)$-*left contraction* (*respectively*, *fuzzy* $(\mathcal{F}_{\eta,X}^{R-\mathcal{K}_{C;\mathcal{A}}},\lambda)$-*right contraction*);

(A2) (X,T) *is* $\mathcal{K}_{C;\mathcal{A}}$ *left* (*respectively*, *right*) $\mathcal{G}$-*admissible in a point* w^0; *and either*

(A3)

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Big\{\lim_{m\to\infty}*_{i=m}^{m+p-1}K_\alpha\big(w^0,T\big(w^0\big),tC_\alpha^{m-1}/\big(p\lambda_\alpha^i\big)\big)=1\Big\}$$

(*respectively,*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} *_{i=m}^{m+p-1} K_\alpha\left(T\left(w^0\right), w^0, tC_\alpha^{m-1}/\left(p\lambda_\alpha^i\right)\right) = 1\right\}\right),$$

or

(A4) $\forall_{\alpha\in\mathcal{A}}\{\lim_{t\to\infty} K_\alpha(w^0, T(w^0), t) = 1\}$ (*respectively,* $\forall_{\alpha\in\mathcal{A}}\{\lim_{t\to\infty} K_\alpha(T(w^0), w^0, t) = 1\}$).

(B) *There exist a left* (*respectively, right*) $\mathcal{W}$*-family* $\mathcal{K}_{C;\mathcal{A}}$ *generated by* $\mathcal{M}_{C;\mathcal{A}}$ *and a point* $w^0 \in X$ *such that*:

(B1) (X, T) *is a fuzzy* $(\mathcal{F}_{\eta,X}^{L-\mathcal{K}_{C;\mathcal{A}}}, \lambda)$*-left contraction* (*respectively, fuzzy* $(\mathcal{F}_{\eta,X}^{R-\mathcal{K}_{C;\mathcal{A}}}, \lambda)$*-right contraction*) *on* X;

(B2) (X, T) *is* $\mathcal{K}_{C;\mathcal{A}}$ *left* (*respectively, right*) $\mathcal{W}$*-admissible in a point* w^0;

(B3) *for each* $\alpha \in \mathcal{A}$, *there exists a sequence* $(a_m^{(\alpha)} : m \in \mathbb{N}) \subset (0;1)$ *for which* $\forall_{\alpha\in\mathcal{A}}\{\sum_{m=1}^{\infty} a_m^{(\alpha)} = 1\}$,

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} *_{i=m}^{\infty} K_\alpha\left(w^0, T\left(w^0\right), ta_i^{(\alpha)} C_\alpha^{m-1}/\lambda_\alpha^i\right) = 1\right\}$$

(*respectively,*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} *_{i=m}^{\infty} K_\alpha\left(T\left(w^0\right), w^0, ta_i^{(\alpha)} C_\alpha^{m-1}/\lambda_\alpha^i\right) = 1\right\}\right)$$

and, in addition, either

$$\forall_{\alpha\in\mathcal{A}}\left\{K_\alpha\left(w^0, T\left(w^0\right), \cdot\right) : (0;\infty) \to (0;1] \textit{ is nondecreasing}\right\}$$

(*respectively,*

$$\forall_{\alpha\in\mathcal{A}}\left\{K_\alpha\left(T\left(w^0\right), w^0, \cdot\right) : (0;\infty) \to \left(0;1\right] \textit{ is nondecreasing}\right\}\right)$$

or

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{x\in W^0=\{T^{[m]}(w^0):m\in\mathbb{N}\}}\left\{K_\alpha(x, x, t) = 1\right\}. \tag{10.1}$$

Then the following statements hold:

(C) *There exists a point* $w \in X$ *such that the sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* (*respectively, right*) $\mathcal{M}_{C;\mathcal{A}}$*-convergent to* w.

(D) *Suppose that the single-valued dynamic system* $(X, T^{[k]})$ *is left* (*respectively, right*) $\mathcal{M}_{C;\mathcal{A}}$*-closed on* X *for some* $k \in \mathbb{N}$. *We have the following*:

(D1) $\mathrm{Fix}(T^{[k]}) \neq \emptyset$;

(D2) *there exists a point* $w \in \mathrm{Fix}(T^{[k]})$ *such that a sequence* $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ *is left* (*respectively, right*) $\mathcal{M}_{C;\mathcal{A}}$*-convergent to* w;

(D3) *if* $v \in \mathrm{Fix}(T^{[k]})$ *and* $\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\{\lim_{m\to\infty} K_\alpha(v, T(v), t(C_\alpha/\lambda_\alpha)^m) = 1\}$, *then* $\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\{K_\alpha(v, T(v), t) = 1\}$;

(D4) *if* $v \in \mathrm{Fix}(T^{[k]})$ *and* $\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\{\lim_{m\to\infty} K_\alpha(T(v), v, t(C_\alpha/\lambda_\alpha)^m) = 1\}$, *then* $\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\{K_\alpha(T(v), v, t) = 1\}$.

(E) *Suppose that a single-valued dynamic system* $(X, T^{[k]})$ *is left* (*respectively, right*) $\mathcal{M}_{C;\mathcal{A}}$*-closed on* X *for some* $k \in \mathbb{N}$ *and that the family* $\mathcal{M}_{C;\mathcal{A}}$ *is separating on* X. *We have the following*:

(E1) *If*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{v\in\mathrm{Fix}(T^{[k]})}\Big\{\lim_{m\to\infty}K_\alpha\big(v,T(v),t(C_\alpha/\lambda_\alpha)^m\big) = \lim_{m\to\infty}K_\alpha\big(T(v),v,t(C_\alpha/\lambda_\alpha)^m\big)=1\Big\}, \tag{10.2}$$

then

$$\mathrm{Fix}\big(T^{[k]}\big)=\mathrm{Fix}(T)\neq\varnothing; \tag{10.3}$$

there exists a point $w\in\mathrm{Fix}(T)$ *such that a sequence* $(w^m=T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})$ *is left* (*respectively, right*) $\mathcal{M}_{C;\mathcal{A}}$*-convergent to* w; *and*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{v\in\mathrm{Fix}(T^{[k]})=\mathrm{Fix}(T)}\big\{K_\alpha(v,v,t)=1\big\}. \tag{10.4}$$

(E2) *If*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{u,v\in\mathrm{Fix}(T^{[k]})}\Big\{\lim_{m\to\infty}K_\alpha\big(v,u,t(C_\alpha/\lambda_\alpha)^m\big) = \lim_{m\to\infty}K_\alpha\big(u,v,t(C_\alpha/\lambda_\alpha)^m\big)=1\Big\}, \tag{10.5}$$

then there exists a point $w\in X$ *satisfying*

$$\mathrm{Fix}\big(T^{[k]}\big)=\mathrm{Fix}(T)=\{w\}; \tag{10.6}$$

the sequence $(w^m=T^{[m]}(w^0):m\in\{0\}\cup\mathbb{N})$ *is left* (*respectively, right*) $\mathcal{M}_{C;\mathcal{A}}$*-convergent to* w; *and*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\big\{K_\alpha(w,w,t)=1\big\}. \tag{10.7}$$

Proof By Theorem 9.1 we prove only (D3), (D4), (E1), and (E2).

PART 1. *Proof of* (D3). Suppose that there exist $v_0\in\mathrm{Fix}(T^{[k]})$, $\alpha_0\in\mathcal{A}$, and $t_0\in(0;\infty)$ such that

$$\lim_{m\to\infty}K_{\alpha_0}\big(v_0,T(v_0),t_0(C_{\alpha_0}/\lambda_{\alpha_0})^m\big)=1 \tag{10.8}$$

and

$$K_{\alpha_0}\big(v_0,T(v_0),t_0\big)<1. \tag{10.9}$$

Of course, for each $m\in\mathbb{N}$, $v_0=T^{[k]}(v_0)=T^{[2k]}(v_0)=T^{[2mk]}(v_0)$ and $T(v_0)=T^{[2k]}(T(v_0))=T^{[2mk]}(T(v_0))$. Hence, for $\eta\in\{1,2\}$, by Definition 6.3(A) and (10.9),

$$\forall_{m\in\mathbb{N}}\Big\{1>K_{\alpha_0}\big(v_0,T(v_0),t_0\big)=K_{\alpha_0}\big(T^{[2mk]}(v_0),T^{[2mk]}\big(T(v_0)\big),t_0\big) \geq F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha_0;\eta,X}\big(T^{[2mk]}(v_0),T^{[2mk]}\big(T(v_0),t_0\big)\big)$$

$$
\begin{aligned}
&\geq K_{\alpha_0}\bigl(T^{[2mk-1]}(v_0), T^{[2mk-1]}\bigl(T(v_0)\bigr), t_0 C_{\alpha_0}/\lambda_{\alpha_0}\bigr)\\
&\geq F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha_0;\eta,X}\bigl(T^{[2mk-1]}(v_0), T^{[2mk-1]}\bigl(T(v_0)\bigr), t_0 C_{\alpha_0}/\lambda_{\alpha_0}\bigr)\\
&\geq K_{\alpha_0}\bigl(T^{[2mk-2]}(v_0), T^{[2mk-2]}\bigl(T(v_0)\bigr), t_0 (C_{\alpha_0}/\lambda_{\alpha_0})^2\bigr) \geq \cdots\\
&\geq K_{\alpha_0}\bigl(v_0, T(v_0), t_0 (C_{\alpha_0}/\lambda_{\alpha_0})^{2mk}\bigr)\bigr\}.
\end{aligned}
$$

By Definition 1.1(v) and property (10.8), this gives $1 > K_{\alpha_0}(v_0, T(v_0), t_0) \geq \lim_{m\to\infty} K_{\alpha_0}(v_0, T(v_0), t_0 h^{2mk}_{\alpha_0}) = 1$. By (10.8) this is impossible. Therefore, (D3) holds.

PART 2. *Proof of* (D4). Suppose that there exist $v_0 \in \mathrm{Fix}(T^{[k]})$, $\alpha_0 \in \mathcal{A}$, and $t_0 \in (0;\infty)$ such that

$$
\lim_{m\to\infty} K_{\alpha_0}\bigl(T(v_0), v_0, t_0 (C_{\alpha_0}/\lambda_{\alpha_0})^m\bigr) = 1 \tag{10.10}
$$

and

$$
K_{\alpha_0}\bigl(T(v_0), v_0, t_0\bigr) < 1. \tag{10.11}
$$

Then, by Definition 6.3(A) and (10.11), using the fact that, for each $m \in \mathbb{N}$, $v_0 = T^{[k]}(v_0) = T^{[2k]}(v_0) = T^{[mk]}(v_0)$, we get, for $\eta \in \{1, 2\}$, that

$$
\begin{aligned}
1 &> K_{\alpha_0}\bigl(T(v_0), v_0, t_0\bigr) = K_{\alpha_0}\bigl(T^{[mk+1]}(v_0), T^{[mk]}(v_0), t_0\bigr)\\
&\geq F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha_0;\eta,X}\bigl(T^{[mk+1]}(v_0), T^{[mk]}(v_0), t_0\bigr)\\
&\geq K_{\alpha_0}\bigl(T^{[mk]}(v_0), T^{[mk-1]}(v_0), t_0 (C_{\alpha_0}/\lambda_{\alpha_0})\bigr)\\
&\geq F^{L-\mathcal{K}_{C;\mathcal{A}}}_{\alpha_0;\eta,X}\bigl(T^{[mk]}(v_0), T^{[mk-1]}(v_0), t_0 (C_{\alpha_0}/\lambda_{\alpha_0})\bigr)\\
&\geq K_{\alpha_0}\bigl(T^{[mk-1]}(v_0), T^{[mk-2]}(v_0), t_0 (C_{\alpha_0}/\lambda_{\alpha_0})^2\bigr)\\
&\geq \cdots \geq K_{\alpha_0}\bigl(T(v_0), v_0, t_0 (C_{\alpha_0}/\lambda_{\alpha_0})^{mk}\bigr).
\end{aligned}
$$

Hence, by Definition 1.1(v) and by (10.10),

$$
\begin{aligned}
1 &> K_{\alpha_0}\bigl(T(v_0), v_0, t_0\bigr)\\
&\geq \lim_{m\to\infty} K_{\alpha_0}\bigl(T(v_0), v_0, t_0 (C_{\alpha_0}/\lambda_{\alpha_0})^{mk}\bigr) = 1,
\end{aligned}
$$

which is impossible. Therefore, (D4) holds.

PART 3. *Proof of* (E1). We first observe that

$$
\forall_{v\in \mathrm{Fix}(T^{[k]})}\bigl\{T(v) = v\bigr\};
$$

in other words, $\mathrm{Fix}(T^{[k]}) = \mathrm{Fix}(T)$. In fact, if $v \in \mathrm{Fix}(T^{[k]})$ and $T(v) \neq v$, then, since the family $\mathcal{M}_{C;\mathcal{A}} = \{M_\alpha, \alpha \in \mathcal{A}\}$ is separating on X, by (2.2) we get that $T(v) \neq v \Rightarrow \exists_{\alpha\in\mathcal{A}}\exists_{t\in(0;\infty)}\{M_\alpha(T(v), v, t) < 1 \vee M_\alpha(v, T(v), t) < 1\}$. In view of Theorem 4.1, this implies $T(v) \neq v \Rightarrow \exists_{\alpha\in\mathcal{A}}\exists_{t\in(0;\infty)}\{K_\alpha(T(v), v) < 1 \vee K_\alpha(v, T(v)) < 1\}$. However, by (10.2), (D3), and (D4) this is impossible. Therefore, (10.3) holds.

