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Stability of general multi-Euler-Lagrange quadratic functional equations in non-Archimedean fuzzy normed spaces

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Abstract

In this paper we prove the generalized Hyers-Ulam stability of the system defining general Euler-Lagrange quadratic mappings in non-Archimedean fuzzy normed spaces over a field with valuation using the direct and the fixed point methods.

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Keywords: stability of general multi-Euler-Lagrange quadratic functional equation; direct method; fixed point method; non-Archimedean fuzzy normed space

1 Introduction

Let $\mathbb{K}$ be a field. A valuation mapping on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $r, s \in \mathbb{K}$ the following conditions are satisfied: (i) $|r| \geq 0$ and equality holds if and only if $r = 0$; (ii) $|rs| = |r| \cdot |s|$; (iii) $|r + s| \leq |r| + |s|$.

A field endowed with a valuation mapping will be called a valued field. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$. In the following we will assume that $|\cdot|$ is non-trivial, *i.e.*, there is an $r_0 \in \mathbb{K}$ such that $|r_0| \neq 0, 1$.

If the condition (iii) in the definition of a valuation mapping is replaced with a strong triangle inequality (ultrametric): $|r + s| \leq \max\{|r|, |s|\}$, then the valuation $|\cdot|$ is said to be non-Archimedean. In any non-Archimedean field we have $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Throughout this paper, we assume that $\mathbb{K}$ is a valued field, $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces over $\mathbb{K}$, $a, b \in \mathbb{K}$ are fixed with $\lambda := a^2 + b^2 \neq 0, 1$ ($\lambda_1 := 2a \neq 0, 1$ if $a = b$) and n is a positive integer. Moreover, $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{R}$ (respectively, $\mathbb{Q}$) denotes the set of all reals (respectively, rationals).

A mapping $f : \mathcal{X}^n \to \mathcal{Y}$ is called a general multi-Euler-Lagrange quadratic mapping if it satisfies the general Euler-Lagrange quadratic equations in each of their n arguments:

$$\begin{aligned} &f(x_1, \ldots, x_{i-1}, ax_i + bx_i', x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, bx_i - ax_i', x_{i+1}, \ldots, x_n) \\ &\quad = (a^2 + b^2)\big[f(x_1, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)\big] \end{aligned} \tag{1.1}$$

for all $i = 1, \ldots, n$ and all $x_1, \ldots, x_{i-1}, x_i, x_i', x_{i+1}, \ldots, x_n \in \mathcal{X}$. Letting $x_i = x_i' = 0$ in (1.1), we get $f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$. Putting $x_i' = 0$ in (1.1), we have

$$f(x_1, \ldots, x_{i-1}, ax_i, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, bx_i, x_{i+1}, \ldots, x_n) = \lambda f(x_1, \ldots, x_n). \tag{1.2}$$

Replacing x_i by ax_i and x_i' by bx_i in (1.1), respectively, we obtain

$$\begin{aligned} & f(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n) \\ & \quad = \lambda\big[f(x_1, \ldots, x_{i-1}, ax_i, x_{i+1}, \ldots, x_n) + f(x_1, \ldots, x_{i-1}, bx_i, x_{i+1}, \ldots, x_n)\big]. \end{aligned} \tag{1.3}$$

From (1.2) and (1.3), one gets

$$f(x_1, \ldots, x_{i-1}, \lambda x_i, x_{i+1}, \ldots, x_n) = \lambda^2 f(x_1, \ldots, x_n) \tag{1.4}$$

for all $i = 1, \ldots, n$ and all $x_1, \ldots, x_n \in \mathcal{X}$. If $a = b$ in (1.1), then we have

$$\begin{aligned} & f\big(x_1, \ldots, x_{i-1}, a\big(x_i + x_i'\big), x_{i+1}, \ldots, x_n\big) + f\big(x_1, \ldots, x_{i-1}, a\big(x_i - x_i'\big), x_{i+1}, \ldots, x_n\big) \\ & \quad = 2a^2\big[f(x_1, \ldots, x_n) + f\big(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n\big)\big]. \end{aligned} \tag{1.5}$$

Letting $x_i' = x_i$ in (1.5), we obtain

$$f(x_1, \ldots, x_{i-1}, \lambda_1 x_i, x_{i+1}, \ldots, x_n) = \lambda_1^2 f(x_1, \ldots, x_n) \tag{1.6}$$

for all $i = 1, \ldots, n$ and all $x_1, \ldots, x_n \in \mathcal{X}$.

The study of stability problems for functional equations is related to a question of Ulam [30] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [13]. The result of Hyers was generalized by Aoki [2] for approximate additive mappings and by Rassias [27] for approximate linear mappings by allowing the Cauchy difference operator $CDf(x, y) = f(x + y) - [f(x) + f(y)]$ to be controlled by $\epsilon(\|x\|^p + \|y\|^p)$. In 1994, a further generalization was obtained by Găvruța [9], who replaced $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. We refer the reader to see, for instance, [1, 4–7, 14–16, 18, 20, 22, 23, 25, 26, 28, 31–37] for more information on different aspects of stability of functional equations. On the other hand, for some outcomes on the stability of multi-quadratic and Euler-Lagrange-type quadratic mappings we refer the reader to [7, 11, 24].

The main purpose of this paper is to prove the generalized Hyers-Ulam stability of multi-Euler-Lagrange quadratic functional equation (1.1) in complete non-Archimedean fuzzy normed spaces over a field with valuation using the direct and the fixed point methods.

2 Preliminaries

We recall the notion of non-Archimedean fuzzy normed spaces over a field with valuation and some preliminary results (see for instance [3, 22, 23, 31, 32]). For more details the reader is referred to [3, 22].

Definition 2.1 Let $\mathcal{X}$ be a linear space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \to [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\|$, $r \in \mathbb{K}$, $x \in \mathcal{X}$;
(iii) the strong triangle inequality

$$\|x + y\| \le \max\{\|x\|, \|y\|\}, \quad x, y \in \mathcal{X}.$$

Then $(\mathcal{X}, \|\cdot\|)$ is called a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

In 1897, Hensel discovered the p-adic numbers as a number-theoretical analogue of power series in complex analysis. Let p be a prime number. For any nonzero rational number a, there exists a unique integer r such that $a = p^r m/n$, where m and n are integers not divisible by p. Then $|a|_p := p^{-r}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(a,b) = |a - b|_p$ is denoted by $\mathbb{Q}_p$ which is called the p-adic number field. Note that if $p > 2$, then $|2^n|_p = 1$ for each integer n but $|2|_2 < 1$.

During the last three decades, p-adic numbers have gained the interest of physicists for their research, in particular, into problems deriving from quantum physics, p-adic strings, and superstrings (see for instance [21]).

A triangular norm (shorter t-norm, [29]) is a binary operation $T : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions: (a) T is commutative and associative; (b) $T(a,1) = a$ for all $a \in [0,1]$; (c) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$. Basic examples of continuous t-norms are the Łukasiewicz t-norm T_L, $T_L(a,b) = \max\{a + b - 1, 0\}$, the product t-norm T_P, $T_P(a,b) = ab$ and the strongest triangular norm T_M, $T_M(a,b) = \min\{a,b\}$. A t-norm is called continuous if it is continuous with respect to the product topology on the set $[0,1] \times [0,1]$.

A t-norm T can be extended (by associativity) in a unique way to an m-array operation taking for $(x_1,\dots,x_m) \in [0,1]^m$, the value $T(x_1,\dots,x_m)$ defined recurrently by $T_{i=1}^0 x_i = 1$ and $T_{i=1}^m x_i = T(T_{i=1}^{m-1} x_i, x_m)$ for $m \in \mathbb{N}$. T can also be extended to a countable operation, taking for any sequence $\{x_i\}_{i\in\mathbb{N}}$ in $[0,1]$, the value $T_{i=1}^\infty x_i$ is defined as $\lim_{m\to\infty} T_{i=1}^m x_i$. The limit exists since the sequence $\{T_{i=1}^m x_i\}_{m\in\mathbb{N}}$ is non-increasing and bounded from below. $T_{i=m}^\infty x_i$ is defined as $T_{i=1}^\infty x_{m+i}$.

Definition 2.2 A t-norm T is said to be of Hadžić-type (H-type, we denote by $T \in \mathcal{H}$) if a family of functions $\{T_{i=1}^m(t)\}$ for all $m \in \mathbb{N}$ is equicontinuous at $t = 1$, that is, for all $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that

$$t > 1 - \delta \quad \Longrightarrow \quad T_{i=1}^m(t) > 1 - \varepsilon \quad \text{for all } m \in \mathbb{N}.$$

The t-norm T_M is a t-norm of Hadžić-type. Other important triangular norms we refer the reader to [12].

Proposition 2.3 (see [12]) (1) *If $T = T_P$ or $T = T_L$, then*

$$\lim_{m\to\infty} T_{i=1}^\infty x_{m+i} = 1 \quad \Longleftrightarrow \quad \sum_{i=1}^\infty (1 - x_i) < \infty.$$

(2) *If T is of Hadžić-type, then*

$$\lim_{m\to\infty} T_{i=m}^{\infty} x_i = \lim_{m\to\infty} T_{i=1}^{\infty} x_{m+i} = 1$$

for every sequence $\{x_i\}_{i\in\mathbb{N}}$ *in* $[0,1]$ *such that* $\lim_{i\to\infty} x_i = 1$.

