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The modified degenerate q-Bernoulli polynomials arising from p-adic invariant integral on $\mathbb{Z}_p$

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Abstract

Dolgy et al. introduced the modified degenerate Bernoulli polynomials, which are different from Carlitz's degenerate Bernoulli polynomials (see Dolgy et al. in Adv. Stud. Contemp. Math. (Kyungshang) 26(1):1-9, 2016). In this paper, we study some explicit identities and properties for the modified degenerate q-Bernoulli polynomials arising from the p-adic invariant integral on $\mathbb{Z}_p$.

MSC: 11B68; 11S40; 11S80

Keywords: degenerate Bernoulli polynomials; modified degenerate q-Bernoulli polynomials

1 Introduction

For a fixed prime number p, $\mathbb{Z}_p$ refers to the ring of p-adic integers, $\mathbb{Q}_p$ to the field of p-adic rational numbers, and $\mathbb{C}_p$ to the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let q be in $\mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$ and $q^x = \exp(x \log q)$ for $|x|_p < 1$. Then the q-analogue of x is defined to be $[x]_q = \frac{1-q^x}{1-q}$.

The Bernoulli polynomials are given by the generating function

$$\left(\frac{t}{e^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{(see [1–25])}. \tag{1.1}$$

When $x = 0$, $B_n = B_n(0)$ are called Bernoulli numbers.

Carlitz [4, 5, 8] defined the degenerate Bernoulli polynomials as follows:

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda)\frac{t^n}{n!}. \tag{1.2}$$

When $x = 0$, $\beta_n(0|\lambda) = \beta_n(\lambda)$ are called Carlitz's degenerate Bernoulli numbers.

From (1.2) we note that

$$\sum_{n=0}^{\infty} \lim_{\lambda\to 0} \beta_n(x|\lambda)\frac{t^n}{n!} = \lim_{\lambda\to 0} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$\begin{aligned} &= \left(\frac{t}{e^t - 1}\right) e^{xt} \\ &= \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \end{aligned} \tag{1.3}$$

Using the derivation given in (1.3), we have

$$\lim_{\lambda \to 0} \beta_n(x|\lambda) = B_n(x) \quad (n \geq 0). \tag{1.4}$$

Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic invariant integral on $\mathbb{Z}_p$ (also called the Volkenborn integral on $\mathbb{Z}_p$) is defined by

$$\int_{\mathbb{Z}_p} f(x)\, d\mu_0(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{n=0}^{p^N - 1} f(x) \quad \bigl(\text{see [1, 9, 10, 15, 17]}\bigr). \tag{1.5}$$

By using the formula defined in (1.1) we note that

$$\int_{\mathbb{Z}_p} f_1(x)\, du_0(x) - \int_{\mathbb{Z}_p} f(x)\, du_0(x) = f'(0) \tag{1.6}$$

and

$$\int_{\mathbb{Z}_p} f_n(x)\, du_0(x) - \int_{\mathbb{Z}_p} f(x)\, du_0(x) = \sum_{l=0}^{n-1} f'(l), \tag{1.7}$$

where $f_n(x) = f(x+n)$ $(n \in \mathbb{N})$; see [1, 9, 10, 15, 17].

Thus, by (1.6) we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t}\, du_0(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.8}$$

The modified degenerate Bernoulli polynomials are recently revisited by Dolgy et al., and they are formulated with the p-adic invariant integral on $\mathbb{Z}_p$ to be

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+\lambda)^{(\frac{x+y}{\lambda})t}\, du_0(x) &= \frac{t}{(1+\lambda)^{\frac{t}{\lambda}} - 1} \left(\frac{\log(1+\lambda)}{\lambda}\right) (1+\lambda)^{\frac{xt}{\lambda}} \\ &= \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!} \quad \bigl(\text{see [1]}\bigr), \end{aligned} \tag{1.9}$$

where $\lambda \in \mathbb{C}_p$ with $|\lambda|_p < p^{-\frac{1}{p-1}}$.

When $x = 0$, we call $\beta_{n,\lambda}(0) = \beta_{n,\lambda}$ the modified degenerate Bernoulli numbers.

Recently, Kim introduced p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_q(f) &= \int_{\mathbb{Z}_p} f(x)\, d\mu_q(x) \\ &= \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N - 1} f(x) q^x \quad (\text{see [17]}). \end{aligned} \tag{1.10}$$

The degenerate q-Bernoulli polynomials are also defined by Kim as follows.

$$\sum_{n=0}^{\infty} \beta_{n,q,\lambda}(x)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} \, d\mu_q(y) \quad \text{(see [20])}. \tag{1.11}$$

The generating functions of Stirling numbers are given by

$$\bigl(\log(1+t)\bigr)^n = n! \sum_{l=n}^{\infty} S_1(l,n)\frac{t^l}{l!} \quad (n \geq 0) \tag{1.12}$$

and

$$\bigl(e^t - 1\bigr)^n = n! \sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \quad (n \geq 0), \tag{1.13}$$

where $S_1(l,n)$ are the Stirling numbers of the first kind, and $S_2(l,n)$ are the Stirling numbers of the second kind.