Next, we see that (10.4) holds. In fact, by Definition 2.1(A) and properties (D3), (D4), and (10.2) we conclude that

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{v\in\mathrm{Fix}(T^{[k]})}\big\{K_\alpha(v,v,t)\\
&\quad\geq K_\alpha\big(v,T(v),t/(2C_\alpha)\big)*K_\alpha\big(T(v),v,t/(2C_\alpha)\big)\\
&\quad=1*1=1\big\}.
\end{aligned}$$

PART 4. *Proof of* (E2). Finally, let us observe that $\mathrm{Fix}(T)$ is a singleton. We argue by contradiction and so suppose that

$$u,w\in\mathrm{Fix}(T)\quad\text{and}\quad u\neq w. \tag{10.12}$$

Then, since the family $\mathcal{M}_{C;\mathcal{A}}=\{M_\alpha,\alpha\in\mathcal{A}\}$ is separating on X, we get $\exists_{\alpha\in\mathcal{A}}\exists_{t\in(0;\infty)}\{M_\alpha(u,w,t)<1\vee M_\alpha(w,u,t)<1\}$. By applying Definition 4.3 and Theorem 4.1 we see that this implies

$$\exists_{\alpha\in\mathcal{A}}\exists_{t\in(0;\infty)}\big\{K_\alpha(u,w,t)<1\vee K_\alpha(w,u,t)<1\big\}. \tag{10.13}$$

On the other hand, for $\eta\in\{1,2\}$, by Definition 6.3(A) we conclude that

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\big\{K_\alpha(u,w,t)\\
&\quad=K_\alpha\big(T(u),T(w),t\big)\\
&\quad\geq F_{\alpha;\eta,X}^{L-\mathcal{K}_{C;\mathcal{A}}}\big(T(u),T(w),t\big)\geq K_\alpha\big(u,w,t(C_\alpha/\lambda_\alpha)\big)\\
&\quad=K_\alpha\big(T(u),T(w),t(C_\alpha/\lambda_\alpha)\big)\\
&\quad\geq F_{\alpha;\eta,X}^{L-\mathcal{K}_{C;\mathcal{A}}}\big(T(u),T(w),t(C_\alpha/\lambda_\alpha)\big)\\
&\quad\geq K_\alpha\big(u,w,t(C_\alpha/\lambda_\alpha)^2\big)\geq\cdots\geq K_\alpha\big(u,w,t(C_\alpha/\lambda_\alpha)^m\big)\big\}
\end{aligned}$$

and

$$\begin{aligned}
&\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\big\{K_\alpha(w,u,t)\\
&\quad=K_\alpha\big(T(w),T(u),t\big)\\
&\quad\geq F_{\alpha;\eta,X}^{L-\mathcal{K}_{C;\mathcal{A}}}\big(T(w),T(u),t\big)\geq K_\alpha\big(w,u,t(C_\alpha/\lambda_\alpha)\big)\\
&\quad=K_\alpha\big(T(w),T(u),t(C_\alpha/\lambda_\alpha)\big)\\
&\quad\geq F_{\alpha;\eta,X}^{L-\mathcal{K}_{C;\mathcal{A}}}\big(T(w),T(u),t(C_\alpha/\lambda_\alpha)\big)\\
&\quad\geq K_\alpha\big(w,u,t(C_\alpha/\lambda_\alpha)^2\big)\geq\cdots\geq K_\alpha\big(w,u,t(C_\alpha/\lambda_\alpha)^m\big)\big\}.
\end{aligned}$$

From this, using (10.5), we have

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{K_\alpha(u,w,t)\geq\lim_{m\to\infty}K_\alpha\big(u,w,t(C_\alpha/\lambda_\alpha)^m\big)=1\Big\} \tag{10.14}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\Big\{K_\alpha(w,u,t)\geq\lim_{m\to\infty}K_\alpha\big(w,u,t(C_\alpha/\lambda_\alpha)^m\big)=1\Big\},\tag{10.15}$$

respectively. Thus, we obtain that (10.12) implies (10.13)-(10.15), which is impossible. Therefore, $\mathrm{Fix}(T)$ is a singleton.

Thus, (10.6) and (10.7) hold. □

11 Interaction of quasi-triangular spaces and some fuzzy quasi-triangular spaces

In this section we provide a background relations between fuzzy quasi-triangular spaces and quasi-triangular spaces.

First, we define quasi-triangular spaces $(X,\mathcal{D}_{C;\mathcal{A}})$, and, next, we construct some fuzzy quasi-triangular spaces $(X,\mathcal{M}_{C;\mathcal{A}}^{\mathcal{D}_{C;\mathcal{A}}},*)$ determined by $(X,\mathcal{D}_{C;\mathcal{A}})$.

Let X be a (nonempty) set. A *distance* on X is a map $d:X\times X\to[0;\infty)$. The set X, together with distances on X, is called *distance spaces*.

Definition 11.1 Let X be a (nonempty) set, $\mathcal{A}$ be an index set, and $C=\{C_\alpha\}_{\alpha\in\mathcal{A}}\in[1;\infty)^{\mathcal{A}}$.

(A) We say that a family $\mathcal{D}_{C;\mathcal{A}}=\{d_\alpha:X\times X\to[0,\infty),\alpha\in\mathcal{A}\}$ of distances d_α, $\alpha\in\mathcal{A}$, is a *quasi-triangular family* on X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\big\{d_\alpha(u,w)\leq C_\alpha\big[d_\alpha(u,v)+d_\alpha(v,w)\big]\big\}.\tag{11.1}$$

A *quasi-triangular space* $(X,\mathcal{D}_{C;\mathcal{A}})$ is a set X together with a quasi-triangular family $\mathcal{D}_{C;\mathcal{A}}$.

(B) We say that a family $\mathcal{D}_{\mathcal{A}}=\{d_\alpha:X\times X\to[0,\infty),\alpha\in\mathcal{A}\}$ of distances d_α, $\alpha\in\mathcal{A}$, is a *triangular family* on X if

$$\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\big\{d_\alpha(u,w)\leq d_\alpha(u,v)+d_\alpha(v,w)\big\}.\tag{11.2}$$

A *triangular space* $(X,\mathcal{D}_{\mathcal{A}})$ is a set X together with a triangular family $\mathcal{D}_{\mathcal{A}}$.

(C) Let $(X,\mathcal{D}_{C;\mathcal{A}})$ be the quasi-triangular space. We say that $\mathcal{D}_{C;\mathcal{A}}$ is *separating on* X if

$$\forall_{u,w\in X}\big\{u\neq w\Rightarrow\exists_{\alpha\in\mathcal{A}}\big\{d_\alpha(u,w)>0\vee d_\alpha(w,u)>0\big\}\big\}.\tag{11.3}$$

We have the following useful result.

Theorem 11.1 *Let* $(X,\mathcal{D}_{C;\mathcal{A}})$, $\mathcal{D}_{C;\mathcal{A}}=\{d_\alpha:X^2\to[0,\infty):\alpha\in\mathcal{A}\}$, *be a quasi-triangular space*, $*$ *be a continuous t-norm defined by* $a*b=a\cdot b$, *and* $\mathcal{M}_{C;\mathcal{A}}^{\mathcal{D}_{C;\mathcal{A}}}=\{M_\alpha^{\mathcal{D}_{C;\mathcal{A}}}:X\times X\times(0;\infty)\to(0,1],\alpha\in\mathcal{A}\}$, *where*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{u,w\in X}\big\{M_\alpha^{\mathcal{D}_{C;\mathcal{A}}}(u,w,t)=t/\big[t+d_\alpha(u,w)\big]\big\}.\tag{11.4}$$

Then we have the following:

(A1) $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\forall_{t,s\in(0;\infty)}\{M_\alpha^{\mathcal{D}_{C;\mathcal{A}}}(u,v,t)*M_\alpha^{\mathcal{D}_{C;\mathcal{A}}}(v,w,s)\leq M_\alpha^{\mathcal{D}_{C;\mathcal{A}}}(u,w,C_\alpha(t+s))\}$.

(A2) $\forall_{\alpha\in\mathcal{A}}\forall_{u,w\in X}\{M_\alpha^{\mathcal{D}_{C;\mathcal{A}}}(u,w,\cdot):(0;\infty)\to(0;1]$ *is nondecreasing*$\}$.

Proof To prove (A1), suppose that

$$\exists_{\alpha_0\in\mathcal{A}}\exists_{u_0,v_0,w_0\in X}\exists_{t_0,s_0\in(0;\infty)}\bigl\{M_{\alpha_0}^{\mathcal{D}_{C;\mathcal{A}}}(u_0,v_0,t_0)*M_{\alpha_0}^{\mathcal{D}_{C;\mathcal{A}}}(v_0,w_0,s_0) > M_{\alpha_0}^{\mathcal{D}_{C;\mathcal{A}}}\bigl(u_0,w_0,C_{\alpha_0}(t_0+s_0)\bigr)\bigr\}.$$

By (11.4) and the definition of $*$, this means that

$$0 > t_0 s_0\bigl\{C_{\alpha_0}\bigl[d_{\alpha_0}(u_0,v_0)+d_{\alpha_0}(v_0,w_0)\bigr]-d_{\alpha_0}(u_0,w_0)\bigr\} + C_{\alpha_0}t_0^2 d_{\alpha_0}(v_0,w_0)+C_{\alpha_0}s_0^2 d_{\alpha_0}(u_0,v_0).$$

Hence, by (11.1) we have that $0 > C_{\alpha_0}t_0^2 d_{\alpha_0}(v_0,w_0) + C_{\alpha_0}s_0^2 d_{\alpha_0}(u_0,v_0)$, a contradiction. Therefore, (A1) holds.

Since

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{u,w\in X}\bigl\{\partial M_{\alpha}^{\mathcal{D}_{C;\mathcal{A}}}(u,w,t)/\partial t = d_\alpha(u,w)/\bigl[t+d_\alpha(u,w)\bigr]^2\geq 0\bigr\},$$

we see that this bound implies property (A2). □

Remark 11.1 Definition 11.1(A) and Theorem 11.1 show, in particular, that $(X,\mathcal{M}_{C;\mathcal{A}}^{\mathcal{D}_{C;\mathcal{A}}},*)$ is a fuzzy quasi-triangular space satisfying additional property (A2).

This suggests the following definition.

Definition 11.2 Let $(X,\mathcal{D}_{C;\mathcal{A}})$, $\mathcal{D}_{C;\mathcal{A}}=\{d_\alpha : X\times X\to[0,\infty):\alpha\in\mathcal{A}\}$, be a quasi-triangular space, and $*$ be a continuous t-norm defined by $a*b=a\cdot b$. We say that $(X,\mathcal{M}_{C;\mathcal{A}}^{\mathcal{D}_{C;\mathcal{A}}},*)$ is a *fuzzy quasi-triangular space determined by* $(X,\mathcal{D}_{C;\mathcal{A}})$.

Now, for given quasi-triangular spaces $(X,\mathcal{D}_{C;\mathcal{A}})$, we define left (right) $\mathcal{G}$-families and $\mathcal{W}$-families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{D}_{C;\mathcal{A}}$.

Definition 11.3 Let $(X,\mathcal{D}_{C;\mathcal{A}})$, $\mathcal{D}_{C;\mathcal{A}}=\{d_\alpha : X\times X\to[0,\infty):\alpha\in\mathcal{A}\}$, be a quasi-triangular space.

(A) The family $\mathcal{J}_{C;\mathcal{A}}=\{J_\alpha:\alpha\in\mathcal{A}\}$ of distances $J_\alpha : X\times X\to[0,\infty)$, $\alpha\in\mathcal{A}$, is said to be the *left* (*right*) $\mathcal{G}$*-family generated by* $\mathcal{D}_{C;\mathcal{A}}$ if:

($\mathcal{J}\mathcal{G}1$) $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{J_\alpha(u,w)\leq C_\alpha[J_\alpha(u,v)+J_\alpha(v,w)]\}$.

($\mathcal{J}\mathcal{G}2$) For any sequences $(u_m:m\in\mathbb{N})$ and $(w_m:m\in\mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathcal{A}}\forall_{p\in\mathbb{N}}\Bigl\{\lim_{m\to\infty}J_\alpha(u_m,u_{m+p})=0\Bigr\} \tag{11.5}$$

$$\Bigl(\forall_{\alpha\in\mathcal{A}}\forall_{p\in\mathbb{N}}\Bigl\{\lim_{m\to\infty}J_\alpha(u_{m+p},u_m)=0\Bigr\}\Bigr) \tag{11.6}$$

and

$$\forall_{\alpha\in\mathcal{A}}\Bigl\{\lim_{m\to\infty}J_\alpha(w_m,u_m)=0\Bigr\} \tag{11.7}$$

$$\Bigl(\forall_{\alpha\in\mathcal{A}}\Bigl\{\lim_{m\to\infty}J_\alpha(u_m,w_m)=0\Bigr\}\Bigr), \tag{11.8}$$

the following holds:

$$\forall_{\alpha\in\mathcal{A}}\left\{\lim_{m\to\infty} d_\alpha(w_m,u_m)=0\right\} \tag{11.9}$$

$$\left(\forall_{\alpha\in\mathcal{A}}\left\{\lim_{m\to\infty} d_\alpha(u_m,w_m)=0\right\}\right). \tag{11.10}$$

(B) $\mathbb{J}^{L;\mathcal{G}}_{(X,\mathcal{D}_{C;\mathcal{A}})}$ ($\mathbb{J}^{R;\mathcal{G}}_{(X,\mathcal{D}_{C;\mathcal{A}})}$) is the set of all left (right) $\mathcal{G}$-families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{D}_{C;\mathcal{A}}$.

Definition 11.4 Let $(X,\mathcal{D}_{C;\mathcal{A}})$, and $\mathcal{D}_{C;\mathcal{A}}=\{d_\alpha : X\times X\to[0,\infty) : \alpha\in\mathcal{A}\}$ be a quasi-triangular space.

(A) The family $\mathcal{J}_{C;\mathcal{A}}=\{J_\alpha:\alpha\in\mathcal{A}\}$ of distances $J_\alpha : X\times X\to[0,\infty)$, $\alpha\in\mathcal{A}$, is said to be the *left* (*right*) $\mathcal{W}$*-family generated by* $\mathcal{D}_{C;\mathcal{A}}$ if:

($\mathcal{JW}$1) $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{J_\alpha(u,w)\le C_\alpha[J_\alpha(u,v)+J_\alpha(v,w)]\}$.

($\mathcal{JW}$2) For any sequences $(u_m:m\in\mathbb{N})$ and $(w_m:m\in\mathbb{N})$ in X satisfying

$$\forall_{\alpha\in\mathcal{A}}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty}\sup_{n>m}J_\alpha(u_m,u_n)=0\right\} \tag{11.11}$$

$$\left(\forall_{\alpha\in\mathcal{A}}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty}\sup_{n>m}J_\alpha(u_n,u_m)=0\right\}\right) \tag{11.12}$$

and

$$\forall_{\alpha\in\mathcal{A}}\left\{\lim_{m\to\infty} J_\alpha(w_m,u_m)=0\right\} \tag{11.13}$$

$$\left(\forall_{\alpha\in\mathcal{A}}\left\{\lim_{m\to\infty} J_\alpha(u_m,w_m)=0\right\}\right), \tag{11.14}$$

the following holds:

$$\forall_{\alpha\in\mathcal{A}}\left\{\lim_{m\to\infty} d_\alpha(w_m,u_m)=0\right\} \tag{11.15}$$

$$\left(\forall_{\alpha\in\mathcal{A}}\left\{\lim_{m\to\infty} d_\alpha(u_m,w_m)=0\right\}\right). \tag{11.16}$$

(B) $\mathbb{J}^{L;\mathcal{W}}_{(X,\mathcal{D}_{C;\mathcal{A}})}$ ($\mathbb{J}^{R;\mathcal{W}}_{(X,\mathcal{D}_{C;\mathcal{A}})}$) is the set of all left (right) $\mathcal{W}$-families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{D}_{C;\mathcal{A}}$.

