ON A $(2,2)$-RATIONAL RECURSIVE SEQUENCE

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Received 26 April 2004 and in revised form 3 November 2004

We investigate the asymptotic behavior of the recursive difference equation $y_{n+1} = (\alpha + \beta y_n)/(1 + y_{n-1})$ when the parameters $\alpha < 0$ and $\beta \in \mathbb{R}$. In particular, we establish the boundedness and the global stability of solutions for different ranges of the parameters α and β. We also give a summary of results and open questions on the more general recursive sequences $y_{n+1} = (a + by_n)/(A + By_{n-1})$, when the parameters $a, b, A, B \in \mathbb{R}$ and $abAB \neq 0$.

1. Introduction

The monograph by Kulenović and Ladas [10] presents a wealth of up-to-date results on the boundedness, global stability, and the periodicity of solutions of all rational difference equations of the form

$$x_{n+1} = \frac{a + bx_n + cx_{n-1}}{A + Bx_n + Cx_{n-1}}, \tag{1.1}$$

where the parameters a, b, c, A, B, C, and the initial conditions x_{-1} and x_0 are nonnegative real numbers. The nonnegativity of the parameters and the initial conditions ensures the existence of the sequence $\{x_n\}$ for all positive integers n.

The techniques and results developed to understand the dynamics of (1.1) are instrumental in exploring the dynamics of many biological models and other applications. As simple as (1.1) may seem, many open problems and conjectures remain to be investigated. One of these questions suggested in both [7, 10] is to study (1.1) when some of the parameters are negative. To this effect, there have been a few papers that dealt with negative parameters. See, for example, [1, 2, 3, 4, 11, 12]. In [1], Aboutaleb et al. studied the equation

$$x_{n+1} = \frac{a + bx_n}{A + Bx_{n-1}}, \tag{1.2}$$

Advances in Difference Equations 2005:3 (2005) 319–332
DOI: 10.1155/ADE.2005.319

where b is the only negative parameter. The purpose of this paper is to complete the study of (1.2) for all parameters a, b, A, and B such that $abAB \neq 0$ as a first step in understanding the dynamics of (1.1) without the nonnegativity requirement. Understanding the wild and rich dynamics exhibited by this more general version of (1.1) is our ultimate goal and motivation.

From now on, we will assume that $a, b, A, B \in \mathbb{R}$ and $abAB \neq 0$. The change of variables $y_n = Bx_n/A$ reduces (1.2) to

$$y_{n+1} = \frac{\alpha + \beta y_n}{1 + y_{n-1}}, \tag{1.3}$$

where $\alpha = aB/A^2$ and $\beta = b/A$.

The case ($\alpha > 0$ and $\beta > 0$) has been studied extensively, see, for example, [5, 6, 7, 8, 9, 10]. The cases ($\alpha > 0$ and $-1 < \beta < 0$), and ($\alpha = 0$ and $\beta < 0$) were studied in [1].

In this paper, we will study the case ($\alpha < 0$ and $\beta \in \mathbb{R}$), and for convenience, we will make α positive and write

$$y_{n+1} = \frac{-\alpha + \beta y_n}{1 + y_{n-1}}, \quad \alpha > 0,\ \beta \in \mathbb{R},\ \beta \neq 0. \tag{1.4}$$

Equation (1.4) has two real fixed points when $0 < 4\alpha < (\beta - 1)^2$, namely,

$$\overline{y}_{1,2} = \frac{(\beta - 1) \pm \sqrt{(\beta - 1)^2 - 4\alpha}}{2}. \tag{1.5}$$

The fixed points will be both positive if $\beta > 1$, and both negative if $\beta < 1$. When $4\alpha = (\beta - 1)^2$, (1.4) can be rewritten as

$$y_{n+1} = \frac{4\beta y_n - (\beta - 1)^2}{4(1 + y_{n-1})}, \tag{1.6}$$

and has a unique fixed point $\overline{y} = (\beta - 1)/2$. The case ($\alpha = 0$ and $\beta = 1$) is covered, for example, in [7, 10]. Finally, when $4\alpha > (\beta - 1)^2$, (1.4) has two complex fixed points

$$\overline{y}_{1,2} = \frac{(\beta - 1) \pm i\sqrt{4\alpha - (\beta - 1)^2}}{2}. \tag{1.7}$$

The following theorem establishes the stability of the real fixed points of the rational recursion (1.4).

Theorem 1.1. (i) *When* $0 < 4\alpha < (\beta - 1)^2$ *and* $\beta > 0$, *the fixed point* $\overline{y}_1$ *is stable and* $\overline{y}_2$ *is unstable. Moreover,* $\overline{y}_2$ *is a repeller if* $4\alpha < (1 - 3\beta)(1 + \beta)$ *and a saddle if* $4\alpha > (1 - 3\beta)(1 + \beta)$.

