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Supergravity as generalised geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M theory

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Abstract: We reformulate eleven-dimensional supergravity, including fermions, in terms of generalised geometry, for spacetimes that are warped products of Minkowski space with a d-dimensional manifold M with $d \leq 7$. The reformulation has an $E_{d(d)} \times \mathbb{R}^+$ structure group and it has a local $\tilde{H}_d$ symmetry, where $\tilde{H}_d$ is the double cover of the maximally compact subgroup of $E_{d(d)}$. The bosonic degrees for freedom unify into a generalised metric, and, defining the generalised analogue D of the Levi-Civita connection, one finds that the corresponding equations of motion are the vanishing of the generalised Ricci tensor. To leading order, we show that the fermionic equations of motion, action and supersymmetry variations can all be written in terms of D. Although we will not give the detailed decompositions, this reformulation is equally applicable to type IIA or IIB supergravity restricted to a $(d-1)$-dimensional manifold. For completeness we give explicit expressions in terms of $\tilde{H}_4 = \mathrm{Spin}(5)$ and $\tilde{H}_7 = \mathrm{SU}(8)$ representations for $d = 4$ and $d = 7$.

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1 Introduction

In this paper we describe the reformulation of eleven-dimensional supergravity in terms of generalised geometry. The analysis is restricted to spacetimes that are warped products of Minkowski space with a d-dimensional manifold M with $d \leq 7$, with all fields taken to be independent of the $(11-d)$-dimensional flat space. We work to leading order in the fermions. The generalised geometry has a $E_{d(d)} \times \mathbb{R}^+$ structure group and the theory has a local $\tilde{H}_d$ symmetry, where $\tilde{H}_d$ is the double cover of the maximally compact subgroup of $E_{d(d)}$. Although we will not give the detailed decompositions, this reformulation is equally applicable to type IIA or IIB supergravity restricted to a $(d-1)$-dimensional manifold.

This completes a programme started in [1], where we studied the formulation of type II supergravity in terms of a generalised geometry with an $O(d, d) \times \mathbb{R}^+$ structure group first proposed by Hitchin and Gualtieri [2, 3]. Such reformulations were first given in the related "doubled" formalism in a series of papers by Hohm, Hull, Kwak and Zweibach [4–6], building in part on work by Siegel [7, 8]. This was extended to the RR sector in [9, 10] (along with [1]). In such Double Field Theory, rather than extending only the tangent space as in generalised geometry, one conjectures that the spacetime is doubled and, to relate to supergravity, then imposes a constraint that there is dependence on only half the coordinates. Interestingly, even when applied to conventional supergravity backgrounds, the doubled space cannot be assumed to have a conventional manifold structure [11].

In [1], we focussed in particular on the notion of *generalised connections*. We showed that there exists a class of torsion-free, metric-compatible generalised connections, analogues of the Levi-Civita connection. (These objects, the corresponding curvature tensors and the relation to the NSNS sector of supergravity, were first discussed in the "doubled" formalism by Siegel [7, 8]. Related "semi-covariant" derivatives, and the corresponding curvature tensors were defined independently by Jeon, Lee and Park in [13, 14].) We then used these results to write down a gravitational theory for generalised geometry and showed that this was precisely type II supergravity with its local "double Lorentz symmetry" $O(9,1)\times O(1,9)$ manifest. In the following paper [12], we defined the analogous concepts in $E_{d(d)}\times\mathbb{R}^+$ generalised geometry, also known as exceptional or extended generalised geometry [15, 16]. We showed that this construction can be used to describe eleven-dimensional supergravity restricted to d dimensions, that is, to the warped product of Minkowski space and a d-dimensional manifold. The action and field equations for the bosonic sector are simply the $E_{d(d)}\times\mathbb{R}^+$ generalised version of Einstein's gravity

$$S_{\mathrm{B}}=\frac{1}{2\kappa^2}\int|\mathrm{vol}_G|\,R\,,\qquad R_{AB}=0, \tag{1.1}$$

where $|\mathrm{vol}_G|$ is the volume density associated to the generalised metric G, and R_{AB} and R are respectively the generalised Ricci tensor and scalar associated to a torsion-free, metric compatible generalised connection D.

The present paper serves as a direct continuation of the $E_{d(d)}\times\mathbb{R}^+$ construction, by including the fermion fields in the description of the supergravity. We show how the same generalised connection describes the supersymmetry algebra, and can be used to obtain the fermion dynamics. The equations of motion for the two fermionic fields present in the theory, ψ and ρ, are simply

$$\begin{aligned}\not{D}\psi+\frac{11-d}{9-d}D\curlywedge\rho&=0,\\ D\curlyvee\psi+\not{D}\rho&=0,\end{aligned} \tag{1.2}$$

where $\not{D}$, $D\curlywedge$ and $D\curlyvee$ are particular $\tilde{H}_d$-covariant operators built from the generalised connection that project, in the first line, onto the ψ representation, and in the second line, onto the ρ representation. The geometry therefore naturally produces a supergravity theory with the larger symmetries manifest, including the fermions to first order.

While the formalism is powerful enough to describe the theory in different dimensions in all generality, the fact that the structures appearing in each case are so diverse forces us to introduce fairly abstract notation. It can therefore be helpful to look at explicit constructions. We work out two examples with the full local symmetry manifest, the relatively simple $d=4$ case and the more interesting $d=7$ case. For instance, in the latter case, when the local group is SU(8) the projections in (1.2) become

$$\begin{aligned}-\frac{1}{12}\epsilon_{\alpha\beta\gamma\delta\delta'\theta_1\theta_2\theta_3}D^{\delta\delta'}\psi^{\theta_1\theta_2\theta_3}+2\bar{D}_{[\alpha\beta}\bar{\rho}_{\gamma]}&=0,\\ -\frac{1}{2}\bar{D}_{\beta\gamma}\psi^{\alpha\beta\gamma}+D^{\alpha\beta}\bar{\rho}_\beta&=0,\end{aligned} \tag{1.3}$$

which is a remarkably compact form compared with the usual supergravity expressions.

There are many precursors and related approaches to the geometrical reformulation discussed here [17–32], as well as several, more recent, papers that have developed or applied ideas of exceptional generalised geometry [33–43]. Let us comment briefly on a couple of related approaches of particular relevance here. In 1986 de Wit and Nicolai [18] already showed that a local SU(8) symmetry (and a global $E_{7(7)}$ symmetry on an extension of the usual vielbein) could be realised directly in $d=11$ supergravity, including the linearised fermions, assuming only the existence of a product structure on the tangent bundle $\mathrm{SO}(3,1) \times \mathrm{SO}(7) \subset \mathrm{SO}(10,1)$. The current paper can be viewed as a geometrical framework from which to interpret these results. In the bosonic sector, a more recent approach is the work of Berman and Perry and collaborators [30–32], using the M-theory extension of double field theory [44–46]. These authors were able to find a non-manifestly covariant form of the action simply in terms of derivatives of the generalised metric G. Again the work of [12] and this paper puts this result on a covariant geometrical footing, demonstrating that the action is none other than the generalised Ricci tensor. One might also expect a connection to special cases of the much broader non-linear E_{11} proposal of West [20–22], along the lines of the relation between Riemannian geometry and the non-linear realisation of gravity due to Borisov and Ogievetsky [47]. In a remarkably detailed construction [28, 29], including the fermions and using West's approach, Hillmann indeed considered a "generalised $E_{7(7)}$ coset dynamics", on a sixty-dimensional spacetime [25]. By demanding that, upon truncating to 4+7 dimensions, the theory possess Diff(7) invariance, the author managed to show that the construction reproduces the results of de Wit and Nicolai [18]. In generalised geometry one does not increase the dimension of the underlying manifold, so clearly the two formalisms are technically distinct, and the precise relation between them is yet to be investigated.

The paper is thus structured as follows. In section 2 we give a quick review eleven-dimensional supergravity and its restrictions to d dimensions. In section 3 we provide a very brief summary of the key points in [12]. In section 4 we introduce the fermion fields and provide a complete rewrite of supergravity in terms of the generalised geometry formalism. In section 5 we work out the explicit $d = 4, 7$ cases. We conclude with a short discussion in section 6. We also include a number of appendices to fix our conventions and also give the details of the various spinor decompositions and group actions that we use.

2 Eleven-dimensional supergravity and its restriction

2.1 $N = 1$, $D = 11$ supergravity

Let us start by reviewing the action, equations of motion and supersymmetry variations of eleven-dimensional supergravity, to leading order in the fermions, following the conventions of [48] (see also appendices A and C).

The fields are simply

$$\{g_{MN}, \mathcal{A}_{MNP}, \psi_M\}, \tag{2.1}$$

where g_{MN} is the metric, $\mathcal{A}_{MNP}$ the three-form potential and ψ_M is the gravitino. The bosonic action is given by

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2} \int \left(\sqrt{-g}\, \mathcal{R} - \frac{1}{2} \mathcal{F} \wedge * \mathcal{F} - \frac{1}{6} \mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F} \right), \tag{2.2}$$

where $\mathcal{R}$ is the Ricci scalar and $\mathcal{F} = \mathrm{d}\mathcal{A}$. This leads to the equations of motion

$$\begin{aligned} \mathcal{R}_{MN} - \frac{1}{12} \left(\mathcal{F}_{MP_1P_2P_3} \mathcal{F}_N{}^{P_1P_2P_3} - \frac{1}{12} g_{MN} \mathcal{F}^2 \right) &= 0, \\ \mathrm{d} * \mathcal{F} + \frac{1}{2} \mathcal{F} \wedge \mathcal{F} &= 0, \end{aligned} \tag{2.3}$$

where $\mathcal{R}_{MN}$ is the Ricci tensor.

Taking Γ^M to be the Cliff$(10,1;\mathbb{R})$ gamma matrices, the fermionic action, to quadratic order in ψ_M, is given by

$$\begin{aligned} S_{\mathrm{F}} = \frac{1}{\kappa^2}\int \sqrt{-g}\Big(&\bar{\psi}_M \Gamma^{MNP}\nabla_N \psi_P + \frac{1}{96}\mathcal{F}_{P_1\dots P_4}\bar{\psi}_M \Gamma^{MP_1\dots P_4 N}\psi_N \\ &+ \frac{1}{8}\mathcal{F}_{P_1\dots P_4}\bar{\psi}^{P_1}\Gamma^{P_2P_3}\psi^{P_4}\Big), \end{aligned} \tag{2.4}$$

the gravitino equation of motion is

$$\Gamma^{MNP}\nabla_N\psi_P + \frac{1}{96}\left(\Gamma^{MNP_1\dots P_4}\mathcal{F}_{P_1\dots P_4} + 12\mathcal{F}^{MN}{}_{P_1P_2}\Gamma^{P_1P_2}\right)\psi_N = 0. \tag{2.5}$$

The supersymmetry variations of the bosons are

$$\begin{aligned} \delta g_{MN} &= 2\bar{\varepsilon}\Gamma_{(M}\psi_{N)}, \\ \delta \mathcal{A}_{MNP} &= -3\bar{\varepsilon}\Gamma_{[MN}\psi_{P]}, \end{aligned} \tag{2.6}$$

while the supersymmetry variation of the gravitino is

$$\delta\psi_M = \nabla_M\varepsilon + \frac{1}{288}\left(\Gamma_M{}^{N_1\dots N_4} - 8\delta_M{}^{N_1}\Gamma^{N_2N_3N_4}\right)\mathcal{F}_{N_1\dots N_4}\varepsilon, \tag{2.7}$$

where ε is the supersymmetry parameter.

2.2 Restricted action, equations of motion and supersymmetry

We will be interested in "restrictions" of eleven-dimensional supergravity where the space-time is assumed to be a warped product $\mathbb{R}^{10-d,1}\times M$ of Minkowski space with a d-dimensional spin manifold M, with $d\le 7$. The metric is taken to have the form

$$\mathrm{d}s^2_{11} = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathbb{R}^{10-d,1}) + \mathrm{d}s^2_d(M), \tag{2.8}$$

where $\mathrm{d}s^2(\mathbb{R}^{10-d,1})$ is the flat metric on $\mathbb{R}^{10-d,1}$ and $\mathrm{d}s^2_d(M)$ is a general metric on M. The warp factor Δ and all the other fields are assumed to be independent of the flat $\mathbb{R}^{10-d,1}$ space. In this sense we restrict the full eleven-dimensional theory to M. We will split the eleven-dimensional indices as external indices $\mu = 0,1,\dots,c-1$ and internal indices $m = 1,\dots,d$ where $c+d=11$.

In the restricted theory, the surviving fields include the obvious internal components of the eleven-dimensional fields (namely the metric g and three-form A) as well as the warp factor Δ. If $d=7$, the eleven-dimensional Hodge dual of the 4-form F can have a purely internal 7-form component. This leads one to introduce, in addition, a dual six-form potential $\tilde{A}$ on M which is related to the seven-form field strength $\tilde{F}$ by

$$\tilde{F} = \mathrm{d}\tilde{A} - \frac{1}{2}A\wedge F. \tag{2.9}$$

The Bianchi identities satisfied by $F=\mathrm{d}A$ and $\tilde{F}$ are then

$$\begin{aligned} \mathrm{d}F &= 0, \\ \mathrm{d}\tilde{F} + \frac{1}{2}F\wedge F &= 0. \end{aligned} \tag{2.10}$$

With these definitions one can see that F and $\tilde{F}$ are related to the components of the eleven-dimensional 4-form field strength $\mathcal{F}$ by

$$F_{m_1\dots m_4} = \mathcal{F}_{m_1\dots m_4}, \qquad\qquad \tilde{F}_{m_1\dots m_7} = (*\mathcal{F})_{m_1\dots m_7}, \tag{2.11}$$

where $*\mathcal{F}$ is the eleven-dimensional Hodge dual. The field strengths F and $\tilde{F}$ are invariant under the gauge transformations of the potentials given by

$$\begin{aligned} A' &= A + \mathrm{d}\Lambda, \\ \tilde{A}' &= \tilde{A} + \mathrm{d}\tilde{\Lambda} - \frac{1}{2}\mathrm{d}\Lambda \wedge A, \end{aligned} \tag{2.12}$$

for some two-form Λ and five-form $\tilde{\Lambda}$. There is an intricate hierarchy of further coupled gauge transformations of Λ and $\tilde{\Lambda}$, discussed in more detail in [16] and [12] and which formally defines a form of "gerbe" [49].

In order to diagonalise the kinetic terms in the fermionic Lagrangian, one introduces the standard field redefinition of the external components of the gravitino

$$\psi'_\mu = \psi_\mu + \frac{1}{c-2}\Gamma_\mu \Gamma^m \psi_m. \tag{2.13}$$

We then denote its trace as

$$\rho = \frac{c-2}{c}\Gamma^\mu \psi'_\mu, \tag{2.14}$$

and allow this to be non-zero and dependant on the internal coordinates (this is the partner of the warp factor Δ). Although the restriction to d-dimensions breaks the Lorentz symmetry to $\mathrm{Spin}(10-d,1)\times\mathrm{Spin}(d) \subset \mathrm{Spin}(10,1)$, we do not make an explicit decomposition of the spinor indices under $\mathrm{Spin}(10-d,1)\times\mathrm{Spin}(d)$. Instead we keep expressions in terms of eleven-dimensional gamma matrices. This is helpful in what follows since it allows us to treat all dimensions in a uniform way.

In summary, the surviving degrees of freedom after the restriction to d dimensions are

$$\{g_{mn}, A_{mnp}, \tilde{A}_{m_1\dots m_6}, \Delta; \psi_m, \rho\}. \tag{2.15}$$

One can then define the internal space bosonic action

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int \sqrt{g}\,\mathrm{e}^{c\Delta}\Big(\mathcal{R} + c(c-1)(\partial\Delta)^2 - \frac{1}{2}\frac{1}{4!}F^2 - \frac{1}{2}\frac{1}{7!}\tilde{F}^2\Big), \tag{2.16}$$

where the associated equations of motion

$$\begin{aligned} \mathcal{R}_{mn} - c\nabla_m\nabla_n\Delta - c(\partial_m\Delta)(\partial_n\Delta) - \frac{1}{2}\frac{1}{4!}\Big(4F_{mp_1p_2p_3}F_n{}^{p_1p_2p_3} - \frac{1}{3}g_{mn}F^2\Big)& \\ -\frac{1}{2}\frac{1}{7!}\Big(7\tilde{F}_{mp_1\dots p_6}\tilde{F}_n{}^{p_1\dots p_6} - \frac{2}{3}g_{mn}\tilde{F}^2\Big) &= 0, \\ \mathcal{R} - 2(c-1)\nabla^2\Delta - c(c-1)(\partial\Delta)^2 - \frac{1}{2}\frac{1}{4!}F^2 - \frac{1}{2}\frac{1}{7!}\tilde{F}^2 &= 0, \\ \mathrm{d}*(\mathrm{e}^{c\Delta}F) - \mathrm{e}^{c\Delta}F\wedge *\tilde{F} &= 0, \\ \mathrm{d}*(\mathrm{e}^{c\Delta}\tilde{F}) &= 0, \end{aligned} \tag{2.17}$$

are those obtained by substituting the field ansatz into (2.3). Similarly, to quadratic order in fermions, the action for the fermion fields is

$$
\begin{aligned}
S_{\mathrm{F}} = -\frac{1}{\kappa^2(c-2)^2}\int\sqrt{g}\,\mathrm{e}^{c\Delta}\Big[&(c-4)\bar{\psi}_m\Gamma^{mnp}\nabla_n\psi_p \\
&-c(c-3)\bar{\psi}^m\Gamma^n\nabla_n\psi_m - c\left(\bar{\psi}^m\Gamma_n\nabla_m\psi^n + \bar{\psi}^m\Gamma_m\nabla_n\psi^n\right) \\
&-\frac{1}{4}\frac{1}{2!}(2c^2-5c+4)\bar{\psi}_m F^{mn}{}_{pq}\Gamma^{pq}\psi_n + \frac{1}{4}c(c-3)\bar{\psi}_m \not{F}\psi^m \\
&+\frac{1}{2}\frac{1}{3!}c\bar{\psi}_m F^m{}_{pqr}\Gamma^{npqr}\psi_n + \frac{1}{4}\frac{1}{4!}(c-4)\bar{\psi}_m F_{p_1\dots p_4}\Gamma^{mnp_1\dots p_4}\psi_n \\
&-\frac{1}{4}\frac{1}{5!}(2c^2-5c+4)\bar{\psi}_m\tilde{F}^{mn}{}_{p_1\dots p_5}\Gamma^{p_1\dots p_5}\psi_n \\
&+\frac{1}{4}\frac{1}{6!}c(c-1)\bar{\psi}_m\tilde{F}^m{}_{p_1\dots p_6}\Gamma^{np_1\dots p_6}\psi_n \\
&+c(c-1)\big(\bar{\psi}^m\nabla_m\rho - \bar{\rho}\nabla^m\psi_m\big) + c\big(\bar{\psi}_m\Gamma^{mn}\nabla_n\rho - \bar{\rho}\Gamma^{mn}\nabla_m\psi_n\big) \\
&-c(c-1)(c-2)\bar{\psi}^m(\partial_m\Delta)\rho - c(c-2)\bar{\psi}_m\Gamma^{mn}(\partial_n\Delta)\rho \\
&+\frac{1}{2}\frac{1}{3!}c(c-1)\bar{\rho}F^m{}_{pqr}\Gamma^{pqr}\psi_m - \frac{1}{2}\frac{1}{4!}c\bar{\rho}\Gamma^m{}_{p_1\dots p_4}F^{p_1\dots p_4}\psi_m \\
&-\frac{1}{2}\frac{1}{6!}c(c-1)\bar{\psi}_m\tilde{F}^m{}_{p_1\dots p_6}\Gamma^{p_1\dots p_6}\rho \\
&+c(c-1)\big(\bar{\rho}\Gamma^m\nabla_m\rho + \frac{1}{4}\bar{\rho}\not{F}\rho - \frac{1}{4}\bar{\rho}\not{\tilde{F}}\rho\big)\Big].
\end{aligned}
\tag{2.18}
$$