Remark 11.2 The left (right) $\mathcal{G}$-families $\mathcal{J}_{C;\mathcal{A}}$ and left (right) $\mathcal{W}$-families $\mathcal{J}_{C;\mathcal{A}}$ generated by $\mathcal{D}_{C;\mathcal{A}}$ are substantial generalizations of $\mathcal{D}_{C;\mathcal{A}}$. Indeed, note that:

(A) From Definitions 11.3 and 11.4 it follows that $\mathcal{D}_{C;\mathcal{A}}\in\mathbb{J}^{L;\mathcal{G}}_{(X,\mathcal{D}_{C;\mathcal{A}})}\cap\mathbb{J}^{R;\mathcal{G}}_{(X,\mathcal{D}_{C;\mathcal{A}})}$ and $\mathcal{D}_{C;\mathcal{A}}\in\mathbb{J}^{L;\mathcal{W}}_{(X,\mathcal{D}_{C;\mathcal{A}})}\cap\mathbb{J}^{R;\mathcal{W}}_{(X,\mathcal{D}_{C;\mathcal{A}})}$, respectively.

(B) From construction of the family $\mathcal{J}_{C;\mathcal{A}}$ given further in Theorem 11.2 it follows that $\mathbb{J}^{L;\mathcal{G}}_{(X,\mathcal{D}_{C;\mathcal{A}})}\setminus\{\mathcal{D}_{C;\mathcal{A}}\}\neq\varnothing$, $\mathbb{J}^{R;\mathcal{G}}_{(X,\mathcal{D}_{C;\mathcal{A}})}\setminus\{\mathcal{D}_{C;\mathcal{A}}\}\neq\varnothing$, $\mathbb{J}^{L;\mathcal{W}}_{(X,\mathcal{D}_{C;\mathcal{A}})}\setminus\{\mathcal{D}_{C;\mathcal{A}}\}\neq\varnothing$, and $\mathbb{J}^{R;\mathcal{W}}_{(X,\mathcal{D}_{C;\mathcal{A}})}\setminus\{\mathcal{D}_{C;\mathcal{A}}\}\neq\varnothing$.

Theorem 11.2 *Let* $(X,\mathcal{D}_{C;\mathcal{A}})$, *and* $\mathcal{D}_{C;\mathcal{A}}=\{d_\alpha : X\times X\to[0,\infty) : \alpha\in\mathcal{A}\}$ *be a quasi-triangular space. Assume that* $E\subset X$ *is a set such that*

$$\forall_{\alpha\in\mathcal{A}}\left\{\delta_\alpha(E)=\sup\left\{d_\alpha(u,w):u,w\in E\right\}\neq 0\right\} \tag{11.17}$$

and let $\{\mu_\alpha\}_{\alpha\in\mathcal{A}} \in (0;\infty)^{\mathcal{A}}$ *satisfy*

$$\forall_{\alpha\in\mathcal{A}}\{\mu_\alpha \geq \delta_\alpha(E)/(2C_\alpha)\}. \tag{11.18}$$

Let $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha, \alpha \in \mathcal{A}\}$ *where, for each* $\alpha \in \mathcal{A}$ *and for each* $u, w \in X$,

$$J_\alpha(u,w) = \begin{cases} d_\alpha(u,w) & \text{if } E \cap \{u,w\} = \{u,w\}, \\ \mu_\alpha & \text{if } E \cap \{u,w\} \neq \{u,w\}. \end{cases} \tag{11.19}$$

Then the family $\mathcal{J}_{C;\mathcal{A}} = \{J_\alpha : \alpha \in \mathcal{A}\}$ *of distances* $J_\alpha : X \times X \to [0,\infty)$, $\alpha \in \mathcal{A}$, *is the left and right* $\mathcal{G}$*-family and the left and right* $\mathcal{W}$*-family generated by* $\mathcal{D}_{C;\mathcal{A}}$.

Proof Indeed, suppose that condition ($\mathcal{JG}1$) or ($\mathcal{JW}1$) does not hold, that is,

$$\exists_{\alpha_0\in\mathcal{A}}\exists_{u_0,v_0,w_0\in X}\{J_{\alpha_0}(u_0,w_0) > C_{\alpha_0}[J_{\alpha_0}(u_0,v_0) + J_{\alpha_0}(v_0,w_0)]\}. \tag{11.20}$$

Then (11.1) and (11.19) imply $\{u_0,v_0,w_0\} \cap E \neq \{u_0,v_0,w_0\}$, and the following Cases A-D hold.

Case A. If $\{u_0,w_0\} \subset E$, then $v_0 \notin E$, and, by (11.19) we see that (11.20) is of the form $d_{\alpha_0}(u_0,w_0) > 2C_{\alpha_0}\mu_{\alpha_0}$. Next, by (11.18) this implies $d_{\alpha_0}(u_0,w_0) > 2C_{\alpha_0}\mu_{\alpha_0} \geq \delta_{\alpha_0}(E)$. By (11.17) this is impossible.

Case B. If $u_0 \in E$ and $w_0 \notin E$, then (11.20) and (11.19) give $\mu_{\alpha_0} > C_{\alpha_0}[d_{\alpha_0}(u_0,v_0) + \mu_{\alpha_0}] \geq C_{\alpha_0}\mu_{\alpha_0}$ whenever $v_0 \in E$ or $\mu_{\alpha_0} > C_{\alpha_0}[\mu_{\alpha_0} + \mu_{\alpha_0}] = 2C_{\alpha_0}\mu_{\alpha_0}$ whenever $v_0 \notin E$. Since $C_{\alpha_0} \geq 1$, this is impossible.

Case C. If $u_0 \notin E$ and $w_0 \in E$, then (11.20) and (11.19) give $\mu_{\alpha_0} > C_{\alpha_0}[\mu_{\alpha_0} + d_{\alpha_0}(v_0,w_0)] \geq C_{\alpha_0}\mu_{\alpha_0}$ whenever $v_0 \in E$ or $\mu_{\alpha_0} > C_{\alpha_0}[\mu_{\alpha_0} + \mu_{\alpha_0}] = 2C_{\alpha_0}\mu_{\alpha_0}$ whenever $v_0 \notin E$. This is impossible.

Case D. If $u_0 \notin E$ and $w_0 \notin E$, then (11.20) and (11.19) give $\mu_{\alpha_0} > C_{\alpha_0}[\mu_{\alpha_0} + \mu_{\alpha_0}] = 2C_{\alpha_0}\mu_{\alpha_0}$ for $v_0 \in X$. This is impossible.

Therefore, $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\{J_\alpha(u,w) \leq C_\alpha[J_\alpha(u,v) + J_\alpha(v,w)]\}$, that is, conditions ($\mathcal{JG}1$) and ($\mathcal{JW}1$) hold.

Assume now that the sequences $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ in X satisfy (11.5) and (11.7) or (11.11) and (11.13). We see that (11.9) holds and is equal to (11.15). Indeed, (11.7) is equal to (11.13) and implies

$$\forall_{\alpha\in\mathcal{A}}\forall_{0<\varepsilon<\mu_\alpha}\exists_{m_0=m_0(\alpha)\in\mathbb{N}}\forall_{m\geq m_0}\{J_\alpha(w_m,u_m) < \varepsilon\}. \tag{11.21}$$

Denoting $m' = \min\{m_0(\alpha) : \alpha \in \mathcal{A}\}$, we see, by (11.21) and (11.17)-(11.19), that $\forall_{m\geq m'}\{E \cap \{w_m,u_m\} = \{w_m,u_m\}\}$. Then, in view of (11.19), this implies $\forall_{\alpha\in\mathcal{A}}\forall_{0<\varepsilon<\mu_\alpha}\exists_{m'\in\mathbb{N}}\forall_{m\geq m'}\{d_\alpha(w_m, u_m) = J_\alpha(w_m,u_m) < \varepsilon\}$. Hence, we obtain that the sequences $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ satisfy (11.9) and (11.15). Thus, we see that $\mathcal{J}_{C;\mathcal{A}}$ is left $\mathcal{G}$- and $\mathcal{W}$-family generated by $\mathcal{D}_{C;\mathcal{A}}$.

In a similar way, we show that if $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ in X satisfy (11.6) and (11.8) or (11.12) and (11.14), then $(u_m : m \in \mathbb{N})$ and $(w_m : m \in \mathbb{N})$ in X satisfy (11.10) and (11.16). Therefore, $\mathcal{J}_{C;\mathcal{A}}$ is right $\mathcal{G}$- and $\mathcal{W}$-family generated by $\mathcal{D}_{C;\mathcal{A}}$. We have proved that $\mathcal{J}_{C;\mathcal{A}} \in \mathbb{J}^{L;\mathcal{G}}_{(X,\mathcal{D}_{C;\mathcal{A}})} \cap \mathbb{J}^{R;\mathcal{G}}_{(X,\mathcal{D}_{C;\mathcal{A}})} \cap \mathbb{J}^{L;\mathcal{W}}_{(X,\mathcal{D}_{C;\mathcal{A}})} \cap \mathbb{J}^{R;\mathcal{W}}_{(X,\mathcal{D}_{C;\mathcal{A}})}$. □

Finally, we see that further analysis of quasi-triangular spaces $(X,\mathcal{D}_{C;\mathcal{A}})$ and families $\mathcal{J}_{C;\mathcal{A}}$ yields the following theorem.

Theorem 11.3 *Let $(X,\mathcal{D}_{C;\mathcal{A}})$, $\mathcal{D}_{C;\mathcal{A}}=\{d_\alpha : X\times X\to[0,\infty) : \alpha\in\mathcal{A}\}$ be a quasi-triangular space, the family $\mathcal{J}_{C;\mathcal{A}}=\{J_\alpha:\alpha\in\mathcal{A}\}$ of distances $J_\alpha : X\times X\to[0,\infty)$, $\alpha\in\mathcal{A}$, be the left (right) $\mathcal{G}$-family or the left (right) $\mathcal{W}$-family generated by $\mathcal{D}_{C;\mathcal{A}}$, and $*$ be the continuous t-norm defined by $a*b=a\cdot b$. Let $\mathcal{K}^{\mathcal{J}_{C;\mathcal{A}}}_{C;\mathcal{A}}=\{K^{\mathcal{J}_{C;\mathcal{A}}}_\alpha : X\times X\times(0;\infty)\to(0;1],\alpha\in\mathcal{A}\}$, where*

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{u,w\in X}\left\{K^{\mathcal{J}_{C;\mathcal{A}}}_\alpha(u,w,t)=t/\left[t+J_\alpha(u,w)\right]\right\}. \tag{11.22}$$

Then we have the following:

(B1) $\forall_{\alpha\in\mathcal{A}}\forall_{u,v,w\in X}\forall_{t,s\in(0;\infty)}\{K^{\mathcal{J}_{C;\mathcal{A}}}_\alpha(u,v,t)*K^{\mathcal{J}_{C;\mathcal{A}}}_\alpha(v,w,s)\leq K^{\mathcal{J}_{C;\mathcal{A}}}_\alpha(u,w,C_\alpha(t+s))\}$.

(B2) $\forall_{\alpha\in\mathcal{A}}\forall_{u,w\in X}\{K^{\mathcal{J}_{C;\mathcal{A}}}_\alpha(u,w,\cdot):(0;\infty)\to(0;1]$ *is nondecreasing*$\}$.

(B3) *If $\mathcal{J}_{C;\mathcal{A}}=\{J_\alpha:\alpha\in\mathcal{A}\}$ is the left (right) $\mathcal{G}$-family generated by $\mathcal{D}_{C;\mathcal{A}}$, then* $\mathcal{K}^{\mathcal{J}_{C;\mathcal{A}}}_{C;\mathcal{A}}\in\mathbb{K}^{L;\mathcal{G}}_{(X,\mathcal{M}^{\mathcal{D}_{C;\mathcal{A}}}_{C;\mathcal{A}},*)}$ $(\mathcal{K}^{\mathcal{J}_{C;\mathcal{A}}}_{C;\mathcal{A}}\in\mathbb{K}^{R;\mathcal{G}}_{(X,\mathcal{M}^{\mathcal{D}_{C;\mathcal{A}}}_{C;\mathcal{A}},*)})$.

(B4) *If $\mathcal{J}_{C;\mathcal{A}}=\{J_\alpha:\alpha\in\mathcal{A}\}$ is the left (right) $\mathcal{W}$-family generated by $\mathcal{D}_{C;\mathcal{A}}$, then* $\mathcal{K}^{\mathcal{J}_{C;\mathcal{A}}}_{C;\mathcal{A}}\in\mathbb{K}^{L;\mathcal{W}}_{(X,\mathcal{M}^{\mathcal{D}_{C;\mathcal{A}}}_{C;\mathcal{A}},*)}$ $(\mathcal{K}^{\mathcal{J}_{C;\mathcal{A}}}_{C;\mathcal{A}}\in\mathbb{K}^{R;\mathcal{W}}_{(X,\mathcal{M}^{\mathcal{D}_{C;\mathcal{A}}}_{C;\mathcal{A}},*)})$.

Proof The proofs of (B1) and (B2) are analogous to those of (A1) and (A2) in Theorem 11.1 and are omitted.