Definition 2.4 (see [22]) Let $\mathcal{X}$ be a linear space over a valued field $\mathbb{K}$ and T be a continuous t-norm. A function $N : \mathcal{X} \times \mathbb{R} \to [0,1]$ is said to be a non-Archimedean fuzzy Menger norm on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $s, t \in \mathbb{R}$:

(N1) $N(x,t) = 0$ for all $t \le 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$, $t > 0$;
(N3) $N(cx,t) = N(x,t/|c|)$ if $c \neq 0$;
(N4) $N(x+y,\max\{s,t\}) \ge T(N(x,s),N(y,t))$, $s,t > 0$;
(N5) $\lim_{t\to\infty} N(x,t) = 1$.

If N is a non-Archimedean fuzzy Menger norm on $\mathcal{X}$, then the triple $(\mathcal{X}, N, T)$ is called a non-Archimedean fuzzy normed space. It should be noticed that from the condition (N4) it follows that

$$N(x,t) \ge T\big(N(0,t),N(x,s)\big) = N(x,s)$$

for every $t > s > 0$ and $x, y \in \mathcal{X}$, that is, $N(x,\cdot)$ is non-decreasing for every x. This implies $N(x,s+t) \ge N(x,\max\{s,t\})$. If (N4) holds, then so does

(N6) $N(x+y,s+t) \ge T(N(x,s),N(y,t))$.

We repeatedly use the fact $N(-x,t) = N(x,t)$, $x \in \mathcal{X}$, $t > 0$, which is deduced from (N3). We also note that Definition 2.4 is more general than the definition of a non-Archimedean Menger norm in [23, 31], where only fields with a non-Archimedean valuation have been considered.

Definition 2.5 Let $(\mathcal{X}, N, T)$ be a non-Archimedean fuzzy normed space. Let $\{x_m\}_{m\in\mathbb{N}}$ be a sequence in $\mathcal{X}$. Then $\{x_m\}_{m\in\mathbb{N}}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that $\lim_{m\to\infty} N(x_m - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_m\}_{m\in\mathbb{N}}$ and we denote it by $\lim_{m\to\infty} x_m = x$. The sequence $\{x_m\}_{m\in\mathbb{N}}$ in $\mathcal{X}$ is said to be a Cauchy sequence if $\lim_{m\to\infty} N(x_{m+p} - x_m, t) = 1$ for all $t > 0$ and $p = 1, 2, \ldots$. If every Cauchy sequence in $\mathcal{X}$ is convergent, then the space is called a complete non-Archimedean fuzzy normed space.

Example 2.6 Let $(\mathcal{X}, \|\cdot\|)$ be a real (or non-Archimedean) normed space. For each $k > 0$, consider

$$N_k(x,t) = \begin{cases} \frac{t}{t+k\|x\|}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

Then $(\mathcal{X}, N_k, T_M)$ is a non-Archimedean fuzzy normed space.

Example 2.7 (see [22]) Let $(\mathcal{X}, \|\cdot\|)$ be a real normed space. Then the triple $(\mathcal{X}, N, T_P)$, where

$$N(x,t) = \begin{cases} e^{-\|x\|/t}, & t > 0, \\ 0, & t \le 0 \end{cases}$$

is a non-Archimedean fuzzy normed space. Moreover, if $(\mathcal{X}, \|\cdot\|)$ is complete, then $(\mathcal{X}, N, T_P)$ is complete and therefore it is a complete non-Archimedean fuzzy normed space over an Archimedean valued field.

Let Ω be a set. A function $d : \Omega \times \Omega \to [0,\infty]$ is called a generalized metric on Ω if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$; (2) $d(x,y) = d(y,x)$, $x,y \in \Omega$; (3) $d(x,y) \le d(x,z) + d(y,z)$, $x,y,z \in \Omega$.

For explicitly later use, we recall the following result by Diaz and Margolis [8].

Theorem 2.8 *Let* (Ω, d) *be a complete generalized metric space and* $J : \Omega \to \Omega$ *be a strictly contractive mapping with Lipschitz constant* $0 < L < 1$, *that is*

$$d(Jx, Jy) \le L d(x,y), \quad x,y \in \Omega.$$

If there exists a nonnegative integer m_0 *such that* $d(J^{m_0}x, J^{m_0+1}x) < \infty$ *for an* $x \in \Omega$, *then*

(1) *the sequence* $\{J^m x\}_{m\in\mathbb{N}}$ *converges to a fixed point* x^* *of* J;

(2) x^* *is the unique fixed point of* J *in the set* Ω^*,

$$\Omega^* := \{ y \in \Omega | d(J^{m_0}x, y) < \infty \};$$

(3) *if* $y \in \Omega^*$, *then*

$$d(y, x^*) \le \frac{1}{1-L} d(y, Jy).$$

3 Stability of the functional equation (1.1): a direct method

Throughout this section, using a direct method, we prove the stability of Eq. (1.1) in complete non-Archimedean fuzzy normed spaces.

Theorem 3.1 *Let* $\mathbb{K}$ *be a valued field,* $\mathcal{X}$ *be a vector space over* $\mathbb{K}$ *and* $(\mathcal{Y}, N, T)$ *be a complete non-Archimedean fuzzy normed space over* $\mathbb{K}$. *Assume also that, for every* $i \in \{1,2,\ldots,n\}$, $\Psi_i : \mathcal{X}^{n+1} \times [0,\infty) \to [0,1]$ *is a mapping such that*

$$\begin{aligned}
&\lim_{j\to\infty} \Psi_i(\lambda^j x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n, |\lambda|^{2j} t) \\
&\quad = \cdots \\
&\quad = \lim_{j\to\infty} \Psi_i(x_1, \ldots, x_{i-2}, \lambda^j x_{i-1}, x_i, x_i', x_{i+1}, \ldots, x_n, |\lambda|^{2j} t) \\
&\quad = \lim_{j\to\infty} \Psi_i(x_1, \ldots, x_{i-1}, \lambda^j x_i, \lambda^j x_i', x_{i+1}, \ldots, x_n, |\lambda|^{2j} t) \\
&\quad = \lim_{j\to\infty} \Psi_i(x_1, \ldots, x_i, x_i', \lambda^j x_{i+1}, x_{i+2}, \ldots, x_n, |\lambda|^{2j} t) = \cdots \\
&\quad = \lim_{j\to\infty} \Psi_i(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_{n-1}, \lambda^j x_n, |\lambda|^{2j} t) = 1
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
&\lim_{k\to\infty} T_{j=k}^{\infty} T\big(\Psi_i\big(x_1,\ldots,x_{i-1},\lambda^j x_i,0,x_{i+1},\ldots,x_n,|\lambda|^{2j+1}t\big),\\
&\quad \Psi_i\big(x_1,\ldots,x_{i-1},a\lambda^j x_i,b\lambda^j x_i,x_{i+1},\ldots,x_n,|\lambda|^{2(j+1)}t\big)\big)\\
&\quad = \lim_{k\to\infty} T_{j=1}^{\infty} T\big(\Psi_i\big(x_1,\ldots,x_{i-1},\lambda^{k+j-1}x_i,0,x_{i+1},\ldots,x_n,|\lambda|^{2k+2j-1}t\big),\\
&\quad \Psi_i\big(x_1,\ldots,x_{i-1},a\lambda^{k+j-1}x_i,b\lambda^{k+j-1}x_i,x_{i+1},\ldots,x_n,|\lambda|^{2k+2j}t\big)\big) = 1 \qquad (3.2)
\end{aligned}$$

for all $x_1,\ldots,x_i,x_i',x_{i+1},\ldots,x_n \in \mathcal{X}$ *and* $t > 0$. *If* $f : \mathcal{X}^n \to \mathcal{Y}$ *is a mapping satisfying*

$$f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n) = 0, \qquad (3.3)$$

and

$$\begin{aligned}
&N\big(f\big(x_1,\ldots,x_{i-1},ax_i+bx_i',x_{i+1},\ldots,x_n\big)+f\big(x_1,\ldots,x_{i-1},bx_i-ax_i',x_{i+1},\ldots,x_n\big)\\
&\quad -\big(a^2+b^2\big)\big[f(x_1,\ldots,x_n)+f\big(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n\big)\big],t\big)\\
&\quad \ge \Psi_i\big(x_1,\ldots,x_i,x_i',x_{i+1},\ldots,x_n,t\big) \qquad (3.4)
\end{aligned}$$

for all $x_1,\ldots,x_i,x_i',x_{i+1},\ldots,x_n \in \mathcal{X}$, $i \in \{1,2,\ldots,n\}$ *and* $t \in [0,\infty)$, *then for every* $i \in \{1,2,\ldots,n\}$ *there exists a unique general multi-Euler-Lagrange quadratic mapping* $Q_i : \mathcal{X}^n \to \mathcal{Y}$ *such that*

$$\begin{aligned}
&N\big(f(x_1,\ldots,x_n)-Q_i(x_1,\ldots,x_n),t\big)\\
&\quad \ge T_{j=1}^{\infty} T\big(\Psi_i\big(x_1,\ldots,x_{i-1},\lambda^{j-1}x_i,0,x_{i+1},\ldots,x_n,|\lambda|^{2j-1}t\big),\\
&\quad \Psi_i\big(x_1,\ldots,x_{i-1},a\lambda^{j-1}x_i,b\lambda^{j-1}x_i,x_{i+1},\ldots,x_n,|\lambda|^{2j}t\big)\big) \qquad (3.5)
\end{aligned}$$

for all $x_1,\ldots,x_n \in \mathcal{X}$ *and* $t > 0$.