(ii) *When* $4\alpha = (\beta - 1)^2$, *then the unique fixed point* $\overline{y} = (\beta - 1)/2$ *is unstable.*

(iii) *When* $0 < 4\alpha < (\beta - 1)^2$ *and* $\beta < 0$, *the fixed point* $\bar{y}_1$ *is asymptotically stable if* $4\alpha < (1 - 3\beta)(1 + \beta)$ *and unstable if* $4\alpha > (1 - 3\beta)(1 + \beta)$. *The fixed point* $\bar{y}_2$ *is a repeller.*

Proof. (i) Linearizing around a fixed point $\overline{y}$, we obtain the characteristic equation

$$\lambda^2 - \frac{\beta}{1+\overline{y}}\lambda + \frac{\overline{y}}{1+\overline{y}} = 0. \tag{1.8}$$

Stability at a fixed point $\overline{y}$ of (1.4) requires that

$$\left|\frac{\beta}{1+\overline{y}}\right| - 1 < \frac{\overline{y}}{1+\overline{y}} < 1. \tag{1.9}$$

When $\beta > 0$, one can easily check that $1+\overline{y} > 0$. Thus we only have to check that $\beta - 1 < 2\overline{y} < 1 + 2\overline{y}$, which is clearly satisfied for $\overline{y}_1 = (\beta - 1 + \sqrt{(\beta-1)^2 - 4\alpha})/2$ and violated for $\overline{y}_2 = (\beta - 1 - \sqrt{(\beta-1)^2 - 4\alpha})/2$, whenever $4\alpha < (\beta-1)^2$.

(ii) The linearized stability analysis in the case $4\alpha = (\beta-1)^2$ yields the eigenvalues

$$\lambda_1 = \frac{\beta - 1}{\beta + 1}, \qquad \lambda_2 = 1. \tag{1.10}$$

While the norm of λ_1 is less than one, the linearized stability test remains inconclusive. The proof of the instability of the fixed point $\overline{y} = (\beta-1)/2$ will be established in Section 3.

(iii) When $\beta < 0$, inequality (1.9) holds if

$$\overline{y} > -1, \qquad |\beta| < 1 + 2\overline{y}. \tag{1.11}$$

These two inequalities will in turn hold for $\overline{y} = \overline{y}_1$ when

$$4\alpha < (1 - 3\beta + 1)(1+\beta). \tag{1.12}$$

However, when $\overline{y} = \overline{y}_2$, we have that $\beta > 1 + 2\overline{y}_2$ and it is easy to check that the fixed point $\overline{y}_2$ is a repeller. □

The rest of the paper is organized as follows. In Section 2, we briefly state results about the case $0 < \beta < 1$ and $0 < 4\alpha \le (\beta-1)^2$. When $\beta > 1$, Sections 3 and 4, respectively, treat the cases $4\alpha = (\beta-1)^2$ and $0 < 4\alpha < (\beta-1)^2$. Sections 5 and 6 establish the boundedness of solutions of (1.4) as well as the global stability of one of the fixed points. Finally, the last part of the paper is meant to be a summary of results and open problems concerning (1.3).

2. The case $0 < \beta < 1$ and $0 < \alpha \le (\beta-1)^2/4$

When $0 < \beta < 1$, and $0 < 4\alpha \le (\beta-1)^2$, the change of variable $y_n = \overline{y}_2 - \overline{y}_2\delta_n$ in (1.4) leads to the difference equation

$$\delta_{n+1} = \frac{p\delta_n + \delta_{n-1}}{q + \delta_{n-1}}, \tag{2.1}$$

where

$$p = -\frac{\beta}{\overline{y}_2} > 0, \qquad q = -\frac{1+\overline{y}_2}{\overline{y}_2} > 0. \tag{2.2}$$

A simple calculation shows that

$$p+1-q=\frac{-\sqrt{(\beta-1)^2-4\alpha}}{2\overline{y}_2}\geq 0,\qquad p>q. \tag{2.3}$$

A straightforward application of the work in [10, Section 6.8, page 109] leads to the following theorem.

Theorem 2.1. *If $0<\beta<1$, and $0<4\alpha\leq(\beta-1)^2$, then the equilibrium point $\overline{y}_1$ is asymptotically stable. Moreover, if y_k and y_{k+1} are in the interval $[\overline{y}_2,+\infty)$ for some $k\geq -1$, and $y_k+y_{k+1}>2\overline{y}_2$, then $y_n\to\overline{y}_1$ as $n\to\infty$.*

A closer examination of recursion (2.1) shows that one can take advantage of the invariability of the first quadrant to extend the basin of attraction of the fixed point $\overline{y}_1$ to a much wider range.