We prove (B3) and (B4). In this aim, let $\mathcal{J}_{C;\mathcal{A}}$ be the left $\mathcal{G}$-family or left $\mathcal{W}$-family generated by $\mathcal{D}_{C;\mathcal{A}}$. If the sequences $(u_m : m\in\mathbb{N})$ and $(w_m : m\in\mathbb{N})$ in X satisfy

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty}K^{\mathcal{J}_{C;\mathcal{A}}}_\alpha(u_m,u_{m+p},t)=1\right\} \tag{11.23}$$

or

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty}\inf_{n>m}K^{\mathcal{J}_{C;\mathcal{A}}}_\alpha(u_m,u_n,t)=1\right\} \tag{11.24}$$

and

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty}K^{\mathcal{J}_{C;\mathcal{A}}}_\alpha(w_m,u_m,t)=1\right\}, \tag{11.25}$$

then, by (11.22), we see that (11.23)-(11.25) imply, respectively,

$$\forall_{\alpha\in\mathcal{A}}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty}J_\alpha(u_m,u_{m+p})=0\right\} \tag{11.26}$$

or

$$\forall_{\alpha\in\mathcal{A}}\left\{\lim_{m\to\infty}\sup_{n>m}J_\alpha(u_m,u_n)=0\right\} \tag{11.27}$$

and

$$\forall_{\alpha\in\mathcal{A}}\left\{\lim_{m\to\infty}J_\alpha(w_m,u_m)=0\right\}. \tag{11.28}$$

Next, we see that in view of (11.26)-(11.28), properties (11.5) and (11.7) or (11.11) and (11.13) hold, and by Definition 11.3 or Definition 11.4, respectively, this implies that

$$\forall_{\alpha\in\mathcal{A}}\left\{\lim_{m\to\infty}d_\alpha(w_m,u_m)=0\right\}. \tag{11.29}$$

However, by Theorem 11.1 and Remark 11.1 (see (11.4)) we get that (11.29) yield

$$\forall_{\alpha\in\mathcal{A}}\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty}M_\alpha^{\mathcal{D}_{C;\mathcal{A}}}(w_m,u_m,t)=1\right\}. \tag{11.30}$$

Hence, it follows that if $\mathcal{J}_{C;\mathcal{A}}$ is the left $\mathcal{G}$-family generated by $\mathcal{D}_{C;\mathcal{A}}$, then by Definition 4.1 the consequence of (B1), (11.23), (11.25), and (11.30) is that $\mathcal{K}_{C;\mathcal{A}}^{\mathcal{J}_{C;\mathcal{A}}}\in\mathbb{K}^{L;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}}^{\mathcal{D}_{C;\mathcal{A}}},*)}$. Moreover, if $\mathcal{J}_{C;\mathcal{A}}$ is the left $\mathcal{W}$-family generated by $\mathcal{D}_{C;\mathcal{A}}$, then, by Definition 4.2, the consequence of (11.24), (11.25), and (11.30) is $\mathcal{K}_{C;\mathcal{A}}^{\mathcal{J}_{C;\mathcal{A}}}\in\mathbb{K}^{L;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}}^{\mathcal{D}_{C;\mathcal{A}}},*)}$.

If $\mathcal{J}_{C;\mathcal{A}}$ is the right $\mathcal{G}$-family or right $\mathcal{W}$-family generated by $\mathcal{D}_{C;\mathcal{A}}$, then the proof that $\mathcal{K}_{C;\mathcal{A}}^{\mathcal{J}_{C;\mathcal{A}}}\in\mathbb{K}^{R;\mathcal{G}}_{(X,\mathcal{M}_{C;\mathcal{A}}^{\mathcal{D}_{C;\mathcal{A}}},*)}$ or $\mathcal{K}_{C;\mathcal{A}}^{\mathcal{J}_{C;\mathcal{A}}}\in\mathbb{K}^{R;\mathcal{W}}_{(X,\mathcal{M}_{C;\mathcal{A}}^{\mathcal{D}_{C;\mathcal{A}}},*)}$, respectively, is analogous and is omitted. □

Remark 11.3 The results obtained show that the Definitions 4.1 and 4.2 are correct.

12 Examples of fuzzy quasi-triangular spaces

In this section we construct examples of some fuzzy quasi-triangular spaces in the case where $*$ is the continuous t-norm defined by $a*b=a\cdot b$.

Example 12.1 Let $X=[0;3]$, and let $\mathcal{M}_{\{8\};\{1\}}=\{M:X\times X\times(0;\infty)\to(0;1]\}$ be of the form

$$M(u,w,t)=\begin{cases}1 & \text{if } u\geq w,\\ t/[t+(w-u)^4] & \text{if } u<w.\end{cases} \tag{12.1}$$

(1) $(X,\mathcal{M}_{\{8\};\{1\}},*)$ *is a fuzzy quasi-triangular space*. Indeed, we take

$$d(u,w)=\begin{cases}0 & \text{if } u\geq w,\\ (w-u)^4 & \text{if } u<w,\end{cases} \tag{12.2}$$

where $u,w\in X$. Note that $\forall_{u,v,w\in X}\{d(u,w)\leq 8[d(u,v)+d(v,w)]\}$; this inequality is a consequence of (12.2) and the following Cases A-D.

Case A. If $u,v,w\in[0;3]$ and $v\leq u<w$, then $d(u,v)=0$ and $w-u\leq w-v$. This gives $d(u,w)=(w-u)^4\leq(w-v)^4<8(w-v)^4=8[d(u,v)+d(v,w)]$.

Case B. If $u,v,w\in[0;3]$, $u<w$, and $u\leq v\leq w$, then $d(u,w)=(w-u)^4$ and $f(v_0)=\min_{u\leq v\leq w}f(v)=(w-u)^4$ where, for $u\leq v\leq w$, $f(v)=8[d(u,v)+d(v,w)]=8[(v-u)^4+(w-v)^4]$ and $v_0=(u+w)/2$.

Case C.

$$\sup_{u,w\in(0;3);u<w}d(u,w)=\sup_{u,w\in(0;3);u<w}(w-u)^4=3^4=81 \quad\text{and}$$

$$\begin{aligned}&\sup_{u,w\in(0;3);u<w}\min_{u\leq v\leq w}8\bigl[d(u,v)+d(v,w)\bigr]\\ &\quad=\sup_{u,w\in(0;3);u<w}\min_{u\leq v\leq w}8\bigl[(v-u)^4+(w-v)^4\bigr]\\ &\quad=8\bigl[(3/2-0)^4+(3-3/2)^4\bigr]=81.\end{aligned}$$

Case D. If $u,v,w\in[0;3]$ and $u<w\leq v$, then $d(v,w)=0$ and $w-u\leq v-u$. This gives $d(u,w)=(w-u)^4\leq(v-u)^4<8(v-u)^4=8[d(u,v)+d(v,w)]$.

By Definition 11.1, denoting $\mathcal{D}_{\{8\};\{1\}} = \{d\}$, we have that $(X, \mathcal{D}_{\{8\};\{1\}})$ is a quasi-triangular space. Next, Theorem 11.1 implies that $(X, \mathcal{M}_{\{8\};\{1\}}, *)$, where $M = t/(t+d)$, is a fuzzy quasi-triangular space.

(2) $\mathcal{M}_{\{8\};\{1\}} = \{M\}$ *is asymmetric*. Indeed, we have that $\forall_{t\in(0;\infty)}\{1 = M(3,1,t) \neq M(1,3,t) = t/(t+16)\}$.

(3) *For the constant sequence of the form* $(u_m = 2 : m \in \mathbb{N}) \subset X$, *the sets* $LIM^{L-\mathcal{M}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})}$ *and* $LIM^{R-\mathcal{M}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})}$ *are not singletons*. Indeed, by (12.1) and Definition 3.1 we have that $LIM^{L-\mathcal{M}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [2;3]$, $LIM^{R-\mathcal{M}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [0;2]$.

Example 12.2 Let $(X, \mathcal{M}_{\{8\};\{1\}}, *)$ be such as in Example 12.1. Moreover, let $E \subset X$, $E \neq \emptyset$, $E \neq X$, $\delta(E) = \sup\{d(u,w) : u, w \in E\} \neq \emptyset$, and γ satisfy $\gamma > \delta(E)/16$; here d is defined by (12.2). Define $\mathcal{K}_{\{8\};\{1\}} = \{K : X \times X \times (0;\infty) \to (0;1]\}$ where

$$K(u,w,t) = \begin{cases} 1 & \text{if } u \geq w \text{ and } E \cap \{u,w\} = \{u,w\}, \\ t/[t+(w-u)^4] & \text{if } u < w \text{ and } E \cap \{u,w\} = \{u,w\}, \\ t/(t+\gamma) & \text{if } E \cap \{u,w\} \neq \{u,w\}. \end{cases} \tag{12.3}$$

(1) $\mathcal{K}_{\{8\};\{1\}} = \{K\}$ *is the left and right* $\mathcal{G}$*-family and a left and right* $\mathcal{W}$*-family generated by* $\mathcal{M}_{\{8\};\{1\}} = \{M\}$ *in a fuzzy quasi-triangular space* $(X, \mathcal{M}_{\{8\};\{1\}}, *)$. Indeed, define $\mathcal{J}_{\{8\};\{1\}} = \{J\}$ where, for each $u, w \in X$,

$$J(u,w) = \begin{cases} d(u,w) & \text{if } E \cap \{u,w\} = \{u,w\}, \\ \gamma & \text{if } E \cap \{u,w\} \neq \{u,w\}. \end{cases}$$

By Theorem 11.2, $\mathcal{J}_{\{8\};\{1\}}$ is the left and right $\mathcal{G}$-family and left and right $\mathcal{W}$-family generated by $\mathcal{D}_{\{8\};\{1\}} = \{d\}$ in quasi-triangular space $(X, \mathcal{D}_{\{8\};\{1\}})$. By Theorem 11.3 and Definitions 4.1 and 4.2 this implies that $\mathcal{K}_{\{8\};\{1\}} = \{K\}$ defined by $\forall_{t\in(0;\infty)}\forall_{u,w\in X}\{K(u,w,t) = t/[t+J(u,w)]\}$ is the left and right $\mathcal{G}$-family and left and right $\mathcal{W}$-family generated by $\mathcal{M}_{\{8\};\{1\}} = \{M\}$ in a fuzzy quasi-triangular space $(X, \mathcal{M}_{\{8\};\{1\}}, *)$.

(2) $(X, \mathcal{K}_{\{8\};\{1\}}, *)$ *is a fuzzy quasi-triangular space*. We see that $\mathcal{K}_{\{8\};\{1\}}$ is a fuzzy quasi-triangular family on X.

(3) $\mathcal{K}_{\{8\};\{1\}} = \{K\}$ *on the diagonal is not equal to one*. Indeed, if $u \in X \setminus E$, then $\forall_{t\in(0;\infty)}\{K(u,u,t) = t/(t+\gamma) < 1\}$.

(4) $\mathcal{K}_{\{8\};\{1\}} = \{K\}$ *is asymmetric*. Indeed, since $\delta(E) > 0$, for each $u, w \in E$, $u > w$, by (12.3) we have that $\forall_{t\in(0;\infty)}\{1 = K(u,w,t) > K(w,u,t) = t/[t+(w-u)^4]\}$.

(5) *For the constant sequence of the form* $(u_m = 2 : m \in \mathbb{N}) \subset X$, *the sets* $LIM^{L-\mathcal{K}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})}$ *and* $LIM^{R-\mathcal{K}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})}$ *are not singletons*. Indeed, by (12.3) and Definition 5.1 we have that $LIM^{L-\mathcal{K}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [2;3]$, $LIM^{R-\mathcal{K}_{\{8\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [0;2]$.

Example 12.3 Let X be a set (nonempty), $A \subset X$, $A \neq \emptyset$, $A \neq X$, $\gamma > 0$, and let $\mathcal{M}_{\{1\};\{1\}} = \{M : X \times X \times (0;\infty) \to (0;1]\}$ be of the form

$$M(u,w,t) = \begin{cases} 1 & \text{if } A \cap \{u,w\} = \{u,w\}, \\ t/(t+\gamma) & \text{if } A \cap \{u,w\} \neq \{u,w\}. \end{cases} \tag{12.4}$$

(1) $(X, \mathcal{M}_{\{1\};\{1\}}, *)$ *is a fuzzy triangular space*. This is a consequence of Theorem 11.1. In fact, we see that $M(u,w,t) = t/[t+d(u,w)]$ for $(u,w,t) \in X \times X \times (0;\infty)$ where $d : X \times X \to$

$[0;\infty)$ is of the form

$$d(u,w) = \begin{cases} 0 & \text{if } A \cap \{u,w\} = \{u,w\}, \\ \gamma & \text{if } A \cap \{u,w\} \neq \{u,w\}. \end{cases} \tag{12.5}$$

Indeed, we see that d defined by (12.5) satisfies

$$\forall_{u,v,w \in X} \bigl\{ d(u,w) \leq d(u,v) + d(v,w) \bigr\}. \tag{12.6}$$

Otherwise, $\exists_{u_0,v_0,w_0 \in X}\{d(u_0,w_0) > d(u_0,v_0) + d(v_0,w_0)\}$. It is clear that then $d(u_0,w_0) = \gamma$, $d(u_0,v_0) = 0$, and $d(v_0,w_0) = 0$. In conclusion, $A \cap \{u_0,w_0\} \neq \{u_0,w_0\}$, $A \cap \{u_0,v_0\} = \{u_0,v_0\}$, and $A \cap \{v_0,w_0\} = \{v_0,w_0\}$. This is impossible. Therefore, (12.6) holds.

(2) $\mathcal{M}_{\{1\};\{1\}} = \{M\}$ *on the diagonal is not equal to one*. Indeed, if $u \in X \backslash A$, then $\forall_{t \in (0;\infty)}\{M(u,u,t) = t/[t+\gamma] < 1\}$. Therefore, the condition $\forall_{t \in (0;\infty)}\forall_{u \in X}\{M(u,u,t) = 1\}$ does not hold.

(3) $\mathcal{M}_{\{1\};\{1\}} = \{M\}$ *is symmetric*. This follows from (12.4).

(4) *We observe that* $LIM^{L-\mathcal{M}_{\{1\};\{1\}}}_{(u_m:m\in\mathbb{N})} = LIM^{R-\mathcal{M}_{\{1\};\{1\}}}_{(u_m:m\in\mathbb{N})} = A$ *for each sequence* $(u_m : m \in \mathbb{N}) \subset A$. We conclude this from (12.4).

Example 12.4 Let $X = [0;3]$, and let $\mathcal{M}_{\{4\};\{1\}} = \{M : X \times X \times (0;\infty) \to (0;1]\}$ be of the form

$$M(u,w,t) = \begin{cases} 1 & \text{if } u \geq w, \\ t/[t + (w-u)^3] & \text{if } u < w. \end{cases} \tag{12.7}$$

(1) $(X, \mathcal{M}_{\{4\};\{1\}}, *)$ *is a fuzzy quasi-triangular space*. Indeed, using Theorem 11.1, we see that $M(u,w,t) = t/[t + d(u,w)]$ for $(u,w,t) \in X^2 \times (0;\infty)$ where $d : X^2 \to [0;\infty)$ is of the form

$$d(u,w) = \begin{cases} 0 & \text{if } u \geq w, \\ (w-u)^3 & \text{if } u < w, \end{cases} \tag{12.8}$$

satisfying

$$\forall_{u,v,w \in X} \bigl\{ d(u,w) \leq 4\bigl[d(u,v) + d(v,w)\bigr]\bigr\}; \tag{12.9}$$

by (12.8) inequality (12.9) is a consequence of the following Cases A-C.

Case A. If $v \leq u < w$, then $d(u,v) = 0$, $w - u \leq w - v$, and, consequently, $d(u,w) = (w - u)^3 \leq (w-v)^3 < 4(w-v)^3 = 4d(v,w) = 4[d(u,v) + d(v,w)]$.