Proof Fix $x_1,\ldots,x_n \in \mathcal{X}$, $j \in \mathbb{N}\cup\{0\}$, $i \in \{1,2,\ldots,n\}$ and $t > 0$. Putting $x_i' = 0$ in (3.4), we get

$$\begin{aligned}
&N\big(f(x_1,\ldots,x_{i-1},ax_i,x_{i+1},\ldots,x_n)+f(x_1,\ldots,x_{i-1},bx_i,x_{i+1},\ldots,x_n)\\
&\quad -\lambda f(x_1,\ldots,x_n),t\big) \ge \Psi_i(x_1,\ldots,x_i,0,x_{i+1},\ldots,x_n,t). \qquad (3.6)
\end{aligned}$$

Replacing x_i by ax_i and x_i' by bx_i in (3.4), respectively, we have

$$\begin{aligned}
&N\big(f(x_1,\ldots,x_{i-1},\lambda x_i,x_{i+1},\ldots,x_n)-\lambda f(x_1,\ldots,x_{i-1},ax_i,x_{i+1},\ldots,x_n)\\
&\quad -\lambda f(x_1,\ldots,x_{i-1},bx_i,x_{i+1},\ldots,x_n),t\big) \ge \Psi_i(x_1,\ldots,x_{i-1},ax_i,bx_i,x_{i+1},\ldots,x_n,t). \qquad (3.7)
\end{aligned}$$

From (3.6) and (3.7), one gets

$$\begin{aligned}
&N\left(\frac{1}{\lambda^2}f(x_1,\ldots,x_{i-1},\lambda x_i,x_{i+1},\ldots,x_n)-f(x_1,\ldots,x_n),t\right)\\
&\quad \ge T\big(\Psi_i(x_1,\ldots,x_i,0,x_{i+1},\ldots,x_n,|\lambda|t),\\
&\quad \Psi_i\big(x_1,\ldots,x_{i-1},ax_i,bx_i,x_{i+1},\ldots,x_n,|\lambda|^2t\big)\big). \qquad (3.8)
\end{aligned}$$

Therefore one can get

$$\begin{aligned}
&N\biggl(\frac{1}{\lambda^{2(k+p)}}f\bigl(x_1,\ldots,x_{i-1},\lambda^{k+p}x_i,x_{i+1},\ldots,x_n\bigr)-\frac{1}{\lambda^{2k}}f\bigl(x_1,\ldots,x_{i-1},\lambda^{k}x_i,x_{i+1},\ldots,x_n\bigr),t\biggr)\\
&\quad\geq T_{j=k}^{k+p-1}T\bigl(\Psi_i\bigl(x_1,\ldots,x_{i-1},\lambda^j x_i,0,x_{i+1},\ldots,x_n,|\lambda|^{2j+1}t\bigr),\\
&\qquad\Psi_i\bigl(x_1,\ldots,x_{i-1},a\lambda^j x_i,b\lambda^j x_i,x_{i+1},\ldots,x_n,|\lambda|^{2(j+1)}t\bigr)\bigr),
\end{aligned}$$

and thus from (3.2) it follows that $\{\frac{1}{\lambda^{2j}}f(x_1,\ldots,x_{i-1},\lambda^j x_i,x_{i+1},\ldots,x_n)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in a complete non-Archimedean fuzzy normed space. Hence, we can define a mapping $Q_i:\mathcal{X}^n\to\mathcal{Y}$ such that

$$\lim_{j\to\infty}N\biggl(\frac{1}{\lambda^{2j}}f\bigl(x_1,\ldots,x_{i-1},\lambda^j x_i,x_{i+1},\ldots,x_n\bigr)-Q_i(x_1,\ldots,x_n),t\biggr)=1.$$

Next, for each $k\in\mathbb{N}$ with $k\geq 1$, we have

$$\begin{aligned}
&N\biggl(f(x_1,\ldots,x_n)-\frac{1}{\lambda^{2k}}f\bigl(x_1,\ldots,x_{i-1},\lambda^{k}x_i,x_{i+1},\ldots,x_n\bigr),t\biggr)\\
&\quad\geq T_{j=1}^{k}N\biggl(\frac{1}{\lambda^{2(j-1)}}f\bigl(x_1,\ldots,x_{i-1},\lambda^{j-1}x_i,x_{i+1},\ldots,x_n\bigr)\\
&\qquad-\frac{1}{\lambda^{2j}}f\bigl(x_1,\ldots,x_{i-1},\lambda^{j}x_i,x_{i+1},\ldots,x_n\bigr),t\biggr)\\
&\quad\geq T_{j=1}^{k}T\bigl(\Psi_i\bigl(x_1,\ldots,x_{i-1},\lambda^{j-1}x_i,0,x_{i+1},\ldots,x_n,|\lambda|^{2j-1}t\bigr),\\
&\qquad\Psi_i\bigl(x_1,\ldots,x_{i-1},a\lambda^{j-1}x_i,b\lambda^{j-1}x_i,x_{i+1},\ldots,x_n,|\lambda|^{2j}t\bigr)\bigr).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&N\bigl(f(x_1,\ldots,x_n)-Q_i(x_1,\ldots,x_n),t\bigr)\\
&\quad\geq T\biggl(N\biggl(f(x_1,\ldots,x_n)-\frac{1}{\lambda^{2k}}f\bigl(x_1,\ldots,x_{i-1},\lambda^{k}x_i,x_{i+1},\ldots,x_n\bigr),t\biggr),\\
&\qquad N\biggl(\frac{1}{\lambda^{2k}}f\bigl(x_1,\ldots,x_{i-1},\lambda^{k}x_i,x_{i+1},\ldots,x_n\bigr)-Q_i(x_1,\ldots,x_n),t\biggr)\biggr)\\
&\quad\geq T\bigl(T_{j=1}^{k}T\bigl(\Psi_i\bigl(x_1,\ldots,x_{i-1},\lambda^{j-1}x_i,0,x_{i+1},\ldots,x_n,|\lambda|^{2j-1}t\bigr),\\
&\qquad\Psi_i\bigl(x_1,\ldots,x_{i-1},a\lambda^{j-1}x_i,b\lambda^{j-1}x_i,x_{i+1},\ldots,x_n,|\lambda|^{2j}t\bigr)\bigr),\\
&\qquad N\bigl(\lambda^{-2k}f\bigl(x_1,\ldots,x_{i-1},\lambda^{k}x_ix_{i+1},\ldots,x_n\bigr)-Q_i(x_1,\ldots,x_n),t\bigr)\bigr).
\end{aligned}$$

Letting $k\to\infty$ in this inequality, we obtain (3.5). Now, fix also $x_i'\in\mathcal{X}$, from (3.1) and (3.4) it follows that

$$\begin{aligned}
&N\bigl(Q_i\bigl(x_1,\ldots,x_{i-1},ax_i+bx_i',x_{i+1},\ldots,x_n\bigr)+Q_i\bigl(x_1,\ldots,x_{i-1},bx_i-ax_i',x_{i+1},\ldots,x_n\bigr)\\
&\qquad-\bigl(a^2+b^2\bigr)\bigl[Q_i(x_1,\ldots,x_n)+Q_i\bigl(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n\bigr)\bigr],t\bigr)\\
&\quad\geq T(N\bigl(Q_i\bigl(x_1,\ldots,x_{i-1},ax_i+bx_i',x_{i+1},\ldots,x_n\bigr)\\
&\qquad-\lambda^{-2j}f\bigl(x_1,\ldots,x_{i-1},\lambda^j\bigl(ax_i+bx_i'\bigr),x_{i+1},\ldots,x_n\bigr),t\bigr),
\end{aligned}$$

$$\begin{aligned}
&N\big(Q_i(x_1,\ldots,x_{i-1},bx_i-ax_i',x_{i+1},\ldots,x_n)\\
&\quad-\lambda^{-2j}f\big(x_1,\ldots,x_{i-1},\lambda^j(bx_i-ax_i'),x_{i+1},\ldots,x_n\big),t\big),\\
&N\big(-\lambda Q_i(x_1,\ldots,x_n)+\lambda^{-2j+1}f\big(x_1,\ldots,x_{i-1},\lambda^jx_i,x_{i+1},\ldots,x_n\big),t\big),\\
&N\big(-\lambda Q_i(x_1,\ldots,x_{i-1},x_i',x_{i+1},\ldots,x_n)+\lambda^{-2j+1}f\big(x_1,\ldots,x_{i-1},\lambda^jx_i',x_{i+1},\ldots,x_n\big),t\big),\\
&\Psi_i\big(x_1,\ldots,x_{i-1},\lambda^jx_i,\lambda^jx_i',x_{i+1},\ldots,x_n,|\lambda|^{2j}t\big)\to 1\quad(j\to\infty).
\end{aligned}$$