This action leads to the equation of motion for ψ_m,

$$
\begin{aligned}
0 = &(c-4)\Gamma_m{}^{np}\left(\nabla_n + \frac{c}{2}\partial_n\Delta\right)\psi_p - c(c-3)\Gamma^n\left(\nabla_n + \frac{c}{2}\partial_n\Delta\right)\psi_m \\
&-c\Gamma_n\left(\nabla_m + \frac{c}{2}\partial_m\Delta\right)\psi^n - c\Gamma_m\left(\nabla_n + \frac{c}{2}\partial_n\Delta\right)\psi^n \\
&-\frac{1}{4}(2c^2-5c+4)\frac{1}{2!}F_{mnpq}\Gamma^{pq}\psi^n + \frac{1}{4}c(c-3)\not{F}\psi_m \\
&+\frac{1}{2}\frac{1}{3!}cF_{(m}{}^{p_1p_2p_3}\Gamma_{n)p_1p_2p_3}\psi^n + \frac{1}{4}\frac{1}{4!}(c-4)\Gamma_{mn}{}^{p_1\dots p_4}F_{p_1\dots p_4}\psi^n \\
&-\frac{1}{4}\frac{1}{5!}(2c^2-5c+4)\tilde{F}_{mnp_1\dots p_5}\Gamma^{p_1\dots p_5}\psi^n + \frac{1}{4}\frac{1}{6!}c(c-1)\tilde{F}_{(m}{}^{p_1\dots p_6}\Gamma_{n)p_1\dots p_6}\psi^n \\
&+c\Gamma_m{}^n\left(\nabla_n + \partial_n\Delta\right)\rho + c(c-1)\left(\nabla_m + \partial_m\Delta\right)\rho \\
&+\frac{1}{4}\frac{1}{3!}c(c-1)F_{mp_1p_2p_3}\Gamma^{p_1p_2p_3}\rho + \frac{1}{4}\frac{1}{4!}c\Gamma_{mp_1\dots p_4}F^{p_1\dots p_4}\rho \\
&-\frac{1}{4}\frac{1}{6!}c(c-1)\tilde{F}_{mn_1\dots n_6}\Gamma^{n_1\dots n_6}\rho,
\end{aligned}
\tag{2.19}
$$

and the equation of motion for ρ,

$$
\begin{aligned}
0 = &\left[\not{\nabla} + \frac{c}{2}(\not{\partial}\Delta) + \frac{1}{4}\not{F} - \frac{1}{4}\not{\tilde{F}}\right]\rho \\
&-\left[\nabla_m + (c-1)\partial_m\Delta\right]\psi^m - \frac{1}{c-1}\Gamma^{mn}\left[\nabla_m + (c-1)\partial_m\Delta\right]\psi_n \\
&+\frac{1}{4}\frac{1}{3!}F^m{}_{p_1p_2p_3}\Gamma^{p_1p_2p_3}\psi_m - \frac{1}{4}\frac{1}{4!}\frac{1}{c-1}\Gamma^m{}_{p_1\dots p_4}F^{p_1\dots p_4}\psi_m \\
&+\frac{1}{4}\frac{1}{6!}\tilde{F}^m{}_{p_1\dots p_6}\Gamma^{p_1\dots p_6}\psi_m.
\end{aligned}
\tag{2.20}
$$

Turning to the supersymmetry transformations, we find that the variations of the fermion fields are given by

$$\begin{aligned}\delta\rho &= \left[\slashed{\nabla} - \frac{1}{4}\slashed{F} - \frac{1}{4}\slashed{\tilde{F}} + \frac{c-2}{2}(\slashed{\partial}\Delta)\right]\varepsilon, \\ \delta\psi_m &= \left[\nabla_m + \frac{1}{288}F_{n_1\dots n_4}\left(\Gamma_m{}^{n_1\dots n_4} - 8\delta_m{}^{n_1}\Gamma^{n_2n_3n_4}\right) - \frac{1}{12}\frac{1}{6!}\tilde{F}_{mn_1\dots n_6}\Gamma^{n_1\dots n_6}\right]\varepsilon,\end{aligned} \tag{2.21}$$

and the variations of the bosons by

$$\begin{aligned}\delta g_{mn} &= 2\bar{\varepsilon}\Gamma_{(m}\psi_{n)}, \\ (c-2)\delta\Delta + \delta\log\sqrt{g} &= \bar{\varepsilon}\rho, \\ \delta A_{mnp} &= -3\bar{\varepsilon}\Gamma_{[mn}\psi_{p]}, \\ \delta\tilde{A}_{m_1\dots m_6} &= 6\bar{\varepsilon}\Gamma_{[m_1\dots m_5}\psi_{m_6]}.\end{aligned} \tag{2.22}$$

This completes our summary of the reduced theory.

In what follows the fermionic fields will be reinterpreted as representations of larger symmetry groups $\tilde{H}_d \supset \mathrm{Spin}(d)$. To mark that distinction, the fermions that have appeared in this section will be denoted by $\varepsilon^{\text{sugra}}$, ρ^{sugra} and ψ^{sugra}. Absent this label, the fields are to be viewed as "generalised" objects transforming under $\tilde{H}_d$, as will be clarified in section 4.1.

3 Review of $E_{d(d)} \times \mathbb{R}^+$ generalised geometry

We now give a brief summary of the key points in the construction of the $E_{d(d)} \times \mathbb{R}^+$ generalised geometry and connections, relevant to reductions of eleven-dimensional supergravity, as discussed in [12].

3.1 Generalised bundles and frames

Let M be a d-dimensional spin manifold with $d \leq 7$.[1] The generalised tangent space E is isomorphic to the sum [15, 16]

$$E \simeq TM \oplus \Lambda^2T^*M \oplus \Lambda^5T^*M \oplus (T^*M \otimes \Lambda^7T^*M), \tag{3.1}$$

where for $d < 7$ some of these terms will of course be absent. Physically the terms in the sum can be thought of as corresponding to different brane charges, namely, momentum, M2-brane, M5-brane and Kaluza-Klein monopole charge. The bundle is actually given by a series of extensions which are defined via the patching data of the three-form and six-form connections. (Specifying particular three- and six-form connections defines an isomorphism (3.1).) The patching structure matches the supergravity symmetries (2.12) (see [15, 16]). In this way the bundle E encodes all the topological information of the supergravity background: the twisting of the tangent space TM as well as that of the form-field potentials.

The fibre E_x of the generalised vector bundle at $x \in M$ forms a representation space of $E_{d(d)} \times \mathbb{R}^+$. These are listed in table 1. The definition of the $E_{d(d)} \times \mathbb{R}^+$ group and

[1] We actually only consider $4 \leq d \leq 7$, as for lower dimensions the relevant structures simplify to a point that generalised geometry has little to add to the usual Riemannian description.

$E_{d(d)}$ group	E rep.	ad $\tilde{F}$ rep.
$E_{7(7)}$	$\mathbf{56_1}$	$\mathbf{133_0} + \mathbf{1_0}$
$E_{6(6)}$	$\mathbf{27'_1}$	$\mathbf{78_0} + \mathbf{1_0}$
$E_{5(5)} \simeq \mathrm{Spin}(5,5)$	$\mathbf{16^c_1}$	$\mathbf{45_0} + \mathbf{1_0}$
$E_{4(4)} \simeq \mathrm{SL}(5,\mathbb{R})$	$\mathbf{10'_1}$	$\mathbf{24_0} + \mathbf{1_0}$

Table 1. Generalised tangent space and frame bundle representations where the subscript denotes the $\mathbb{R}^+$ weight, where $\mathbf{1_1} \simeq (\det T^*M)^{1/(9-d)}$.

its explicit action on E_x is given in appendix D. Crucially, the patching used to define E is compatible with the $E_{d(d)} \times\mathbb{R}^+$ action. This means that one can define a generalised structure bundle as a sub-bundle of the general frame bundle F for E. Let $\{\hat{E}_A\}$ be a basis for E_x, where the label A runs over the dimension of the generalised tangent space as listed in table 1. As usual, a choice of coordinates on a patch U defines a particular such basis where

$$\{\hat{E}_A\} = \{\partial/\partial x^m\} \cup \{\mathrm{d}x^m \wedge \mathrm{d}x^n\} \cup \{\mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_5}\} \cup \{\mathrm{d}x^m \otimes \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_7}\}. \quad (3.2)$$

We will denote the components of a generalised vector V in such a coordinate frame by an index M, namely $V^M = (v^m, \omega_{mn}, \sigma_{m_1\dots m_5}, \tau_{m,m_1\dots m_7})$.

The *generalised structure bundle* is then the unique $E_{d(d)} \times\mathbb{R}^+$ principle sub-bundle $\tilde{F} \subset F$ compatible with the patching. Concretely, it can be written as

$$\tilde{F} = \big\{(x, \{\hat{E}_A\}) : x \in M, \text{ and } \{\hat{E}_A\} \text{ is a } E_{d(d)} \times\mathbb{R}^+ \text{ basis of } \tilde{E}_x\big\}, \quad (3.3)$$

where an $E_{d(d)} \times\mathbb{R}^+$ basis is any choice of frame that is related to the coordinate frame by an $E_{d(d)} \times\mathbb{R}^+$ transformation as defined in appendix D. By construction, this is a principle bundle with fibre $E_{d(d)} \times\mathbb{R}^+$.

A special class of $E_{d(d)} \times\mathbb{R}^+$ frames are those defined by a splitting of the generalised tangent space E, that is, an isomorphism of the form (3.1). As we mentioned, this is equivalent to introducing the three- and six-form gauge potentials, A and $\tilde{A}$. Then, given a generic basis $\{\hat{e}_a\}$ for TM, $\{e^a\}$ as the dual basis on T^*M and a scalar function Δ, one has that a *conformal split frame* $\{\hat{E}_A\}$ for $\tilde{E}$ has the general form (see appendix D for notation)

$$\begin{aligned}
\hat{E}_a &= \mathrm{e}^{\Delta}\Big(\hat{e}_a + i_{\hat{e}_a}A + i_{\hat{e}_a}\tilde{A} + \frac{1}{2}A \wedge i_{\hat{e}_a}A \\
&\qquad + jA \wedge i_{\hat{e}_a}\tilde{A} + \frac{1}{6}jA \wedge A \wedge i_{\hat{e}_a}A\Big), \\
\hat{E}^{ab} &= \mathrm{e}^{\Delta}\left(e^{ab} + A \wedge e^{ab} - j\tilde{A} \wedge e^{ab} + \frac{1}{2}jA \wedge A \wedge e^{ab}\right), \\
\hat{E}^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(e^{a_1\dots a_5} + jA \wedge e^{a_1\dots a_5}\right), \\
\hat{E}^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta} e^{a,a_1\dots a_7}.
\end{aligned} \quad (3.4)$$

In this frame, the components of the generalised vector

$$V = v^a\hat{E}_a + \frac{1}{2}\omega_{ab}\hat{E}^{ab} + \frac{1}{5!}\sigma_{a_1\dots a_5}\hat{E}^{a_1\dots a_5} + \frac{1}{7!}\tau_{a,a_1\dots a_7}\hat{E}^{a,a_1\dots a_7} \quad (3.5)$$

can be used to construct

$$V^{(A,\tilde{A},\Delta)} = v^a \hat{e}_a + \frac{1}{2}\omega_{ab}e^{ab} + \frac{1}{5!}\sigma_{a_1\dots a_5}e^{a_1\dots a_5} + \frac{1}{7!}\tau_{a,a_1\dots a_7}e^{a,a_1\dots a_7} \\ \in \Gamma(TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M)), \tag{3.6}$$

thus realising the isomorphism (3.1).

Given the generalised structure bundle one can then define vector bundles associated to any given representation of $E_{d(d)} \times \mathbb{R}^+$. We refer to sections of such bundles as *generalised tensors*.

3.2 The Dorfman derivative

The generalised tangent space admits a generalisation of the Lie derivative which ultimately will encode the local bosonic symmetries of the supergravity. Given $V = v + \omega + \sigma + \tau \in \Gamma(E)$, one can define an operator L_V, the *Dorfman derivative*, which combines the action of an infinitesimal diffeomorphism generated by v and A- and $\tilde{A}$-field gauge transformations generated by ω and σ. In components, acting on $V' \in \Gamma(E)$, it is given by

$$L_V V' = \mathcal{L}_v v' + \left(\mathcal{L}_v \omega' - i_{v'}\mathrm{d}\omega\right) + \left(\mathcal{L}_v \sigma' - i_{v'}\mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega\right) \\ + \left(\mathcal{L}_v \tau' - j\sigma' \wedge \mathrm{d}\omega - j\omega' \wedge \mathrm{d}\sigma\right), \tag{3.7}$$

where $\mathcal{L}_v$ is the conventional Lie derivative. It can also be written in an $E_{d(d)} \times \mathbb{R}^+$ form, using coordinate indices M, as

$$L_V V'^M = V^N \partial_N V'^M - (\partial \times_{\mathrm{ad}} V)^M{}_N V'^N, \tag{3.8}$$

where the action of the partial derivative operator has been embedded into the dual generalised tangent space via the map $T^*M \to E^*$ so that

$$\partial_M = \begin{cases} \partial_m & \text{for } M = m \\ 0 & \text{otherwise} \end{cases}, \tag{3.9}$$

and $\times_{\mathrm{ad}}$ is the projection to the adjoint representation of $E_{d(d)} \times \mathbb{R}^+$

$$\times_{\mathrm{ad}} : E^* \otimes E \to \mathrm{ad}\,\tilde{F}, \tag{3.10}$$

as defined in (D.9). By taking the appropriate adjoint action on the given $E_{d(d)} \times \mathbb{R}^+$ representation, the Dorfman derivative can be naturally extended to an arbitrary generalised tensor.

3.3 Generalised $E_{d(d)} \times \mathbb{R}^+$ connections and torsion

Generalised connections are first-order linear differential operators D, analogues of conventional connections on TM, which can be written in the form, given $W = W^A \hat{E}_A \in \Gamma(E)$ in frame indices

$$D_M W^A = \partial_M W^A + \Omega_M{}^A{}_B W^B, \tag{3.11}$$

$E_{d(d)}$ group	torsion rep.
$E_{7(7)}$	$\mathbf{912}_{-1} + \mathbf{56}_{-1}$
$E_{6(6)}$	$\mathbf{351}'_{-1} + \mathbf{27}_{-1}$
$E_{5(5)} \simeq \mathrm{Spin}(5,5)$	$\mathbf{144}^c_{-1} + \mathbf{16}^c_{-1}$
$E_{4(4)} \simeq \mathrm{SL}(5,\mathbb{R})$	$\mathbf{40}_{-1} + \mathbf{15}'_{-1} + \mathbf{10}_{-1}$

Table 2. Generalised torsion representations.

where Ω is a section of E^* (denoted by the M index) taking values in $\operatorname{ad}\tilde{F}$ (denoted by the A and B frame indices), and as such, the action of D then extends naturally to any generalised $E_{d(d)} \times \mathbb{R}^+$ tensor. Note that unlike a conventional connection, the index M runs over the whole of E^* and so one can take the derivative not only in a vector direction but also along two-forms, five-forms and so on.

Let α be a generalised tensor and $L^D_V \alpha$ be its Dorfman derivative (3.8) with ∂ replaced by D. The *generalised torsion* of the generalised connection D can be defined as a linear map $T : \Gamma(E) \to \Gamma(\operatorname{ad}(\tilde{F}))$ given by

$$T(V) \cdot \alpha = L^D_V \alpha - L_V \alpha, \tag{3.12}$$

for any $V \in \Gamma(E)$ and where $T(V)$ acts via the adjoint representation on α. Remarkably one finds that the torsion is an element of only particular irreducible representations of $E^* \otimes \operatorname{ad}\tilde{F}$ as listed in table 2. As discussed in [12], these are exactly the representations that appear in the embedding tensor formulation of gauged supergravities [50, 51], including gaugings [52] of the so-called "trombone" symmetry [53].

We can construct a simple example of a generalised connection with torsion as follows. Let ∇ be a conventional torsion-free connection. Given a conformal split frame, it can be lifted to a generalised connection acting on E by taking

$$D^\nabla_M V = \begin{cases} (\nabla_m v^a)\hat{E}_a + \frac{1}{2}(\nabla_m \omega_{ab})\hat{E}^{ab} & \\ \quad + \frac{1}{5!}(\nabla_m \sigma_{a_1\dots a_5})\hat{E}^{a_1\dots a_5} + \frac{1}{7!}(\nabla_m \tau_{a,a_1\dots a_7})\hat{E}^{a,a_1\dots a_7} & \text{for } M = m \\ 0 & \text{otherwise} \end{cases} . \tag{3.13}$$

By construction D^∇ depends on a choice of A, $\tilde{A}$ and Δ used to define the frame as well of ∇. The generalised torsion of D^∇ is then given by

$$T(V) = \mathrm{e}^\Delta \big(- i_v \mathrm{d}\Delta + v \otimes \mathrm{d}\Delta - i_v F + \mathrm{d}\Delta \wedge \omega - i_v \tilde{F} + \omega \wedge F + \mathrm{d}\Delta \wedge \sigma \big), \tag{3.14}$$

using the notation of (D.2). For other examples of generalised connections, with and without torsion, see also [33, 35].

3.4 Generalised G structures

In what follows we will be interested in further refinements of the generalised frame bundle $\tilde{F}$. We define a *generalised G structure* P as a $G \subset E_{d(d)} \times \mathbb{R}^+$ principle sub-bundle of the generalised structure bundle $\tilde{F}$, that is

$$P \subset \tilde{F} \text{ with fibre } G. \tag{3.15}$$

$E_{d(d)}$ group	$\tilde{H}_d$ group	$E \simeq E^*$	$\mathrm{ad}\, P^\perp = \mathrm{ad}\, \tilde{F} / \mathrm{ad}\, P$
$E_{7(7)}$	$\mathrm{SU}(8)$	$\mathbf{28} + \bar{\mathbf{28}}$	$\mathbf{35} + \bar{\mathbf{35}} + \mathbf{1}$
$E_{6(6)}$	$USp(8)$	$\mathbf{27}$	$\mathbf{42} + \mathbf{1}$
$E_{5(5)} \simeq \mathrm{Spin}(5,5)$	$\mathrm{Spin}(5) \times \mathrm{Spin}(5)$	$(\mathbf{4}, \mathbf{4})$	$(\mathbf{5}, \mathbf{5}) + (\mathbf{1}, \mathbf{1})$
$E_{4(4)} \simeq \mathrm{SL}(5, \mathbb{R})$	$\mathrm{Spin}(5)$	$\mathbf{10}$	$\mathbf{14} + \mathbf{1}$

Table 3. Double covers of the maximal compact subgroups of $E_{d(d)}$ and H_d representations of the generalised tangent spaces and coset bundles.

It picks out a special subset of frames that are related by G transformations. Typically one can also define P by giving a set of nowhere vanishing generalised tensors $\{K_{(a)}\}$, invariant under the action of G. By definition, the invariant tensors parametrise, at each point $x \in M$, an element of the coset

$$\{K_{(a)}\}\big|_x \in \frac{E_{d(d)} \times \mathbb{R}^+}{G}. \tag{3.16}$$

A generalised connection D is said to be compatible with the G structure P if it preserves all the invariant tensors

$$DK_{(a)} = 0 \tag{3.17}$$

or, equivalently, if the derivative acts only in the G sub-bundle P.

A special class of generalised G structures are those characterised by the maximal compact subgroup H_d of $E_{d(d)}$. In the next section we shall see how the extra data present in an H_d structure allows one to naturally describe eleven-dimensional supergravity.

4 Supergravity as H_d generalised gravity

We now turn to the main result of this paper. We give a complete rewriting in the language of generalised geometry of the restricted eleven-dimensional supergravity, to leading order in fermions. This will result in a unified formulation which has the larger bosonic symmetries of the theory manifest. Specifically, the local symmetry of the theory is $\mathrm{Spin}(10-d,1) \times \tilde{H}_d$ where $\tilde{H}_d$ is the double-cover of the maximal compact subgroup of $E_{d(d)}$.

4.1 Supergravity degrees of freedom and H_d structures

4.1.1 Bosons

As discussed in [12], the bosonic supergravity fields define a generalised H_d structure P, where H_d is the maximally compact subgroup of $E_{d(d)}$. These, or rather their double covers[2] $\tilde{H}_d$, are listed in table 3.