Case B. If $u < w$ and $u \leq v \leq w$, then $d(u,w) = (w-u)^3$ and $f(v_0) = \min_{u \leq v \leq w} f(v) = (w-u)^3$ where $v_0 = (u+w)/2$ is a minimum point of the map $f(v) = 4[d(u,v) + d(v,w)] = 4(w - u)[w^2 + wu + u^2 + 3v^2 - 3v(w+u)]$.

Case C. If $u < w \leq v$, then $d(v,w) = 0$, and, consequently, $d(u,w) = (w-u)^3 \leq (v-u)^3 < 4(v-u)^3 = 4d(u,v) = 4[d(u,v) + d(v,w)]$.

(2) $\mathcal{M}_{\{4\};\{1\}} = \{M\}$ *on the diagonal is equal yo one*. In fact, by (12.7) it is clear that $\forall_{t \in (0;\infty)}\forall_{u \in X}\{M(u,u,t) = 1\}$.

(3) $\mathcal{M}_{\{4\};\{1\}} = \{M\}$ *is asymmetric*. Indeed, we have that $\forall_{t \in (0;\infty)}\{1 = M(3,0,t) \neq M(0,3,t) = t/(t+27)\}$.

(4) *We observe that* $LIM^{L-\mathcal{M}_{\{4\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [2;3]$ *and* $LIM^{R-\mathcal{M}_{\{4\};\{1\}}}_{(u_m:m\in\mathbb{N})} = [0;2]$ *for a sequence* $(u_m = 2 : m \in \mathbb{N})$. We conclude this from (12.7).

Example 12.5 Let $X = \mathbb{R}$, and let $\mathcal{M}_{\{4\};\{1\}} = \{M : \mathbb{R} \times \mathbb{R} \times (0;\infty) \to (0;1]\}$ be of the form

$$M(u,w,t) = \begin{cases} 1 & \text{if } u \geq w, \\ t/[t+(w-u)^3] & \text{if } u < w, \end{cases} \quad u,w \in \mathbb{R}, t \in (0;\infty). \tag{12.10}$$

We will consider the sequence $(x_m = \sum_{s=1}^{m} 1/s : m \in \mathbb{N}) \subset X$.

(1) $(X, \mathcal{M}_{\{4\};\{1\}}, *)$ *is a fuzzy quasi-triangular space.* The proof is analogous to those in Example 12.4(1) and is omitted.

(2) $(x_m : m \in \mathbb{N})$ *is an* $\mathcal{M}_{\{4\};\{1\}}$ *left and right* $\mathcal{G}$*-sequence in* X. Indeed, by (12.10) and Definition 3.2 we see that $\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\{\lim_{m\to\infty} M(x_m, x_{m+p}, t) = \lim_{m\to\infty} t/[t + (\sum_{s=m+1}^{m+p} 1/s)^3] = 1\}$ and $\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\{\lim_{m\to\infty} M(x_{m+p}, x_m, t) = 1\}$.

(3) $(x_m : m \in \mathbb{N})$ *is not an* $\mathcal{M}_{\{4\};\{1\}}$ *left* $\mathcal{W}$*-sequence in* X *and is an* $\mathcal{M}_{\{4\};\{1\}}$ *right* $\mathcal{W}$*-sequence in* X. Indeed, by (12.10) and Definition 3.2 we have that

$$\forall_{t\in(0;\infty)} \left\{ \lim_{m\to\infty} \inf_{n>m} M(x_m, x_n, t) = \lim_{m\to\infty} \inf_{n>m} t \Big/ \left[t + \left(\sum_{s=m+1}^{n} 1/s \right)^3 \right] \right.$$
$$\left. = \lim_{m\to\infty} t \Big/ \left[t + \left(\sum_{s=m+1}^{\infty} 1/s \right)^3 \right] = 0 \right\}$$

and $\forall_{t\in(0;\infty)}\{\lim_{m\to\infty} \inf_{n>m} M(x_n, x_m, t) = 1\}$.

(4) $(x_m : m \in \mathbb{N})$ *is not left* $\mathcal{M}_{\{4\};\{1\}}$*-convergent in* X *and is right* $\mathcal{M}_{\{4\};\{1\}}$*-convergent in* X. Indeed, by (12.10) and Definition 3.1 we see that

$$\forall_{t\in(0;\infty)}\forall_{x\in X}\left\{ \lim_{m\to\infty} M(x, x_m, t) = \lim_{m\to\infty} t/\left[t + (x_m - x)^3\right] = 0 \right\}$$

and $\forall_{t\in(0;\infty)}\forall_{x\in X}\{\lim_{m\to\infty} M(x_m, x, t) = 1\}$.

13 Examples illustrating Theorem 9.1

Let, in the sequel, $*$ be the continuous t-norm defined by $a * b = a \cdot b$.

Example 13.1 Let $X = [0;3)$, and let $\mathcal{M}_{\{8\};\{1\}} = \{M : X \times X \times (0;\infty) \to (0;1]\}$ be of the form (12.1). Define a set-valued dynamic system (X, T) by

$$T(u) = \begin{cases} [0;1] & \text{if } u \in [0;1] \cup (2;3), \\ (2;3) & \text{if } u \in (1;2]. \end{cases} \tag{13.1}$$

Let

$$E = [0;1] \cup (2;3), \tag{13.2}$$

let $\gamma > 128$ be arbitrary and fixed, and let $\mathcal{K}_{\{8\};\{1\}} = \{K\}$, where, for $u, w \in X$ and $t \in (0;\infty)$,

$$K(u,w,t) = \begin{cases} 1 & \text{if } u \geq w \text{ and } E \cap \{u,w\} = \{u,w\}, \\ t/[t+(w-u)^4] & \text{if } u < w \text{ and } E \cap \{u,w\} = \{u,w\}, \\ t/(t+\gamma) & \text{if } E \cap \{u,w\} \neq \{u,w\}. \end{cases} \tag{13.3}$$

(1) $(X, \mathcal{M}_{\{8\};\{1\}}, *)$ *is a fuzzy quasi-triangular space*. See Example 12.1(1).

(2) $\mathcal{K}_{\{8\};\{1\}} = \{K\}$ *is the left and right* $\mathcal{G}$*-family and left and right* $\mathcal{W}$*-family generated by* $\mathcal{M}_{\{8\};\{1\}} = \{M\}$ *in a fuzzy quasi-triangular space* $(X, \mathcal{M}_{\{8\};\{1\}}, *)$. See Example 12.2; we have that $\gamma > 128 > \delta(E)/16 = 81/16$.

(3) *For each* $\lambda \in (128/\gamma; 1)$, (X, T) *is a fuzzy* $(\mathcal{F}^{L-\mathcal{K}_{\{8\};\{1\}}}_{1,2^X}, \lambda)$*-left contraction and a fuzzy* $(\mathcal{F}^{R-\mathcal{K}_{\{8\};\{1\}}}_{1,2^X}, \lambda)$*-right contraction*. Indeed, we prove that, for $C = 8$ and $\lambda \in (128/\gamma; 1)$,

$$\forall_{t\in(0;\infty)}\forall_{x,y\in X}\left\{F_{\mathcal{K}}\big(T(x), T(y), \lambda t\big) \geq M(x, y, Ct)\right\},$$

where, by Definition 6.1 for $\eta = 1$,

$$\begin{aligned}\forall_{t\in(0;\infty)}&\Big\{F_{\mathcal{K}}(U, W, t) = F^{L-\mathcal{K}_{\{8\};\{1\}}}_{1;1,2^X}(U, W, t) = F^{R-\mathcal{K}_{\{8\};\{1\}}}_{1;1,2^X}(U, W, t)\\ &= \min\Big\{\inf_{u\in U} K(u, W, t), \inf_{w\in W} K(U, w, t)\Big\}\Big\}, \quad U, W \in 2^X.\end{aligned}$$

With this aim, we consider the following Cases A-C.

Case A. If $x, y \in [0; 1] \cup (2; 3)$, then $T(x) = T(y) = [0; 1] = U \subset E$, and, by (13.3), $\inf_{u\in U}\{\sup_{w\in U} K(u, w, t)\} = \inf_{u\in U}\{K(u, u, t) = 1\} = 1$ and, consequently, $\forall_{t\in(0;\infty)} \forall_{\lambda\in(128/\gamma;1)}\{F_{\mathcal{K}}(T(x), T(y), \lambda t) = 1 \geq K(x, y, 8t)\}$.

Case B. If $x, y \in [1; 2]$, then $T(x) = T(y) = (2; 3) = U \subset E$and, by (13.3), $\inf_{u\in U}\{\sup_{w\in U} K(u, w, t)\} = \inf_{u\in U}\{K(u, u, t) = 1\} = 1$. Consequently, $\forall_{t\in(0;\infty)}\forall_{\lambda\in(128/\gamma;1)}\{F_{\mathcal{K}}(T(x), T(y), \lambda t) = 1 \geq K(x, y, 8t)\}$.

Case C. If $x \in [0; 1] \cup (2; 3)$ and $y \in (1; 2]$, then $T(x) = [0; 1] = U \subset E$, $T(y) = (2; 3) = W \subset E$, and, by (13.3),

$$\begin{aligned}\inf_{u\in U}\Big\{\sup_{w\in W} K(u, w, t)\Big\} &= \inf_{u\in U}\Big\{\sup_{w\in W} t/\big[t + (w - u)^4\big] = t/\big[t + (2 - u)^4\big]\Big\}\\ &= t/\big(t + 2^4\big)\end{aligned}$$

and $\inf_{w\in W}\{\sup_{u\in U} K(u, w, t)\} = \inf_{w\in W}\{\sup_{u\in U} t/[t + (w - u)^4]\} = \inf_{w\in W}\{t/[t + (w - 1)^4]\} = t/(t + 2^4)$. Thus, $F_{\mathcal{K}}(T(x), T(y), \lambda t) = \min\{t/(t + 2^4/\lambda), t/(t + 2^4/\lambda)\} = t/(t + 2^4/\lambda)$. Moreover, we have $E \cap \{x, y\} \neq \{x, y\}$. This gives, by (13.3), that $K(x, y, 8t) = t/(t + \gamma/8)$. Hence, $t/(t + 2^4/\lambda) \geq t/(t + \gamma/8)$ whenever $2^4/\lambda \leq \gamma/8$. Therefore, if $\gamma > 128$, then $\lambda \in (128/\gamma; 1)$.

(4) *Property* (A3) *holds, that is*, $\forall_{t\in(0;\infty)}\forall_{x\in X}\forall_{\beta\in(0;\infty)}\exists_{y\in T(x)}\{K(x, T(x), t) \leq K(x, y, t(1 + \beta))\}$. Indeed, this follows from the following Cases A-C.

Case A. Let $x \in [0; 1]$ and $\beta \in (0; \infty)$ be arbitrary and fixed, and let $y = x \in T(x) = [0; 1]$. By (13.1)-(13.3) and (6.1) we get that $\forall_{t\in(0;\infty)}\{K(x, y, t(1 + \beta)) = K(x, x, t(1 + \beta)) = 1\}$. Therefore, $\forall_{t\in(0;\infty)}\{K(x, T(x), t) \leq K(x, y, t(1 + \beta)) = 1\}$.

Case B. Let $x \in (2; 3)$, $\beta \in (0; \infty)$, and $y \in T(x) = [0; 1]$ be arbitrary and fixed. By (13.1)-(13.3) and (6.1), since $x > y$, we get $\forall_{t\in(0;\infty)}\{K(x, y, t(1 + \beta)) = 1\}$. Therefore, $\forall_{t\in(0;\infty)}\{K(x, T(x), t) \leq K(x, y, t(1 + \beta))\}$.

Case C. Let $x \in (1; 2]$, $\beta \in (0; \infty)$, and $y \in T(x) = (2; 3)$ be arbitrary and fixed. By (13.1)-(13.3) and (6.1), since $\forall_{v\in T(x)}\{E \cap \{x, v\} \neq \{x, v\}\}$, we get $\forall_{t\in(0;\infty)}\{K(x, y, t(1 + \beta)) = t(1 + \beta)/[t(1 + \beta) + \gamma] = t/[t + \gamma/(1 + \beta)]\}$ and $\forall_{t\in(0;\infty)}\{K(x, T(x), t) = \sup\{M(x, v, t) : v \in T(x)\} = t/(t + \gamma)\}$. This gives $\forall_{t\in(0;\infty)}\{K(x, T(x), t) \leq K(x, y, t(1 + \beta))\}$.

(5) (X,T) *is* $\mathcal{K}_{\{8\};\{1\}}$ *left and right* $\mathcal{G}$*-admissible and left and right* $\mathcal{W}$*-admissible on* X. By Definition 7.1, assuming that $w^0 \in X$ is arbitrary and fixed, we must prove that each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X,T) starting at w^0 and satisfying

$$\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\left\{\lim_{m\to\infty} K\left(w^m, w^{m+p}, t\right) = 1\right\}$$

or

$$\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty}\inf_{n>m} K\left(w^m, w^n, t\right) = 1\right\}$$

is left and right $\mathcal{K}_{\{8\};\{1\}}$-convergent. In fact, since, by (13.1), $\forall_{m\geq 1}\{w^m \in T(w^{m-1}) \subset [0;1]\}$; therefore, by (13.1)-(13.3) we get that $1 \in LIM^{L-\mathcal{K}_{\{8\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$ and $0 \in LIM^{R-\mathcal{K}_{\{8\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

(6) (X,T) *is left and right* $\mathcal{M}_{\{8\};\{1\}}$*-closed*. Indeed, since $T(X) = E$, if $(x_m : m \in \mathbb{N}) \subset T(X)$ is a left or right $\mathcal{M}_{\{8\};\{1\}}$-converging sequence in X and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m \in T(u_m)\}$, then, by (13.1)-(13.3) we have that $\exists_{m_0\in\mathbb{N}}\forall_{m\geq m_0}\{x_m \in [0;1]\}$, $1 \in LIM^{L-\mathcal{M}_{\{8\};\{1\}}}_{(x_m:m\in\mathbb{N})}$, $0 \in LIM^{R-\mathcal{M}_{\{8\};\{1\}}}_{(x_m:m\in\{0\}\cup\mathbb{N})}$, $1 \in T(1)$, and $0 \in T(0)$.