The following diagram illustrates the variations of several types of q-Bernoulli polynomials and numbers. The definitions of the q-Bernoulli polynomials and the degenerate q-Bernoulli polynomials applied in the given diagram are provided by Carlitz [4, 5, 8] and Kim [20], respectively. In this paper, we investigate some of the explicit identities to characterize the modified degenerate q-Bernoulli polynomials used in the diagram

$\int_{\mathbb{Z}_p} e^{[x+y]_q t} \, d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x)\frac{t^n}{n!}$
(q-Bernoulli polynomials)

$\int_{\mathbb{Z}_p} q^{-x} e^{[x+y]_q t} \, d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!}$
(modified q-Bernoulli polynomials)

$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{[x+y]_q}{\lambda}} \, d\mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q,\lambda}(x)\frac{t^n}{n!}$
(degenerate q-Bernoulli polynomials)

$\int_{\mathbb{Z}_p} q^{-y} (1+\lambda)^{\frac{[x+y]_q}{\lambda}t} \, d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,q,\lambda}(x)\frac{t^n}{n!}$
(modified degenerate q-Bernoulli polynomials)

A few studies have identified some of the properties of the degenerate q-Bernoulli polynomials and numbers. This paper defines the modified q-Bernoulli polynomials and numbers arising from the p-adic invariant integral on $\mathbb{Z}_p$ and introduces additional characteristic properties of these polynomials and numbers, which are defined from the generating functions and p-adic invariant integral on $\mathbb{Z}_p$.

2 The modified degenerate *q*-Bernoulli polynomials and numbers

In the following discussions, we assume that $\lambda, t \in \mathbb{C}_p$ with $0 < |\lambda| \leq 1$ and $|t|_p < p^{-\frac{1}{p-1}}$. Then, as $|\lambda t|_p < p^{-\frac{1}{p-1}}$, $|\log(1+\lambda t)|_p = |\lambda t|_p$, and hence $|\frac{1}{\lambda}\log(1+\lambda t)|_p = |t|_p < p^{-\frac{1}{p-1}}$, it makes sense to take the limit as $\lambda \to 0$.

Following (1.3), we define the modified degenerate q-Bernoulli polynomials given by the generating function

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y) = \sum_{n=0}^{\infty} \widetilde{B}_{n,q,\lambda}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x=0$, $\widetilde{B}_{n,q,\lambda}(0) = \widetilde{B}_{n,q,\lambda}$ are called the modified degenerate q-Bernoulli numbers.

Note that

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y) \\
&\quad= \int_{\mathbb{Z}_p} q^{-y}e^{[x+y]_q t}\,d\mu_q(y) \\
&\quad= \sum_{n=0}^{\infty} B_{n,q}(x)\frac{t^n}{n!},
\end{aligned} \tag{2.2}$$

where $B_{n,q}(x)$ are the modified Carlitz q-Bernoulli polynomials.

Now, we consider

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y) \\
&\quad= \int_{\mathbb{Z}_p} q^{-y}e^{\frac{[x+y]_q}{\lambda}t\log(1+\lambda)}\,d\mu_q(y) \\
&\quad= \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n \int_{\mathbb{Z}_p} q^{-y}[x+y]_q^n\,d\mu_q(y)\frac{t^n}{n!} \\
&\quad= \sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n B_{n,q}(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

By the definitions provided in (2.1), (2.2), and (2.3) we are able to derive the following theorem.

Theorem 2.1 *For $n \geq 0$, $\widetilde{B}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}_{n,q,\lambda}(x) = \left(\frac{\log(1+\lambda)}{\lambda}\right)^n B_{n,q}(x). \tag{2.4}$$

Note that $(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l$ $(n \geq 0)$, where S_1 are the Stirling numbers of the first kind.

Then, by using (2.1) we are able to state

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t}\,d\mu_q(y) \\
&\quad= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} q^{-y}\binom{\frac{[x+y]_q}{\lambda}t}{n}\lambda^n\,d\mu_q(y) \\
&\quad= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} q^{-y}\lambda^n\sum_{l=0}^{n} S_1(n,l)\left(\frac{[x+y]_q}{\lambda}\right)^l\frac{t^l}{n!}\,d\mu_q(y)
\end{aligned}$$

$$= \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} S_1(n,l)\lambda^{n-l} \frac{t^l}{n!} \int_{\mathbb{Z}_p} q^{-y}[x+y]_q^l \, d\mu_q(y)$$
$$= \sum_{l=0}^{\infty} \left(\sum_{n=l}^{\infty} S_1(n,l)\lambda^{n-l} \frac{l!}{n!} B_{l,q}(x) \right) \frac{t^l}{l!}. \tag{2.5}$$

Given the descriptions in (2.1) and (2.5), we have another theorem.