Next, fix $k\in\{1,\ldots,n\}\backslash\{i\}$, $x_k'\in\mathcal{X}$, and assume, without loss of generality, that $k<i$ (the same arguments apply to the case where $k>i$). From (3.1) and (3.4), it follows that

$$\begin{aligned}
&N\big(Q_i(x_1,\ldots,x_{k-1},ax_k+bx_k',x_{k+1},\ldots,x_n)+Q_i(x_1,\ldots,x_{k-1},bx_k-ax_k',x_{k+1},\ldots,x_n)\\
&\qquad-(a^2+b^2)\big[Q_i(x_1,\ldots,x_n)+Q_i(x_1,\ldots,x_{k-1},x_k',x_{k+1},\ldots,x_n)\big],t\big)\\
&\quad\geq T\big(N\big(Q_i(x_1,\ldots,x_{k-1},ax_k+bx_k',x_{k+1},\ldots,x_n)\\
&\qquad-\lambda^{-2j}f\big(x_1,\ldots,x_{k-1},ax_k+bx_k',x_{k+1},\ldots,x_{i-1},\lambda^jx_i,x_{i+1},\ldots,x_n\big),t\big),\\
&\qquad N\big(Q_i(x_1,\ldots,x_{k-1},bx_k-ax_k',x_{k+1},\ldots,x_n)\\
&\qquad-\lambda^{-2j}f\big(x_1,\ldots,x_{k-1},bx_k-ax_k',x_{k+1},\ldots,x_{i-1},\lambda^jx_i,x_{i+1},\ldots,x_n\big),t\big),\\
&\qquad N\big(-\lambda Q_i(x_1,\ldots,x_n)+\lambda^{-2j+1}f\big(x_1,\ldots,x_{i-1},\lambda^jx_i,x_{i+1},\ldots,x_n\big),t\big),\\
&\qquad N\big(-\lambda Q_i(x_1,\ldots,x_{k-1},x_k',x_{k+1},\ldots,x_n)\\
&\qquad+\lambda^{-2j+1}f\big(x_1,\ldots,x_{k-1},x_k',x_{k+1},\ldots,x_{i-1},\lambda^jx_i,x_{i+1},\ldots,x_n\big),t\big),\\
&\qquad\Psi_k\big(x_1,\ldots,x_k,x_k',x_{k+1},\ldots,x_{i-1},\lambda^jx_i,x_{i+1},\ldots,x_n,|\lambda|^{2j}t\big)\big)\to 1\quad(j\to\infty).
\end{aligned}$$

Hence the mapping Q_i is a general multi-Euler-Lagrange quadratic mapping. Let us finally assume that $Q_i':\mathcal{X}^n\to\mathcal{Y}$ is another multi-Euler-Lagrange quadratic mapping satisfying (3.5). Then, by (1.4), (3.5) and (3.2), it follows that

$$\begin{aligned}
&N\big(Q_i(x_1,\ldots,x_n)-Q_i'(x_1,\ldots,x_n),t\big)\\
&\quad=N\big(Q_i(x_1,\ldots,x_{i-1},\lambda^kx_i,x_{i+1},\ldots,x_n)-Q_i'(x_1,\ldots,x_{i-1},\lambda^kx_i,x_{i+1},\ldots,x_n),|\lambda|^{2k}t\big)\\
&\quad\geq T\big(T_{j=1}^{\infty}T\big(\Psi_i\big(x_1,\ldots,x_{i-1},\lambda^{k+j-1}x_i,0,x_{i+1},\ldots,x_n,|\lambda|^{2k+2j-1}t\big),\\
&\qquad\Psi_i\big(x_1,\ldots,x_{i-1},a\lambda^{k+j-1}x_i,b\lambda^{k+j-1}x_i,x_{i+1},\ldots,x_n,|\lambda|^{2k+2j}t\big)\big),\\
&\qquad T_{j=1}^{\infty}T\big(\Psi_i\big(x_1,\ldots,x_{i-1},\lambda^{k+j-1}x_i,0,x_{i+1},\ldots,x_n,|\lambda|^{2k+2j-1}t\big),\\
&\qquad\Psi_i\big(x_1,\ldots,x_{i-1},a\lambda^{k+j-1}x_i,b\lambda^{k+j-1}x_i,x_{i+1},\ldots,x_n,|\lambda|^{2k+2j}t\big)\big)\big)\\
&\quad\to 1\quad(k\to\infty)
\end{aligned}$$

and therefore $Q_i=Q_i'$. □

For $a=b$, we get the following result.

Theorem 3.2 *Let $\mathbb{K}$ be a valued field, $\mathcal{X}$ be a vector space over $\mathbb{K}$ and $(\mathcal{Y},N,T)$ be a complete non-Archimedean fuzzy normed space over $\mathbb{K}$. Assume also that, for every*

$i \in \{1,2,\dots,n\}$, $\Psi_i : \mathcal{X}^{n+1} \times [0,\infty) \to [0,1]$ *is a mapping such that*

$$\begin{aligned}
&\lim_{j\to\infty} \Psi_i\big(\lambda_1^j x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n,|\lambda_1|^{2j}t\big)\\
&\quad= \cdots\\
&\quad= \lim_{j\to\infty} \Psi_i\big(x_1,\dots,x_{i-2},\lambda_1^j x_{i-1},x_i,x_i',x_{i+1},\dots,x_n,|\lambda_1|^{2j}t\big)\\
&\quad= \lim_{j\to\infty} \Psi_i\big(x_1,\dots,x_{i-1},\lambda_1^j x_i,\lambda_1^j x_i',x_{i+1},\dots,x_n,|\lambda_1|^{2j}t\big)\\
&\quad= \lim_{j\to\infty} \Psi_i\big(x_1,\dots,x_i,x_i',\lambda_1^j x_{i+1},x_{i+2},\dots,x_n,|\lambda_1|^{2j}t\big) = \cdots\\
&\quad= \lim_{j\to\infty} \Psi_i\big(x_1,\dots,x_i,x_i',x_{i+1},\dots,x_{n-1},\lambda_1^j x_n,|\lambda_1|^{2j}t\big) = 1
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
&\lim_{k\to\infty} T_{j=k}^{\infty} \Psi_i\big(x_1,\dots,x_{i-1},\lambda_1^j x_i,\lambda_1^j x_i,x_{i+1},\dots,x_n,|\lambda_1|^{2j+2}t\big)\\
&\quad= \lim_{k\to\infty} T_{j=1}^{\infty} \Psi_i\big(x_1,\dots,x_{i-1},\lambda_1^{k+j-1} x_i,\lambda_1^{k+j-1} x_i,x_{i+1},\dots,x_n,|\lambda_1|^{2k+2j}t\big) = 1
\end{aligned} \tag{3.10}$$

for all $x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n \in \mathcal{X}$ *and* $t > 0$. *If* $f : \mathcal{X}^n \to \mathcal{Y}$ *is a mapping satisfying (3.3) and*

$$\begin{aligned}
&N\big(f\big(x_1,\dots,x_{i-1},a\big(x_i+x_i'\big),x_{i+1},\dots,x_n\big) + f\big(x_1,\dots,x_{i-1},a\big(x_i-x_i'\big),x_{i+1},\dots,x_n\big)\\
&\qquad - 2a^2\big[f(x_1,\dots,x_n) + f\big(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_n\big)\big],t\big)\\
&\quad\ge \Psi_i\big(x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n,t\big)
\end{aligned} \tag{3.11}$$

for all $x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n \in \mathcal{X}$, $i \in \{1,2,\dots,n\}$ *and* $t \in [0,\infty)$, *then for every* $i \in \{1,2,\dots,n\}$ *there exists a unique general multi-Euler-Lagrange quadratic mapping* $Q_i : \mathcal{X}^n \to \mathcal{Y}$ *satisfying the functional equation (1.5) and such that*

$$\begin{aligned}
&N\big(f(x_1,\dots,x_n) - Q_i(x_1,\dots,x_n),t\big)\\
&\quad\ge T_{j=1}^{\infty} \Psi_i\big(x_1,\dots,x_{i-1},\lambda_1^{j-1} x_i,\lambda_1^{j-1} x_i,x_{i+1},\dots,x_n,|\lambda_1|^{2j}t\big)
\end{aligned} \tag{3.12}$$

for all $x_1,\dots,x_n \in \mathcal{X}$ *and* $t > 0$.