An H_d structure on $\tilde{F}$ is the direct analogy of a metric structure, where one considers the set of orthonormal frames related by $O(d)$ transformations. The choice of such a

[2]Since the underlying manifold M is assumed to possess a spin structure, we are free to promote to the double cover.

structure is parametrised, at each point on the manifold, by a Riemannian metric g, a three-form A and a six-form $\tilde{A}$ gauge fields, and a scalar Δ, that is

$$\{g, A, \tilde{A}, \Delta\} \in \frac{E_{d(d)} \times \mathbb{R}^+}{H_d}. \tag{4.1}$$

These are precisely the set of bosonic fields in the restricted theory. The corresponding coset representations are listed in table 3.

One can construct elements of the structure bundle $P \subset \tilde{F}$ concretely, that is, identify the analogues of "orthonormal" frames, as follows. It is always possible to choose an H_d frame in a conformal split form (3.4), where now one takes $\hat{e}_a$ to be an orthonormal basis of TM for the metric g. Any other frame is then related by an H_d transformation. (The action of H_d on the generalised tangent space is given explicitly in (D.4) and (E.3).)

As in the Riemannian case, one can also also construct a *generalised metric*, which is invariant under a change of H_d frame. Given $V = V^A \hat{E}_A \in \Gamma(E)$, expanded in an H_d basis, one defines

$$G(V, V) = v^2 + \frac{1}{2!}\omega^2 + \frac{1}{5!}\sigma^2 + \frac{1}{7!}\tau^2, \tag{4.2}$$

where $v^2 = v_a v^a$, $\omega^2 = \omega_{ab}\omega^{ab}$, $\sigma^2 = \sigma_{a_1\dots a_5}\sigma^{a_1\dots a_5}$, $\tau^2 = \tau_{a,a_1\dots a_7}\tau^{a,a_1\dots a_7}$, and indices are contracted using the flat frame metric δ_{ab}. (Note that G allows us to identify $E \simeq E^*$.) It is easy to show, given the transformation (D.4), that this is an H_d invariant, independent of the choice of H_d frame. Thus it can be evaluated in the conformal split representative (3.4) and one sees explicitly that the metric is defined by the fields g, A, $\tilde{A}$ and Δ that determine the coset element. Explicit expressions for the generalised metric in terms of the supergravity fields in the coordinate frame have been worked out, for example, in [30–32]. The fact that there is always a singlet present in the coset representations, as can be seen from table 3, implies that there is always a density that is H_d (and $E_{d(d)}$) invariant, corresponding to the choice of $\mathbb{R}^+$ factor and which we denote as $|\mathrm{vol}_G|$. In a coordinate frame it is given by[3]

$$|\mathrm{vol}_G| = \sqrt{g}\, \mathrm{e}^{(9-d)\Delta}. \tag{4.3}$$

As described in [12, 54], the infinitesimal bosonic symmetry transformation is naturally encoded as the Dorfman derivative by $V \in \Gamma(E)$

$$\delta_V G = L_V G, \tag{4.4}$$

and the algebra of these transformations is given by $[L_U, L_V] = L_{L_U V} = -L_{L_V U} = L_{[\![U,V]\!]}$ where the Courant bracket $[\![U, V]\!]$ is the antisymmetrisation of the Dorfman derivative.

4.1.2 Fermions

The fermionic degrees of freedom form spinor representations of $\tilde{H}_d$, the double cover[4] of H_d [18, 19, 55]. Let S and J denote the bundles associated to the representations of

[3] In general, $|\mathrm{vol}_G|$ can be related to the determinant of the metric by $\det G = |\mathrm{vol}_G|^{-\dim E/(9-d)}$.

[4] Note that, as discussed in appendix B.1, $\tilde{H}_d$ can be defined abstractly for all $d \leq 8$ as the subgroup of $\mathrm{Cliff}(d;\mathbb{R})$ preserving a particular involution of the algebra.

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$\tilde{H}_d$	S	J
SU(8)	$\mathbf{8}+\bar{\mathbf{8}}$	$\mathbf{56}+\bar{\mathbf{56}}$
USp(8)	$\mathbf{8}$	$\mathbf{48}$
Spin(5) × Spin(5)	$(\mathbf{4},\mathbf{1})+(\mathbf{1},\mathbf{4})$	$(\mathbf{4},\mathbf{5})+(\mathbf{5},\mathbf{4})$
Spin(5)	$\mathbf{4}$	$\mathbf{16}$

Table 4. Spinor and gravitino representations in each dimension.

d	$\hat{S}^-$	$\hat{S}^+$	$\hat{J}^-$	$\hat{J}^+$
7	$(\mathbf{2},\mathbf{8})+(\bar{\mathbf{2}},\bar{\mathbf{8}})$	$(\mathbf{2},\bar{\mathbf{8}})+(\bar{\mathbf{2}},\mathbf{8})$	$(\mathbf{2},\mathbf{56})+(\bar{\mathbf{2}},\bar{\mathbf{56}})$	$(\mathbf{2},\bar{\mathbf{56}})+(\bar{\mathbf{2}},\mathbf{56})$
6	$(\mathbf{4},\mathbf{8})$	$(\mathbf{4},\mathbf{8})$	$(\mathbf{4},\mathbf{48})$	$(\mathbf{4},\mathbf{48})$
5	$(\mathbf{4},\mathbf{4},\mathbf{1})+(\bar{\mathbf{4}},\mathbf{1},\mathbf{4})$	$(\mathbf{4},\mathbf{1},\mathbf{4})+(\bar{\mathbf{4}},\mathbf{4},\mathbf{1})$	$(\mathbf{4},\mathbf{4},\mathbf{5})+(\bar{\mathbf{4}},\mathbf{5},\mathbf{4})$	$(\mathbf{4},\mathbf{5},\mathbf{4})+(\bar{\mathbf{4}},\mathbf{4},\mathbf{5})$
4	$(\mathbf{8},\mathbf{4})$	$(\mathbf{8},\mathbf{4})$	$(\mathbf{8},\mathbf{16})$	$(\mathbf{8},\mathbf{16})$

Table 5. Spinor and gravitino as $\text{Spin}(10-d,1)\times\tilde{H}_d$ representations. Note that when d is even the positive and negative representations are actually equivalent.

$\tilde{H}_d$ listed in table 4. The fermion fields ψ, ρ and the supersymmetry parameter ε of the restricted theory are sections

$$\psi\in\Gamma(J),\qquad\qquad \rho\in\Gamma(S),\qquad\qquad \varepsilon\in\Gamma(S). \tag{4.5}$$

However, the restricted fermions also transform as spinors of the flat $\mathbb{R}^{10-d,1}$ space. As discussed in section 2, the simplest formulation is to view them as eleven-dimensional spinors and use the embedding $\text{Spin}(10-d,1)\times\tilde{H}_d\subset\text{Cliff}(10,1;\mathbb{R})$ described in appendix E.3[5]. This will allow us to write expressions directly comparable to the ones in section 2. There is a complication, in that there are actually two distinct ways of realising the action of $\tilde{H}_d$ on the $\text{Cliff}(10,1;\mathbb{R})$ spinor bundle $\hat{S}$, related by a change of sign of the gamma matrices. Given $\chi^\pm\in\Gamma(\hat{S})$ and $N\in\Gamma(\text{ad}\,P)$ we have the two actions

$$N\cdot\hat{\chi}^\pm=\frac{1}{2}\left(\frac{1}{2!}n_{ab}\Gamma^{ab}\pm\frac{1}{3!}b_{abc}\Gamma^{abc}-\frac{1}{6!}\tilde{b}_{a_1\dots a_6}\Gamma^{a_1\dots a_6}\right)\hat{\chi}^\pm. \tag{4.6}$$

If one denotes as $\hat{S}^\pm$ the bundle of spinors transforming under the two actions, one finds, for even d, that the two representations are equivalent, and $\hat{S}\simeq\hat{S}^+\simeq\hat{S}^-$. However for odd d they are distinct and the spinor bundle decomposes $\hat{S}\simeq\hat{S}^+\oplus\hat{S}^-$. The same applies to spin-$\frac{3}{2}$ bundles $\hat{J}^\pm$. The $\text{Spin}(10-d,1)\times\tilde{H}_d$ representations of the corresponding four bundles listed in table 5 (see also [56]).

[5]The alternative is to decompose the eleven-dimensional spinors which necessarily leads to dimension dependent expressions, as can be seen from appendix C. That approach is therefore better suited for the explicit constructions we will be examining in the next section. For now we maintain the discussion completely general.

Finally, we find that the supergravity fields of section 2 can be identified as follows,

$$\begin{aligned}\hat{\varepsilon}^- &= \mathrm{e}^{-\Delta/2}\,\varepsilon^{\text{sugra}} &&\in \Gamma(\hat{S}^-),\\ \hat{\rho}^+ &= \mathrm{e}^{\Delta/2}\,\rho^{\text{sugra}} &&\in \Gamma(\hat{S}^+),\\ \hat{\psi}^-_a &= \mathrm{e}^{\Delta/2}\,\psi^{\text{sugra}}_a &&\in \Gamma(\hat{J}^-).\end{aligned} \tag{4.7}$$

Note that, due to the warping of the metric, the precise maps between the fermion fields as viewed in the geometry and in the supergravity description involve a conformal rescaling. This is of course purely conventional, since one could just as easily perform field redefinitions at the supergravity level. We chose, however, to maintain the conventions in section 2 as familiar as possible and make the identification at this point.

4.2 Supergravity operators

The differential operators and curvatures that appear in the supergravity equations will be built out of generalised connections D which are simultaneously torsion-free and H_d compatible, in analogy to the Levi-Civita connection. Recall that a generalised connection is said to be compatible with the H_d structure if $DG = 0$ or, equivalently, if the derivative acts only in the H_d sub-bundle. We proved in [12] that there always exists such a torsion-free, metric compatible connection but, unlike the Levi-Civita connection, it is not unique.

To see this, let ∇ be the Levi-Civita connection for the metric g and D^∇ its lift to an action on E as in (3.13). Since ∇ is compatible with the $O(d) \subset H_d$ subgroup, it is necessarily an H_d-compatible connection. However, D^∇ is not torsion-free, as can be seen from (3.14). To construct a torsion-free compatible connection one simply modifies D^∇. A generic generalised connection D can always be written as

$$D_M W^A = D^\nabla_M W^A + \Sigma_M{}^A{}_B W^B. \tag{4.8}$$

If D is compatible with the H_d structure then

$$\Sigma \in \Gamma(E^* \otimes \operatorname{ad} P), \tag{4.9}$$

that is, it is a generalised covector taking values in the adjoint of H_d. In [12] we showed that one can always find a suitable Σ such that the torsion of D vanishes, but the solution is not unique. Contracting with $V \in \Gamma(E)$ so $\Sigma(V) \in \Gamma(\operatorname{ad} P)$ and using the basis for the adjoint of H_d given in (E.2) and (E.3), one finds that in a conformal split frame

$$\begin{aligned}\Sigma(V)_{ab} &= \mathrm{e}^{\Delta}\left(\frac{2(7-d)}{d-1} v_{[a}\partial_{b]}\Delta + \frac{1}{4!}\omega_{cd}F^{cd}{}_{ab} + \frac{1}{7!}\sigma_{c_1\dots c_5}\tilde{F}^{c_1\dots c_5}{}_{ab} + Q(V)_{ab}\right),\\ \Sigma(V)_{abc} &= \mathrm{e}^{\Delta}\left(\frac{6}{(d-1)(d-2)}(\mathrm{d}\Delta\wedge\omega)_{abc} + \frac{1}{4}v^d F_{dabc} + Q(V)_{abc}\right),\\ \Sigma(V)_{a_1\dots a_6} &= \mathrm{e}^{\Delta}\left(\frac{1}{7}v^b\tilde{F}_{ba_1\dots a_6} + Q(V)_{a_1\dots a_6}\right),\end{aligned} \tag{4.10}$$

where $Q \in \Gamma(E^* \otimes \operatorname{ad} P)$ is the undetermined part of the connection — it projects to zero under the map to the torsion representations. Clearly, requiring metricity and vanishing torsion is not enough to specify a single generalised connection.

Although D is ambiguous, one can define projections of D which result in unique operators. We identified four such maps in [12], and they turn out to be directly related to the representations of the fermion fields. Since we are interested in comparing with the supergravity expressions, we can take the embedding (E.12) and consider the natural action of D on the $\mathrm{Spin}(10-d,1) \times \tilde{H}_d$ representations listed in table 5. Following the notation of (E.17) we define the projected operators

$$\begin{aligned} \not{D} &: \hat{S}^{\pm} \to \hat{S}^{\mp}, & \not{D} &: \hat{J}^{\pm} \to \hat{J}^{\mp}, \\ D \curlywedge &: \hat{S}^{\pm} \to \hat{J}^{\pm}, & D \curlyvee &: \hat{J}^{\pm} \to \hat{S}^{\pm}. \end{aligned} \tag{4.11}$$

We can check that they are indeed independent of Q by decomposing under $\mathrm{Spin}(d) \subset \tilde{H}_d$ and taking the torsion-free connection (4.10). Using the formulae for the projections given in (E.18) and (E.19), and already applying the operators to the supersymmetry parameter $\hat{\varepsilon}^-$ in (4.7), we then find

$$\begin{aligned} \not{D}\hat{\varepsilon}^- &= \mathrm{e}^{\Delta/2}\left(\not{\nabla} + \frac{9-d}{2}(\not{\partial}\Delta) - \frac{1}{4}\not{F} - \frac{1}{4}\not{\tilde{F}}\right)\varepsilon^{\text{sugra}}, \\ (D \curlywedge \hat{\varepsilon}^-)_a &= \mathrm{e}^{\Delta/2}\bigg(\nabla_a + \frac{1}{288}(\Gamma_a{}^{b_1\dots b_4} - 8\delta_a{}^{b_1}\Gamma^{b_2b_3b_4})F_{b_1\dots b_4} \\ &\qquad - \frac{1}{12}\frac{1}{6!}\tilde{F}_{ab_1\dots b_6}\Gamma^{b_1\dots b_6}\bigg)\varepsilon^{\text{sugra}}. \end{aligned} \tag{4.12}$$

From derivatives of elements $\Gamma(\hat{J}^{\pm})$ we obtain the second set of unique operators which using (E.20) and (E.21) as applied to $\hat{\psi}^-$ of (4.7), take the form

$$\begin{aligned} D \curlyvee \hat{\psi}^- &= \mathrm{e}^{3\Delta/2}\bigg[\nabla^a - \frac{1}{10-d}\Gamma^{ab}\nabla_b + (10-d)\partial^a\Delta - \Gamma^{ab}\partial_b\Delta \\ &\qquad - \frac{1}{4}\frac{1}{3!}F^a{}_{b_1b_2b_3}\Gamma^{b_1b_2b_3} + \frac{1}{4}\frac{1}{10-d}\frac{1}{4!}\Gamma^a{}_{b_1\dots b_4}F^{b_1\dots b_4} \\ &\qquad - \frac{1}{4}\frac{1}{6!}\tilde{F}^a{}_{b_1\dots b_6}\Gamma^{b_1\dots b_6}\bigg]\psi_a^{\text{sugra}}, \\ \left(\not{D}\hat{\psi}^-\right)_a &= -\mathrm{e}^{3\Delta/2}\bigg[\Gamma^c\left(\nabla_c + \frac{11-d}{2}\partial_c\Delta\right)\delta_a{}^b + \frac{2}{9-d}\Gamma^b\left(\nabla_a + \frac{11-d}{2}\partial_a\Delta\right) \\ &\qquad - \frac{1}{12}\left(3 + \frac{2}{9-d}\right)\not{F}\delta_a{}^b + \frac{1}{3}\frac{10-d}{9-d}\frac{1}{2!}F_a{}^b{}_{cd}\Gamma^{cd} \\ &\qquad - \frac{1}{3}\frac{1}{9-d}\frac{1}{3!}F_a{}^{c_1\dots c_3}\Gamma^b{}_{c_1\dots c_3} + \frac{1}{6}\frac{10-d}{9-d}\frac{1}{3!}F^{bc_1\dots c_3}\Gamma_{ac_1\dots c_3} \\ &\qquad - \frac{1}{6}\frac{1}{9-d}\frac{1}{4!}F_{c_1\dots c_4}\Gamma_a{}^{bc_1\dots c_4} + \frac{1}{4}\frac{1}{5!}\tilde{F}_a{}^b{}_{c_1\dots c_5}\Gamma^{c_1\dots c_5}\bigg]\psi_b^{\text{sugra}}. \end{aligned} \tag{4.13}$$

These four operators, all constructed from the same connection, will now enable us to rewrite all the supergravity equations of section 2.2.

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4.3 Supergravity equations from generalised geometry

4.3.1 Supersymmetry algebra

Comparing with (2.21), we immediately see that the operators (4.11) give precisely the supersymmetry variations of the two fermion fields

$$
\begin{aligned}
\delta\hat{\psi}^- &= D \curlywedge \hat{\varepsilon}^-, \\
\delta\hat{\rho}^+ &= \not{D}\hat{\varepsilon}^-.
\end{aligned} \tag{4.14}
$$

Since the bosons arrange themselves into the generalised metric, one expects that their supersymmetry variations (2.22) are given by the variation of G. In fact, the most convenient object to consider is $G^{-1}\delta G$ which is naturally a section of the bundle $\mathrm{ad}(P)^\perp$, listed in table 3. One has the isomorphism (E.5)

$$
\mathrm{ad}(P)^\perp \simeq \mathbb{R} \oplus S^2T^*M \oplus \Lambda^3T^*M \oplus \Lambda^6T^*M \tag{4.15}
$$

and we can identify the component variations of the generalised metric, as written in the split frame, as

$$
\begin{aligned}
(G^{-1}\delta G) &= -2\delta\Delta, \\
(G^{-1}\delta G)_{ab} &= \delta g_{ab} \\
(G^{-1}\delta G)_{abc} &= -\delta A_{abc}, \\
(G^{-1}\delta G)_{a_1\dots a_6} &= -\delta\tilde{A}_{a_1\dots a_6}.
\end{aligned} \tag{4.16}
$$

One finds that the supersymmetry variations of the bosons (2.22) can be written in the $\tilde{H}_d$ covariant form

$$
G^{-1}\delta G = (\hat{\psi}^- \times_{\mathrm{ad}\, P^\perp} \hat{\varepsilon}^-) + (\hat{\rho}^+ \times_{\mathrm{ad}\, P^\perp} \hat{\varepsilon}^-), \tag{4.17}
$$

where $\times_{\mathrm{ad}\, P^\perp}$ denotes the projection to $\mathrm{ad}(P)^\perp$ given in (E.15) and (E.16).

4.3.2 Generalised curvatures and the equations of motion

To realise the fermionic equations of motion one uses the unique projections (4.11). We can then formulate the two equations (2.19) and (2.20) as, respectively,

$$
\begin{aligned}
-\not{D}\hat{\psi}^- - \frac{11-d}{9-d} D \curlywedge \hat{\rho}^+ &= 0, \\
-\not{D}\hat{\rho}^+ - D \curlyvee \hat{\psi}^- &= 0.
\end{aligned} \tag{4.18}
$$

Note that $\hat{\rho}^+$ is embedded with a different conformal factor to $\hat{\varepsilon}^-$ and also is a section of $\hat{S}^+$ rather than $\hat{S}^-$. This means we have

$$
\begin{aligned}
\not{D}\hat{\rho}^+ &= -\mathrm{e}^{3\Delta/2}\left(\not{\nabla} + \frac{11-d}{2}(\not{\partial}\Delta) + \frac{1}{4}\not{F} - \frac{1}{4}\not{\tilde{F}}\right)\rho^{\mathrm{sugra}} \\
(D \curlywedge \hat{\rho}^+)_a &= \mathrm{e}^{3\Delta/2}\Big[(\nabla_a + \partial_a\Delta) - \frac{1}{288}(\Gamma_a{}^{b_1\dots b_4} - 8\delta_a{}^{b_1}\Gamma^{b_2b_3b_4})F_{b_1\dots b_4} \\
&\qquad\qquad - \frac{1}{12}\frac{1}{6!}\tilde{F}_{ab_1\dots b_6}\Gamma^{b_1\dots b_6}\Big]\rho^{\mathrm{sugra}}
\end{aligned} \tag{4.19}
$$

From these we can find the generalised Ricci tensor R_{AB}, following [12]. Recall that the supersymmetric variation of the fermionic equations of motion vanishes up to the bosonic equations of motion (4.18). Anticipating that the bosonic equations of motion will correspond to $R_{AB} = 0$, one way to define generalised Ricci tensor is via the variation of (4.18) under (4.14). By construction this gives R_{AB} as a section of $\operatorname{ad} P^{\perp} \subset E^* \otimes E^*$, the same space as variations of the generalised metric δG, in complete analogy to the conventional metric and Ricci tensor. Defining R_{AB} as an $\tilde{H}_d$ tensor we write

$$
\begin{aligned}
-\not{D}(D \curlywedge \hat{\varepsilon}^-) - \frac{11-d}{9-d} D \curlywedge (\not{D}\hat{\varepsilon}^-) &= R^0 \cdot \hat{\varepsilon}^-, \\
D \curlyvee (D \curlywedge \hat{\varepsilon}^-) + \not{D}(\not{D}\hat{\varepsilon}^-) &= R\,\hat{\varepsilon}^-,
\end{aligned}
\tag{4.20}
$$

for any $\hat{\varepsilon}^- \in \Gamma(\hat{S}^-)$ and where R and R^0_{AB} are the scalar and non-scalar parts of R_{AB} respectively. The action of R^0_{AB} on $\hat{\varepsilon}^-$ that appears of the right-hand side of (4.20) is given explicitly in (E.11).