Theorem 2.2. *Let $\delta_{-1}=1-(y_{-1}/\overline{y}_2)$ and let $\delta_0=1-(y_0/\overline{y}_2)$. Then, $y_n\to\overline{y}_1$ as $n\to\infty$ if one of the following conditions is satisfied.*
(i) $\delta_{-1}>-q$ *and* $\delta_0>\sup(-\delta_{-1}/p,-p\delta_{-1}/(p^2+q+\delta_{-1}),-q)$.
(ii) $\delta_{-1}>-q$ *and* $-\delta_{-1}/p<\delta_0<\inf(-p\delta_{-1}/(p^2+q+\delta_{-1}),-q)$.
(iii) $-(p^2+q)<\delta_{-1}<-q$ *and* $-p\delta_{-1}/(p^2+q+\delta_{-1})<\delta_0<\inf(-\delta_{-1}/p,-q)$.
(iv) $-(p^2+q)<\delta_{-1}<-q$ *and* $-q<\delta_0<\inf(-\delta_{-1}/p,-p\delta_{-1}/(p^2+q+\delta_{-1}))$.
(v) $\delta_{-1}<-(p^2+q)$ *and* $\delta_0<\inf(-\delta_{-1}/p,-p\delta_{-1}/(p^2+q+\delta_{-1}),-q)$.
(vi) $\delta_{-1}<-(p^2+q)$ *and* $\sup(-q,-p\delta_{-1}/(p^2+q+\delta_{-1}))<\delta_0<-\delta_{-1}/p$.

Proof. In all of the above cases, it is easy to check that both

$$\delta_1=\frac{p\delta_0+\delta_{-1}}{q+\delta_{-1}}>0,\qquad \delta_2=\frac{(p^2+q+\delta_{-1})\delta_0+p\delta_{-1}}{(q+\delta_{-1})(q+\delta_0)}>0. \tag{2.4}$$

The rest follows from Theorem 2.1. □

We end this section with a theorem giving different bound estimates for positive solutions of recursion (2.1). In particular, this theorem shows that positive solutions quickly get absorbed in the interval $[q/p,p/q]$.

Theorem 2.3. *Let $p>q>0$, let $t=\log_{q/p}(pq)$, and consider $\{\delta_n\}_{n=-1}^{\infty}$ a positive solution of (2.1). Assume that for $n\geq 0$,*

$$\delta_n=\left(\frac{q}{p}\right)^r,\qquad \delta_{n-1}=\left(\frac{q}{p}\right)^s. \tag{2.5}$$

Then, the following statements are true:
(i) *if* $r\geq 1$, *then* $(q/p)^{r-1}\leq\delta_{n+1}\leq 1$;
(ii) *if* $r\leq 1$, *then* $1\leq\delta_{n+1}\leq(q/p)^{r-1}$;
(iii) *if* $r-2s+t\leq 0$, *then* $(1/p)(q/p)^{s-1}\leq\delta_{n+1}\leq p(q/p)^{r-s}$;
(iv) *if* $r-2s+t\geq 0$, *then* $p(q/p)^{r-s}\leq\delta_{n+1}\leq(1/p)(q/p)^{s-1}$.

Proof. We will prove (i) and (iii) only. To prove (i), notice that if $r \geq 1$, then $(q/p)^{r-1} \leq 1$. Thus we can write

$$p\left(\frac{q}{p}\right)^r \leq q, \qquad p\left(\frac{q}{p}\right)^r + \left(\frac{q}{p}\right)^s \leq q + \left(\frac{q}{p}\right)^s, \tag{2.6}$$

which leads to the conclusion that $\delta_{n+1} \leq 1$. On the other hand, we also have

$$\left(\frac{q}{p}\right)^{r+s-1} \leq \left(\frac{q}{p}\right)^s, \qquad q\left(\frac{q}{p}\right)^r + \left(\frac{q}{p}\right)^{r+s-1} \leq p\left(\frac{q}{p}\right)^{r-1} + \left(\frac{q}{p}\right)^s. \tag{2.7}$$

Dividing both sides of the inequality by $q + (q/p)^s$ completes the proof of (i).

To prove (iii), notice that if $r - 2s + t \leq 0$, then

$$pq\left(\frac{q}{p}\right)^{r-2s} \geq 1 \quad \text{or equivalently,} \quad p^2\left(\frac{q}{p}\right)^{r-2s+1} \geq 1. \tag{2.8}$$

Thus

$$p\left(\frac{q}{p}\right)^r + \left(\frac{q}{p}\right)^s \geq \left(\frac{q}{p}\right)^s + \frac{1}{p}\left(\frac{q}{p}\right)^{2s-1} = \frac{1}{p}\left(\frac{q}{p}\right)^{s-1}\left[q + \left(\frac{q}{p}\right)^s\right], \tag{2.9}$$

and consequently $\delta_{n+1} \geq (1/p)(q/p)^{s-1}$. The second part of the inequality follows from a similar manipulation. □

3. The case $4\alpha = (\beta - 1)^2$ and $\beta > 1$

In this section, we present a sequence of lemmas showing the instability of the unique fixed point $\overline{y} = (\beta - 1)/2$. We also prove the existence of a convergent subsequence and establish the existence of an invariant domain. For the proofs of the lemmas, we will focus on the case $\beta > 1$.