Proof Fix $x_1,\dots,x_n \in \mathcal{X}$, $j \in \mathbb{N} \cup \{0\}$, $i \in \{1,2,\dots,n\}$ and $t > 0$. Putting $x_i' = x_i$ in (3.11), we get

$$\begin{aligned}
&N\bigg(\frac{1}{\lambda_1^2} f(x_1,\dots,x_{i-1},\lambda_1 x_i,x_{i+1},\dots,x_n) - f(x_1,\dots,x_n),t\bigg)\\
&\quad\ge \Psi_i\big(x_1,\dots,x_i,x_i,x_{i+1},\dots,x_n,|\lambda_1|^2 t\big).
\end{aligned} \tag{3.13}$$

Hence,

$$\begin{aligned}
&N\bigg(\frac{1}{\lambda_1^{2(j+1)}} f\big(x_1,\dots,x_{i-1},\lambda_1^{j+1} x_i,x_{i+1},\dots,x_n\big) - \frac{1}{\lambda_1^{2j}} f\big(x_1,\dots,x_{i-1},\lambda_1^j x_i,x_{i+1},\dots,x_n\big),t\bigg)\\
&\quad\ge \Psi_i\big(x_1,\dots,x_{i-1},\lambda_1^j x_i,\lambda_1^j x_i,x_{i+1},\dots,x_n,|\lambda_1|^{2j+2}t\big).
\end{aligned}$$

Therefore one can get

$$\begin{aligned}
&N\left(\frac{1}{\lambda_1^{2(k+p)}}f(x_1,\ldots,x_{i-1},\lambda_1^{k+p}x_i,x_{i+1},\ldots,x_n)-\frac{1}{\lambda_1^{2k}}f(x_1,\ldots,x_{i-1},\lambda^k x_i,x_{i+1},\ldots,x_n),t\right)\\
&\quad\geq T_{j=k}^{k+p-1}N\left(\frac{1}{\lambda_1^{2(j+1)}}f(x_1,\ldots,x_{i-1},\lambda_1^{j+1}x_i,x_{i+1},\ldots,x_n)\right.\\
&\qquad\left.-\frac{1}{\lambda_1^{2j}}f(x_1,\ldots,x_{i-1},\lambda_1^j x_i,x_{i+1},\ldots,x_n),t\right)\\
&\quad\geq T_{j=k}^{k+p-1}\Psi_i(x_1,\ldots,x_{i-1},\lambda_1^j x_i,\lambda_1^j x_i,x_{i+1},\ldots,x_n,|\lambda_1|^{2j+2}t),
\end{aligned}$$

and thus by (3.10) it follows that $\{\frac{1}{\lambda_1^{2j}}f(x_1,\ldots,x_{i-1},\lambda_1^j x_i,x_{i+1},\ldots,x_n)\}_{j\in\mathbb{N}}$ is a Cauchy sequence in a complete non-Archimedean fuzzy normed space. Hence, we can define a mapping $Q_i:\mathcal{X}^n\to\mathcal{Y}$ such that

$$\lim_{j\to\infty}N\left(\frac{1}{\lambda_1^{2j}}f(x_1,\ldots,x_{i-1},\lambda_1^j x_i,x_{i+1},\ldots,x_n)-Q_i(x_1,\ldots,x_n),t\right)=1.$$

Using (3.13) and induction, one can show that for any $k\in\mathbb{N}$ we have

$$\begin{aligned}
&N\left(f(x_1,\ldots,x_n)-\frac{1}{\lambda_1^{2k}}f(x_1,\ldots,x_{i-1},\lambda_1^k x_i,x_{i+1},\ldots,x_n),t\right)\\
&\quad\geq T_{j=1}^{k}\Psi_i(x_1,\ldots,x_{i-1},\lambda_1^{j-1}x_i,\lambda_1^{j-1}x_i,x_{i+1},\ldots,x_n,|\lambda_1|^{2j}t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&N(f(x_1,\ldots,x_n)-Q_i(x_1,\ldots,x_n),t)\\
&\quad\geq T\big(T_{j=1}^{k}\Psi_i(x_1,\ldots,x_{i-1},\lambda_1^{j-1}x_i,\lambda_1^{j-1}x_i,x_{i+1},\ldots,x_n,|\lambda_1|^{2j}t),\\
&\qquad N(\lambda_1^{-2k}f(x_1,\ldots,x_{i-1},\lambda_1^k x_i,x_{i+1},\ldots,x_n)-Q_i(x_1,\ldots,x_n),t)\big).
\end{aligned}$$

Letting $k\to\infty$ in this inequality, we obtain (3.12). The rest of the proof of this theorem is omitted as being similar to the corresponding that of Theorem 3.1. □

Let $(\mathcal{Y},N,T)$ be a complete non-Archimedean fuzzy normed space over a non-Archimedean field $\mathbb{K}$. In any such space, a sequence $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy if and only if $\{x_{k+1}-x_k\}_{k\in\mathbb{N}}$ converges to zero. Analysis similar to that in the proof of Theorem 3.2 gives the following.

Theorem 3.3 *Let $\mathbb{K}$ be a non-Archimedean field, $\mathcal{X}$ be a vector space over $\mathbb{K}$ and $(\mathcal{Y},N,T)$ be a complete non-Archimedean fuzzy normed space over $\mathbb{K}$. Assume also that, for every $i\in\{1,2,\ldots,n\}$, $\Psi_i:\mathcal{X}^{n+1}\times[0,\infty)\to[0,1]$ is a mapping such that (3.9) holds and*

$$\lim_{k\to\infty}T_{j=1}^{\infty}\Psi_i(x_1,\ldots,x_{i-1},\lambda_1^{k+j-1}x_i,\lambda_1^{k+j-1}x_i,x_{i+1},\ldots,x_n,|\lambda_1|^{2k+2j}t)=1$$

for all $x_1,\ldots,x_i,x_i',x_{i+1},\ldots,x_n\in\mathcal{X}$ and $t>0$. If $f:\mathcal{X}^n\to\mathcal{Y}$ is a mapping satisfying (3.3) and (3.11), then for every $i\in\{1,2,\ldots,n\}$ there exists a unique general multi-Euler-Lagrange quadratic mapping $Q_i:\mathcal{X}^n\to\mathcal{Y}$ satisfying (1.5) and (3.12).

Remark 3.4 Let $a, b \in \mathbb{N}$ and $\mathcal{X}$ be a commutative group, Theorems 3.1-3.3 also hold. For $a = 1$, consider the non-Archimedean fuzzy normed space $(\mathcal{Y}, N_1, T_M)$ defined as in Example 2.6, Theorem 3.3 yields Theorem 2 in [7]. If $a = b = \pm\frac{1}{\sqrt{2}} \in \mathbb{K}$, then $\lambda_1 = 2a = \pm\sqrt{2} \neq 1$ in Theorems 3.2-3.3 and $\lambda = a^2 + b^2 = 1$ which is a singular case $\lambda = 1$ of Theorem 3.1.

Analysis similar to that in the proof of Theorem 3.1 gives the following.

Theorem 3.5 *Let $\mathbb{K}$ be a valued field, $\mathcal{X}$ be a vector space over $\mathbb{K}$ and $(\mathcal{Y}, N, T)$ be a complete non-Archimedean fuzzy normed space over $\mathbb{K}$. Assume also that, for every $i \in \{1, 2, \ldots, n\}$, $\Psi_i : \mathcal{X}^{n+1} \times [0, \infty) \to [0, 1]$ is a mapping such that*

$$\begin{aligned}
&\lim_{j\to\infty} \Psi_i\bigl(\lambda^{-j}x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n, |\lambda|^{-2j}t\bigr)\\
&\quad = \cdots\\
&\quad = \lim_{j\to\infty} \Psi_i\bigl(x_1, \ldots, x_{i-2}, \lambda^{-j}x_{i-1}, x_i, x_i', x_{i+1}, \ldots, x_n, |\lambda|^{-2j}t\bigr)\\
&\quad = \lim_{j\to\infty} \Psi_i\bigl(x_1, \ldots, x_{i-1}, \lambda^{-j}x_i, \lambda^{-j}x_i', x_{i+1}, \ldots, x_n, |\lambda|^{-2j}t\bigr)\\
&\quad = \lim_{j\to\infty} \Psi_i\bigl(x_1, \ldots, x_i, x_i', \lambda^{-j}x_{i+1}, x_{i+2}, \ldots, x_n, |\lambda|^{-2j}t\bigr) = \cdots\\
&\quad = \lim_{j\to\infty} \Psi_i\bigl(x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_{n-1}, \lambda^{-j}x_n, |\lambda|^{-2j}t\bigr) = 1
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
&\lim_{k\to\infty} T_{j=k}^{\infty} T\bigl(\Psi_i\bigl(x_1, \ldots, x_{i-1}, \lambda^{-j-1}x_i, 0, x_{i+1}, \ldots, x_n, |\lambda|^{-2j-1}t\bigr),\\
&\qquad \Psi_i\bigl(x_1, \ldots, x_{i-1}, a\lambda^{-j-1}x_i, b\lambda^{-j-1}x_i, x_{i+1}, \ldots, x_n, |\lambda|^{-2j}t\bigr)\bigr)\\
&\quad = \lim_{k\to\infty} T_{j=1}^{\infty} T\bigl(\Psi_i\bigl(x_1, \ldots, x_{i-1}, \lambda^{-k-j-1}x_i, 0, x_{i+1}, \ldots, x_n, |\lambda|^{-2k-2j-1}t\bigr),\\
&\qquad \Psi_i\bigl(x_1, \ldots, x_{i-1}, a\lambda^{-k-j-1}x_i, b\lambda^{-k-j-1}x_i, x_{i+1}, \ldots, x_n, |\lambda|^{-2k-2j}t\bigr)\bigr) = 1
\end{aligned}\tag{3.15}$$

for all $x_1, \ldots, x_i, x_i', x_{i+1}, \ldots, x_n \in \mathcal{X}$ and $t > 0$. If $f : \mathcal{X}^n \to \mathcal{Y}$ is a mapping satisfying (3.3) and (3.4), then for every $i \in \{1, 2, \ldots, n\}$ there exists a unique general multi-Euler-Lagrange quadratic mapping $Q_i : \mathcal{X}^n \to \mathcal{Y}$ such that

$$\begin{aligned}
&N\bigl(f(x_1, \ldots, x_n) - Q_i(x_1, \ldots, x_n), t\bigr)\\
&\quad \geq T_{j=1}^{\infty} T\bigl(\Psi_i\bigl(x_1, \ldots, x_{i-1}, \lambda^{-j-1}x_i, 0, x_{i+1}, \ldots, x_n, |\lambda|^{-2j-1}t\bigr),\\
&\qquad \Psi_i\bigl(x_1, \ldots, x_{i-1}, a\lambda^{-j-1}x_i, b\lambda^{-j-1}x_i, x_{i+1}, \ldots, x_n, |\lambda|^{-2j}t\bigr)\bigr)
\end{aligned}$$

for all $x_1, \ldots, x_n \in \mathcal{X}$ and $t > 0$.