In components, using the notation of (E.5), we find

$$
\begin{aligned}
R &= \mathrm{e}^{2\Delta}\left[\mathcal{R} - 2(c-1)\nabla^2\Delta - c(c-1)(\partial\Delta)^2 - \frac{1}{2}\frac{1}{4!}F^2 - \frac{1}{2}\frac{1}{7!}\tilde{F}^2\right] \\
R_{ab} &= \mathrm{e}^{2\Delta}\Bigg[\mathcal{R}_{ab} - c\nabla_a\nabla_b\Delta - c(\partial_a\Delta)(\partial_b\Delta) \\
&\qquad - \frac{1}{2}\frac{1}{4!}\left(4F_{ac_1c_2c_3}F_b{}^{c_1c_2c_3} - \frac{1}{3}g_{ab}F^2\right) \\
&\qquad - \frac{1}{2}\frac{1}{7!}\left(7\tilde{F}_{ac_1\dots c_6}\tilde{F}_b{}^{c_1\dots c_6} - \frac{2}{3}g_{ab}\tilde{F}^2\right)\Bigg], \\
R_{abc} &= \frac{1}{2}\mathrm{e}^{2\Delta} * \left[\mathrm{e}^{-c\Delta}\mathrm{d} * (\mathrm{e}^{c\Delta}F) - F \wedge *\tilde{F}\right]_{abc}, \\
R_{a_1\dots a_6} &= \frac{1}{2}\mathrm{e}^{2\Delta} * \left[\mathrm{e}^{-c\Delta}\mathrm{d} * (\mathrm{e}^{c\Delta}\tilde{F})\right]_{a_1\dots a_6},
\end{aligned}
\tag{4.21}
$$

where $c = 11 - d$. The generalised Ricci tensor is manifestly uniquely determined and comparing with (2.17) we see that the bosonic equations of motion become simply

$$
R_{AB} = 0. \tag{4.22}
$$

The bosonic action (2.16) is given by the generalised curvature scalar, integrated with the volume form (4.3)

$$
S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int |\mathrm{vol}_G|\, R. \tag{4.23}
$$

Finally, the fermionic action can be written using the natural invariant pairings of the terms in (4.18) with the fermionic fields. Using the expressions (E.14) and (E.13) for the spinor bilinears, we find that (2.18) can be rewritten as

$$
\begin{aligned}
S_{\mathrm{F}} = \frac{1}{\kappa^2}\int |\mathrm{vol}_G| \Bigg[&-\langle\hat{\psi}^-, \not{D}\hat{\psi}^-\rangle - \frac{c}{c-2}\langle\hat{\psi}^-, D \curlywedge \hat{\rho}^+\rangle \\
&+ \frac{c(c-1)}{(c-2)^2}\langle\hat{\rho}^+, D \curlyvee \hat{\psi}^-\rangle + \frac{c(c-1)}{(c-2)^2}\langle\hat{\rho}^+, \not{D}\hat{\rho}^+\rangle\Bigg].
\end{aligned}
\tag{4.24}
$$

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5 Explicit H_d constructions

In the previous section, we gave the generic construction of the supergravity in terms of generalised geometry, valid in all $d \leq 7$. The theory has a local $\tilde{H}_d$ symmetry, however this might not be not totally explicit since we used a Cliff$(10,1;\mathbb{R})$ formulation for the fermionic fields.

For completeness, we now demonstrate for two examples, in $d=4$ and $d=7$, how one can write our expressions with indices which transform directly under the manifest local Spin(5) and SU(8) symmetries of the previous section. Correspondingly, in this section we treat the fermions slightly differently from the previous ones. Whereas before we kept all spinors as Cliff$(10,1;\mathbb{R})$ objects, we now want to make their $\tilde{H}_d$ nature more explicit. In order to make this possible, one has to decompose the eleven-dimensional spinors following the procedures outlined in appendix C and embed the Cliff$(d;\mathbb{R})$ expressions into $\tilde{H}_d$ representations, according to appendix E.1. We will then keep the external spinor indices of the fermion fields hidden and treat them as sections of the genuine $\tilde{H}_d$ bundles S and J.

5.1 $d=4$ and $\tilde{H}_4 = \mathrm{Spin}(5)$

5.1.1 $\mathrm{GL}^+(5,\mathbb{R})$ generalised geometry

In four dimensions, we have $E_{4(4)} \times \mathbb{R}^+ \simeq \mathrm{SL}(5,\mathbb{R}) \times \mathbb{R}^+ \simeq \mathrm{GL}^+(5,\mathbb{R})$. We can then write the generalised geometry explicitly in terms of indices $i,j,k,\dots = 1,\dots,5$ transforming under $\mathrm{GL}^+(5,\mathbb{R})$.

Generalised vectors V transform in the antisymmetric **10** representation. We can introduce a basis $\{\hat{E}_{ii'}\}$ (locally a section of the generalised structure bundle $\tilde{F}$) transforming under $\mathrm{GL}^+(5,\mathbb{R})$ so that

$$V = \frac{1}{2} V^{ii'} \hat{E}_{ii'}. \tag{5.1}$$

In the conformal split frame (3.4), we can identify [15, 38]

$$\begin{aligned} \hat{E}_{a5} &= \mathrm{e}^{\Delta} \left(\hat{e}_a + i_{\hat{e}_a} A \right), \\ \hat{E}_{ab} &= \frac{1}{2} \mathrm{e}^{\Delta} \epsilon_{abcd} e^{cd}, \end{aligned} \tag{5.2}$$

where ϵ is the numerical totally antisymmetric symbol. Equivalently

$$\begin{aligned} V^{a5} &= v^a, \\ V^{ab} &= \frac{1}{2} \epsilon^{abcd} \omega_{cd}, \end{aligned} \tag{5.3}$$

where v^a and ω_{ab} are as in (3.6). In this frame the partial derivative (3.9) $\partial_{ii'}$ has the form

$$\begin{aligned} \partial_{a5} &= \frac{1}{2} \mathrm{e}^{\Delta} \partial_a, \\ \partial_{ab} &= 0. \end{aligned} \tag{5.4}$$

Note that there is also a generalised tensor bundle W which transforms in the fundamental **5** representation of $\mathrm{GL}^+(5,\mathbb{R})$. One finds

$$W \simeq (\det T^*M)^{1/2} \otimes (TM \oplus \det TM), \tag{5.5}$$

and a choice of basis $\{\hat{E}_{ii'}\}$ defines a basis $\{\hat{E}_i\}$ of W where $K = K^i\hat{E}_i \in \Gamma(W)$, such that

$$\hat{E}_{ii'} = \hat{E}_i \wedge \hat{E}_{i'}, \tag{5.6}$$

since $E \simeq \Lambda^2 W$, and where we use the four-dimensional isomorphism $\det T^*M \otimes \Lambda^2 TM \simeq \Lambda^2 T^*M$.

With this notation we can then use the $\mathrm{GL}^+(5,\mathbb{R})$ adjoint action explicitly to write the Dorfman derivative (3.8) of a generalised vector. It takes its simplest form in the coordinate frame (3.2), where it reads

$$L_V W^{ij} = V^{kk'}\partial_{kk'}W^{ij} + 4\left(\partial_{kk'}V^{k[i}\right)W^{j]k'} + \left(\partial_{kk'}V^{kk'}\right)W^{ij}. \tag{5.7}$$

This form of the $d = 4$ Dorfman derivative was given, without the $\mathbb{R}^+$ action, in [38].[6] We can then write a generic generalised connection as

$$D_{ii'}V^{jj'} = \partial_{ii'}V^{jj'} + \Omega_{ii'}{}^{j}{}_{k}V^{kj'} + \Omega_{ii'}{}^{j'}{}_{k}V^{jk}, \tag{5.8}$$

where the j and k indices of $\Omega_{ii'}{}^{j}{}_{k}$ parametrise an element of the adjoint of $\mathrm{GL}^+(5,\mathbb{R})$.

5.1.2 Spin(5) structures and supergravity

In four dimensions $H_d \simeq \mathrm{SO}(5)$ and we define the sub-bundle $P \subset \tilde{F}$ of SO(5) frames as the set of frames where the generalised metric (4.2) can be written as

$$G(V,W) = \frac{1}{2}\delta_{ij}\delta_{i'j'}V^{ii'}W^{jj'}, \tag{5.9}$$

where δ_{ij} is the flat SO(5) metric with which we can raise and lower indices frame indices. Equivalently we can think of the generalised metric as defining orthonormal frames on the **5**-representation bundle W.

Upon decomposing the fermionic fields of the supergravity according to C.1, one finds that they embed into the spinor and traceless vector-spinor representations of Spin(5). Our conventions regarding $\mathrm{Cliff}(4;\mathbb{R})$ and $\mathrm{Cliff}(5;\mathbb{R})$ algebras are given in appendix B.2.2 and we leave Spin(5) spinor indices implicit throughout. We define

$$\begin{aligned}
\varepsilon &= \mathrm{e}^{-\Delta/2}\varepsilon^{\text{sugra}} && \in \Gamma(S),\\
\rho &= \mathrm{e}^{\Delta/2}\gamma^{(4)}\rho^{\text{sugra}} && \in \Gamma(S),\\
\psi_i &= \begin{cases} \mathrm{e}^{\Delta/2}\gamma^{(4)}\left(\delta^b{}_a - \frac{2}{5}\gamma_a\gamma^b\right)\psi_b^{\text{sugra}} & \text{for } i = a\\ -\frac{3}{5}\mathrm{e}^{\Delta/2}\gamma^a\psi_a^{\text{sugra}} & \text{for } i = 5\end{cases} && \in \Gamma(J).
\end{aligned} \tag{5.10}$$

Crucially, note the appearance of conformal factors in the definitions, in similar fashion to (4.7). Recall also that in four dimensions we have $S \simeq S^+ \simeq S^-$, where the action by $\gamma^{(4)}$ in the second line of (5.10) realises the second isomorphism.

[6] For the antisymmetrisation of $L_V W$ (which is simply the Courant bracket for two-forms [2]) in $\mathrm{SL}(5,\mathbb{R})$ indices see also [57, 58].

A generalised connection is compatible with the generalised metric (5.9) if $DG = 0$. In terms of the connection (5.8) in frame indices this implies

$$\Omega_{ii'\,jj'} = -\Omega_{ii'\,j'j} \tag{5.11}$$

where indices are lowered using the SO(5) metric δ_{ij}. For such SO(5)-connections, we can define the generalised spinor derivative, given $\chi \in \Gamma(S)$

$$D_{ii'}\chi = \left(\partial_{ii'} + \frac{1}{4}\Omega_{ii'jj'}\hat{\gamma}^{jj'}\right)\chi. \tag{5.12}$$

An example of such a generalised connection is the one (3.13) defined by the Levi-Civita connection ∇, where, acting on $\chi \in \Gamma(S)$, we have

$$D^{\nabla}_{ii'}\chi = \begin{cases} \frac{1}{2}\mathrm{e}^{\Delta}\left(\partial_a + \frac{1}{4}\omega_{a\,bc}\hat{\gamma}^{bc}\right)\chi & \text{if } i = a \text{ and } i' = 5 \\ 0 & \text{if } i = a \text{ and } i' = b \end{cases}, \tag{5.13}$$

where $\omega_{a\,bc}$ is the usual spin-connection.

We can construct a torsion-free compatible connection D, by shifting D^{∇} by an additional connection piece $\Sigma_{[ii'][jj']}$, such that its action on $\chi \in \Gamma(S)$ is given by

$$D_{ii'}\chi = D^{\nabla}_{ii'}\chi + \frac{1}{4}\Sigma_{ii'jj'}\hat{\gamma}^{jj'}\chi. \tag{5.14}$$

The connection is torsion-free if

$$\Sigma_{ii'jj'} = \frac{1}{2}\left(\delta_{j[i}\Sigma_{i']j'} - \delta_{j'[i}\Sigma_{i']j}\right) + Q_{ii'jj'}, \tag{5.15}$$

where $Q_{ii'jj'}$ is the undetermined part — traceless and symmetric under exchange of pairs of indices, so it transforms in the **35** of SO(5), see [12] — and

$$\begin{aligned} \Sigma_{a5} &= -\Sigma_{5a} = -2\mathrm{e}^{\Delta}\partial_a\Delta, \\ \Sigma_{ab} &= \frac{1}{12}\mathrm{e}^{\Delta}F\delta_{ab}, \\ \Sigma_{55} &= -\frac{7}{12}\mathrm{e}^{\Delta}F, \end{aligned} \tag{5.16}$$

with $F = \frac{1}{4!}\epsilon^{abcd}F_{abcd}$. The projections (4.11) can be written in Spin(5) indices as

$$\begin{aligned} \not{D}\varepsilon &= -\hat{\gamma}^{ij}D_{ij}\varepsilon, \\ (D \curlywedge \varepsilon)_i &= 2\left(\hat{\gamma}^{j}D_{ij}\varepsilon - \frac{1}{5}\hat{\gamma}_i\hat{\gamma}^{jj'}D_{jj'}\varepsilon\right), \\ (\not{D}\psi)_i &= -\hat{\gamma}^{jj'}D_{jj'}\psi_i + \frac{12}{5}D_{ij}\psi^j - \frac{8}{5}\hat{\gamma}_i{}^{j}D_{jj'}\psi^{j'}, \\ D \curlyvee \psi &= -\frac{5}{3}\hat{\gamma}^{i}D_{ij}\psi^j. \end{aligned} \tag{5.17}$$

and are unique, independent of $Q_{ii'jj'}$.

The supersymmetry variations of the fermions (4.14) can then be written in a manifestly Spin(5) covariant form

$$\begin{aligned} \delta\psi_i &= (D \curlywedge \varepsilon)_i = 2\left(\hat{\gamma}^j D_{ij}\varepsilon - \frac{1}{5}\hat{\gamma}_i\hat{\gamma}^{jj'}D_{jj'}\varepsilon\right), \\ \delta\rho &= \not{D}\varepsilon = -\hat{\gamma}^{ij}D_{ij}\varepsilon, \end{aligned} \tag{5.18}$$

whereas the variation of the bosons (4.17) is given by

$$\delta G_{[ii'][jj']} = \frac{1}{2}\left(\delta H_{i[j}\delta_{j']i'} - \delta H_{i'[j}\delta_{j']i}\right), \tag{5.19}$$

with

$$\delta H_{ij} = -2\bar{\varepsilon}\hat{\gamma}_{(i}\psi_{j)} - \frac{1}{5}\delta_{ij}\bar{\varepsilon}\rho. \tag{5.20}$$

Turning to the equations of motion, from (4.18), we find that the fermionic equations take the form

$$\begin{aligned} -\frac{14}{5}\left(\hat{\gamma}^j D_{ij}\rho - \frac{1}{5}\hat{\gamma}_i\hat{\gamma}^{jj'}D_{jj'}\rho\right) - \hat{\gamma}^{jj'}D_{jj'}\psi_i + \frac{12}{5}D_{ij}\psi^j - \frac{8}{5}\hat{\gamma}_i{}^j D_{jj'}\psi^{j'} &= 0, \\ \hat{\gamma}^{ij}D_{ij}\rho + \frac{5}{3}\hat{\gamma}^i D_{ij}\psi^j &= 0. \end{aligned} \tag{5.21}$$

The generalised Ricci tensor (4.20), after some rearrangement and gamma matrix algebra, can be written as

$$\begin{aligned} R^0_{ij}\hat{\gamma}^j\varepsilon &= \frac{4}{5}\hat{\gamma}^j\left[D_{ik}, D_j{}^k\right]\varepsilon - 2\hat{\gamma}^{jkl}\left[D_{ij}, D_{kl}\right]\varepsilon - \frac{56}{25}\hat{\gamma}_i{}^{jk}\left[D_{jl}, D_k{}^l\right]\varepsilon \\ &\quad - \frac{16}{5}\hat{\gamma}^{jkl}D_{[ij}D_{kl]}\varepsilon + \frac{8}{5}\hat{\gamma}_i{}^{j_1\dots j_4}D_{[j_1j_2}D_{j_3j_4]}\varepsilon, \\ \frac{5}{24}R\varepsilon &= \frac{5}{3}\hat{\gamma}^{ii'jj'}D_{ii'}D_{jj'}\varepsilon - \frac{5}{3}\hat{\gamma}^{ij}[D_{ik}, D_j{}^k]\varepsilon. \end{aligned} \tag{5.22}$$

Note that in this form one can clearly see that the curvatures cannot be obtained simply from the commutator of two generalised covariant derivatives. Instead, one must consider additional terms resulting from a specific symmetric projection of the connections, as observed in section **3.3** of [12].

The bosonic action (4.23) is

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int |\mathrm{vol}_G|\, R. \tag{5.23}$$

While the fermionic action (4.24) can be written as

$$\begin{aligned} S_{\mathrm{F}} = \frac{1}{\kappa^2}\int |\mathrm{vol}_G| \Bigg(&- \bar{\psi}^i\left(-\hat{\gamma}^{jk}D_{jk}\psi_i + \frac{12}{5}D_{ij}\psi^j - \frac{8}{5}\hat{\gamma}_i{}^j D_{jk}\psi^k\right) \\ &- \frac{14}{5}\bar{\psi}^i(\hat{\gamma}^j D_{ij}\rho - \frac{1}{5}\hat{\gamma}_i\hat{\gamma}^{jj'}D_{jj'}\rho) \\ &- \frac{14}{5}(\bar{\rho}\hat{\gamma}^i D_{ij}\psi^j) - \frac{42}{25}(\bar{\rho}\hat{\gamma}^{ij}D_{ij}\rho)\Bigg), \end{aligned} \tag{5.24}$$

where we use the Spin(5) covariant spinor conjugate (see appendix A). It is also important to note that there are two sets of suppressed indices on the spinors in this expression. These are the SU(2) indices for the five-dimensional symplectic Majorana spinors and the external Spin(6, 1) indices, which must be summed over. For full details of the spinor conventions used, see appendices B.2.2 and C.1.

We have now rewritten all of the supergravity equations with manifest Spin(5) symmetry following the prescription of section 4.3.

5.2 $d = 7$ and $\tilde{H}_7 = \mathrm{SU}(8)$

5.2.1 $E_{7(7)} \times \mathbb{R}^+$ generalised geometry

We follow the standard approach [18] of describing $E_{7(7)}$ in terms of its SL(8, $\mathbb{R}$) subgroup, following the notation of [16][7]. We denote indices transforming under SL(8, $\mathbb{R}$) by $i, j, k, \dots = 1, \dots, 8$.