Lemma 3.1. *Every negative semicycle (except perhaps the first one) has at least two elements. Moreover, if $y_{k+1} > 0$ is the first element in a negative semicycle, then $y_{k+2} < y_{k+1}$.*

Proof. Consider the equation

$$y_{k+2} = \frac{4\beta y_{k+1} - (\beta - 1)^2}{4(1 + y_k)} = y_{k+1} + \frac{4y_{k+1}(\beta - 1 - y_k)}{4(1 + y_k)} - \frac{(\beta - 1)^2}{4(1 + y_k)}. \tag{3.1}$$

When $0 < y_{k+1} < (\beta - 1)/2$ and $y_k > (\beta - 1)/2$, it is easy to see that $4y_{k+1}(\beta - 1 - y_k) < (\beta - 1)^2$, and thus $y_{k+2} < y_{k+1}$ as required. On the other hand, if $y_{k+1} < 0$, then so is y_{k+2}. □

Lemma 3.2. *If* $y_0 < y_{-1} < (\beta - 1)/2$, *then there exists* $k \geq -1$ *such that* $y_{k+1} < y_k < -1$.

Proof. There are three cases to be discussed
Case 1. When $y_0 < y_{-1} < -1$, the lemma is trivial and $k = -1$.
Case 2. If $y_0 < -1 < y_{-1} < (\beta - 1)/2$, then

$$y_1 = \frac{4\beta y_0 - (\beta - 1)^2}{4(1 + y_{-1})} = y_0 + \frac{4y_0(\beta - 1 - y_{-1})}{4(1 + y_{-1})} - \frac{(\beta - 1)^2}{4(1 + y_{-1})}. \tag{3.2}$$

The second and third terms of the above equality are both negative. Hence, $y_1 < y_0 < -1$ and $k = 0$.
Case 3. If $-1 < y_0 < y_{-1} < (\beta - 1)/2$, then let $y_0 = -\delta + (\beta - 1)/2$ and $y_{-1} = -\kappa\delta + (\beta - 1)/2$, where $0 < \delta < (\beta + 1)/2$ and $0 < \kappa < 1$.

We then have that

$$y_1 = \frac{4\beta y_0 - (\beta - 1)^2}{4(1 + y_{-1})} = y_0 - \frac{4\kappa\delta^2 + 2(\beta - 1)\delta(1 - \kappa)}{4(1 + y_{-1})}, \tag{3.3}$$

and thus $y_1 < y_0$. If $y_1 < -1$, then we are back to Case 2, otherwise in the same way as above we can establish that $y_2 < y_1$ and so on to obtain $y_{n+1} < y_n < \cdots < y_2 < y_1$. If the sequence is bounded below by -1, then it has to converge, creating a contradiction with the fact that $(\beta - 1)/2$ is the only fixed point. The sequence cannot reach the value -1 either, for otherwise the relation $(\beta + 1)^2 = 4\delta(\kappa - 1)$ must hold, which is again a contradiction with the choice of $\delta(\kappa - 1) < 0$. The only scenario left is for the sequence to cross the value -1 for the first time at $y_{n+1} < -1 < y_n$, in which case we are back again to Case 2. □

Theorem 3.3. *The equilibrium point* $\overline{y} = (\beta - 1)/2$ *is unstable.*

Proof. Let $0 < \epsilon \ll 1$ and take $y_{-1} = \overline{y} + \epsilon$ and $y_0 = \overline{y} - \epsilon$. By Lemma 3.1, we obtain that $y_1 < y_0$. By Lemma 3.2, there exists k such that $y_k < -1$ and this proves that $\overline{y} = (\beta - 1)/2$ is unstable. □

While unstable, our numerical investigations show that the fixed point $\overline{y} = (\beta - 1)/2$ is a global attractor for a substantial set of initial conditions; a fact that unfortunately we cannot prove. Instead, we will establish a bounded invariant region for which $\overline{y}$ is indeed a global attractor. To this end, we start by studying positive semicycles.

Lemma 3.4. $y_0 < y_{-1} < -1$, *then* $y_1 > \beta$ *and* $y_2 < 0$.

Proof. By assumption, $y_0/y_{-1} > 1$ and $0 < (1 + 1/y_{-1}) < 1$. Hence,

$$\begin{gathered} y_1 = \frac{4\beta y_0 - (\beta - 1)^2}{4(1 + y_{-1})} = \frac{y_0}{y_{-1}} \frac{4\beta - (\beta - 1)^2/y_0}{4(1 + 1/y_{-1})} > \beta - \frac{(\beta - 1)^2}{4y_0} > \beta, \\ y_2 = \frac{4\beta y_1 - (\beta - 1)^2}{4(1 + y_0)}. \end{gathered} \tag{3.4}$$

The numerator in the expression of y_2 is always greater than $3\beta^2 + 2\beta - 1 > 0$, and the denominator is negative. □

Corollary 3.5. *If $y_0 < y_{-1} < -1$, then the next positive semicycle has exactly one element.*

Lemma 3.6. *A necessary condition for a positive semicycle to have more than one element is that two consecutive elements y_k, y_{k+1} of the previous negative semicycle satisfy $y_k < y_{k+1}$.*