Corollary 3.6 *Let $\mathbb{K}$ be a non-Archimedean field with $0 < |\lambda| < 1$, $\mathcal{X}$ be a normed space over $\mathbb{K}$ and let $(\mathcal{Y}, N, T)$ be a complete non-Archimedean fuzzy normed space over $\mathbb{K}$ under a t-norm $T \in \mathcal{H}$. Assume also that $\delta > 0$ and $\alpha : [0, \infty) \to [0, \infty)$ is a function such that $\alpha(|\lambda|^{-1}) < |\lambda|^{-1}$ and $\alpha(|\lambda|^{-1}t) \leq \alpha(|\lambda|^{-1})\alpha(t)$ for all $t \in [0, \infty)$.*

If $f : \mathcal{X}^n \to \mathcal{Y}$ is a mapping satisfying (3.3) and

$$\begin{aligned} &N\big(f\big(x_1,\dots,x_{i-1},ax_i+bx_i',x_{i+1},\dots,x_n\big)+f\big(x_1,\dots,x_{i-1},bx_i-ax_i',x_{i+1},\dots,x_n\big) \\ &\quad -\big(a^2+b^2\big)\big[f(x_1,\dots,x_n)+f\big(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_n\big)\big],t\big) \\ &\geq \frac{t}{t+\delta\alpha(\|x_i\|)\alpha(\|x_i'\|)} \end{aligned} \tag{3.16}$$

for all $x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n \in \mathcal{X}$, $i \in \{1,2,\dots,n\}$ and $t \in [0,\infty)$, then for every $i \in \{1,2,\dots,n\}$ there exists a unique general multi-Euler-Lagrange quadratic mapping $Q_i : \mathcal{X}^n \to \mathcal{Y}$ such that

$$\begin{aligned} &N\big(f(x_1,\dots,x_n)-Q_i(x_1,\dots,x_n),t\big) \\ &\geq T_{j=1}^{\infty} T\left(\frac{t}{t+\delta|\lambda|^{2j+1}\alpha(|\lambda|^{-1})^{j+1}\alpha(\|x_i\|)\alpha(0)}, \frac{t}{t+\delta|\lambda|^{2j}\alpha(|\lambda|^{-1})^{2j+2}\alpha(\|ax_i\|)\alpha(\|bx_i\|)}\right) \end{aligned}$$

for all $x_1,\dots,x_n \in \mathcal{X}$ and $t > 0$.

Proof Fix $i \in \{1,2,\dots,n\}$, $x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n \in \mathcal{X}$ and $t \in [0,\infty)$. Let $\Psi_i : \mathcal{X}^{n+1} \times [0,\infty) \to [0,1]$ be defined by $\Psi_i(x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n,t) := \frac{t}{t+\delta\alpha(\|x_i\|)\alpha(\|x_i'\|)}$. Then we can apply Theorem 3.5 to obtain the result. □

Remark 3.7 Let $0 < |\lambda| < 1$ and $p \in (0,1)$. Then the mapping $\alpha : [0,\infty) \to [0,\infty)$ given by $\alpha(t) := t^p$, $t \in [0,\infty)$ satisfies $\alpha(|\lambda|^{-1}) < |\lambda|^{-1}$ and $\alpha(|\lambda|^{-1}t) \leq \alpha(|\lambda|^{-1})\alpha(t)$ for all $t \in [0,\infty)$.

4 Stability of the functional equation (1.1): a fixed point method

Throughout this section, we prove the stability of Eq. (1.1) in complete non-Archimedean fuzzy normed spaces using the fixed point method.

Theorem 4.1 *Let $\mathbb{K}$ be a valued field, $\mathcal{X}$ be a vector space over $\mathbb{K}$ and $(\mathcal{Y},N,T)$ be a complete non-Archimedean fuzzy normed space over $\mathbb{K}$. Assume also that, for every $i \in \{1,2,\dots,n\}$, $\Psi_i : \mathcal{X}^{n+1} \times [0,\infty) \to [0,1]$ is a mapping such that (3.1) holds and*

$$\begin{aligned} &\Psi_i\big(x_1,\dots,x_{i-1},\lambda x_i,\lambda x_i,x_{i+1},\dots,x_n,|\lambda|^2 L_i t\big) \\ &\geq \Psi_i(x_1,\dots,x_i,x_i,x_{i+1},\dots,x_n,t), \quad x_1,\dots,x_n \in \mathcal{X} \end{aligned} \tag{4.1}$$

for an $L_i \in (0,1)$. If $f : \mathcal{X}^n \to \mathcal{Y}$ is a mapping satisfying (3.3) and (3.4), then for every $i \in \{1,2,\dots,n\}$ there exists a unique general multi-Euler-Lagrange quadratic mapping $Q_i : \mathcal{X}^n \to \mathcal{Y}$ such that

$$\begin{aligned} &N\big(f(x_1,\dots,x_n)-Q_i(x_1,\dots,x_n),t\big) \\ &\geq T\big(\Psi_i\big(x_1,\dots,x_i,0,x_{i+1},\dots,x_n,|\lambda|(1-L_i)t\big), \\ &\quad \Psi_i\big(x_1,\dots,x_{i-1},ax_i,bx_i,x_{i+1},\dots,x_n,|\lambda|^2(1-L_i)t\big)\big) \end{aligned} \tag{4.2}$$

for all $x_1,\dots,x_n \in \mathcal{X}$ and $t > 0$.

Proof Fix an $i \in \{1, 2, \ldots, n\}$. Consider the set $\Omega := \{g : \mathcal{X}^n \to \mathcal{Y}\}$ and introduce the generalized metric on Ω:

$$\begin{aligned} d_i(g,h) = \inf\bigl\{C \in [0,\infty] : & N\bigl(g(x_1,\ldots,x_n) - h(x_1,\ldots,x_n), Ct\bigr) \\ & \geq T\bigl(\Psi_i(x_1,\ldots,x_i,0,x_{i+1},\ldots,x_n,|\lambda|t), \\ & \Psi_i\bigl(x_1,\ldots,x_{i-1},ax_i,bx_i,x_{i+1},\ldots,x_n,|\lambda|^2t\bigr)\bigr), \\ & x_1,\ldots,x_n \in \mathcal{X}, t > 0\bigr\}, \quad g,h \in \Omega. \end{aligned}$$

A standard verification (see for instance [19]) shows that (Ω, d_i) is a complete generalized metric space. We now define an operator $J_i : \Omega \to \Omega$ by

$$J_i g(x_1,\ldots,x_n) = \frac{1}{\lambda^2} g(x_1,\ldots,x_{i-1},\lambda x_i,x_{i+1},\ldots,x_n), \quad g \in \Omega, x_1,\ldots,x_n \in \mathcal{X}.$$