Generalised vectors transform in the **56** representation of $E_{7(7)}$, which under SL(8, $\mathbb{R}$) decomposes into the sum $\mathbf{28} + \mathbf{28}'$ of bivectors and two-forms. We can introduce a basis $\{\hat{E}_{ii'}, \check{E}^{ii'}\}$ transforming under $E_{7(7)}$ and write a generalised vector as

$$V = \frac{1}{2} V^{ii'} \hat{E}_{ii'} + \frac{1}{2} \tilde{V}_{ii'} \check{E}^{ii'}. \tag{5.25}$$

In the conformal split frame (3.4), we can identify

$$\begin{aligned} V^{a8} &= v^a, & V^{ab} &= \frac{1}{5!} \epsilon^{abc_1 \dots c_5} \sigma_{c_1 \dots c_5}, \\ \tilde{V}_{a8} &= \frac{1}{7!} \epsilon^{b_1 \dots b_7} \tau_{a, b_1 \dots b_7}, & \tilde{V}_{ab} &= \omega_{ab}, \end{aligned} \tag{5.26}$$

where v^a, ω_{ab}, etc. are as in (3.6), with the obvious corresponding identification of $\hat{E}_{a8}$ etc. The partial derivative ∂_μ is lifted into E^*, with a conformal factor due to the form of the conformal split frame, as

$$\partial_{a8} = \frac{1}{2} \mathrm{e}^{\Delta} \partial_a, \qquad \partial_{ab} = 0, \qquad \tilde{\partial}^{ii'} = 0. \tag{5.27}$$

In this notation, the Dorfman derivative (3.8), the antisymmetrisation of which is the "exceptional Courant bracket" of [16], can then be written in the coordinate frame (3.2) as

$$\begin{aligned} (L_V W)^{ii'} &= V^{jj'} \partial_{jj'} W^{ii'} + 4 W^{j[i} \partial_{jj'} V^{i']j'} \\ &\quad + W^{ii'} \partial_{jj'} V^{jj'} - \frac{1}{4} \epsilon^{ii'jj'kk'll'} \tilde{W}_{jj'} \partial_{kk'} \tilde{V}_{ll'}, \\ (L_V W)_{ii'} &= V^{jj'} \partial_{jj'} \tilde{W}_{ii'} - 4 \tilde{W}_{j[i} \partial_{i']j'} V^{jj'} - 6 W^{jj'} \partial_{[jj'} \tilde{V}_{ii']}, \end{aligned} \tag{5.28}$$

where $\epsilon^{i_1 \dots i_8}$ is the totally antisymmetric symbol preserved by SL(8, $\mathbb{R}$).

[7] Note however that when it comes to spinors, here we take instead $\gamma^{(7)} = -\mathrm{i}$, the opposite choice to that in [16], and we also use a different normalisation of our SU(8) indices.

A generic $E_{7(7)}\times\mathbb{R}^+$ generalised connection $D = (D_{ii'}, \tilde{D}^{ii'})$ acting on $V \in \Gamma(E)$ takes the form

$$\begin{aligned}
D_{ii'}V^{jj'} &= \partial_{ii'}V^{jj'} + \Omega_{ii'}{}^{j}{}_{k}V^{kj'} + \Omega_{ii'}{}^{j'}{}_{k}V^{jk} + {*\Omega_{ii'}}{}^{jj'kk'}\tilde{V}_{kk'}, \\
D_{ii'}\tilde{V}_{jj'} &= \partial_{ii'}\tilde{V}_{jj'} - \Omega_{ii'}{}^{k}{}_{j}\tilde{V}_{kj'} - \Omega_{ii'}{}^{k}{}_{j'}\tilde{V}_{jk} + \Omega_{ii'\,jj'kk'}V^{kk'}, \\
\tilde{D}^{ii'}V^{jj'} &= \tilde{\partial}^{ii'}V^{jj'} + \tilde{\Omega}^{ii'\,j}{}_{k}V^{kj'} + \tilde{\Omega}^{ii'\,j'}{}_{k}V^{jk} + {*\tilde{\Omega}}^{ii'\,jj'kk'}\tilde{V}_{kk'}, \\
\tilde{D}^{ii'}\tilde{V}_{jj'} &= \tilde{\partial}^{ii'}\tilde{V}_{jj'} - \tilde{\Omega}^{ii'\,k}{}_{j}\tilde{V}_{kj'} - \tilde{\Omega}^{ii'\,k}{}_{j'}\tilde{V}_{jk} + \tilde{\Omega}^{ii'}{}_{jj'kk'}V^{kk'},
\end{aligned} \tag{5.29}$$

where ${*\Omega_{ii'}}{}^{jj'kk'} = \frac{1}{4}\epsilon^{jj'kk'll'mm'}\Omega_{ii'\,ll'mm'}$ and similarly for ${*\tilde{\Omega}}^{ii'jj'kk'}$.

5.2.2 SU(8) structures and supergravity

In seven dimensions $H_d = \mathrm{SU}(8)/\mathbb{Z}_2$ and the common subgroup of H_d and the $\mathrm{SL}(8,\mathbb{R})$ subgroup that we used to define $E_{7(7)}$ is SO(8). We define the sub-bundle $P \subset \tilde{F}$ of $\mathrm{SU}(8)/\mathbb{Z}_2$ frames as the set of frames where the generalised metric (4.2) can be written as

$$G(V,W) = \frac{1}{2}\big(\delta_{ij}\delta_{i'j'}V^{ii'}W^{jj'} + \delta^{ij}\delta^{i'j'}\tilde{V}_{ii'}\tilde{W}_{jj'}\big), \tag{5.30}$$

where δ_{ij} is the flat SO(8) metric. To write sections of E with manifest SU(8) indices $\alpha,\beta,\gamma,\ldots = 1,\ldots,8$ one uses the SO(8) gamma matrices

$$\begin{aligned}
V^{\alpha\beta} &= \mathrm{i}(\hat{\gamma}_{ij})^{\alpha\beta}\big(V^{ij} + \mathrm{i}\tilde{V}^{ij}\big), \\
\bar{V}_{\alpha\beta} &= -\mathrm{i}(\hat{\gamma}^{ij})_{\alpha\beta}\big(V_{ij} - \mathrm{i}\tilde{V}_{ij}\big).
\end{aligned} \tag{5.31}$$

where, $\hat{\gamma}^{ij}$ are defined in (B.21) and, when restricted to the Spin(8) subgroup $\alpha,\beta,\ldots$ indices are raised and lowered using the intertwiner $\tilde{C}$ (see appendix B.2).

The eleven-dimensional supergravity fermion fields can be decomposed into complex seven-dimensional spinors following the discussion in C.4. Using the embedding $\mathrm{Spin}(7)\subset \mathrm{Spin}(8)\subset\mathrm{SU}(8)$, discussed in detail in appendix B.2.3, they can be identified as SU(8) representations as follows. For the spinors we simply have

$$\begin{aligned}
\varepsilon^{\alpha} &= \mathrm{e}^{-\Delta/2}(\varepsilon^{\text{sugra}})^{\alpha} && \in \Gamma(S^-), \\
\bar{\rho}_{\alpha} &= \mathrm{i}\mathrm{e}^{\Delta/2}\tilde{C}_{\alpha\beta}(\gamma^{(7)}\rho^{\text{sugra}})^{\beta} && \in \Gamma(S^+).
\end{aligned} \tag{5.32}$$

Note the need to include the conformal factors in the definitions and also that, though we write $\bar{\rho}$ since it is embedded into the $\bar{\mathbf{8}}$ representation of SU(8), $\bar{\rho}_\alpha$ is defined in terms of the un-conjugated ρ^{sugra}. The $\mathbf{8}$ and $\bar{\mathbf{8}}$ representations are simply the fundamental and anti-fundamental so are related by conjugation so that $\bar{\varepsilon}_\alpha = (\varepsilon^\beta)^* A_{\dot{\beta}\alpha}$, using the SU(8)-invariant intertwiner A (see appendix B.2).

For the 56-dimensional vector-spinor we proceed in two steps, first embedding into Spin(8) by writing

$$\begin{aligned}
\psi^{\mathrm{Spin}(8)}_{a8} &= \frac{1}{4}\mathrm{e}^{\Delta/2}\left(\delta^{b}{}_{a} + \frac{1}{2}\gamma_a\gamma^b\right)\psi^{\text{sugra}}_b, \\
\psi^{\mathrm{Spin}(8)}_{ab} &= -\frac{1}{2}\mathrm{e}^{\Delta/2}\gamma^{(7)}\left(\gamma_{[a}\delta_{b]}{}^{c} - \frac{1}{4}\gamma_{ab}\gamma^c\right)\psi^{\text{sugra}}_c,
\end{aligned} \tag{5.33}$$

and then into SU(8) as

$$\psi^{\alpha\beta\gamma} = \frac{1}{3}\mathrm{i}(\hat{\gamma}^{ii'})^{[\alpha\beta}(\psi^{\mathrm{Spin}(8)}_{ii'})^{\gamma]} \in \Gamma(J^-). \tag{5.34}$$

A generalised connection is compatible with the generalised metric (5.30) if $DG = 0$. For such connections, we can define the generalised spinor derivative via the adjoint action of SU(8) given in [16]. Acting on $\chi \in \Gamma(S^-)$ we have

$$\begin{aligned} D_{ii'}\chi &= \partial_{ii'}\chi + \frac{1}{4}\Omega_{ii'jj'}\hat{\gamma}^{jj'}\chi - \frac{1}{48}\mathrm{i}\Omega_{ii'k_1\dots k_4}\hat{\gamma}^{k_1\dots k_4}\chi, \\ \tilde{D}_{ii'}\chi &= \tilde{\partial}_{ii'}\chi + \frac{1}{4}\tilde{\Omega}_{ii'jj'}\hat{\gamma}^{jj'}\chi - \frac{1}{48}\mathrm{i}\tilde{\Omega}_{ii'k_1\dots k_4}\hat{\gamma}^{k_1\dots k_4}\chi. \end{aligned} \tag{5.35}$$

where we have used the SO(8) metric δ_{ij} to lower indices. An example of such a generalised connection is the one (3.13) defined by the Levi-Civita connection ∇

$$\begin{aligned} D^{\nabla}_{ii'}\chi &= \begin{cases} \frac{1}{2}\mathrm{e}^{\Delta}\left(\partial_a + \frac{1}{4}\omega_{a\,bc}\hat{\gamma}^{bc}\right)\chi & \text{if } i = a \text{ and } i' = 8 \\ 0 & \text{if } i = a \text{ and } i' = b \end{cases}, \\ \tilde{D}^{\nabla}_{ii'}\chi &= 0. \end{aligned} \tag{5.36}$$

where $\omega_{a\,bc}$ is the usual spin-connection.

We can construct a torsion-free compatible connection D, by shifting D^{∇} by an additional connection piece Σ, such that its action on $\chi \in \Gamma(S^-)$ is given by

$$\begin{aligned} D_{ii'}\chi &= D^{\nabla}_{ii'}\chi + \frac{1}{4}\Sigma_{ii'jj'}\hat{\gamma}^{jj'}\chi - \frac{1}{48}\mathrm{i}\Sigma_{ii'k_1\dots k_4}\hat{\gamma}^{k_1\dots k_4}\chi, \\ \tilde{D}_{ii'}\chi &= \tilde{D}^{\nabla}_{ii'}\chi + \frac{1}{4}\tilde{\Sigma}_{ii'jj'}\hat{\gamma}^{jj'}\chi - \frac{1}{48}\mathrm{i}\tilde{\Sigma}_{ii'k_1\dots k_4}\hat{\gamma}^{k_1\dots k_4}\chi. \end{aligned} \tag{5.37}$$

where, in the conformal split frame,

$$\begin{aligned} \Sigma_{ii'jj'} &= -\frac{1}{3}\mathrm{e}^{\Delta}\delta_{ij}\tilde{K}_{i'j'} + \frac{1}{42}\mathrm{e}^{\Delta}\tilde{F}\delta_{ij}\delta_{i'j'} - \mathrm{e}^{\Delta}\delta_{ij}\partial_{i'j'}\Delta + Q_{ii'jj'}, \\ \tilde{\Sigma}_{ii'jj'} &= \frac{1}{3}\mathrm{e}^{\Delta}K_{ii'jj'} - \frac{1}{6}\mathrm{e}^{\Delta}K_{jj'ii'} + \tilde{Q}_{ii'jj'}, \\ \Sigma_{i_1\dots i_6} &= Q_{i_1\dots i_6}, \\ \tilde{\Sigma}_{i_1\dots i_6} &= \tilde{Q}_{i_1\dots i_6}. \end{aligned} \tag{5.38}$$

In this expression primed and unprimed indices are antisymmetrised implicitly, $(Q, \tilde{Q})$ are the undetermined components[8], $\tilde{F} = \frac{1}{7!}\epsilon^{a_1\dots a_7}\tilde{F}_{a_1\dots a_7}$ and

$$\begin{aligned} K_{ii'jj'} &= \begin{cases} (*F)_{abc} & \text{for } (i, i', j, j') = (a, b, c, 8) \\ 0 & \text{otherwise} \end{cases}, \\ \tilde{K}_{ij} &= \begin{cases} \tilde{F} & \text{for } (i, j) = (8, 8) \\ 0 & \text{otherwise} \end{cases}, \end{aligned} \tag{5.39}$$

[8]These are sections of the $\mathbf{1280} + \overline{\mathbf{1280}}$ representations of SU(8), see [12].

give the embedding of the supergravity fluxes. The connection can be rewritten in SU(8) indices through

$$\begin{aligned} D^{\alpha\beta} &= \mathrm{i}(\hat{\gamma}^{ij})^{\alpha\beta}\big(D_{ij} + \mathrm{i}\tilde{D}_{ij}\big), \\ \bar{D}_{\alpha\beta} &= -\mathrm{i}(\hat{\gamma}_{ij})_{\alpha\beta}\big(D^{ij} - \mathrm{i}\tilde{D}^{ij}\big). \end{aligned} \tag{5.40}$$

With these definitions, we can now give the explicit form of the unique operators (4.11) in SU(8) indices

$$\begin{aligned} (D \curlywedge \varepsilon)^{\alpha\beta\gamma} &= D^{[\alpha\beta}\varepsilon^{\gamma]}, \\ (\not{D}\varepsilon)_\alpha &= -\bar{D}_{\alpha\beta}\varepsilon^\beta, \\ (\not{D}\psi)_{\alpha\beta\gamma} &= -\frac{1}{12}\,\epsilon_{\alpha\beta\gamma\delta\delta'\theta_1\theta_2\theta_3}D^{\delta\delta'}\psi^{\theta_1\theta_2\theta_3}, \\ (D \curlyvee \psi)^\alpha &= \frac{1}{2}\bar{D}_{\beta\gamma}\psi^{\alpha\beta\gamma}, \end{aligned} \tag{5.41}$$

where $\epsilon_{\alpha_1\ldots\alpha_8}$ is the totally antisymmetric symbol preserved by SU(8).

From the first two we can immediately read off the supersymmetry variations of the fermions (4.14)

$$\delta\psi^{\alpha\beta\gamma} = D^{[\alpha\beta}\varepsilon^{\gamma]}, \qquad\qquad \delta\bar{\rho}_\alpha = -\bar{D}_{\alpha\beta}\varepsilon^\beta, \tag{5.42}$$

while the variations of the bosons (4.17) can be packaged as

$$\delta G_{AB} = \begin{pmatrix} \delta G_{\alpha\beta\gamma\delta} & \delta G_{\alpha\beta}{}^{\gamma\delta} \\ \delta G^{\alpha\beta}{}_{\gamma\delta} & \delta G^{\alpha\beta\gamma\delta} \end{pmatrix} = \frac{1}{|\mathrm{vol}_G|}\begin{pmatrix} \delta\bar{H}_{\alpha\beta\gamma\delta} & 0 \\ 0 & \delta H^{\alpha\beta\gamma\delta} \end{pmatrix} - G_{AB}\,\delta\log|\mathrm{vol}_G| \tag{5.43}$$

with

$$\begin{aligned} \delta H^{\alpha\beta\gamma\delta} &= -\frac{3}{16}\left(\varepsilon^{[\alpha}\psi^{\beta\gamma\delta]} + \frac{1}{4!}\epsilon^{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'}\bar{\varepsilon}_{\alpha'}\bar{\psi}_{\beta'\gamma'\delta'}\right) \\ \delta\log|\mathrm{vol}_G| &= \bar{\rho}_\alpha\varepsilon^\alpha + \rho^\alpha\bar{\varepsilon}_\alpha \end{aligned} \tag{5.44}$$

The fermion equations of motion (4.18) are

$$\begin{aligned} -\frac{1}{12}\,\epsilon_{\alpha\beta\gamma\delta\delta'\theta_1\theta_2\theta_3}D^{\delta\delta'}\psi^{\theta_1\theta_2\theta_3} + 2\bar{D}_{[\alpha\beta}\bar{\rho}_{\gamma]} &= 0, \\ D^{\alpha\beta}\bar{\rho}_\beta - \frac{1}{2}\bar{D}_{\beta\gamma}\psi^{\alpha\beta\gamma} &= 0. \end{aligned} \tag{5.45}$$

As before, the curvatures can be obtained by taking the supersymmetry variations of the fermion equations of motion and after some algebra one obtains the expressions

$$\begin{aligned} R^0_{\alpha\beta\gamma\delta}\varepsilon^\delta &= -2\left(\bar{D}_{[\alpha\beta}\bar{D}_{\gamma\delta]} + \frac{1}{4!}\,\epsilon_{\alpha\beta\gamma\delta\epsilon\epsilon'\theta\theta'}D^{\epsilon\epsilon'}D^{\theta\theta'}\right)\varepsilon^\delta - \left[\bar{D}_{[\alpha\beta}, \bar{D}_{\gamma]\delta}\right]\varepsilon^\delta, \\ \frac{1}{6}R\varepsilon^\alpha &= -\frac{2}{3}\left(\{D^{\alpha\gamma}, \bar{D}_{\beta\gamma}\} - \frac{1}{8}\delta^\alpha{}_\beta\{D^{\gamma\delta}, \bar{D}_{\gamma\delta}\}\right)\varepsilon^\beta \\ &\quad - \frac{1}{3}\left([D^{\alpha\gamma}, \bar{D}_{\beta\gamma}] - \frac{1}{8}\delta^\alpha{}_\beta[D^{\gamma\delta}, \bar{D}_{\gamma\delta}]\right)\varepsilon^\beta - \frac{1}{8}\left[D^{\beta\gamma}, \bar{D}_{\beta\gamma}\right]\varepsilon^\alpha. \end{aligned} \tag{5.46}$$

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The vanishing of these then corresponds to the bosonic equations of motion (4.22). As for $d = 4$, we again observe that the curvatures contain terms symmetric in the two connections, in the representations identified in [12].

The bosonic action (4.23) takes the form

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2} \int |\mathrm{vol}_G|\, R, \tag{5.47}$$

while the fermion action (4.24) is

$$\begin{aligned} S_{\mathrm{F}} = \frac{3}{2\kappa^2} \int |\mathrm{vol}_G| \Big(&\frac{1}{4!} \epsilon_{\alpha_1\alpha_2\alpha_3\beta\beta'\gamma_1\gamma_2\gamma_3} \psi^{\alpha_1\alpha_2\alpha_3} D^{\beta\beta'} \psi^{\gamma_1\gamma_2\gamma_3} \\ &+ \bar{\rho}_\alpha \bar{D}_{\beta\gamma} \psi^{\alpha\beta\gamma} - \psi^{\alpha\beta\gamma} \bar{D}_{\alpha\beta} \bar{\rho}_\gamma - 2\bar{\rho}_\alpha D^{\alpha\beta} \bar{\rho}_\beta + \mathrm{cc} \Big). \end{aligned} \tag{5.48}$$

This completes the rewriting of the seven-dimensional theory with explicit local SU(8) symmetry following from the natural generalised geometry construction of section 4.3.

6 Conclusions and discussion

As promised at the end of [12] we have provided a reformulation of eleven-dimensional supergravity, including the fermions to leading order, such that its larger bosonic symmetries are manifest. This was accomplished by writing down an analogue of Einstein gravity for $E_{d(d)} \times \mathbb{R}^+$ generalised geometry, the fermion fields embedding directly into representations of the local symmetry group $\tilde{H}_d$. To summarise, the supergravity is described by a simple set of equations which are manifestly diffeomorphism, gauge and $\tilde{H}_d$-covariant

$$\begin{array}{cc} \text{Equations of Motion} & \text{Supersymmetry} \\ \left\{ \begin{aligned} \not{D}\psi + \frac{11-d}{9-d} D \curlywedge \rho &= 0, \\ D \curlyvee \psi + \not{D}\rho &= 0, \\ R_{AB} &= 0, \end{aligned} \right. & \left\{ \begin{aligned} &\delta\psi = D \curlywedge \varepsilon, \\ &\delta\rho = \not{D}\varepsilon, \\ &\delta G = (\psi \times_{\mathrm{ad}\, P^\perp} \varepsilon) + (\rho \times_{\mathrm{ad}\, P^\perp} \varepsilon). \end{aligned} \right. \end{array} \tag{6.1}$$

This reinforces the conclusion of [1] that generalised geometry is a natural framework with which to formulate supergravity.