Proof. Let y_k, y_{k+1} be two elements in a negative semicycle. If $y_{k+1} < y_k < (\beta-1)/2$, then by Lemmas 3.2 and 3.4, the following positive semicycle has exactly one element. Thus y_{k+1} must be greater or equal to y_k. The cases $y_{k+1} = y_k = -1$ and $y_{k+1} = y_k = (\beta-3)/4$ are not to be considered because y_{k+3} does not exist for these choices. If $y_{k+1} = y_k$, then

$$y_{k+2} = y_{k+1} - \frac{\left(y_{k+1} - (\beta-1)/2\right)^2}{\left(1+y_{k+1}\right)} = \frac{\beta-1}{2} + \frac{(\beta+1)\left(y_{k+1} - (\beta-1)/2\right)}{2y_{k+1}}. \tag{3.5}$$

If $y_{k+1} > -1$, then $y_{k+2} < y_{k+1}$ and the next positive semicycle will have exactly one element. If $y_{k+1} < -1$, then obviously $y_{k+2} > y_{k+1}$ as required. The second part of the above equality guarantees that y_{k+2} is still less than $(\beta-1)/2$. □

Lemma 3.7. *Assume that*

(i) $0 < M < 1/(2\beta-2)$,

(ii)

$$c \in \left(\frac{\beta - \sqrt{1-2(\beta-1)M}}{\beta+1+2M}, \frac{\beta + \sqrt{1-2(\beta-1)M}}{\beta+1+2M}\right), \tag{3.6}$$

(iii) $cM < \delta < M$.

Then,

$$c\delta < \frac{2\beta\delta - (\beta-1)M}{\beta+1+2M} < \delta. \tag{3.7}$$

Proof. That $(2\beta\delta - (\beta-1)M)/(\beta+1+2M) < \delta$ follows from a straightforward manipulation of the fact that $\delta < M$. To prove that $c\delta < (2\beta\delta - (\beta-1)M)/(\beta+1+2M)$, notice that if condition (ii) of the lemma is satisfied, then $c^2(\beta+1+2M) - 2\beta c + \beta - 1 < 0$ and

$$c\delta < \frac{\delta\left(2\beta - (\beta-1)/c\right)}{\beta+1+2M}. \tag{3.8}$$

Since $\delta/c > M$, the desired inequality is established. □

Theorem 3.8. *If $y_{-1} = \bar{y} + \delta < y_0 = \bar{y} + M$, and δ and M satisfy the conditions of Lemma 3.7, then $y_n \to (\beta-1)/2$ as $n \to \infty$.*

Proof. Let $y_n = \bar{y} + \delta_n$. The conditions imposed on δ and M imply that $0 < c\delta_{-1} < \delta_0 < \delta_{-1} < 1/(2(\beta-1))$. Moreover, the sequence $\{\delta_n\}$ satisfies the recurrence relation

$$\delta_{n+1} = \frac{2\beta\delta_n - (\beta-1)\delta_{n-1}}{\beta+1+2\delta_{n-1}} \tag{3.9}$$

which has 0 as its unique fixed point. Using the previous lemma, we obtain that $c\delta_0 < \delta_1 < \delta_0$ and by induction that $c\delta_n < \delta_{n+1} < \delta_n$. Thus $\{\delta_n\}$ is a bounded positive decreasing sequence whose only possible limit is 0. Hence, $\{y_n\}$ converges to $(\beta-1)/2$. □

4. The case $4\alpha < (\beta-1)^2$ and $\beta > 1$

As discussed in Section 1, the point

$$\frac{\beta-1}{2} < \bar{y} = \frac{\beta-1+\sqrt{(\beta-1)^2-4\alpha}}{2} < \beta-1 \tag{4.1}$$

is a stable fixed point of (1.4). The change of the variable $y_n = \bar{y}+\delta_n$ yields the recurrence equation

$$\delta_{n+1} = \frac{\beta\delta_n - \bar{y}\delta_{n-1}}{1+\bar{y}+\delta_{n-1}}. \tag{4.2}$$

Obviously, $\bar{\delta} = 0$ is a stable fixed point of (4.2).

Lemma 4.1. *If* $(1+\bar{y}) < \delta_{n-1} < 0$ *and* $\delta_n \geq 0$, *then*

(i) *the positive semicycle containing* δ_n *has at least 3 elements,*
(ii) $\delta_{n+1} > \delta_n$,
(iii) *if* $\bar{y} > (\sqrt{\beta^2+1}-1)/2$, *then the ratios* $\{\delta_{k+1}/\delta_k\}$ *are strictly decreasing.*

Proof. Parts (i) and (ii) of the lemma follow straight from the identities

$$\begin{aligned} \delta_{n+1} &= \delta_n + \frac{(\beta-1-\bar{y})\delta_n - \delta_{n-1}(\bar{y}+\delta_n)}{1+\bar{y}+\delta_{n-1}} > \delta_n, \\ \delta_{n+2} &= \frac{\beta\delta_{n+1}-\bar{y}\delta_n}{1+\bar{y}+\delta_n} > \frac{(\beta-\bar{y})\delta_n}{1+\bar{y}+\delta_n} > 0. \end{aligned} \tag{4.3}$$