Let $g, h \in \Omega$ and $C_{g,h} \in [0,\infty]$ with $d_i(g,h) \leq C_{g,h}$. Then

$$\begin{aligned} N\bigl(g(x_1,\ldots,x_n) - h(x_1,\ldots,x_n), C_{g,h}t\bigr) \geq T\bigl(&\Psi_i(x_1,\ldots,x_i,0,x_{i+1},\ldots,x_n,|\lambda|t), \\ &\Psi_i\bigl(x_1,\ldots,x_{i-1},ax_i,bx_i,x_{i+1},\ldots,x_n,|\lambda|^2t\bigr)\bigr), \end{aligned}$$

which together with (4.1) gives

$$\begin{aligned} N\bigl(J_i g(x_1,\ldots,x_n) - J_i h(x_1,\ldots,x_n), t\bigr) \geq T\biggl(&\Psi_i\biggl(x_1,\ldots,x_i,0,x_{i+1},\ldots,x_n,\frac{|\lambda|t}{L_i C_{g,h}}\biggr), \\ &\Psi_i\biggl(x_1,\ldots,x_{i-1},ax_i,bx_i,x_{i+1},\ldots,x_n,\frac{|\lambda|^2 t}{L_i C_{g,h}}\biggr)\biggr), \end{aligned}$$

and consequently, $d_i(J_i g, J_i h) \leq L_i C_{g,h}$, which means that the operator J_i is strictly contractive. Moreover, from (3.8) it follows that

$$\begin{aligned} N\bigl(J_i f(x_1,\ldots,x_n) - f(x_1,\ldots,x_n), t\bigr) \geq T\bigl(&\Psi_i(x_1,\ldots,x_i,0,x_{i+1},\ldots,x_n,|\lambda|t), \\ &\Psi_i\bigl(x_1,\ldots,x_{i-1},ax_i,bx_i,x_{i+1},\ldots,x_n,|\lambda|^2t\bigr)\bigr) \end{aligned}$$

and thus $d_i(J_i f, f) \leq 1 < \infty$. Therefore, by Theorem 2.8, J_i has a unique fixed point $Q_i : \mathcal{X}^n \to \mathcal{Y}$ in the set $\Omega^* = \{g \in \Omega : d(f,g) < \infty\}$ such that

$$\frac{1}{\lambda^2} Q_i(x_1,\ldots,x_{i-1},\lambda x_i,x_{i+1},\ldots,x_n) = Q_i(x_1,\ldots,x_n) \tag{4.3}$$

and

$$Q_i(x_1,\ldots,x_n) = \lim_{j\to\infty} \frac{1}{\lambda^{2j}} f\bigl(x_1,\ldots,x_{i-1},\lambda^j x_i,x_{i+1},\ldots,x_n\bigr).$$

Furthermore, from the fact that $f \in \Omega^*$, Theorem 2.8, and $d_i(J_i f, f) \leq 1$, we get

$$d_i(f, Q_i) \leq \frac{1}{1 - L_i} d_i(J_i f, f) \leq \frac{1}{1 - L_i}$$

and (4.2) follows. Similar to the proof of Theorem 3.1, one can prove that the mapping Q_i is also general multi-Euler-Lagrange quadratic.

Let us finally assume that $Q'_i : \mathcal{X}^n \to \mathcal{Y}$ is a general multi-Euler-Lagrange quadratic mapping satisfying condition (4.2). Then Q'_i fulfills (4.3), and therefore, it is a fixed point of the operator J_i. Moreover, by (4.2), we have $d_i(f, Q'_i) \le \frac{1}{1-L_i} < \infty$, and consequently $Q'_i \in \Omega^*$. Theorem 2.8 shows that $Q'_i = Q_i$. □

Similar to Theorem 4.1, one can prove the following result.

Theorem 4.2 *Let $\mathbb{K}$ be a valued field, $\mathcal{X}$ be a vector space over $\mathbb{K}$ and $(\mathcal{Y}, N, T)$ be a complete non-Archimedean fuzzy normed space over $\mathbb{K}$. Assume also that, for every $i \in \{1,2,\dots,n\}$, $\Psi_i : \mathcal{X}^{n+1} \times [0,\infty) \to [0,1]$ is a mapping such that (3.14) holds and*

$$\begin{aligned}&\Psi_i\big(x_1,\dots,x_{i-1},\lambda^{-1}x_i,\lambda^{-1}x_i,x_{i+1},\dots,x_n,|\lambda|^{-2}L_i t\big)\\&\quad\ge \Psi_i(x_1,\dots,x_i,x_i,x_{i+1},\dots,x_n,t),\quad x_1,\dots,x_n \in \mathcal{X}\end{aligned}$$

for an $L_i \in (0,1)$. If $f : \mathcal{X}^n \to \mathcal{Y}$ is a mapping satisfying (3.3) and (3.4), then for every $i \in \{1,2,\dots,n\}$ there exists a unique general multi-Euler-Lagrange quadratic mapping $Q_i : \mathcal{X}^n \to \mathcal{Y}$ such that

$$\begin{aligned}N\big(f(x_1,\dots,x_n) - Q_i(x_1,\dots,x_n),t\big) \ge T\big(&\Psi_i\big(x_1,\dots,x_i,0,x_{i+1},\dots,x_n,|\lambda|\big(L_i^{-1}-1\big)t\big),\\&\Psi_i\big(x_1,\dots,x_{i-1},ax_i,bx_i,x_{i+1},\dots,x_n,|\lambda|^2\big(L_i^{-1}-1\big)t\big)\big)\end{aligned}$$

for all $x_1,\dots,x_n \in \mathcal{X}$ and $t > 0$.

Remark 4.3 Similar to the proof of Corollary 3.6, one can deduce from Theorem 4.2 an analog of Corollary 3.6.

As applications of Theorems 4.1 and 4.2 , we get the following corollaries.

Corollary 4.4 *Let $\mathcal{X}$ be a real normed space, $\mathcal{Y}$ be a real Banach space and $(\mathcal{Y}, N, T_P)$ be the complete non-Archimedean fuzzy normed space defined as in the second example in the preliminaries. Let $\delta, r, p, q \in (0,\infty)$ such that $r, s := p+q \in (2,\infty)$, or $r, s \in (0,2)$. If $f : \mathcal{X}^n \to \mathcal{Y}$ is a mapping satisfying (3.3) and (3.4), where*

$$\Psi_i\big(x_1,\dots,x_i,x'_i,x_{i+1},\dots,x_n,t\big) := \frac{t}{t+\delta[\|x_1\|^r\cdots\|x_{i-1}\|^r(\|x_i\|^p\|x'_i\|^q)\|x_{i+1}\|^r\cdots\|x_n\|^r]},$$

then for every $i \in \{1,2,\dots,n\}$ there exists a unique general multi-Euler-Lagrange quadratic mapping $Q_i : \mathcal{X}^n \to \mathcal{Y}$ such that

$$\begin{aligned}&e^{-\frac{\|f(x_1,\dots,x_n)-Q_i(x_1,\dots,x_n)\|}{t}}\\&\quad\ge \frac{|\lambda^s-\lambda^2|t}{|\lambda^s-\lambda^2|t+\delta|a|^p|b|^q(\|x_1\|^r\cdots\|x_{i-1}\|^r\|x_i\|^s\|x_{i+1}\|^r\cdots\|x_n\|^r)}\end{aligned}\tag{4.4}$$

for all $x_1,\dots,x_n \in \mathcal{X}$ and $t > 0$.

Proof Fix $i \in \{1,2,\dots,n\}$, $x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n \in \mathcal{X}$, $t \in [0,\infty)$ and assume that $\lambda > 1$, $r,s \in (2,\infty)$ (the same arguments apply to the case where $\lambda < 1$, $r,s \in (0,2)$). Then we can choose $L_i = \lambda^{2-s} < 1$ and apply Theorem 4.2 to obtain the result. For $\lambda > 1$, $r,s \in (0,2)$, or $\lambda < 1$, $r,s \in (2,\infty)$, the corollary follows from Theorem 4.1. □

Corollary 4.5 *Let $\mathcal{X}$ be a real normed space and $\mathcal{Y}$ be a real Banach space (or $\mathcal{X}$ be a non-Archimedean normed space and $\mathcal{Y}$ be a complete non-Archimedean normed space over a non-Archimedean field $\mathbb{K}$, respectively). Let $\delta > 0$ and $r \in (0,2) \cup (2,\infty)$. If $f : \mathcal{X}^n \to \mathcal{Y}$ is a mapping satisfying (3.3) and*

$$\begin{aligned}&\big\|f(x_1,\dots,x_{i-1},ax_i+bx_i',x_{i+1},\dots,x_n)+f(x_1,\dots,x_{i-1},bx_i-ax_i',x_{i+1},\dots,x_n),\\&\quad-(a^2+b^2)\big[f(x_1,\dots,x_n)+f(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_n)\big]\big\|\\&\le\delta\big[\|x_1\|^r\cdots\|x_{i-1}\|^r\big(\|x_i\|^r+\|x_i'\|^r\big)\|x_{i+1}\|^r\cdots\|x_n\|^r\big],\end{aligned}$$

then for every $i \in \{1,2,\dots,n\}$ there exists a unique general multi-Euler-Lagrange quadratic mapping $Q_i : \mathcal{X}^n \to \mathcal{Y}$ such that

$$\big\|f(x_1,\dots,x_n)-Q_i(x_1,\dots,x_n)\big\|\le\frac{\max\{|\lambda|,|a|^r+|b|^r\}\delta(\|x_1\|^r\cdots\|x_n\|^r)}{||\lambda|^r-|\lambda|^2|}$$

for all $x_1,\dots,x_n \in \mathcal{X}$.

Proof Consider the non-Archimedean fuzzy normed space $(\mathcal{Y},N_1,T_M)$ defined as in the first example in the preliminaries, Ψ_i be defined by

$$\Psi_i(x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n,t):=\frac{t}{t+\delta[\|x_1\|^r\cdots\|x_{i-1}\|^r(\|x_i\|^r+\|x_i'\|^r)\|x_{i+1}\|^r\cdots\|x_n\|^r]},$$

and apply Theorems 4.1 and 4.2.