It is important to note that these equations are equally applicable to the reformulation of ten-dimensional type IIA or IIB supergravity restricted to warped products of Minkowski space and a $d-1$-dimensional manifold, where all the bosonic degrees (NSNS and RR) are unified into the generalised metric G. Matching to the familiar forms of the type II supergravity requires identifying the appropriate $\mathrm{GL}(d-1,\mathbb{R})$ and $\mathrm{Spin}(d-1)$ subgroups of $E_{d(d)}$ and $\tilde{H}_d$ respectively.[9] Alternatively by identifying the appropriate $O(d-1,d-1)$ subgroup one can decompose to match to the $O(d-1,d-1)\times\mathbb{R}^+$ generalised geometrical description of [1].

[9]Some of the details are given in appendix B of [12].

A surprising outcome of our work is the observation that, despite the fact that the geometric construction is entirely bosonic, supersymmetry is deeply integrated in the formalism — torsion-free, metric-compatible connections describe the variation of the fermions and the equations of motion of the fermions close under supersymmetry on the bosonic generalised curvatures. This relation between generalised geometry and supersymmetry is clearly something that warrants further exploration. One might try to formulate other supergravities, such as six-dimensional $N = (1, 0)$, which should provide further evidence of this connection. It is of course also interesting to see how one might extend the generalised geometry to make supersymmetry manifest.

One problem that generalised geometry is particularly well suited to tackle is that of describing generic supersymmetric vacua with flux [33, 35, 40]. It turns out that the Killing spinor equations can be shown to be equivalent to integrability conditions on the generalised connection D. One then expects that its special holonomy $G \subset \tilde{H}_d$ can be used to classify flux backgrounds. The language we developed in section 5.2 will be especially useful for studying compactifications of eleven-dimensional supergravity down to four-dimensional spacetime, something we will elaborate on in upcoming work.

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A Conventions in d dimensions

A.1 Tensor notation

We use the indices $m, n, p, \dots$ as the coordinate indices and $a, b, c \dots$ for the tangent space indices. We take symmetrisation of indices with weight one. Given a polyvector $w \in \Lambda^p TM$ and a form $\lambda \in \Lambda^q T^*M$, we write in components

$$\begin{aligned} w &= \frac{1}{p!} w^{m_1 \dots m_p} \frac{\partial}{\partial x^{m_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{m_p}}, \\ \lambda &= \frac{1}{q!} \lambda_{m_1 \dots m_q} \mathrm{d}x^{m_1} \wedge \dots \wedge \mathrm{d}x^{m_q}, \end{aligned} \tag{A.1}$$

so that wedge products and contractions are given by

$$
\begin{aligned}
(w\wedge w')^{m_1\dots m_{p+p'}} &= \frac{(p+p')!}{p!p'!} w^{[m_1\dots m_p} u^{m_{p+1}\dots m_{p+p'}]}, \\
(\lambda\wedge\lambda')_{m_1\dots m_{q+q'}} &= \frac{(q+q')!}{q!q'!} \lambda_{[m_1\dots m_q}\mu_{m_{p+1}\dots m_{q+q'}]}, \\
(w\lrcorner\lambda)_{a_1\dots a_{q-p}} &:= \frac{1}{p!} w^{c_1\dots c_p}\lambda_{c_1\dots c_p a_1\dots a_{q-p}} \qquad \text{if } p\le q, \\
(w\lrcorner\lambda)^{a_1\dots a_{p-q}} &:= \frac{1}{q!} w^{a_1\dots a_{p-q}c_1\dots c_q}\lambda_{c_1\dots c_q} \qquad \text{if } p\ge q.
\end{aligned}
\tag{A.2}
$$

Given the tensors $t\in TM\otimes\Lambda^7 TM$, $\tau\in T^*M\otimes\Lambda^7 T^*M$ and $a\in TM\otimes T^*M$ with components

$$
\begin{aligned}
t &= \frac{1}{7!} w^{m,m_1\dots m_7}\frac{\partial}{\partial x^m}\otimes\frac{\partial}{\partial x^{m_1}}\wedge\dots\wedge\frac{\partial}{\partial x^{m_7}}, \\
\tau &= \frac{1}{7!}\tau_{m,m_1\dots m_7}\mathrm{d}x^m\otimes\mathrm{d}x^{m_1}\wedge\dots\wedge\mathrm{d}x^{m_q}, \\
a &= a^m{}_n\frac{\partial}{\partial x^m}\otimes\mathrm{d}x^n,
\end{aligned}
\tag{A.3}
$$

we also use the "j-notation" from [12, 16], defining

$$
\begin{aligned}
(w\lrcorner\tau)_{a_1\dots a_{8-p}} &:= \frac{1}{(p-1)!} w^{c_1\dots c_p}\tau_{c_1,c_2\dots c_p a_1\dots a_{8-p}}, \\
(t\lrcorner\lambda)^{a_1\dots a_{8-q}} &:= \frac{1}{(q-1)!} t^{c_1,c_2\dots c_q a_1\dots a_{8-q}}\lambda_{c_1\dots c_q}, \\
(t\lrcorner\tau) &:= \frac{1}{7!} t^{a,b_1\dots b_7}\tau_{a,b_1\dots b_7}, \\
(jw\wedge w')^{a,a_1\dots a_7} &:= \frac{7!}{(p-1)!(8-p)!} w^{a[a_1\dots a_{p-1}} w'^{a_p\dots a_7]}, \\
(j\lambda\wedge\lambda')_{a,a_1\dots a_7} &:= \frac{7!}{(q-1)!(8-q)!}\lambda_{a[a_1\dots a_{q-1}}\lambda'_{a_q\dots a_7]}, \\
(jw\lrcorner j\lambda)^a{}_b &:= \frac{1}{(p-1)!} w^{ac_1\dots c_{p-1}}\lambda_{bc_1\dots c_{p-1}}, \\
(jt\lrcorner j\tau)^a{}_b &:= \frac{1}{7!} t^{a,c_1\dots c_7}\tau_{b,c_1\dots c_7}.
\end{aligned}
\tag{A.4}
$$

A.2 Metrics, connections and curvatures

The d-dimensional metric g is always positive definite. We define the orientation, $\epsilon_{1\dots d} = \epsilon^{1\dots d} = +1$, and use the conventions

$$
\begin{aligned}
*\lambda_{m_1\dots m_{d-q}} &= \frac{1}{q!}\sqrt{|g|}\epsilon_{m_1\dots m_{d-k}n_1\dots n_q}\lambda^{n_1\dots n_q}, \\
\lambda^2 &= \lambda_{m_1\dots m_q}\lambda^{m_1\dots m_q}.
\end{aligned}
\tag{A.5}
$$

Let $\nabla_m v^n = \partial_m v^n + \omega_m{}^n{}_p v^p$ be a general connection on TM. The torsion $T\in\Gamma(TM\otimes\Lambda^2 T^*M)$ of ∇ is defined by

$$
T(v,w) = \nabla_v w - \nabla_w v - [v,w]\,. \tag{A.6}
$$

or concretely, in coordinate indices,

$$T^m{}_{np} = \omega_n{}^m{}_p - \omega_p{}^m{}_n, \tag{A.7}$$

while, in a general basis where $v = v^a \hat{e}_a$ and $\nabla_m v^a = \partial_m v^a + \omega_m{}^a{}_b v^b$, one has

$$T^a{}_{bc} = \omega_b{}^a{}_c - \omega_c{}^a{}_b + [\hat{e}_b, \hat{e}_c]^a \,. \tag{A.8}$$

The curvature of a connection ∇ is given by the Riemann tensor $\mathcal{R} \in \Gamma(\Lambda^2 T^*M \otimes TM \otimes T^*M)$, defined by $\mathcal{R}(u,v)w = [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w$, or in components

$$\mathcal{R}_{mn}{}^p{}_q w^q = [\nabla_m, \nabla_n] w^p - T^q{}_{mn} \nabla_q w^p. \tag{A.9}$$

The Ricci tensor is the trace of the Riemann curvature

$$\mathcal{R}_{mn} = \mathcal{R}_{pm}{}^p{}_n. \tag{A.10}$$

Given a metric g the Ricci scalar for a metric-compatible connection is defined by

$$\mathcal{R} = g^{mn} \mathcal{R}_{mn}. \tag{A.11}$$

The Levi-Civita connection is the unique connection that is both torsion free ($T = 0$) and metric-compatible ($\nabla g = 0$).

B Clifford algebras and spinors

B.1 Clifford algebras, involutions and $\tilde{H}_d$

The real Clifford algebras Cliff$(p, q; \mathbb{R})$ are generated by gamma matrices satisfying

$$\{\gamma^m, \gamma^n\} = 2g^{mn}, \qquad \gamma^{m_1 \ldots m_k} = \gamma^{[m_1} \ldots \gamma^{m_k]}, \tag{B.1}$$

where g is a d-dimensional metric of signature (p, q). Here we will be primarily interested in Cliff$(d; \mathbb{R})$ = Cliff$(d, 0; \mathbb{R})$ and Cliff$(d-1, 1; \mathbb{R})$. The top gamma matrix is defined as

$$\gamma^{(d)} = \frac{1}{d!} \epsilon_{m_1 \ldots m_d} \gamma^{m_1 \ldots m_d} = \begin{cases} \gamma^0 \gamma^1 \ldots \gamma^{d-1} & \text{for Cliff}(d-1, 1; \mathbb{R}) \\ \gamma^1 \ldots \gamma^d & \text{for Cliff}(d; \mathbb{R}) \end{cases}, \tag{B.2}$$

and one has $\left[\gamma^{(d)}, \gamma^m\right] = 0$ if d is odd, while $\{\gamma^{(d)}, \gamma^m\} = 0$ if d is even, and

$$(\gamma^{(d)})^2 = \begin{cases} 1 & \text{if } p - q = 0, 1 \pmod 4 \\ -1 & \text{if } p - q = 2, 3 \pmod 4 \end{cases} \tag{B.3}$$

We also use Dirac slash notation with weight one so that for $\omega \in \Gamma(\Lambda^k T^*M)$

$$\not{\omega} = \frac{1}{k!} \omega_{m_1 \ldots m_k} \gamma^{m_1 \ldots m_k}. \tag{B.4}$$

The real Clifford algebras are isomorphic to matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$. These are listed in table 6. Note that in odd dimensions the pair $\{1, \gamma^{(d)}\}$ generate

$p-q \pmod 8$	$\mathrm{Cliff}(p,q;\mathbb{R})$
0, 2	$\mathrm{GL}(2^{d/2},\mathbb{R})$
1	$\mathrm{GL}(2^{[d/2]},\mathbb{R})\oplus\mathrm{GL}(2^{[d/2]},\mathbb{R})$
3, 7	$\mathrm{GL}(2^{[d/2]},\mathbb{C})$
4, 6	$\mathrm{GL}(2^{d/2-1},\mathbb{H})$
5	$\mathrm{GL}(2^{[d/2]-1},\mathbb{H})\oplus\mathrm{GL}(2^{[d/2]-1},\mathbb{H})$

Table 6. Real Clifford algebras.

the centre of the algebra, which is isomorphic to $\mathbb{R}\oplus\mathbb{R}$ if $p-q=1 \pmod 4$ and $\mathbb{C}$ if $p-q=3 \pmod 4$. In the first case $\mathrm{Cliff}(p,q;\mathbb{R})$ splits into two pieces with $\gamma^{(d)}$ eigenvalues of ±1. In the second case $\gamma^{(d)}$ plays the role of i under the isomorphism with $\mathrm{GL}(2^{[d/2]},\mathbb{C})$.

There are three involutions of the algebra given by

$$\begin{aligned}
\gamma^{m_1\dots m_k} &\mapsto (-)^k\gamma^{m_1\dots m_k}, \\
\gamma^{m_1\dots m_k} &\mapsto \gamma^{m_k\dots m_1}, \\
\gamma^{m_1\dots m_k} &\mapsto (-)^k\gamma^{m_k\dots m_1},
\end{aligned} \tag{B.5}$$

usually called "reflection", "reversal" and "Clifford conjugation". The first is an automorphism of the algebra, the other two are anti-automorphisms. The reflection involution gives a grading of $\mathrm{Cliff}(p,q;\mathbb{R})=\mathrm{Cliff}^+(p,q;\mathbb{R})\oplus\mathrm{Cliff}^-(p,q;\mathbb{R})$ into odd and even powers of γ^m. The group $\mathrm{Spin}(p,q)$ lies in $\mathrm{Cliff}^+(p,q;\mathbb{R})$.

The involutions can be used to define other subgroups of the Clifford algebra. In particular one has

$$\tilde{H}_{p,q}=\{g\in\mathrm{Cliff}(p,q;\mathbb{R}) : g^tg=1\} \tag{B.6}$$

g^t is the image of g under the reversal involution. For the corresponding Lie algebra we require $a^t+a=0$, and so the algebra is generated by elements in the negative eigenspace of the involution. For $d\leq 8$, this is the set $\{\gamma^{mn},\gamma^{mnp},\gamma^{m_1\dots m_6},\gamma^{m_1\dots m_7}\}$. We see that the maximally compact subgroups $\tilde{H}_d\subset E_{d(d)}$ are given by

$$\tilde{H}_d=\tilde{H}_{d,0} \tag{B.7}$$

for the $\mathrm{Cliff}(d;\mathbb{R})$ algebras[10].

B.2 Representations of $\mathrm{Cliff}(p,q;\mathbb{R})$ and intertwiners

It is usual to consider irreducible complex representations of the gamma matrices acting on spinors. When d is even there is only one such representation. There are then three intertwiners realising the involutions discussed above, namely,

$$\begin{aligned}
\gamma_{(d)}\gamma^m\gamma_{(d)}^{-1} &= -\gamma^m, \\
C\gamma^mC^{-1} &= (\gamma^m)^T, \\
\tilde{C}\gamma^m\tilde{C}^{-1} &= -(\gamma^m)^T,
\end{aligned} \tag{B.8}$$

[10]Note that $\tilde{H}_{7,0}$ is strictly $\mathrm{U}(8)$. Dropping the $\gamma^{(7)}$ generator one gets $\tilde{H}_7=\mathrm{SU}(8)$.

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where $\tilde{C} = C\gamma^{(d)}$. There are four further intertwiners, not all independent, giving

$$\begin{aligned} A\gamma^m A^{-1} &= (\gamma^m)^\dagger, & D\gamma^m D^{-1} &= (\gamma^m)^*, \\ \tilde{A}\gamma^m \tilde{A}^{-1} &= -(\gamma^m)^\dagger, & \tilde{D}\gamma^m \tilde{D}^{-1} &= -(\gamma^m)^*. \end{aligned} \tag{B.9}$$

By construction we see that $\tilde{H}_d$ is the group preserving C.

When d is odd there are two inequivalent irreducible representations with either $\gamma^{(d)} = \pm 1$ when $p - q = 1 \pmod 4$ or $\gamma^{(d)} = \pm \mathrm{i}$ when $p - q = 3 \pmod 4$. Since here $\gamma^{(d)}$ is odd under the reflection, this involution exchanges the two representations. Thus only half of the possible intertwiners exist on each. One has

$$\begin{aligned} C\gamma^m C^{-1} &= (\gamma^m)^T, && \text{if } d = 1 \pmod 4, \\ \tilde{C}\gamma^m \tilde{C}^{-1} &= -(\gamma^m)^T, && \text{if } d = 3 \pmod 4. \end{aligned} \tag{B.10}$$

while

$$\begin{aligned} A\gamma^m A^{-1} &= (\gamma^m)^\dagger, && \text{if } p \text{ is odd}, \\ \tilde{A}\gamma^m \tilde{A}^{-1} &= -(\gamma^m)^\dagger, && \text{if } p \text{ is even}, \\ D\gamma^m D^{-1} &= (\gamma^m)^*, && \text{if } p - q = 1 \pmod 4), \\ \tilde{D}\gamma^m \tilde{D}^{-1} &= -(\gamma^m)^*, && \text{if } p - q = 3 \pmod 4). \end{aligned} \tag{B.11}$$

Note that under reversal $(\gamma^{(d)})^t = (-)^{d(d-1)/2}\gamma^{(d)}$ so when $d = 3 \pmod 4$ the involution exchanges representations and we have no C intertwiner. In particular for $\mathrm{Cliff}(d;\mathbb{R})$ it maps $\gamma^{(d)} = \mathrm{i}$ to $\gamma^{(d)} = -\mathrm{i}$. However, this map can also be realised on each representation separately by the adjoint $A\gamma^m A^{-1} = (\gamma^m)^\dagger$. Hence for $d = 3 \pmod 4$ we can instead define $\tilde{H}_d$ as the group preserving A.

The conjugate intertwiners allow us to define Majorana and symplectic Majorana representations when there is an isomorphism to real and quaternionic matrix algebras respectively. Thus when $p - q = 0, 1, 2 \pmod 8$ one has $DD^* = 1$ and can define a reality condition on the spinors

$$\chi = (D\chi)^*. \tag{B.12}$$

When $p - q = 4, 5, 6 \pmod 8$ one has $DD^* = -1$ one can define a symplectic reality condition. Introducing a pair of SU(2) indices $A, B, \ldots = 1, 2$ on the spinors with the convention for raising and lowering these indices

$$\chi_A = \epsilon_{AB}\chi^B, \qquad \chi^A = \epsilon^{AB}\chi_B, \tag{B.13}$$

the symplectic Majorana condition is

$$\eta^A = \epsilon^{AB}(D\eta^B)^*. \tag{B.14}$$

Note that for $p - q = 0, 6, 7 \pmod 8$ and $p - q = 2, 3, 4 \pmod 8$ one can also define analogous Majorana and symplectic Majorana conditions respectively using $\tilde{D}$.

B.2.1 Cliff$(10,1;\mathbb{R})$

For Cliff$(10,1;\mathbb{R}) \simeq \mathrm{GL}(32,\mathbb{R}) \oplus \mathrm{GL}(32,\mathbb{R})$, following the conventions of [48] we take the representation with

$$\Gamma^{(11)} = \Gamma^0\Gamma^1 \dots \Gamma^{10} = -1. \tag{B.15}$$

The D intertwiner defines Majorana spinors, while $\tilde{C} = -\tilde{C}^T$ defines the conjugate

$$\varepsilon = (D\varepsilon)^*, \qquad\qquad \bar{\varepsilon} = \varepsilon^T\tilde{C}. \tag{B.16}$$

such that

$$\overline{\Gamma^{M_1\dots M_k}\varepsilon} = (-1)^{[(k+1)/2]}\bar{\varepsilon}\Gamma^{M_1\dots M_k}. \tag{B.17}$$

B.2.2 Cliff$(4;\mathbb{R})$ and Spin(5)

For Cliff$(4;\mathbb{R}) \simeq \mathrm{GL}(2,\mathbb{H})$, $D^*D = -1$ and we can use this to introduce symplectic Majorana spinors, while we use $\tilde{C}$ to define the conjugate spinor

$$\chi^A = \epsilon^{AB}(D\chi^B)^*, \qquad \bar{\chi}_A = \epsilon_{AB}(\chi^B)^T\tilde{C} \tag{B.18}$$

The other intertwiner $C = \tilde{C}\gamma^{(4)}$ provides a symplectic inner product on spinors, which is preserved by $\{\gamma^{mn},\gamma^{mnp}\}$, i.e. the $\tilde{H}_4 \cong \mathrm{Spin}(5)$ algebra. The Spin(5) gamma matrix algebra can be realised explicitly by setting

$$\hat{\gamma}^i = \begin{cases} \gamma^a & i = a \\ \gamma^{(4)} & i = 5 \end{cases}, \tag{B.19}$$

and identifying $\gamma^{mnp} = -\epsilon^{mnpq}\gamma_q\gamma^{(4)}$. The same gamma matrices give a representation of Cliff$(5;\mathbb{R})$ (with $\gamma^{(5)} = +1$).