Let $\delta_{k-1} > 0$ and $\delta_k > 0$ be two consecutive elements of a positive semicycle and consider the identity

$$\frac{\delta_{k+1}}{\delta_k} = \frac{\delta_k}{\delta_{k-1}} + \frac{-(1+\bar{y}+\delta_{k-1})\delta_k^2 + \beta\delta_k\delta_{k-1} - \bar{y}\delta_{k-1}^2}{\delta_k\delta_{k-1}(1+\bar{y}+\delta_{k-1})}. \tag{4.4}$$

The discriminant of $-(1+\bar{y}+\delta_{k-1})\delta_k^2 + \beta\delta_k\delta_{k-1} - \bar{y}\delta_{k-1}^2$ viewed as a polynomial of second degree in δ_k is given by

$$\delta_{k-1}^2(\beta^2 - 4\bar{y}(1+\bar{y}+\delta_{k-1})) < \delta_{k-1}^2(\beta^2-4\bar{y}(1+\bar{y})) < 0, \tag{4.5}$$

whenever $\bar{y} > (\sqrt{\beta^2+1}-1)/2$. □

The following lemma is about negative semicycles. Its content is similar to the previous lemma and so we will omit its proof.

LEMMA 4.2. *Let $\delta_{n-1} > 0$, and let $-(1+\bar{y}) < \delta_n < 0$ be the first element in a negative semicycle. Then*

(i) *the negative semicycle has at least 3 elements,*

(ii) $\delta_{n+1} < \delta_n$,

(iii) *if $\bar{y} > (\sqrt{\beta^2+1}-1)/2$, then the sequence $\{\delta_{k+1}/\delta_k\}$ is strictly decreasing.*

The previous two lemmas indicate that if $\bar{y} > (\sqrt{\beta^2+1}-1)/2$, then solutions converging to $\bar{\delta} = 0$ spiral to the fixed point clockwise in the space (δ_n, δ_{n+1}). This allows us to find a basin of attraction of $\bar{\delta} = 0$. In fact, the sequence $\{D_n\}$ given by

$$D_n = \left(\delta_n - a\delta_{n-1}\right)^2 + p\delta_{n-1}^2, \tag{4.6}$$

where

$$a = \frac{\beta}{2(1+\bar{y})}, \qquad p = \frac{(1+2\bar{y})^2 - \beta^2}{4(1+\bar{y})^2}, \tag{4.7}$$

defines a distance between the point (δ_{n-1}, δ_n) and the origin. A rather simple but tedious computation shows that

$$D_{n+1} - D_n = \frac{A(\delta_{n-1})\delta_n^2 + B(\delta_{n-1})\delta_n + C(\delta_{n-1})}{\left(1+\bar{y}+\delta_{n-1}\right)^2}, \tag{4.8}$$

where $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are polynomials of degrees 2, 3, and 4, respectively, satisfying the following conditions.

(i) $A(0) = -(3+4\bar{y})/4$ and $B(0) = C(0) = 0$.

(ii) $A(\delta_{n-1})$ remains negative as long as

$$|\delta_{n-1}| < M_1 = \frac{(1+\bar{y})\left(3+4\bar{y}+2\beta^2 - 2\beta\sqrt{3+4y+\beta^2}\right)}{6+8\bar{y}}. \tag{4.9}$$

(iii) The discriminant

$$B^2 - 4AC = -K\delta_{n-1}^2\left(3+16\bar{y}+16\bar{y}^2 - 4\beta^2 + b\delta_{n-1} + c\delta_{n-1}^2\right) \tag{4.10}$$

is less or equal to zero whenever

$$\begin{aligned} |\delta_{n-1}| < M_2 = {} & \frac{2\sqrt{(3+4\bar{y})\left((1+2\bar{y})^2-\beta^2\right)\left(\bar{y}^2(3+4\bar{y})+\beta^2\right)}}{4(1+\bar{y})^2\beta^2 - (1+2\bar{y})^2(3+4\bar{y})} \\ & - \frac{(1+2\bar{y})(3+10\bar{y}+8\bar{y}^2-2\beta^2)}{4(1+\bar{y})^2\beta^2 - (1+2\bar{y})^2(3+4\bar{y})}. \end{aligned} \tag{4.11}$$

The above analysis shows that if $|\delta_{n-1}| < \inf(M_1, M_2)$, then $D_{n+1} - D_n \le 0$. Hence, the following theorem holds.

THEOREM 4.3. *Let* $M = \inf(M_1, M_2)$, *and let* E_M *be the largest ellipse of the form*

$$(x - ay)^2 + py^2 = constant \tag{4.12}$$

that can be fit within the square S_M *defined by*

$$S_M = \{(x, y) : |x| < M,\ |y| < M\}. \tag{4.13}$$

Then, E_M *is invariant. Moreover, if for some* $k \geq -1$, $(\delta_k, \delta_{k+1}) \in E_M$, *then* $\delta_n \to 0$ *as* $n \to \infty$.