The following example shows that the Hyers-Ulam stability problem for the case of $r = 2$ was excluded in Corollary 4.5. □

Example 4.6 Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x)=\begin{cases}x^2, & \text{for } |x|<1,\\ 1, & \text{for } |x|\ge 1.\end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ be defined by

$$f(x)=\sum_{j=0}^{\infty}\frac{\phi(\alpha^j x)}{\alpha^{2j}}$$

for all $x \in \mathbb{C}$, where $\alpha > \max\{|a|,|b|,1\}$. Then f satisfies the functional inequality

$$\begin{aligned}&\big|f(ax+by)+f(bx-ay)-(a^2+b^2)\big[f(x)+f(y)\big]\big|\\&\le\frac{2\alpha^4(|a|^2+|b|^2+1)}{\alpha^2-1}\big(|x|^2+|y|^2\big)\end{aligned}\tag{4.5}$$

for all $x, y \in \mathbb{C}$, but there do not exist a general multi-Euler-Lagrange quadratic function $Q : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|f(x) - Q(x)| \leq d|x|^2$ for all $x \in \mathbb{C}$.

It is clear that f is bounded by $\frac{\alpha^2}{\alpha^2-1}$ on $\mathbb{C}$. If $|x|^2 + |y|^2 = 0$ or $|x|^2 + |y|^2 \geq \frac{1}{\alpha^2}$, then

$$\left|f(ax+by) + f(bx-ay) - \left(a^2+b^2\right)\left[f(x) + f(y)\right]\right| \leq \frac{2\alpha^4(|a|^2+|b|^2+1)}{\alpha^2-1}\left(|x|^2+|y|^2\right).$$

Now suppose that $0 < |x|^2 + |y|^2 < \frac{1}{\alpha^2}$. Then there exists an integer $k \geq 1$ such that

$$\frac{1}{\alpha^{2(k+2)}} \leq |x|^2 + |y|^2 < \frac{1}{\alpha^{2(k+1)}}. \tag{4.6}$$

Hence

$$\alpha^l|ax+by| < 1, \qquad \alpha^l|bx-ay| < 1, \qquad \alpha^l|x| < 1, \qquad \alpha^l|y| < 1$$

for all $l = 0, 1, \ldots, k-1$. From the definition of f and the inequality (4.6), we obtain that f satisfies (4.5). Now, we claim that the functional equation (1.1) is not stable for $r = 2$ in Corollary 4.5. Suppose, on the contrary, that there exist a general multi-Euler-Lagrange quadratic function $Q : \mathbb{C} \to \mathbb{C}$ and a constant $d > 0$ such that $|f(x) - Q(x)| \leq d|x|^2$ for all $x \in \mathbb{C}$. Then there exists a constant $c \in \mathbb{C}$ such that $Q(x) = cx^2$ for all rational numbers x. So, we obtain that

$$\left|f(x)\right| \leq \left(d + |c|\right)|x|^2 \tag{4.7}$$

for all rational numbers x. Let $s \in \mathbb{N}$ with $s + 1 > d + |c|$. If x is a rational number in $(0, \alpha^{-s})$, then $\alpha^j x \in (0,1)$ for all $j = 0, 1, \ldots, s$, and for this x we get

$$f(x) = \sum_{j=0}^{\infty} \frac{\phi(\alpha^j x)}{\alpha^{2j}} \geq \sum_{j=0}^{s} \frac{\phi(\alpha^j x)}{\alpha^{2j}} = (s+1)x^2 > \left(d + |c|\right)x^2,$$

which contradicts (4.7).

Corollary 4.7 *Let $\mathbb{K}$ be a non-Archimedean field with $0 < |\lambda| < 1$, $\mathcal{X}$ be a normed space over $\mathbb{K}$ and $\mathcal{Y}$ be a complete non-Archimedean normed space over $\mathbb{K}$. Let $\delta, p, q \in (0, \infty)$ such that $p + q \in (0, 2)$. If $f : \mathcal{X}^n \to \mathcal{Y}$ is a mapping satisfying (3.3) and*

$$\begin{aligned} &\left\|f\left(x_1, \ldots, x_{i-1}, ax_i + bx_i', x_{i+1}, \ldots, x_n\right) + f\left(x_1, \ldots, x_{i-1}, bx_i - ax_i', x_{i+1}, \ldots, x_n\right),\right. \\ &\quad \left. - \left(a^2+b^2\right)\left[f(x_1, \ldots, x_n) + f\left(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n\right)\right]\right\| \leq \delta\left(\|x_i\|^p \left\|x_i'\right\|^q\right), \end{aligned}$$

then for every $i \in \{1, 2, \ldots, n\}$ there exists a unique general multi-Euler-Lagrange quadratic mapping $Q_i : \mathcal{X}^n \to \mathcal{Y}$ such that

$$\left\|f(x_1, \ldots, x_n) - Q_i(x_1, \ldots, x_n)\right\| \leq \frac{\delta|a|^p|b|^q\|x_i\|^{p+q}}{|\lambda|^{p+q} - |\lambda|^2} \tag{4.8}$$

for all $x_1, \ldots, x_n \in \mathcal{X}$.

Proof Fix $i \in \{1,2,\dots,n\}$, $x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n \in \mathcal{X}$ and $t \in [0,\infty)$. Let $\Psi_i : \mathcal{X}^{n+1} \times [0,\infty) \to [0,1]$ be defined by $\Psi_i(x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n,t) := \frac{t}{t+\delta(\|x_i\|^p\|x_i'\|^q)}$. Consider the non-Archimedean fuzzy normed space $(\mathcal{Y},N_1,T_M)$ defined as in Example 2.6, and apply Theorem 4.2. □

Corollary 4.8 *Let $\mathcal{X}$ be a real normed space and $\mathcal{Y}$ be a real Banach space. Let $\delta,r,p,q \in (0,\infty)$ such that $r,p+q \in (0,2)$, or $r,p+q \in (2,\infty)$. If $f:\mathcal{X}^n \to \mathcal{Y}$ is a mapping satisfying (3.3) and*

$$\begin{aligned}
&\big\|f(x_1,\dots,x_{i-1},ax_i+bx_i',x_{i+1},\dots,x_n)+f(x_1,\dots,x_{i-1},bx_i-ax_i',x_{i+1},\dots,x_n),\\
&\quad-(a^2+b^2)\big[f(x_1,\dots,x_n)+f(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_n)\big]\big\|\\
&\le \delta\big[\|x_1\|^r\cdots\|x_{i-1}\|^r\big(\|x_i\|^p\|x_i'\|^q\big)\|x_{i+1}\|^r\cdots\|x_n\|^r\big],
\end{aligned}$$

then, for every $i \in \{1,2,\dots,n\}$, there exists a unique general multi-Euler-Lagrange quadratic mapping $Q_i:\mathcal{X}^n \to \mathcal{Y}$ such that

$$\begin{aligned}
&\big\|f(x_1,\dots,x_n)-Q_i(x_1,\dots,x_n)\big\|\\
&\quad\le \frac{\delta|a|^p|b|^q(\|x_1\|^r\cdots\|x_{i-1}\|^r\|x_i\|^{p+q}\|x_{i+1}\|^r\cdots\|x_n\|^r)}{|\lambda^{p+q}-\lambda^2|}
\end{aligned} \tag{4.9}$$

for all $x_1,\dots,x_n \in \mathcal{X}$.

Proof Fix $i \in \{1,2,\dots,n\}$, $x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n \in \mathcal{X}$ and $t \in [0,\infty)$. Let $\Psi_i : \mathcal{X}^{n+1} \times [0,\infty) \to [0,1]$ be defined by

$$\Psi_i(x_1,\dots,x_i,x_i',x_{i+1},\dots,x_n,t) := \frac{t}{t+\delta[\|x_1\|^r\cdots\|x_{i-1}\|^r(\|x_i\|^p\|x_i'\|^q)\|x_{i+1}\|^r\cdots\|x_n\|^r]}.$$

Consider the non-Archimedean fuzzy normed space $(\mathcal{Y},N_1,T_M)$ defined as in Example 2.6, and apply Theorems 4.1 and 4.2. □

Remark 4.9 Theorems 4.1 and 4.2 can be regarded as a generalization of the classical stability result in the framework of normed spaces (see [14]). For $a=b=1$ and $n=1$, Corollary 4.8 yields the main theorem in [17]. The generalized Hyers-Ulam stability problem for the case of $r=p+q=2$ was excluded in Corollary 4.8 (see[10]).

Note that by (4.4) one can get

$$\begin{aligned}
&\big\|f(x_1,\dots,x_n)-Q_i(x_1,\dots,x_n)\big\|\\
&\quad\le \ln\left(1+\frac{\delta|a|^p|b|^q(\|x_1\|^r\cdots\|x_{i-1}\|^r\|x_i\|^{p+q}\|x_{i+1}\|^r\cdots\|x_n\|^r)}{|\lambda^{p+q}-\lambda^2|\cdot t}\right)^t.
\end{aligned}$$

Letting $t \to \infty$ in this inequality, we obtain (4.9). Thus Corollary 4.8 is a singular case of Corollary 4.4. This study indeed presents a relationship between three various disciplines: the theory of non-Archimedean fuzzy normed spaces, the theory of stability of functional equations and the fixed point theory.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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