(7) *Property* (A4) *holds*. Indeed, by (13.1)-(13.3), even for each $t \in (0;\infty)$, $p \in \mathbb{N}$, $i \in \{m,\dots,m+p-1\}$, $\lambda \in (128/\gamma;1)$, $w^0 \in X$, and $w^1 \in T(w^0)$, since $T(w^0) \subset E$, $C = 8$, and $\lim_{m\to\infty}\lambda^m = 0$, if we assume that $\forall_{m\in\mathbb{N}}\{\beta^{(m)} = 1/m^2\}$, then we get that $\lim_{m\to\infty} b_m \in (0;\infty)$, where $\forall_{m\in\mathbb{N}}\{b_m = \prod_{l=1}^m (1+\beta^{(l)})\}$, and

$$\begin{aligned}
&\lim_{m\to\infty} *_{i=m}^{m+p-1} K\left(w^0, w^1, tC^{m-1}/\left(\lambda^i p b_i\right)\right)\\
&\quad = \lim_{m\to\infty}\prod_{i=m}^{m+p-1} K\left(w^0, w^1, tC^{m-1}/\left(\lambda^i p b_i\right)\right) = 1.
\end{aligned}$$

This follows from the following Cases A and B.

Case A. Let $w^0 \in E$ satisfies $w^0 \geq w^1$. Then, by (13.3),

$$\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\forall_{i\in\{m,\dots,m+p-1\}}\left\{K\left(w^0, w^1, tC^{m-1}/\left(\lambda^i p b_i\right)\right) = 1\right\}.$$

Hence,

$$\lim_{m\to\infty}\prod_{i=m}^{m+p-1} K\left(w^0, w^1, tC^{m-1}/\left(\lambda^i p b_i\right)\right) = \prod_{i=m}^{m+p-1} 1 = 1.$$

Case B. Let $w^0 \in E$ satisfies $w^0 < w^1$, or let $w^0 \notin E$. Then, by (13.3), defining constant Q as

$$Q = \begin{cases} (w^1 - w^0)^4 & \text{if } w^0 \in E \text{ satisfies } w^0 < w^1,\\ \gamma & \text{if } w^0 \notin E \end{cases} \tag{13.4}$$

we deduce that

$$\lim_{m\to\infty}\prod_{i=m}^{m+p-1} K\left(w^0, w^1, tC^{m-1}/\left(\lambda^i p b_i\right)\right) = \lim_{m\to\infty}\prod_{i=m}^{m+p-1} t/\left[t + Q\lambda^i p b_i/C^{m-1}\right] = 1$$

since $\forall_{i\in\{m,\dots,m+p-1\}}\{\lim_{m\to\infty}\lambda^i p b_i/C^{m-1} = 0\}$.

Therefore, property (9.3) holds. Analogously, we prove that property (9.4) holds.

(8) *Property* (A5) *holds*. Indeed, by (13.1)-(13.3), even for each $w^0 \in X$ and $w^1 \in T(w^0)$, since $T(w^0) \subset E$, we get

$$\lim_{t\to\infty} K\bigl(w^0, w^1, t\bigr) = \begin{cases} 1 & \text{if } w^0 \in E \text{ satisfies } w^0 \geq w^1, \\ \lim_{t\to\infty} t/[t + (w^1 - w^0)^4] = 1 & \text{if } w^0 \in E \text{ satisfies } w^0 < w^1, \\ \lim_{t\to\infty} t/(t+\gamma) = 1 & \text{if } w^0 \notin E. \end{cases}$$

Thus, (9.5) holds. Proof of (9.6) is analogous.

(9) *Property* (B4) *holds*. Indeed, by (13.1)-(13.3), even for each $t \in (0;\infty)$, $\lambda \in (128/\gamma;1)$, $w^0 \in X$, and $w^1 \in T(w^0)$, since $T(w^0) \subset E$, $C = 8$, and $\lim_{m\to\infty} \lambda^m = 0$, if we assume that $\forall_{m\in\mathbb{N}}\{\beta^{(m)} = 1/m^2\}$ and $\forall_{m\in\mathbb{N}}\{a_m = 1/[m(m+1)]\}$, then we get that $\lim_{m\to\infty} b_m = \lim_{m\to\infty} \prod_{l=1}^{m}(1 + \beta^{(l)}) \in (0;\infty)$ and $\sum_{m=1}^{\infty} a_m = 1$, and also, denoting

$$\forall_{m\in\mathbb{N}}\bigl\{h_m = a_m/\bigl(\lambda^m b_m\bigr)\bigr\},$$

we observe that property (9.9) holds, that is,

$$\lim_{m\to\infty} *_{i=m}^{\infty} M\bigl(w^0, w^1, th_i C^{m-1}\bigr) = \lim_{m\to\infty} \prod_{i=m}^{\infty} M\bigl(w^0, w^1, th_i C^{m-1}\bigr) = 1. \tag{13.5}$$

The proof of (13.5) follows from the following Cases A and B.

Case A. Let $w^0 \in E$ satisfies $w^0 \geq w^1$. Then

$$\lim_{m\to\infty} \prod_{i=m}^{\infty} M\bigl(w^0, w^1, th_i C^{m-1}\bigr) = \lim_{m\to\infty} \prod_{i=m}^{\infty} 1 = 1.$$

Case B. Let $w^0 \in E$ satisfy $w^0 < w^1$, or let $w^0 \notin E$. Then, first, we see that

$$\forall_{m\in\mathbb{N}}\left\{\prod_{i=m}^{\infty} M\bigl(w^0, w^1, th_i C^{m-1}\bigr) = \prod_{i=m}^{\infty} th_i C^{m-1}\big/\bigl(th_i C^{m-1} + Q\bigr) = \prod_{i=m}^{\infty}(1 - P_i)\right\}, \tag{13.6}$$

where $\forall_{i\in\mathbb{N}}\{P_i = [Q/(h_i C^{m-1})]/[t + Q/(h_i C^{m-1})]\}$, and Q is defined by (13.4).

Now, we prove that $\sum_{i=1}^{\infty} P_i$ is convergent. With this aim, we study the limit $\lim_{i\to\infty} P_{i+1}/P_i$. First, we observe that

$$P_{i+1}/P_i = \lambda \frac{i+2}{i}\bigl(1 + \beta^{(i+1)}\bigr)\frac{t + R_i}{t + R_{i+1}}, \tag{13.7}$$

where $R_i = Q/(h_i C^{m-1}) = Qb_i i(i+1)\lambda^i/C^{m-1}$. Next, we see that

$$\lim_{i\to\infty} R_i = 0; \tag{13.8}$$

in fact, we have $\lim_{i\to\infty} R_{i+1}/R_i = \lim_{i\to\infty}(1+\beta^{(i+1)})\lambda(i+2)/i = \lambda < 1$, which gives that the series $\sum_{i=1}^{\infty} R_i$ is convergent, and, consequently, (13.8) holds. The consequence of (13.7) and (13.8) is that $\lim_{i\to\infty} P_{i+1}/P_i = \lambda$, which means that the series $\sum_{i=1}^{\infty} P_i$ is convergent.

Since (13.6) holds, the convergence of $\sum_{i=1}^{\infty} P_i$ implies (13.5). Therefore, (9.9) holds. Analogously, we prove (9.10).

It follows from (13.3) that (9.11) and (9.12) hold. Clearly, (9.13) also holds since $W^0 \subset E$, and thus by (13.3) we get $\forall_{t\in(0;\infty)}\forall_{x\in W^0}\{K(x,x,t)=1\}$.

(10) *For all* $w^0 \in X$, $w^1 \in T(w^0)$, *and* $\lambda \in (128/\gamma;1)$, *all assumptions* (A) *and* (B) *of Theorem* 9.1 *for* $\mathcal{K}_{\{8\};\{1\}}$ *in* $(X,\mathcal{M}_{\{8\};\{1\}},*)$ *hold*. This follows from items (1)-(9).

Claim (a) $\mathrm{Fix}(T) = [0;1]$. (b) *For each* $w^0 \in X$, *each dynamic process* $(w^m : m \in \{0\}\cup\mathbb{N})$ *of the system* (X,T) *starting at* w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1} \in T(w^m)\}$, *is left and right* $\mathcal{M}_{\{8\};\{1\}}$-*convergent to some point of* $\mathrm{Fix}(T)$.

Example 13.2 Let X, $\mathcal{M}_{\{8\};\{1\}} = \{M\}$, $\eta = 1$, and (X,T) be such as in Example 13.1.

(1) $(X,\mathcal{M}_{\{8\};\{1\}},*)$ *is a fuzzy quasi-triangular space*. See Example 13.1.

(2) *For each* $\lambda \in (0;1)$, *the condition*

$$\forall_{t\in(0;\infty)}\forall_{x,y\in X}\bigl\{F_{\mathcal{M}}\bigl(T(x),T(y),\lambda t\bigr) \geq M(x,y,8t)\bigr\}$$

does not hold, *where*

$$\begin{aligned}\forall_{t\in(0;\infty)}\Bigl\{&F_{\mathcal{M}}(U,W,t) = F_{1;1,2^X}^{L-\mathcal{M}_{\{8\};\{1\}}}(U,W,t) = F_{1;1,2^X}^{R-\mathcal{M}_{\{8\};\{1\}}}(U,W,t)\\ &= \min\Bigl\{\inf_{u\in U} M(u,W,t), \inf_{w\in W} M(U,w,t)\Bigr\}\Bigr\}, \quad U,W \in 2^X.\end{aligned}$$

Indeed, suppose that $\exists_{\lambda_0\in(0;1)}\forall_{t\in(0;\infty)}\forall_{x,y\in X}\{F_{\mathcal{M}}(T(x),T(y),\lambda_0 t) \geq M(x,y,8t)\}$. Letting $x_0 = 5/2$ and $y_0 = 2$, by (12.1) it can be shown that $M(x_0,y_0,8t) = 1$, $T(x_0) = [0;1]$, $T(y_0) = (2;3)$, $\inf_{u\in[0;1]} M(u,(2;3),\lambda_0 t) = \inf_{u\in[0;1]}\sup_{w\in(2;3)} \lambda_0 t/[\lambda_0 t + (w-u)^4] = \inf_{u\in[0;1]} \lambda_0 t/[\lambda_0 t + (2-u)^4] = \lambda_0 t/(\lambda_0 t + 2^4)$, and $\inf_{w\in(2;3)} M([0;1],w,\lambda_0 t) = \inf_{w\in(2;3)}\sup_{u\in[0;1]} \lambda_0 t/[\lambda_0 t + (w-u)^4] = \inf_{w\in(2;3)} \lambda_0 t/[\lambda_0 t + (w-1)^4] = \lambda_0 t/(\lambda_0 t + 2^4)$. Therefore, $F_{\mathcal{M}}(T(x_0),T(y_0),\lambda_0 t) = \lambda_0 t/(\lambda_0 t + 2^4) \geq 1 = M(x_0,y_0,8t)$, which is absurd.

Remark 13.1 Let us observe that the main tool is the families $\mathcal{K}_{C;\mathcal{A}}$. We make the following remarks about Examples 13.1 and 13.2 showing how natural these families $\mathcal{K}_{C;\mathcal{A}}$ are. (a) By Example 13.1 we observe that we may apply Theorem 9.1 for set-valued dynamic systems (X,T) in a fuzzy quasi-triangular space $(X,\mathcal{M}_{C;\mathcal{A}},*)$ with left and right $\mathcal{G}$-admissible and left and right $\mathcal{W}$-admissible family $\mathcal{K}_{C;\mathcal{A}}$ generated by $\mathcal{M}_{C;\mathcal{A}}$ where $\mathcal{K}_{C;\mathcal{A}} \neq \mathcal{M}_{C;\mathcal{A}}$. (b) By Example 13.2 we note, however, that we do not apply Theorem 9.1 in a fuzzy quasi-triangular space $(X,\mathcal{M}_{C;\mathcal{A}},*)$ when $\mathcal{K}_{C;\mathcal{A}} = \mathcal{M}_{C;\mathcal{A}}$. (c) From (a) and (b) it follows that, in Theorem 9.1, the existence of families $\mathcal{K}_{C;\mathcal{A}}$ such that $\mathcal{K}_{C;\mathcal{A}} \neq \mathcal{M}_{C;\mathcal{A}}$ are essential.

Example 13.3 Let $X = (0;3)$, $\gamma > 0$, $A = A_1 \cup A_2$, $A_1 = (0;1]$, $A_2 = [2;3)$, $\mathcal{M}_{\{1\};\{1\}} = \{M : X^2 \times (0;\infty) \to (0;1]\}$ where, for all $u,w \in X$ and $t \in (0;\infty)$,

$$M(u,w,t) = \begin{cases} 1 & \text{if } A\cap\{u,w\} = \{u,w\},\\ t/(t+\gamma) & \text{if } A\cap\{u,w\} \neq \{u,w\},\end{cases} \tag{13.9}$$

and let a set-valued dynamic system (X, T) be of the form

$$T(u) = \begin{cases} A_2 & \text{for } u \in A_1, \\ A & \text{for } u \in (1;2), \\ A_1 & \text{for } u \in A_2. \end{cases} \tag{13.10}$$

(1) $(X, \mathcal{M}_{\{1\};\{1\}}, *)$ *is a fuzzy triangular space*; *see Example* 12.3(1).

(2) *For each* $\lambda \in (0;1)$, (X,T) *is a fuzzy* $(\mathcal{F}^{L-\mathcal{M}_{\{1\};\{1\}}}_{1,2^X}, \lambda)$*-left contraction and a fuzzy* $(\mathcal{F}^{R-\mathcal{M}_{\{1\};\{1\}}}_{1,2^X}, \lambda)$*-right contraction.* Indeed, if $x, y \in X$, then $T(x) = U \subset A$, $T(y) = W \subset A$, and, by (13.9), (6.1), Example 12.3(3), and Definition 6.1, for $\eta = 1$,

$$\forall_{t\in(0;\infty)} \Big\{ F_{\mathcal{M}}(U,W,t) = F^{L-\mathcal{M}_{\{1\};\{1\}}}_{1;1,2^X}(U,W,t) \\ = F^{R-\mathcal{M}_{\{1\};\{1\}}}_{1;1,2^X}(U,W,t) = \min\Big\{ \inf_{u\in U} M(u,W,t), \inf_{w\in W} M(U,w,t) \Big\} = 1 \Big\}.$$

Thus, for $C = 1$,

$$\forall_{t\in(0;\infty)} \forall_{\lambda\in(0;1)} \forall_{x,y\in X} \big\{ F_{\mathcal{M}}\big(T(x), T(y), \lambda t\big) = 1 \geq M(x,y,Ct) \big\}.$$

(3) *Property* (A3) *holds, that is,* $\forall_{t\in(0;\infty)} \forall_{x\in X} \forall_{\beta\in(0;\infty)} \exists_{y\in T(x)} \{M(x, T(x), t) \leq M(x,y,t(1+\beta))\}$. Indeed, this follows from the following Cases A and B.