B.2.3 Cliff$(7;\mathbb{R})$ and Spin(8)

For Cliff$(7;\mathbb{R})$ we take the representation with $\gamma^{(7)} = -\mathrm{i}$ and define conjugate spinors

$$\bar{\varepsilon} = \varepsilon^\dagger A. \tag{B.20}$$

This provides a hermitian inner product on spinors, which is preserved by $\tilde{H}_7 \cong \mathrm{SU}(8)$, generated by $\{\gamma^{mn},\gamma^{mnp},\gamma^{m_1\dots m_6}\}$. The intertwiner $\tilde{C} = \tilde{C}^T$ is preserved by a $\mathrm{Spin}(8) \subset \mathrm{SU}(8)$ subgroup. The corresponding generators can be written as

$$\hat{\gamma}^{ij} = \begin{cases} \gamma^{ab} & i = a, j = b \\ +\gamma^a\gamma^{(7)} & i = a, j = 8 \\ -\gamma^b\gamma^{(7)} & i = 8, j = b \end{cases}, \tag{B.21}$$

This representation has negative chirality in the sense that

$$\hat{\gamma}^{i_1\dots i_8} = -\epsilon^{i_1\dots i_8}. \tag{B.22}$$

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We have the useful completeness relations, reflecting SO(8) triality,

$$\hat{\gamma}^{ij}{}_{\alpha\beta}\hat{\gamma}_{ij}{}^{\gamma\delta} = 16\delta^{\gamma\delta}_{\alpha\beta}, \qquad \hat{\gamma}^{ij}{}_{\alpha\beta}\hat{\gamma}_{kl}{}^{\alpha\beta} = 16\delta^{ij}_{kl},$$

where we have used $\tilde{C}$ to raise and lower spinor indices, and Fierz identity, which also serves as our definition of $\epsilon_{\alpha_1\ldots\alpha_8}$,

$$\frac{1}{4!}\epsilon_{\alpha\alpha'\beta\beta'\gamma\gamma'\delta\delta'}\hat{\gamma}^{ij\gamma\gamma'}\hat{\gamma}^{kl\delta\delta'} = 2\hat{\gamma}^{[ij}{}_{[\alpha\alpha'}\hat{\gamma}^{kl]}{}_{\beta\beta']} - \hat{\gamma}^{ij}{}_{[\alpha\alpha'}\hat{\gamma}^{kl}{}_{\beta\beta']}. \tag{B.23}$$

Note that as a representation of the Spin(8) algebra we can impose a reality condition on the spinors $\chi = (D\chi)^*$ using the intertwiner $\tilde{D}$ with $\tilde{D}^*\tilde{D} = +1$. For such a real spinor the two possible definitions of spinor conjugate coincide $\bar{\chi} = \chi^T\tilde{C} = \chi^\dagger A$. In fact there exists a $GL(8,\mathbb{R})$ family of purely imaginary bases of gamma matrices such that $\tilde{D} = 1$ and $A = \tilde{C}$. In such a basis we have $\bar{\varepsilon} = \varepsilon^\dagger\tilde{C} = \varepsilon^\dagger A$ for a general spinor $\varepsilon = \chi_1 + \mathrm{i}\chi_2$. Many of our SU(8) equations are written under a $\mathrm{Spin}(8) = \mathrm{SU}(8)\cap \mathrm{SL}(8,\mathbb{R})$ decomposition in such an imaginary basis, and thus it is natural to raise and lower spinor indices with the Spin(8) invariant $\tilde{C}$.

C Spinor decompositions

C.1 $(10,1) \to (6,1) + (4,0)$

We can decompose the $\mathrm{Cliff}(10,1;\mathbb{R})$ gamma matrices as

$$\Gamma^\mu = \gamma^\mu \otimes \gamma^{(4)}, \qquad \Gamma^m = 1 \otimes \gamma^m, \tag{C.1}$$

and the eleven-dimensional intertwiners as

$$\tilde{C} = \tilde{C}_{(6,1)} \otimes \tilde{C}_{(4)}, \qquad D = D_{(6,1)} \otimes D_{(4)}. \tag{C.2}$$

Introducing a basis of seven-dimensional symplectic Majorana spinors $\{\eta^A_I\}$, we can then decompose a general eleven-dimensional Majorana spinor as

$$\varepsilon = \epsilon_{AB}\left(\eta^A_I \otimes \chi^{BI}\right), \tag{C.3}$$

where $\{\chi^{AI}\}$ are some collection of four-dimensional symplectic Majorana spinors. All of the data of the eleven-dimensional spinor is now contained in χ^{AI}, the extra index I serving as the external Spin(6,1) index.

The eleven-dimensional spinor conjugate can be realised in terms of the four-dimensional spinors χ^{AI} by setting

$$\bar{\chi}_{AI} = \epsilon_{AIBJ}(\chi^{BJ})^T\tilde{C}_{(4)}, \tag{C.4}$$

where $\epsilon_{AIBJ} = (\eta_{AI})^T\tilde{C}_{(6,1)}\eta_{BJ}$.

Clearly from the decomposition (C.1) the action of the internal eleven-dimensional gamma matrices is simply

$$\Gamma^m\varepsilon \leftrightarrow \gamma^m\chi^{AI}, \tag{C.5}$$

and any eleven-dimensional equation with only internal gamma matrices takes the same form in terms of χ^{AI}. Thus, supressing the extra indices on χ, the supergravity equations with fermions in section 2.2 take exactly the same form when written in terms of the four-dimensional spinors.

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C.2 $(10,1) \to (5,1) + (5,0)$

We can use a complex decomposition of the Cliff$(10,1;\mathbb{R})$ gamma matrices as

$$\Gamma^\mu = \gamma^\mu \otimes 1, \qquad \Gamma^m = \gamma^{(6)} \otimes \gamma^m, \tag{C.6}$$

and the eleven-dimensional intertwiners as

$$\tilde{C} = \tilde{C}_{(5,1)} \otimes C_{(5)}, \qquad D = D_{(5,1)} \otimes D_{(5)}. \tag{C.7}$$

We introduce bases of positive and negative chirality symplectic Majorana-Weyl spinors $\{\eta_I^{+A}\}$ and $\{\eta_I^{-A}\}$ and decompose a general eleven-dimensional Majorana spinor as $\varepsilon = \varepsilon^+ + \varepsilon^-$ with

$$\varepsilon^+ = \epsilon_{AB}\left(\eta_I^{+A} \otimes \chi_1^{IB}\right) \qquad \varepsilon^- = \epsilon_{AB}\left(\eta_I^{-A} \otimes \chi_2^{IB}\right), \tag{C.8}$$

where $\{\chi_1^{AI}\}$ and $\{\chi_2^{AI}\}$ are two collections of five-dimensional symplectic Majorana spinors, the extra I indices serving as external Spin$(5,1)$ indices.

The action of the internal eleven-dimensional gamma matrices is then

$$\Gamma^m\varepsilon^+ = \epsilon_{AB}\left(\eta_I^{+A} \otimes \gamma^m\chi_1^{IB}\right) \qquad \Gamma^m\varepsilon^- = \epsilon_{AB}\left(\eta_I^{-A} \otimes (-\gamma^m)\chi_2^{IB}\right), \tag{C.9}$$

so that this induces a different representation of Cliff$(5;\mathbb{R})$ on each of χ_1 and χ_2. To see how to express eleven-dimensional spinor bilinears in this case, we expand

$$\begin{aligned}\bar{\varepsilon}\,\Gamma^{m_1\ldots m_k}\varepsilon' &= \bar{\varepsilon}^+\,\Gamma^{m_1\ldots m_k}\varepsilon'^- + \bar{\varepsilon}^-\,\Gamma^{m_1\ldots m_k}\varepsilon'^+ \\ &= \left((\eta^+_{AI})^T\tilde{C}_{(5,1)}\eta^-_{BJ}\right)\left((\chi_1^{AI})^TC_{(5)}(-1)^k\gamma^{m_1\ldots m_k}\chi_2'^{BJ}\right) \\ &\quad + \left((\eta^-_{AI})^T\tilde{C}_{(5,1)}\eta^+_{BJ}\right)\left((\chi_2^{AI})^TC_{(5)}\gamma^{m_1\ldots m_k}\chi_1'^{BJ}\right),\end{aligned} \tag{C.10}$$

so that we see that this pairs χ_1 with χ_2' and χ_2 with χ_1'. We therefore define slightly different conjugates for χ_1 and χ_2 as

$$\bar{\chi}_{1,AI} = \epsilon_{AIBJ}(\chi_1^{BJ})^TC_{(5)}, \qquad \bar{\chi}_{2,AI} = \epsilon_{BJAI}(\chi_2^{BJ})^TC_{(5)}, \tag{C.11}$$

where $\epsilon_{AIBJ} = (\eta^-_{AI})^T\tilde{C}_{(5,1)}\eta^+_{BJ}$. We can then suppress the extra indices and write

$$\bar{\varepsilon}\,\Gamma^{m_1\ldots m_k}\varepsilon' = \bar{\chi}_2\gamma^{m_1\ldots m_k}\chi_1 + (-1)^k\bar{\chi}_1\gamma^{m_1\ldots m_k}\chi_2. \tag{C.12}$$

Under this decomposition, an equation linear in fermions from section 2.2 becomes two copies of the same equation, one identical copy with "χ_1" and one copy with "χ_2" and the sign of the gamma matrices reversed. The fermion bilinears split into two terms as (C.12).

C.3 $(10,1) \to (4,1) + (6,0)$

We make a complex decomposition of the Cliff$(10,1;\mathbb{R})$ gamma matrices as

$$\Gamma^\mu = \mathrm{i}\gamma^\mu \otimes \gamma^{(6)}, \qquad \Gamma^m = 1 \otimes \gamma^m, \tag{C.13}$$

and the eleven-dimensional intertwiners as

$$\tilde{C} = C_{(4,1)} \otimes \tilde{C}_{(6)}, \qquad D = \tilde{D}_{(4,1)} \otimes D_{(6)}. \tag{C.14}$$

Introducing a basis of five-dimensional symplectic Majorana spinors $\{\eta_I^A\}$ we can then decompose a general eleven-dimensional Majorana spinor as

$$\varepsilon = \epsilon_{AB} \left(\eta_I^A \otimes \chi^{BI}\right), \tag{C.15}$$

where $\{\chi^{AI}\}$ are some collection of six-dimensional symplectic Majorana spinors. All of the data of the eleven-dimensional spinor is now contained in χ^{AI}, the extra index I serving as the external $\mathrm{Spin}(4,1)$ index.

The eleven-dimensional spinor conjugate can be realised in terms of the six-dimensional spinors χ^{AI} by setting

$$\bar{\chi}_{AI} = \epsilon_{AIBJ}(\chi^{BJ})^T \tilde{C}_{(6)}, \tag{C.16}$$

where $\epsilon_{AIBJ} = -(\eta_{AI})^T C_{(4,1)} \eta_{BJ}$.

Clearly from the decomposition (C.1) the action of the internal eleven-dimensional gamma matrices is simply

$$\Gamma^m \varepsilon \leftrightarrow \gamma^m \chi^{AI}, \tag{C.17}$$

and any eleven-dimensional equation with only internal gamma matrices takes the same form in terms of χ^{AI}. Thus, supressing the extra indices on χ, the supergravity equations with fermions in section 2.2 take exactly the same form when written in terms of the six-dimensional spinors.

C.4 $(10,1) \to (3,1) + (7,0)$

We can use a complex decomposition of the $\mathrm{Cliff}(10,1;\mathbb{R})$ gamma matrices as

$$\Gamma^\mu = \gamma^\mu \otimes 1, \qquad \Gamma^m = \mathrm{i}\gamma^{(4)} \otimes \gamma^m, \tag{C.18}$$

and the eleven-dimensional intertwiners as

$$\tilde{C} = \tilde{C}_{(3,1)} \otimes \tilde{C}_{(7)}, \qquad D = D_{(3,1)} \otimes \tilde{D}_{(7)}. \tag{C.19}$$

We take a chiral decomposition of an eleven-dimensional Majorana spinor

$$\varepsilon = \left(\eta_I^+ \otimes \chi^I\right) + (D_{(3,1)}\eta_I^+)^* \otimes (\tilde{D}_{(7)}\chi^I)^*, \tag{C.20}$$

where $\gamma^{(4)}\eta_I^+ = -\mathrm{i}\eta_I^+$ so that $\{\eta_I^+\}$ are a basis of complex Weyl spinors in the external space. The Majorana condition on ε is automatic with no additional constraint on χ^I, which is complex. Again the extra index I on χ provides an external $\mathrm{Spin}(3,1)$ index.

The Clifford action of the internal eleven-dimensional gamma matrices then reduces to the action of the seven-dimensional gamma matrices on χ

$$\Gamma^m \varepsilon = \eta_I^+ \otimes (\gamma^m \chi^I) + (D_{(3,1)}\eta_I^+)^* \otimes (\tilde{D}_{(7)}\gamma^m\chi^I)^*. \tag{C.21}$$

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To see how to write eleven-dimensional spinor bilinears in this language, we expand

$$\begin{aligned}\bar{\varepsilon}\Gamma^{m_1\dots m_k}\varepsilon' &= \Big((\eta_I^+)^T\tilde{C}_{(3,1)}\eta_J^+\Big)\Big((\chi^I)^T\tilde{C}_{(7)}\gamma^{m_1\dots m_k}\chi'^J\Big)\\ &\quad+\Big((\eta_I^+)^T D_{(3,1)}^T\tilde{C}_{(3,1)}D_{(3,1)}\eta_J^+\Big)^*\Big((\chi^I)^T\tilde{D}_{(7)}^T\tilde{C}_{(7)}\tilde{D}_{(7)}\gamma^{m_1\dots m_k}\chi'^J\Big)^*\\ &= \Big(\bar{\chi}_I\gamma^{m_1\dots m_k}\chi'^I\Big)+\Big(\text{cc}\Big),\end{aligned} \tag{C.22}$$

where we have made the definition

$$\bar{\chi}_I = \epsilon_{IJ}(\chi^J)^T\tilde{C}_{(7)}, \tag{C.23}$$

with $\epsilon_{IJ} = -(\eta_I^+)^T\tilde{C}_{(3,1)}\eta_J^+$.

With these definitions, the equations linear in spinors in section 2.2 take the same form when written in terms of χ^I, while the spinor bilinear expressions take the same form with a complex conjugate piece added to them.

D $E_{d(d)}\times\mathbb{R}^+$ and $\mathrm{GL}(d,\mathbb{R})$

In this appendix we review from [12] the construction of $E_{d(d)}\times\mathbb{R}^+$ in terms of $\mathrm{GL}(d,\mathbb{R})$, the basic representations and tensor products.

We will describe the action directly in terms of the bundles that appear in the generalised geometry. We have

$$\begin{aligned} E &\simeq TM\oplus\Lambda^2T^*M\oplus\Lambda^5T^*M\oplus(T^*M\otimes\Lambda^7T^*M),\\ E^* &\simeq T^*M\oplus\Lambda^2TM\oplus\Lambda^5TM\oplus(TM\otimes\Lambda^7TM),\\ \operatorname{ad}\tilde{F} &\simeq \mathbb{R}\oplus(TM\otimes T^*M)\oplus\Lambda^3T^*M\oplus\Lambda^6T^*M\oplus\Lambda^3TM\oplus\Lambda^6TM.\end{aligned} \tag{D.1}$$

The corresponding $E_{d(d)}\times\mathbb{R}^+$ representations are listed in table 1. We write sections as

$$\begin{aligned} V &= v+\omega+\sigma+\tau &&\in E,\\ Z &= \zeta+u+s+t &&\in E^*,\\ R &= c+r+a+\tilde{a}+\alpha+\tilde{\alpha} &&\in \operatorname{ad}\tilde{F},\end{aligned} \tag{D.2}$$

so that $v\in TM$, $\omega\in\Lambda^2T^*M$, $\zeta\in T^*M$, $c\in\mathbb{R}$ etc. If $\{\hat{e}_a\}$ be a basis for TM with a dual basis $\{e^a\}$ on T^*M then there is a natural $\mathrm{gl}(d,\mathbb{R})$ action on each tensor component. For instance

$$(r\cdot v)^a = r^a{}_b v^b, \qquad (r\cdot\omega)_{ab} = -r^c{}_a\omega_{cb} - r^c{}_b\omega_{ac}, \qquad \text{etc.} \tag{D.3}$$

Writing $V' = R\cdot V$ for the adjoint $E_{d(d)}\times\mathbb{R}^+$ action of $R\in\operatorname{ad}\tilde{F}$ on $V\in E$, the components of V', using the notation of appendix A.1, are given by

$$\begin{aligned} v' &= cv + r\cdot v + \alpha\lrcorner\omega - \tilde{\alpha}\lrcorner\sigma,\\ \omega' &= c\omega + r\cdot\omega + v\lrcorner a + \alpha\lrcorner\sigma + \tilde{\alpha}\lrcorner\tau,\\ \sigma' &= c\sigma + r\cdot\sigma + v\lrcorner\tilde{a} + a\wedge\omega + \alpha\lrcorner\tau,\\ \tau' &= c\tau + r\cdot\tau - j\tilde{a}\wedge\omega + ja\wedge\sigma.\end{aligned} \tag{D.4}$$

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Note that, the $E_{d(d)}$ sub-algebra is generated by setting $c = \frac{1}{(9-d)} r^a{}_a$. Similarly, given $Z \in E^*$ we have

$$\begin{aligned} \zeta' &= -c\zeta + r \cdot \zeta - u \lrcorner a + s \lrcorner \tilde{a}, \\ u' &= -cu + r \cdot u - \alpha \lrcorner \zeta - s \lrcorner a + t \lrcorner \tilde{a}, \\ s' &= -cs + r \cdot s - \tilde{\alpha} \lrcorner \zeta - \alpha \wedge u - t \lrcorner a, \\ t' &= -ct + r \cdot t - j\alpha \wedge s - j\tilde{\alpha} \wedge u. \end{aligned} \tag{D.5}$$

Finally the adjoint commutator

$$R'' = [R, R'] \tag{D.6}$$

has components

$$\begin{aligned} c'' &= \frac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a) + \frac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}'), \\ r'' &= [r, r'] + j\alpha \lrcorner ja' - j\alpha' \lrcorner ja - \frac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a)\mathbb{1} \\ &\quad + j\tilde{\alpha}' \lrcorner j\tilde{a} - j\tilde{\alpha} \lrcorner j\tilde{a}' - \frac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}')\mathbb{1}, \\ a'' &= r \cdot a' - r' \cdot a + \alpha' \lrcorner \tilde{a} - \alpha \lrcorner \tilde{a}', \\ \tilde{a}'' &= r \cdot \tilde{a}' - r' \cdot \tilde{a} - a \wedge a', \\ \alpha'' &= r \cdot \alpha' - r' \cdot \alpha + \tilde{\alpha}' \lrcorner a - \tilde{\alpha} \lrcorner a', \\ \tilde{\alpha}'' &= r \cdot \tilde{\alpha}' - r' \cdot \tilde{\alpha} - \alpha \wedge \alpha' \end{aligned} \tag{D.7}$$

Here we have $c'' = \frac{1}{9-d} r''^a{}_a$, as R'' lies in the $E_{d(d)}$ sub-algebra.

We also need the projection

$$\times_{\mathrm{ad}} : E^* \otimes E \to \mathrm{ad}\,\tilde{F}. \tag{D.8}$$

Writing $R = Z \times_{\mathrm{ad}} V$ we have

$$\begin{aligned} c &= -\frac{1}{3} u \lrcorner \omega - \frac{2}{3} s \lrcorner \sigma - t \lrcorner \tau, \\ r &= v \otimes \zeta - ju \lrcorner j\omega + \frac{1}{3}(u \lrcorner \omega)\mathbb{1} - js \lrcorner j\sigma + \frac{2}{3}(s \lrcorner \sigma)\mathbb{1} - jt \lrcorner j\tau, \\ \alpha &= v \wedge u + s \lrcorner \omega + t \lrcorner \sigma, \\ \tilde{\alpha} &= -v \wedge s - t \lrcorner \omega, \\ a &= \zeta \wedge \omega + u \lrcorner \sigma + s \lrcorner \tau, \\ \tilde{a} &= \zeta \wedge \sigma + u \lrcorner \tau. \end{aligned} \tag{D.9}$$

Note in particular that

$$\partial \times_{\mathrm{ad}} V = \partial \otimes v + \mathrm{d}\omega + \mathrm{d}\sigma. \tag{D.10}$$

E $\boldsymbol{H_d}$ and $\boldsymbol{\tilde{H}_d}$

We now turn to the analogous description of H_d in $\mathrm{SO}(d)$ representations. We then give a detailed description of the spinor representations of H_d and provide several important projections of tensor products in this language.