5. Boundedness of solutions of (1.4) when $\beta < 0$

In this section, we assume that $\beta < 0$. For convenience, we assume that $\beta > 0$ and rewrite (1.4) in the form

$$y_{n+1} = -\frac{\alpha + \beta y_n}{1 + y_{n-1}}, \quad \alpha > 0,\ \beta > 0. \tag{5.1}$$

All of the results in this section apply equally to both (5.1) and more generally to difference equations of the type

$$y_{n+1} = -\frac{\alpha + \sum_{i=0}^{k} \beta_i y_{n-i}}{1 + \sum_{j=0}^{k} \gamma_j y_{n-j}}, \tag{5.2}$$

where k is a nonnegative integer and where the coefficients β_i and γ_j are nonnegative real numbers satisfying

$$\sum_{i=0}^{k} \beta_i = \beta > 0, \qquad \sum_{j=0}^{k} \gamma_j = 1. \tag{5.3}$$

THEOREM 5.1. *If* $0 < \beta < 1$ *and* $0 < 4\alpha < (1 - \beta)(3\beta + 1)$, *then for all*

$$\begin{aligned} c &\in \left(\frac{\beta - 1 - \sqrt{(1-\beta)(3\beta+1) - 4\alpha}}{2}, \bar{y}_1 \right), \\ d &\in \left(\frac{-1 + \sqrt{1 - 4(\alpha + \beta c)}}{2}, -\frac{(c^2 + c + \alpha)}{\beta} \right), \end{aligned} \tag{5.4}$$

the interval $[c, d]$ *is invariant. In other words, if* $y_n, y_{n+1}, \ldots,$ *and* $y_{n+k-1} \in [c, d]$ *for some* $n \geq 1$, *then* $y_i \in [c, d]$ *for all* $i \geq n$.

Proof. Let c and d be two real numbers such that $-1 < c < d$ and $\alpha + \beta c > 0$. If both y_n and y_{n+1} belong to the interval $[c,d]$, then

$$-\frac{\alpha+\beta d}{1+c} \leq y_{n+2} \leq -\frac{\alpha+\beta c}{1+d}. \tag{5.5}$$

In order to guarantee that $y_{n+1} \in [c,d]$, the following inequalities must hold:

$$d^2 + d + \alpha + \beta c \geq 0 \geq c^2 + c + \alpha + \beta d. \tag{5.6}$$

The conditions imposed on the parameters α and β guarantee the existence of a feasible region to the system of inequalities (5.6). The rest of the proof follows by induction. □

6. Global stability of (5.1)

In this section, we give a global stability result for solutions of (1.4) with initial conditions in the invariant interval obtained in the previous section.

THEOREM 6.1. *Assume that* $0 < \beta < 1$ *and that* $0 < 4\alpha < (1-\beta)(3\beta+1)$. *If both* y_0 *and* y_1 *are in the interval* $[c,d]$ *as described in Theorem 5.1, then the sequence* $\{y_n\} \to \bar{y}_1$ *as* $n \to \infty$. *Moreover, if* $(y_0 - \bar{y}_1)(y_1 - \bar{y}_1) < 0$, *then the sequence* $\{y_n\}$ *is strictly oscillatory. That is,* $(y_n - \bar{y}_1)(y_{n+1} - \bar{y}_1) < 0$ *for all* $n \geq 1$.

Proof. Choose c and d as described in Theorem 5.1 and consider the function $f : [c,d] \times [c,d] \to [c,d]$ defined by

$$f(x,y) = -\frac{\alpha+\beta x}{1+y}. \tag{6.1}$$

The function f is decreasing in the variable x and increasing in the variable y. Moreover, it is easy to check that the difference equation (5.1) has no prime period-2 solution in the interval $[c,d]$. A straightforward application of [10, Theorem 1.4.6, page 12] gives us that all solutions of (5.1) with initial conditions in $[c,d]$ converge to $\bar{y}_1$.

To see that the sequence is strictly oscillatory, notice that the change of variables $z_n = y_n - \bar{y}_1$ leads to the difference equation

$$z_{n+1} = -\frac{\beta z_n + \bar{y}_1 z_{n-1}}{1 + \bar{y}_1 + z_{n-1}}. \tag{6.2}$$

Now, it suffices to notice that the denominator in the recursion (6.2) is always positive when the initial conditions y_0 and y_1 are in the interval $[c,d]$. In addition, $z_n z_{n-1} < 0$ implies that $z_{n+1} z_n < 0$ and the proof is complete. □

7. Equation (1.3): summary of results and open questions

In this section, we summarize the results about (1.3) when $\alpha\beta \neq 0$, and point out some important open questions that are yet to be answered.

7.1. The first quadrant ($\alpha > 0$ and $\beta > 0$). This quadrant was studied in [5, 7, 9, 10], where the following results were established.