Case A. Let $x \in A$, $\beta \in (0;\infty)$, and $y \in T(x)$ be arbitrary and fixed. By (13.9), (13.10), and (6.1) we get $T(x) \subset A$, $\forall_{t\in(0;\infty)} \forall_{v\in T(x)} \{M(x,v,t(1+\beta)) = 1\}$ and $\forall_{t\in(0;\infty)} \{M(x,T(x),t) = \sup\{M(x,v,t) : v \in T(x)\} = 1\}$. Therefore, $\forall_{t\in(0;\infty)} \{M(x,T(x),t) = M(x,y,t(1+\beta))\}$.

Case B. If $x \in A$, $\beta \in (0;\infty)$, and $y \in T(x)$ are arbitrary and fixed, then, by (13.9), (13.10), and (6.1) we get $T(x) = A$, $\forall_{t\in(0;\infty)} \forall_{v\in T(x)} \{M(x,v,t(1+\beta)) = t(1+\beta)/[t(1+\beta)+\gamma]\}$, and $\forall_{t\in(0;\infty)} \{M(x,T(x),t) = \sup\{M(x,v,t) : v \in T(x)\} = t/[t+\gamma]\}$. Hence, by Theorem 11.1(A2), $\forall_{t\in(0;\infty)} \{M(x,T(x),t) \leq M(x,y,t(1+\beta))\}$.

(4) (X,T) *is* $\mathcal{M}_{\{1\};\{1\}}$ *left and right* $\mathcal{G}$*-admissible and* $\mathcal{W}$*-admissible on* X. By Definition 7.1, assuming that $w^0 \in X$ is arbitrary and fixed, we must prove that each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of (X,T) starting at w^0 and satisfying

$$\forall_{t\in(0;\infty)} \forall_{p\in\mathbb{N}} \Big\{ \lim_{m\to\infty} M\big(w^m, w^{m+p}, t\big) = 1 \Big\}$$

or

$$\forall_{t\in(0;\infty)} \Big\{ \lim_{m\to\infty} \inf_{n>m} M\big(w^m, w^n, t\big) = 1 \Big\}$$

is left and right $\mathcal{M}_{\{1\};\{1\}}$-convergent. Indeed, since, by (13.10), $\forall_{m\geq 1}\{w^m \in T(w^{m-1}) \subset A\}$, by (13.9) we get that $A = LIM^{L-\mathcal{M}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = LIM^{R-\mathcal{M}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$.

(5) $(X, T^{[2]})$ *is left and right* $\mathcal{M}_{\{1\};\{1\}}$*-closed.* Indeed, since $T^{[2]}(X) = A$, if $(x_m : m \in \mathbb{N}) \subset T^{[2]}(X)$ is a left or right $\mathcal{M}_{\{1\};\{1\}}$-converging sequence in X and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m \in T^{[2]}(u_m)\}$, then by (13.9) and (13.10) we have that $\exists_{m_0\in\mathbb{N}} \forall_{m\geq m_0} \{x_m \in A\}$, $A = LIM^{L-\mathcal{M}_{\{1\};\{1\}}}_{(x_m:m\in\mathbb{N})} = LIM^{R-\mathcal{M}_{\{1\};\{1\}}}_{(x_m:m\in\{0\}\cup\mathbb{N})}$, and $\mathrm{Fix}(T^{[2]}) = A$.

(6) *Property* (A4) *holds.* Indeed, by (13.9), even for each $t \in (0;\infty)$, $p \in \mathbb{N}$, $i \in \{m, \dots, m+p-1\}$, $\lambda \in (0;1)$, $w^0 \in X$, and $w^1 \in T(w^0)$, since $T(w^0) \subset A$, $C = 1$, and $\lim_{m\to\infty} \lambda^m = 0$, if we

assume that $\forall_{m\in\mathbb{N}}\{\beta^{(m)} = 1/m^2\}$, then we get that $\lim_{m\to\infty} b_m \in (0;\infty)$, where $\forall_{m\in\mathbb{N}}\{b_m = \prod_{l=1}^{m}(1+\beta^{(l)})\}$, and

$$\begin{aligned}
&\lim_{m\to\infty} *_{i=m}^{m+p-1} M\big(w^0, w^1, t/\big(\lambda^i p b_i\big)\big) \\
&\quad = \lim_{m\to\infty} \prod_{i=m}^{m+p-1} M\big(w^0, w^1, t/\big(\lambda^i p b_i\big)\big) \\
&\quad = \begin{cases} \prod_{i=m}^{m+p-1} 1 = 1 & \text{if } w^0 \in A, \\ \lim_{m\to\infty} \prod_{i=m}^{m+p-1} t/(t+\gamma\lambda^i p b_i) = 1 & \text{if } w^0 \in X\backslash A. \end{cases}
\end{aligned}$$

Thus, (9.3) and (9.4) hold.

(7) *Property* (A5) *holds*. Indeed, by (13.9), even for all $w^0 \in X$ and $w^1 \in T(w^0)$, since $T(w^0) \subset A$, we get

$$\lim_{t\to\infty} M\big(w^0, w^1, t\big) = \begin{cases} 1 & \text{if } w^0 \in A, \\ \lim_{t\to\infty} t/(t+\gamma) = 1 & \text{if } w^0 \in X\backslash A. \end{cases}$$

Thus, (9.5) and (9.6) hold.

(8) *Property* (B4) *holds*. Indeed, by (13.9), even for all $t \in (0;\infty)$, $\lambda \in (0;1)$, $w^0 \in X$, and $w^1 \in T(w^0)$, since $T(w^0) \subset A$, $C = 1$, and $\lim_{m\to\infty} \lambda^m = 0$, if we assume that $\forall_{m\in\mathbb{N}}\{\beta^{(m)} = 1/m^2\}$ and $\forall_{m\in\mathbb{N}}\{a_m = 1/[m(m+1)]\}$, then we get that $\lim_{m\to\infty} b_m \in (0;\infty)$ and $\sum_{m=1}^{\infty} a_m = 1$, and also, denoting

$$\forall_{m\in\mathbb{N}}\big\{h_m = a_m/\big(\lambda^m b_m\big)\big\},$$

we observe that

$$\lim_{m\to\infty} *_{i=m}^{\infty} M\big(w^0, w^1, th_i\big) = \lim_{m\to\infty} \prod_{i=m}^{\infty} M\big(w^0, w^1, th_i\big) = 1. \tag{13.11}$$

This follows from the following Cases A and B.

Case A. Let $w^0 \in A$. Then, by (13.9), $\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\{M(w^0, w^1, th_m) = 1\}$. Consequently, $\lim_{m\to\infty} \prod_{i=m}^{\infty} M(w^0, w^1, th_i) = \lim_{m\to\infty} \prod_{i=m}^{\infty} 1 = 1$, that is, (13.11) holds.

Case B. Let $w^0 \in X\backslash A$. Then, by (13.9), $\forall_{m\in\mathbb{N}}\{\prod_{i=m}^{\infty} M(w^0, w^1, th_i) = \prod_{i=m}^{\infty} t/(t+\gamma/h_i) = \prod_{i=m}^{\infty}(1-Q_i)\}$ where $\forall_{m\in\mathbb{N}}\{Q_m = (\gamma/h_m)/(t+\gamma/h_m)\}$. Let us prove that

$$\lim_{m\to\infty} \prod_{i=m}^{\infty}(1-Q_i) = 1. \tag{13.12}$$

With this aim, it suffices to show that the series $\sum_{m=1}^{\infty} Q_m$ is convergent.

First, let us observe that

$$\forall_{m\in\mathbb{N}}\left\{Q_{m+1}/Q_m = \frac{(t+Z_m)(m+2)}{(t+Z_{m+1})(m+1)}\big(1+\beta^{(m+1)}\big)\lambda\right\}, \tag{13.13}$$

where $\forall_{m\in\mathbb{N}}\{Z_m = \lambda^m \gamma m(m+1)\prod_{l=1}^{m}(1+\beta^{(l)})\}$. Next, we see that

$$\lim_{m\to\infty} Z_{m+1}/Z_m = \lim_{m\to\infty} \frac{m+2}{m+1}\lambda\big(1+\beta^{(m+1)}\big) = \lambda < 1,$$

which implies that the series $\sum_{m=1}^{\infty} Z_m$ is convergent, and, consequently,

$$\lim_{m\to\infty} Z_m = 0. \tag{13.14}$$

Now, using (13.14) in (13.13), we obtain $\lim_{m\to\infty} Q_{m+1}/Q_m = \lambda < 1$. Hence, $\sum_{m=1}^{\infty} Q_m$ is convergent. Thus, we have that (13.12) holds.

Therefore, properties (9.9) and (9.10) hold.

It follows from (13.9) that properties (9.11) and (9.12) hold. Clearly, also (9.13) holds since $W^0 = A$.

(9) *For each $w^0 \in X$ and for each $w^1 \in T(w^0)$, all assumptions* (A) *and* (B) *of Theorem* 9.1 *for $\mathcal{K}_{\{1\};\{1\}} = \mathcal{M}_{\{1\};\{1\}}$ hold*. This follows from (1)-(8).

Claim (a) $\mathrm{Fix}(T) = \varnothing$ *and* $\mathrm{Fix}(T^{[2]}) = A$. (b) *For each $w^0 \in X$, each dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of the system (X, T) starting at w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1} \in T(w^m)\}$, is left and right $\mathcal{M}_{\{1\};\{1\}}$-convergent to each point of* $\mathrm{Fix}(T^{[2]})$.

14 Examples illustrating Theorem 10.1

Let, in the sequel, $*$ be the continuous t-norm defined by $a * b = a \cdot b$.

Example 14.1 Let $X = (0;3)$. Define the single-valued dynamic system (X, T) by

$$T(u) = \begin{cases} 1+u & \text{for } u \in (0;1], \\ u/2 & \text{for } u \in (1;2], \\ 2+u/3 & \text{for } u \in (2;3). \end{cases} \tag{14.1}$$

Let $\gamma > 0$, $A = (1/2;3)$, and $\mathcal{M}_{\{1\};\{1\}} = \{M : X^2 \times (0;\infty) \to (0;1]\}$ where, for all $x, y \in X$ and $t \in (0;\infty)$,

$$M(u,w,t) = \begin{cases} 1 & \text{if } A \cap \{u,w\} = \{u,w\}, \\ t/(t+\gamma) & \text{if } A \cap \{u,w\} \neq \{u,w\}. \end{cases} \tag{14.2}$$

(1) $(X, \mathcal{M}_{\{1\};\{1\}}, *)$ *is a fuzzy triangular space*; *see Example* 12.3(1).

(2) *For each $\lambda \in (0;1)$, (X,T) is a fuzzy $(\mathcal{F}_{1,X}^{L-\mathcal{M}_{\{1\};\{1\}}}, \lambda)$-left contraction and a fuzzy $(\mathcal{F}_{1,X}^{R-\mathcal{M}_{\{1\};\{1\}}}, \lambda)$-right contraction*. Indeed, by Definition 6.3 for $\eta = 1$, denoting

$$\begin{aligned} &\forall_{t\in(0;\infty)}\forall_{u,w\in X}\Bigl\{F_{\mathcal{M}}(u,w,t) = F_{1;1,X}^{L-\mathcal{M}_{\{1\};\{1\}}}(u,w,t) \\ &\quad = F_{1;1,X}^{R-\mathcal{M}_{\{1\};\{1\}}}(w,u,t) = \min\bigl\{M(u,w,t), M(w,u,t)\bigr\}\Bigr\}, \end{aligned}$$

we see that, for all $x, y \in X$, $T(x), T(y) \in A$, and thus, for $C = 1$,

$$\forall_{t\in(0;\infty)}\forall_{x,y\in X}\bigl\{F_{\mathcal{M}}\bigl(T(x),T(y),\lambda t\bigr) = 1 \geq M(x,y,Ct)\bigr\}$$

since

$$M(x,y,Ct) = \begin{cases} 1 & \text{if } A \cap \{x,y\} = \{x,y\}, \\ Ct/(Ct+\gamma) & \text{if } A \cap \{x,y\} \neq \{x,y\}. \end{cases} \tag{14.3}$$

(3) *(X,T) is $\mathcal{M}_{\{1\};\{1\}}$ left and right $\mathcal{G}$-admissible and $\mathcal{M}_{\{1\};\{1\}}$ left and right $\mathcal{W}$-admissible on X.* Indeed, let $w^0 \in X$ be arbitrary and fixed. Since $(w^m = T^{[m]}(w^0) : m \geq 1) \subset A$, we have

$$\forall_{t\in(0;\infty)} \forall_{p\in\mathbb{N}} \Big\{ \lim_{m\to\infty} M\big(w^m, w^{m+p}, t\big) = 1 \Big\},$$
$$\forall_{t\in(0;\infty)} \Big\{ \lim_{m\to\infty} \inf_{n>m} M\big(w^m, w^n, t\big) = 1 \Big\},$$

and

$$\forall_{t\in(0;\infty)} \forall_{w\in A} \Big\{ \lim_{m\to\infty} M\big(w, w^m, t\big) = 1 \Big\}.$$

(4) *The single-valued dynamic system $(X, T^{[2]})$ is left and right $\mathcal{M}_{\{1\};\{1\}}$-closed on X.* Indeed, since

$$T^{[2]}(u) = \begin{cases} 1/2 + u/2 & \text{for } u \in (0;1], \\ 1 + u/2 & \text{for } u \in (1;2], \\ 2 + 2/3 + u/3^2 & \text{for } u \in (2;3), \end{cases} \tag{14.4}$$

we have $T^{[2]}(X) \subset A$, and if $(x_m : m \in \mathbb{N}) \subset T^{[2]}(X)$ is a left or right $\mathcal{M}_{\{1\};\{1\}}$-converging sequence in X and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T^{[2]}(u_m)\}$, then $A = LIM^{L-\mathcal{M}_{\{1\};\{1\}}}_{(x_m:m\in\mathbb{N})} = LIM^{R-\mathcal{M}_{\{1\};\{1\}}}_{(x_m:m\in\{0\}\cup\mathbb{N})}$ and $\mathrm{Fix}(T^{[2]}) = \{1,2\} \subset A$.

(5) *We see that, for each $w^0 \in X$, all assumptions* (A) *and* (B) *of Theorem* 10.1 *in $(X, \mathcal{M}_{\{1\};\{1\}}, *)$ hold and, for $k = 2$, the assumption in statement* (D) *of Theorem* 10.1 *holds. Consequently, the assertions in* (C) *and* (D) *hold.* This follows from items (1)-(4) and (14.1)-(14.4) since $\forall_{m\in\mathbb{N}}\{T^{[m]}(X) \subset A\}$.