E.1 H_d and SO(d)

Given a positive definite metric g on TM, which for convenience we take to be in standard form δ_{ab} in frame indices, we can define a metric on E by

$$G(V,V) = v^2 + \frac{1}{2!}\omega^2 + \frac{1}{5!}\sigma^2 + \frac{1}{7!}\tau^2, \tag{E.1}$$

where $v^2 = v_a v^a$, $\omega^2 = \omega_{ab}\omega^{ab}$, etc as in (A.5). Note that this metric allows us to identify $E \simeq E^*$.

The subgroup of $E_{d(d)} \times \mathbb{R}^+$ that leaves the metric is invariant is H_d, the maximal compact subgroup of $E_{d(d)}$ (see table 3). Geometrically it defines a generalised H_d structure, that is an H_d sub-bundle P of the generalised structure bundle $\tilde{F}$. The corresponding Lie algebra bundle is parametrised by

$$\begin{aligned} \operatorname{ad} P &\simeq \Lambda^2 T^*M \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M, \\ N &= n + b + \tilde{b}, \end{aligned} \tag{E.2}$$

and embeds in $\operatorname{ad}\tilde{F}$ as

$$\begin{aligned} c &= 0, \\ r_{ab} &= n_{ab}, \\ a_{abc} &= -\alpha_{abc} = b_{abc}, \\ \tilde{a}_{a_1\dots a_6} &= \tilde{\alpha}_{a_1\dots a_6} = \tilde{b}_{a_1\dots a_6}, \end{aligned} \tag{E.3}$$

where indices are lowered with the metric g. Note that n_{ab} generates the $O(d) \subset \mathrm{GL}(d,\mathbb{R})$ subgroup that preserves g. Concretely a general group element can be written as

$$H \cdot V = \mathrm{e}^{\alpha+\tilde{\alpha}}\,\mathrm{e}^{a+\tilde{a}}\,h \cdot V, \tag{E.4}$$

where $h \in O(d)$ and a and α and $\tilde{a}$ and $\tilde{\alpha}$ are related as in (E.3).

The generalised tangent space $E \simeq E^*$ forms an irreducible H_d bundle, where the action of H_d just follows from (D.4). The corresponding representations are listed in table 3.

Another important representation of H_d is the compliment of the adjoint of H_d in $E_{d(d)} \times \mathbb{R}^+$, which we denote as $\operatorname{ad} P^\perp$ (see table 3). An element of $\operatorname{ad} P^\perp$ be represented as

$$\begin{aligned} \operatorname{ad} P^\perp &\simeq \mathbb{R} \oplus S^2 F^* \oplus \Lambda^3 F^* \oplus \Lambda^6 F^*, \\ Q &= c + h + q + \tilde{q} \end{aligned} \tag{E.5}$$

and it embeds in $\operatorname{ad}\tilde{F}$

$$\begin{aligned} c &= c, \\ r_{ab} &= h_{ab}, \\ a_{abc} &= \alpha_{abc} = q_{abc}, \\ \tilde{a}_{a_1\dots a_6} &= -\tilde{\alpha}_{a_1\dots a_6} = \tilde{q}_{a_1\dots a_6}. \end{aligned} \tag{E.6}$$

The action of H_d on this representation is given by the $E_{d(d)} \times \mathbb{R}^+$ Lie algebra. Writing $Q' = N \cdot Q$ we have

$$
\begin{aligned}
c' &= -\frac{2}{3} b \lrcorner q - \frac{4}{3}\tilde{b} \lrcorner \tilde{q}, \\
h' &= n \cdot h - jb \lrcorner jq - jq \lrcorner jb - j\tilde{b} \lrcorner j\tilde{q} - j\tilde{q} \lrcorner j\tilde{b} + \left(\frac{2}{3} b \lrcorner q + \frac{4}{3}\tilde{b} \lrcorner \tilde{q}\right) \mathbb{1}, \\
q' &= n \cdot q - h \cdot b + b \lrcorner \tilde{q} + q \lrcorner \tilde{b}, \\
\tilde{q}' &= n \cdot \tilde{q} - h \cdot \tilde{b} - b \wedge q,
\end{aligned}
\tag{E.7}
$$

where we are using the $\mathrm{GL}(d, \mathbb{R})$ adjoint action of h on $\Lambda^3 T^*M$ and $\Lambda^6 T^*M$. The H_d invariant scalar part of Q is given by $c - \frac{1}{9-d} h^a{}_a$, while the remaining irreducible component has $c = \frac{1}{9-d} h^a{}_a$.

E.2 $\tilde{H}_d$ and $\mathrm{Cliff}(d; \mathbb{R})$

The double cover $\tilde{H}_d$ of H_d has a realisation in terms of the Clifford algebra $\mathrm{Cliff}(d; \mathbb{R})$. Let S be the bundle of $\mathrm{Cliff}(d; \mathbb{R})$ spinors. We can identify sections of S as $\tilde{H}_d$ bundles in two different ways, which we denote $S^\pm$. Specifically $\chi^\pm \in S^\pm$ if

$$
N \cdot \chi^\pm = \frac{1}{2}\left(\frac{1}{2!} n_{ab}\gamma^{ab} \pm \frac{1}{3!} b_{abc}\gamma^{abc} - \frac{1}{6!}\tilde{b}_{a_1 \dots a_6}\gamma^{a_1 \dots a_6}\right)\chi^\pm, \tag{E.8}
$$

for $N \in \mathrm{ad}\, P$. As expected, in both cases n generates the $\mathrm{Spin}(d)$ subgroup of $\tilde{H}_d$. The two representations are mapped into each other by $\gamma^a \to -\gamma^a$. As such, they are inequivalent in odd dimensions. However, in even dimensions, since $-\gamma^a = \gamma^{(d)}\gamma^a(\gamma^{(d)})^{-1}$, they are equivalent and one can identify $\chi^- = \gamma^{(d)}\chi^+$. Thus one finds

$$
\begin{aligned}
S &\simeq S^+ \oplus S^- \quad \text{if } d \text{ is odd}, \\
S &\simeq S^+ \simeq S^- \quad \text{if } d \text{ is even}.
\end{aligned}
\tag{E.9}
$$

The different $\tilde{H}_d$ representations are listed explicitly in table 4.

The $\mathrm{Spin}(d)$ vector-spinor bundle J also forms representations of $\tilde{H}_d$. Again we can identify two different actions. If $\varphi_a^\pm \in J^\pm$ we have[11]

$$
\begin{aligned}
N \cdot \varphi_a^\pm = &\frac{1}{2}\left(\frac{1}{2!} n_{bc}\gamma^{bc} \pm \frac{1}{3!} b_{bcd}\gamma^{bcd} - \frac{1}{6!}\tilde{b}_{b_1 \dots b_6}\gamma^{b_1 \dots b_6}\right)\varphi_a^\pm - n^b{}_a\varphi_b^\pm \\
&\mp \frac{2}{3} b_a{}^b{}_c\gamma^c\varphi_b^\pm \mp \frac{1}{3}\frac{1}{2!} b^b{}_{cd}\gamma_a{}^{cd}\varphi_b^\pm \\
&+ \frac{1}{3}\frac{1}{4!}\tilde{b}_a{}^b{}_{c_1 \dots c_4}\gamma^{c_1 \dots c_4}\varphi_b^\pm + \frac{2}{3}\frac{1}{5!}\tilde{b}^b{}_{c_1 \dots c_5}\gamma_a{}^{c_1 \dots c_5}\varphi_b^\pm.
\end{aligned}
\tag{E.10}
$$

Again in even dimension $J^+ \simeq J^-$. The $\tilde{H}_d$ representations are listed explicitly in table 4.

[11] The formula given here matches those found in [59, 60] for levels 0, 1 and 2 of $K(E_{10})$. A similar formula also appears in the context of E_{11} in [61].

Finally will also need the projections $\operatorname{ad} P^\perp \otimes S^\pm \to J^\mp$, which, for $Q \in \operatorname{ad} P^\perp$ and $\chi^\pm \in S^\pm$, is given by

$$\begin{aligned}(Q \times_{J^\mp} \chi^\pm)_a = {} & \frac{1}{2} h_{ab}\gamma^b\chi^\pm \mp \frac{1}{3}\frac{1}{2!} q_{abc}\gamma^{bc}\chi^\pm \pm \frac{1}{6}\frac{1}{3!} q_{bcd}\gamma_a{}^{bcd}\chi^\pm \\ & + \frac{1}{6}\frac{1}{5!}\tilde{q}_{ab_1\dots b_5}\gamma^{b_1\dots b_5}\chi^\pm - \frac{1}{3}\frac{1}{6!}\tilde{q}_{c_1\dots c_6}\gamma_a{}^{c_1\dots c_6}\chi^\pm .\end{aligned} \tag{E.11}$$

E.3 $\tilde{H}_d$ and $\mathrm{Cliff}(10,1;\mathbb{R})$

To describe the reformulation of $D = 11$ supergravity restricted to d dimensions, it is very useful to use the embedding of $\tilde{H}_d$ in $\mathrm{Cliff}(10,1;\mathbb{R})$. This identifies the same action of $\tilde{H}_d$ on spinors given in (E.8) but now using the internal spacelike gamma matrices Γ^a for $a = 1,\dots,d$. Combined with the external spin generators $\Gamma^{\mu\nu}$, this actually gives an action of $\mathrm{Spin}(10-d,1) \times \tilde{H}_d$ on eleven-dimensional spinors. As before the action of $\tilde{H}_d$ can be embedded in two different ways. We write $\hat{\chi}^\pm \in \hat{S}^\pm$ with

$$N \cdot \hat{\chi}^\pm = \frac{1}{2}\Big(\frac{1}{2!} n_{ab}\Gamma^{ab} \pm \frac{1}{3!} b_{abc}\Gamma^{abc} - \frac{1}{6!}\tilde{b}_{a_1\dots a_6}\Gamma^{a_1\dots a_6}\Big)\hat{\chi}^\pm . \tag{E.12}$$

Since the algebra of the $\{\Gamma^a\}$ is the same as $\mathrm{Cliff}(d;\mathbb{R})$ all the equations of the previous section translate directly to this presentation of $\tilde{H}_d$. The advantage of the direct action on eleven-dimensional spinors is that it allows us to write $\tilde{H}_d$ covariant spinor equations in a dimension independent way.

As before we can also identify two realisations $\hat{J}^\pm$ of $\tilde{H}_d$ on the representations with one eleven-dimensional spinor index and one internal vector index which transform as (E.10) (with Γ^a in place of γ^a). The $\mathrm{Spin}(d-1,1) \times \tilde{H}_d$ represenations for $\hat{S}^\pm$ and $\hat{J}^\pm$ are listed explicitly in table 5.

In addition to the projection $\operatorname{ad} P^\perp \otimes \hat{S}^\pm \to \hat{J}^\mp$ given by (E.11) (with Γ^a in place of γ^a) we can identify various other tensor products. We have the singlet projections $\langle\cdot,\cdot\rangle : \hat{S}^\mp \otimes \hat{S}^\pm \to \mathbb{1}$ given by the conventional $\mathrm{Cliff}(10,1;\mathbb{R})$ bilinear, defined using (B.16), so

$$\langle \hat{\chi}^-, \hat{\chi}^+ \rangle = \bar{\hat{\chi}}^- \chi^+ , \tag{E.13}$$

where $\hat{\chi}^\pm \in \hat{S}^\pm$. There is a similar singlet projection $\langle\cdot,\cdot\rangle : \hat{J}^\mp \otimes \hat{J}^\pm \to \mathbb{1}$ given by[12]

$$\langle \hat{\varphi}^\mp, \hat{\varphi}^\pm \rangle = \bar{\hat{\varphi}}_a^\mp \left(\delta^{ab} + \frac{1}{9-d}\Gamma^a\Gamma^b \right) \hat{\varphi}_b^\pm , \tag{E.14}$$

where $\hat{\varphi}^\pm \in \hat{J}^\pm$.

We also have projections from $\hat{S}^\pm \otimes \hat{J}^\pm$ and $\hat{S}^\pm \otimes \hat{S}^\mp$ to $\operatorname{ad} P^\perp$. Given $\hat{\chi}^+ \in \hat{S}^-$ and $\hat{\varphi}^\pm \in \hat{J}^\pm$ we have, using the decomposition (E.5),

$$\begin{aligned}(\hat{\chi}^\pm \times_{\operatorname{ad} P^\perp} \hat{\varphi}^\pm) &= \frac{2}{9-d}\bar{\hat{\chi}}^\pm \Gamma^a \hat{\varphi}_a^\pm , \\ (\hat{\chi}^\pm \times_{\operatorname{ad} P^\perp} \hat{\varphi}^\pm)_{ab} &= 2\bar{\hat{\chi}}^\pm \Gamma_{(a}\hat{\varphi}_{b)}^\pm \\ (\hat{\chi}^\pm \times_{\operatorname{ad} P^\perp} \hat{\varphi}^\pm)_{abc} &= \mp 3\bar{\hat{\chi}}^\pm \Gamma_{[ab}\hat{\varphi}_{c]}^\pm , \\ (\hat{\chi}^\pm \times_{\operatorname{ad} P^\perp} \hat{\varphi}^\pm)_{a_1\dots a_6} &= -6\bar{\hat{\chi}}^\pm \Gamma_{[a_1\dots a_5}\hat{\varphi}_{a_6]}^\pm ,\end{aligned} \tag{E.15}$$

[12] Setting $d = 10$ in this reproduces the corresponding inner product in [59].

Note that the image of this projection does not include the $\tilde{H}_d$ scalar part of $\operatorname{ad} P^{\perp}$, since, from the first two components, $c - \frac{1}{9-d} h^a{}_a = 0$. We also have

$$(\hat{\chi}^+ \times_{\operatorname{ad} P^{\perp}} \hat{\chi}^-) = \frac{2}{9-d} \bar{\hat{\chi}}^- \hat{\chi}^+, \tag{E.16}$$

and all other components of $\operatorname{ad} P^{\perp}$ are set to zero. We see that the image of this map is in the $\tilde{H}_d$ scalar part of $\operatorname{ad} P^{\perp}$.

Finally, we also need the $\tilde{H}_d$ projections for $E \simeq E^*$ acting on $S^{\pm}$ and $J^{\pm}$. Given $V \in E$ it is useful to introduce the notation

$$\begin{aligned} \not{V} &: \hat{S}^{\pm} \to \hat{S}^{\mp}, & V \curlywedge &: \hat{S}^{\pm} \to \hat{J}^{\pm}, \\ V \curlyvee &: \hat{J}^{\pm} \to \hat{S}^{\pm}, & \not{V} &: \hat{J}^{\pm} \to \hat{J}^{\mp}. \end{aligned} \tag{E.17}$$

Given $\hat{\chi}^{\pm} \in \hat{S}^{\pm}$ and $\hat{\varphi}^{\pm}_a \in \hat{J}^{\pm}$ we find

$$(\not{V} \hat{\chi}^{\pm}) = \left(\mp v^a \Gamma_a + \frac{1}{2!} \omega_{ab} \Gamma^{ab} \mp \frac{1}{5!} \sigma_{a_1 \dots a_5} \Gamma^{a_1 \dots a_5} + \frac{1}{6!} \tau^b{}_{b a_1 \dots a_6} \Gamma^{a_1 \dots a_6} \right) \hat{\chi}^{\pm}, \tag{E.18}$$

and

$$\begin{aligned} (V \curlywedge \hat{\chi}^{\pm})_a = \Big(& v_a \pm \frac{2}{3} \Gamma^b \omega_{ab} \mp \frac{1}{3} \frac{1}{2!} \Gamma_a{}^{cd} \omega_{cd} - \frac{1}{3} \frac{1}{4!} \Gamma^{c_1 \dots c_4} \sigma_{a c_1 \dots c_4} \\ & + \frac{2}{3} \frac{1}{5!} \Gamma_a{}^{c_1 \dots c_5} \sigma_{c_1 \dots c_5} \pm \frac{1}{7!} \Gamma^{c_1 \dots c_7} \tau_{a, c_1 \dots c_7} \Big) \hat{\chi}^{\pm}, \end{aligned} \tag{E.19}$$

while

$$\begin{aligned} (V \curlyvee \hat{\varphi}^{\pm}) = {} & v^a \hat{\varphi}^{\pm}_a + \frac{1}{10-d} v_a \Gamma^{ab} \hat{\varphi}^{\pm}_b \pm \frac{1}{10-d} \frac{1}{2!} \omega_{bc} \Gamma^{abc} \hat{\varphi}^{\pm}_a \pm \frac{8-d}{10-d} \omega^a{}_b \Gamma^b \hat{\varphi}^{\pm}_a \\ & - \frac{1}{10-d} \frac{1}{5!} \sigma^{b_1 \dots b_5} \Gamma^a{}_{b_1 \dots b_5} \hat{\varphi}^{\pm}_a - \frac{8-d}{10-d} \frac{1}{4!} \sigma^a{}_{b_1 \dots b_4} \Gamma^{b_1 \dots b_4} \hat{\varphi}^{\pm}_a \\ & \mp \frac{1}{7!} \tau^a{}_{, b_1 \dots b_7} \Gamma^{b_1 \dots b_7} \hat{\varphi}^{\pm}_a \mp \frac{1}{3} \frac{1}{5!} \tau^c{}_{, c}{}^a{}_{b_1 \dots b_5} \Gamma^{b_1 \dots b_5} \hat{\varphi}^{\pm}_a, \end{aligned} \tag{E.20}$$

and finally

$$\begin{aligned} (\not{V} \hat{\varphi}^{\pm})_a = {} & \pm v^c \Gamma_c \hat{\varphi}^{\pm}_a \pm \frac{2}{9-d} \Gamma^c v_a \hat{\varphi}^{\pm}_c - \frac{1}{2!} \omega_{cd} \Gamma^{cd} \hat{\varphi}^{\pm}_a + \frac{4}{3} \omega_a{}^b \hat{\varphi}^{\pm}_b \\ & - \frac{2}{3} \omega_{cd} \Gamma_a{}^c \hat{\varphi}^{\pm d} - \frac{4}{3} \frac{1}{9-d} \omega_{ab} \Gamma^b \Gamma^c \hat{\varphi}^{\pm}_c + \frac{2}{3} \frac{1}{9-d} \frac{1}{2!} \omega_{bc} \Gamma_a{}^{bc} \Gamma^d \hat{\varphi}^{\pm}_d \\ & \pm \frac{1}{5!} \sigma_{c_1 \dots c_5} \Gamma^{c_1 \dots c_5} \hat{\varphi}^{\pm}_a \mp \frac{2}{3} \frac{1}{3!} \sigma_a{}^b{}_{c_1 c_2 c_3} \Gamma^{c_1 c_2 c_3} \hat{\varphi}^{\pm}_b \mp \frac{4}{3} \frac{1}{4!} \sigma^b{}_{c_1 \dots c_4} \Gamma_a{}^{c_1 \dots c_4} \hat{\varphi}^{\pm}_b \\ & \mp \frac{2}{3} \frac{1}{9-d} \frac{1}{4!} \sigma_{a c_1 \dots c_4} \Gamma^{c_1 \dots c_4} \Gamma^d \hat{\varphi}^{\pm}_d \pm \frac{4}{3} \frac{1}{9-d} \frac{1}{5!} \sigma_{c_1 \dots c_5} \Gamma_a{}^{c_1 \dots c_5} \Gamma^d \hat{\varphi}^{\pm}_d \\ & + \frac{1}{7!} \tau_{c, d_1 \dots d_7} \Gamma^c \Gamma^{d_1 \dots d_7} \hat{\varphi}^{\pm}_a + \frac{1}{7!} \tau_{a, c_1 \dots c_7} \Gamma^{c_1 \dots c_7} \Gamma^d \hat{\varphi}^{\pm}_d. \end{aligned} \tag{E.21}$$

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