(i) Positive solutions of (1.3) are bounded and persist.

(ii) The positive equilibrium of (1.3) is globally asymptotically stable when either $0 < \beta < 1$ or $0 < \alpha \leq 2(\beta + 1)$.

However, two questions remain open.

(1) Establishing the forbidden set of (1.3), that is, the set of initial conditions (y_0, y_1) for which the sequence becomes undefined for some $n \geq 2$.

(2) Proving that the positive equilibrium is globally stable for all values of $\alpha > 0$ and $\beta > 0$.

7.2. The second quadrant ($\alpha > 0$ and $\beta < 0$). This quadrant was studied in [1]. However, the range of parameters studied was limited to $-1/4 \leq \beta \leq 0$ and $2\beta^2 \leq \alpha \leq -2\beta$. For this range of parameters, an invariant interval was found and the global attractivity of the positive equilibrium point was established.

The following theorem improves and generalizes the results in [1] to all values of $\alpha > 0$ and $-1 < \beta < 0$. Its proof is also different from the one given in [1].

Theorem 7.1. *For all $\alpha > 0$ and $-1 < \beta < 0$,*

(i) *the interval $[0, -\alpha/\beta]$ is invariant,*

(ii) *let $\bar{y}$ be the positive fixed point of (1.3), $c \in [0, \bar{y}]$, and $(\alpha + \beta c)/(1 + c) \leq d \leq (\alpha - c)/(c - \beta)$. Then, the interval $[c, d]$ is invariant,*

(iii) *the fixed point $\bar{y}$ is a global attractor for the whole interval $[0, -\alpha/\beta]$.*

Proof. We will only prove part (iii). To do so, consider the function $f : [0, -\alpha/\beta] \times [0, -\alpha/\beta] \to [0, -\alpha/\beta]$ defined by

$$f(x, y) = \frac{\alpha + \beta x}{1 + y}. \tag{7.1}$$

Notice that since $\beta \leq 0$, the function f is nonincreasing in each of its arguments. Moreover, if $f(m, m) = M$ and $f(M, M) = m$ for some m and M in $[0, -\alpha/\beta]$, then $m = M$. Applying [10, Theorem 1.4.7, page 13], we obtain that every solution in $[0, -\alpha/\beta]$ converges to $\bar{y}$. □

Still, numerical experiments show that the positive equilibrium $\bar{y}$ of the recursion (1.3) is a global attractor for a wider range of the parameters α and β. In fact, the equilibrium point $\bar{y}$ is asymptotically stable whenever $4\alpha > (3\beta - 1)(\beta + 1)$. Establishing the global stability of the fixed point $\bar{y}$ when $\beta \leq -1$ and $4\alpha > (3\beta - 1)(\beta + 1)$ as well as establishing an invariant region for this range of parameters remain open questions.

7.3. The third quadrant $\alpha < 0$ and $\beta < 0$. This was the subject of Sections 5 and 6 of this paper. When $4\alpha < (3\beta - 1)(\beta + 1)$, we have witnessed *thin* regions delimited by parabolic curves where every solution seems to converge to a periodic solution. Some of the periods we have observed are 11, 15, 19, 22, 23, 24, 26, 30, 32, 40, 44, 52, and 60. A detailed description of the numerical experimentation and its results will be given elsewhere.

7.4. The fourth quadrant $\alpha < 0$ and $\beta > 0$. This quadrant can be divided into three main regions. The first two regions were studied in Sections 3 and 4. The third region remains unstudied.

Region 1 ($-(\beta-1)^2 \le 4\alpha < 0$ and $0 < \beta < 1$). We proved that the interval

$$I = \left[\frac{\beta - 1 - \sqrt{(\beta-1)^2 + 4\alpha}}{2}, +\infty\right) \tag{7.2}$$

is invariant and that all solutions with initial conditions inside this interval converge to $\bar{y} = (\beta - 1 + \sqrt{(\beta-1)^2 + 4\alpha})/2$.

Region 2 ($-(\beta-1)^2 \le 4\alpha < 0$ and $\beta > 1$). When $4\alpha = -(\beta-1)^2$, we proved that even though the fixed point $\bar{y} = (\beta-1)/2$ is unstable, there exists an invariant region for which $\bar{y}$ is a global attractor. We also proved that when $(\beta-1)^2 + 4\alpha > 0$, the larger of the two positive fixed points is asymptotically stable with an ellipsoidal basin of attraction.

Region 3 ($4\alpha < -(\beta-1)^2$). In this region, there are no fixed points, and as far as we know, there are no studies of (1.3) for this range of parameters. This is despite the rich dynamics exhibited in this range. For example, our numerical simulations suggest that there is a region delimited by parabolic-like curves for which all solutions converge to period-5 solutions. Other regions also delimited by parabolic-like curves exist for different periods. Unfortunately, establishing the existence, the global stability, or just the stability of these periodic solutions with the "usual" tools used so far in studying (1.3) seems unlikely. Perhaps, new theoretical tools should be introduced.

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