Claim (a) $\mathrm{Fix}(T) = \varnothing$ *and* $\mathrm{Fix}(T^{[2]}) = \{1,2\}$. (b) *For each $w^0 \in X$, the dynamic process $(w^m : m \in \{0\} \cup \mathbb{N})$ of the system (X,T) starting at w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1} = T(w^m)\}$, is left and right $\mathcal{M}_{\{1\};\{1\}}$-convergent, and*

$$\mathrm{Fix}\big(T^{[2]}\big) \subset A = LIM^{L-\mathcal{M}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = LIM^{R-\mathcal{M}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}.$$

(c) *The property $\forall_{v\in \mathrm{Fix}(T^{[2]})} \forall_{t\in(0;\infty)}\{M(v, T(v), t) = M(T(v), v, t) = 1\}$ holds.* (d) *Assertions in statement* (E) *of Theorem* 10.1 *do not hold; M is not separating on X since, for $x, y \in A$, $x \neq y$, we have $\forall_{t\in(0;\infty)}\{M(x,y,t) = M(y,x,t) = 1\}$.*

Example 14.2 Let $X = [0;3)$, and let $\mathcal{M}_{\{1\};\{1\}} = \{M : X^2 \times (0;\infty) \to (0;1]\}$ where, for $u, w \in X$ and $t \in (0;\infty)$,

$$M(u,w,t) = t/\big(t + |w-u|\big) \quad \text{for } u,w \in X \text{ and } t \in (0;\infty). \tag{14.5}$$

Define the single-valued dynamic system (X,T) by

$$T(u) = \begin{cases} 1 & \text{for } u \in [0;1], \\ 1/2 & \text{for } u \in (1;2], \\ 0 & \text{for } u \in (2;3). \end{cases} \tag{14.6}$$

Further, let $E = [0;1]$, $\gamma > \delta(E)/2 = \sup_{x,y\in E}|x-y|/2 = 1/2$, and $\mathcal{K}_{\{1\};\{1\}} = \{K : X^2 \times (0;\infty) \to (0;1]\}$ where, for $u, w \in X$ and $t \in (0;\infty)$,

$$K(u,w,t) = \begin{cases} t/(t+|w-u|) & \text{if } E \cap \{u,w\} = \{u,w\}, \\ t/(t+\gamma) & \text{if } E \cap \{u,w\} \neq \{u,w\}. \end{cases} \tag{14.7}$$

(1) $(X, \mathcal{M}_{\{1\};\{1\}}, *)$ *is a fuzzy triangular space*; *see Example* 12.3(1).

(2) $\mathcal{K}_{\{1\};\{1\}} = \{K\}$ *is the left and right* $\mathcal{G}$*-family and the left and right* $\mathcal{W}$*-family generated by* $\mathcal{M}_{\{1\};\{1\}} = \{M\}$ *in a fuzzy quasi-triangular space* $(X, \mathcal{M}_{\{1\};\{1\}}, *)$. See Example 12.2; we have that $\gamma > \delta(E)/2 = \sup_{x,y\in E}|x-y|/2 = 1/2$.

(3) *For each* $\lambda \in (1/(2\gamma);1)$, (X,T) *is a fuzzy* $(\mathcal{F}_{2,X}^{L-\mathcal{K}_{\{1\};\{1\}}}, \lambda)$*-left contraction and a fuzzy* $(\mathcal{F}_{2,X}^{R-\mathcal{K}_{\{1\};\{1\}}}, \lambda)$*-right contraction*. Indeed, we prove that, for $C = 1$ and $\lambda \in (1/(2\gamma);1)$,

$$\forall_{t\in(0;\infty)}\forall_{x,y\in X}\bigl\{F_{\mathcal{K}}\bigl(T(x),T(y),\lambda t\bigr) \geq K(x,y,Ct)\bigr\}, \tag{14.8}$$

where, by Definition 6.3, for $\eta = 2$,

$$\begin{aligned}\forall_{t\in(0;\infty)}\forall_{u,w\in X}\bigl\{F_{\mathcal{K}}(u,w,t) &= F_{1;2,X}^{L-\mathcal{K}_{\{1\};\{1\}}}(u,w,t) \\ &= F_{1;2,X}^{R-\mathcal{K}_{\{1\};\{1\}}}(w,u,t) = K(u,w,t)\bigr\}.\end{aligned}$$

With this aim, we consider the following Cases A and B.

Case A. If $\{x,y\} \subset [0;1]$ or $\{x,y\} \subset (1;2)$ or $\{x,y\} \subset (2;3)$, then $T(x) = T(y) \in E$. By (14.6) and (14.7) this gives $F_{\mathcal{K}}(T(x),T(y),\lambda t) = K(T(x),T(y),\lambda t) = 1$. Therefore, (14.8) holds in this case.

Case B. If $\{x,y\} \cap [0;1] \neq \{x,y\}$, $\{x,y\} \cap (1;2) \neq \{x,y\}$, and $\{x,y\} \cap (2;3) \neq \{x,y\}$, then $T(x), T(y) \in E$, $T(x) \neq T(y)$, and (14.8) holds since

$$K\bigl(T(x),T(y),\lambda t\bigr) = \lambda t/(\lambda t + 1/2) \geq t/(t+\gamma) = K(x,y,Ct), \quad C = 1,$$

whenever $\lambda \in (1/(2\gamma);1)$.

(4) (X,T) *is* $\mathcal{K}_{\{1\};\{1\}}$ *left and right* $\mathcal{G}$*-admissible and left and right* $\mathcal{W}$*-admissible on* X. By Definition 7.1, assuming that $w^0 \in X$ is arbitrary and fixed, we must prove that a dynamic process $(w^m = T^{[m]}(w^0) : m \in \{0\} \cup \mathbb{N})$ of (X,T) starting at w^0 and satisfying

$$\forall_{t\in(0;\infty)}\forall_{p\in\mathbb{N}}\Bigl\{\lim_{m\to\infty} K\bigl(w^m, w^{m+p}, t\bigr) = 1\Bigr\} \tag{14.9}$$

or

$$\forall_{t\in(0;\infty)}\Bigl\{\lim_{m\to\infty}\inf_{n>m} K\bigl(w^m, w^n, t\bigr) = 1\Bigr\} \tag{14.10}$$

is left and right $\mathcal{K}_{\{1\};\{1\}}$-convergent. In fact, since, by (14.6), $\forall_{m\geq 2}\{w^m = 1\}$, by (14.6) and (14.7) we get that (14.9), (14.10), and $LIM_{(w^m:m\in\{0\}\cup\mathbb{N})}^{L-\mathcal{K}_{\{1\};\{1\}}} = LIM_{(w^m:m\in\{0\}\cup\mathbb{N})}^{R-\mathcal{K}_{\{8\};\{1\}}} = \{1\}$ hold.

(5) *For each* $w^0 \in X$, *property* (A4) *holds*. Indeed, by (14.7),

$$\forall_{w^0\in X}\Bigl\{\lim_{t\to\infty} K\bigl(w^0, T\bigl(w^0\bigr), t\bigr) = \lim_{t\to\infty} K\bigl(T\bigl(w^0\bigr), w^0, t\bigr) = 1\Bigr\}.$$

(6) *For each* $w^0 \in X$, *property* (B3) *holds*. Indeed, even for each $t \in (0;\infty)$, $\lambda \in (1/(2\gamma);1)$, and $w^0 \in X$, since $T(w^0) \subset E$, $C = 1$, and $\lim_{m\to\infty} \lambda^m = 0$, if we assume that $\forall_{m\in\mathbb{N}}\{a_m = 1/[m(m+1)]\}$, then we get that $\sum_{m=1}^{\infty} a_m = 1$, and also, denoting

$$\forall_{m\in\mathbb{N}}\{h_m = a_m/\lambda^m\},$$

we observe, by (14.7), that $\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\{K(w^0, T(w^0), th_m) = K(T(w^0), w^0, th_m)\}$ and $\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\{\prod_{i=m}^{\infty} K(w^0, T(w^0), th_i) = \prod_{i=m}^{\infty}(1 - Q_i)\}$ where, for all $m \in \mathbb{N}$ and $t \in (0;\infty)$, $Q_m = P/h_m/[t + P/h_m]$ and

$$P = \begin{cases} |w^0 - T(w^0)| & \text{if } w^0 \in E, \\ \gamma & \text{if } w^0 \notin E. \end{cases}$$

We prove that

$$\forall_{t\in(0;\infty)}\left\{\lim_{m\to\infty} *_{i=m}^{\infty} K\big(w^0, T\big(w^0\big), th_i\big)\right.$$
$$\left. = \lim_{m\to\infty}\prod_{i=m}^{\infty} K\big(w^0, T\big(w^0\big), th_i\big) = \lim_{m\to\infty}\prod_{i=m}^{\infty}(1 - Q_i) = 1\right\}. \tag{14.11}$$

With this aim, it is sufficient to prove that the series $\sum_{m=1}^{\infty} Q_m$ is convergent. Indeed, we see that

$$\forall_{t\in(0;\infty)}\forall_{m\in\mathbb{N}}\left\{Q_{m+1}/Q_m = \lambda\frac{m+2}{m}\cdot\frac{t + Z_m}{t + Z_{m+1}}\right\}, \tag{14.12}$$

where $\forall_{m\in\mathbb{N}}\{Z_m = Pm(m+1)\lambda^m\}$. Since $\lim_{m\to\infty} Z_{m+1}/Z_m = \lambda < 1$, the series $\sum_{m=1}^{\infty} Z_m$ is convergent, and, consequently, $\lim_{m\to\infty} Z_m = 0$. By (14.12) this implies that $\lim_{m\to\infty} Q_{m+1}/Q_m = \lambda < 1$, so $\sum_{m=1}^{\infty} Q_m$ is convergent, and (14.11) holds.

Moreover, by (14.7) we see that $\forall_{t\in(0;\infty)}\forall_{w^0\in X}\{K_\alpha(w^0, T(w^0), t) : (0;\infty) \to (0;1]$ is non-decreasing$\}$.

(7) $(X, T^{[2]})$ *is left and right* $\mathcal{M}_{\{1\};\{1\}}$*-closed*. Indeed, since $T^{[2]}(X) = \{1\}$, by Definition 8.2, if $(x_m : m \in \mathbb{N}) \subset T^{[2]}(X)$ is a left or right $\mathcal{M}_{\{1\};\{1\}}$-converging sequence in X and having subsequences $(v_m : m \in \mathbb{N})$ and $(u_m : m \in \mathbb{N})$ satisfying $\forall_{m\in\mathbb{N}}\{v_m = T^{[2]}(u_m)\}$, then we have that $\forall_{m\geq 2}\{x_m = 1\}$, $1 \in LIM^{L-\mathcal{M}_{\{1\};\{1\}}}_{(x_m:m\in\mathbb{N})} = LIM^{R-\mathcal{M}_{\{1\};\{1\}}}_{(x_m:m\in\{0\}\cup\mathbb{N})}$, and $1 = T^{[2]}(1)$.

(8) *We see that, for each* $w^0 \in X$, *all assumptions* (A) *and* (B) *of Theorem* 10.1 *in* $(X, \mathcal{M}_{\{1\};\{1\}}, *)$ *hold and, for* $k = 2$, *the assumptions in statements* (C)-(D) *of Theorem* 10.1 *hold*. This follows from items (1)-(7).

Claim (a) $\mathrm{Fix}(T^{[2]}) = \mathrm{Fix}(T) = \{1\}$. (b) *For each* $w^0 \in X$, *a dynamic process* $(w^m : m \in \{0\} \cup \mathbb{N})$ *of the system* (X, T) *starting at* w^0, $\forall_{m\in\{0\}\cup\mathbb{N}}\{w^{m+1} = T^{[m]}(w^0)\}$, *is left and right* $\mathcal{M}_{\{1\};\{1\}}$*-convergent, and* $\{1\} = LIM^{L-\mathcal{M}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})} = LIM^{R-\mathcal{M}_{\{1\};\{1\}}}_{(w^m:m\in\{0\}\cup\mathbb{N})}$. (c) *The property* $\forall_{t\in(0;\infty)}\{M(1,1,t) = 1\}$ *holds*.

Example 14.3 Let X, $\mathcal{M}_{\{1\};\{1\}} = \{M : X^2 \times (0;\infty) \to (0;1]\}$, and (X, T) be such as in Example 14.2. It is not hard to see that Theorem 10.1 cannot be used for $\mathcal{K}_{\{1\};\{1\}} = \{K : X^2 \times$

$(0;\infty) \to (0;1]\}$ when $K = M$ since the property $\exists_{\lambda\in(0;1)}\forall_{t\in(0;\infty)}\forall_{x,y\in X}\{F_{\mathcal{M}}(T(x),T(y),\lambda t) \geq M(x,y,t)\}$ does not hold, where

$$\forall_{t\in(0;\infty)}\forall_{u,w\in X}\bigl\{F_{\mathcal{M}}(u,w,t) = F_{1;2,X}^{L-\mathcal{M}_{\{1\};\{1\}}}(u,w,t) = F_{1;2,X}^{R-\mathcal{M}_{\{1\};\{1\}}}(w,u,t) = M(u,w,t)\bigr\}.$$

Indeed, using $x_0 = 1$ and $y_0 = 11/10$, we compute that

$$M\bigl(T(x_0),T(y_0),\lambda t\bigr) = \lambda t/(\lambda t + 1/2) \geq t/(t + 1/10) = M(x_0,y_0,t)$$

whenever $\lambda \geq 5$, which is impossible.

Remark 14.1 Before going further, let us observe that $(X,\mathcal{M}_{\{1\};\{1\}},*)$, where $\mathcal{M}_{\{1\};\{1\}} = \{M\}$, and M is defined by (14.5), is a GV-fuzzy metric space. Consequently, Examples 14.2 and 14.3 show that Theorem 10.1 is a new one even in GV-fuzzy metric spaces and that $\mathcal{K}_{C;\mathcal{A}} \neq \mathcal{M}_{C;\mathcal{A}}$ is a useful tool from a practical point of view in the sense that we may construct a fuzzy periodic and fuzzy fixed point theory in very general classes of spaces and maps.

15 Conclusions

It is easy to show, by constructing appropriate examples, that in mathematics, some interesting spaces and fascinating results in these spaces are not optimal and that the answers to many basic problems about them are still missing (*e.g.*, statements that we can give in more general spaces, some hypotheses are not significant, assertions are not deep). From any point of view, this situation is not satisfactory and inspires further investigations. This paper describes and solves some problems in this direction and leads to entirely new concepts of fuzzy spaces and set-valued and single-valued fuzzy contractions and to a new way of looking at fuzzy periodic and fixed point theory.

Competing interests
The author declares that he has no conflict of interests regarding the publication of this paper.

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