Theorem 2.2 *For $n \geq 0$, $\widetilde{B}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}_{n,q,\lambda}(x) = \sum_{n=l}^{\infty} S_1(n,l)\lambda^{n-l} \frac{l!}{n!} B_{l,q}(x). \tag{2.6}$$

We observe that

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t} \, d\mu_q(y)$$
$$= \int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x]_q}{\lambda}t}(1+\lambda)^{\frac{[y]_q}{\lambda}q^x t} \, d\mu_q(y)$$
$$= (1+\lambda)^{\frac{[x]_q}{\lambda}t} \int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[y]_q}{\lambda}q^x t} \, d\mu_q(y)$$
$$= \left(\sum_{l=0}^{\infty} \left(\frac{\log(1+\lambda)}{\lambda} \right)^l [x]_q^l \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \widetilde{B}_{m,q,\lambda} \frac{q^{mx} t^m}{m!} \right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \widetilde{B}_{m,q,\lambda} [x]_q^{n-m} q^{mx} \left(\frac{\log(1+\lambda)}{\lambda} \right)^{n-m} \right) \frac{t^n}{n!}. \tag{2.7}$$

The third theorem is obtained by (2.1) and (2.7) as follows.

Theorem 2.3 *For $n \geq 0$, $\widetilde{B}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}_{n,q,\lambda}(x) = \sum_{m=0}^{n} \binom{n}{m} \widetilde{B}_{m,q,\lambda} [x]_q^{n-m} q^{mx} \left(\frac{\log(1+\lambda)}{\lambda} \right)^{n-m}. \tag{2.8}$$

Remark 2.4

$$\lim_{\lambda \to 0} \widetilde{B}_{m,q,\lambda}(x) = \lim_{\lambda \to 0} \sum_{m=0}^{n} \binom{n}{m} \widetilde{B}_{m,q,\lambda} [x]_q^{n-m} q^{mx} \left(\frac{\log(1+\lambda)}{\lambda} \right)^{n-m}$$
$$= \sum_{m=0}^{n} \binom{n}{m} \widetilde{B}_{m,q} q^{mx}$$
$$= B_{m,q}(x). \tag{2.9}$$

Note that

$$\int_{\mathbb{Z}_p} q^{-y}(1+\lambda)^{\frac{[x+y]_q}{\lambda}t} \, d\mu_q(y)$$
$$= \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{y=0}^{dp^N-1} (1+\lambda)^{\frac{[x+y]_q}{\lambda}t}$$

$$= \lim_{N\to\infty} \frac{1}{[dp^N]_q} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (1+\lambda)^{\frac{[x+a+dy]_q}{\lambda} t}$$

$$= \lim_{N\to\infty} \frac{1}{[d]_q [p^N]_{q^d}} \sum_{a=0}^{d-1} \sum_{y=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{x+a}{d}+y]_{q^d} t}$$

$$= \frac{1}{[d]_q} \sum_{a=0}^{d-1} \lim_{N\to\infty} \frac{1}{[p^N]_{q^d}} \sum_{y=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{x+a}{d}+y]_{q^d} t} q^{-dy} q^{dy}$$

$$= \frac{1}{[d]_q} \sum_{a=0}^{d-1} \left(\int_{\mathbb{Z}_p} q^{-dy} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{x+a}{d}+y]_{q^d} t} \, du_{q^d}(y) \right)$$

$$= \frac{1}{[d]_q} \sum_{a=0}^{d-1} \sum_{n=0}^{\infty} \widetilde{B}_{n,q^d,\lambda}\left(\frac{x+a}{d}\right) \frac{[d]_q^n t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left([d]_q^{n-1} \sum_{a=0}^{d-1} \widetilde{B}_{n,q^d,\lambda}\left(\frac{x+a}{d}\right)\right) \frac{t^n}{n!}, \tag{2.10}$$

where $d \in \mathbb{N}$.

The following theorem is obtained from (2.10).

Theorem 2.5 *For $n \geq 0$ and $d \in \mathbb{N}$, $\widetilde{B}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}_{n,q,\lambda}(x) = [d]_q^{n-1} \sum_{a=0}^{d-1} \widetilde{B}_{n,q^d,\lambda}\left(\frac{x+a}{d}\right). \tag{2.11}$$

Now, we observe that

$$\int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q t} \, du_q(y) = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}. \tag{2.12}$$

We obtain Theorem 2.1 as follows by substituting t by $\log(1+\lambda)^{\frac{t}{\lambda}}$ in (2.12):

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{-y} e^{[x+y]_q \log(1+\lambda)^{\frac{t}{\lambda}}} \, du_q(y) &= \int_{\mathbb{Z}_p} q^{-y} (1+\lambda)^{\frac{[x+y]_q}{\lambda} t} \, du_q(y) \\ &= \sum_{n=0}^{\infty} B_{n,q}(x) \frac{1}{n!} \left(\log(1+\lambda)^{\frac{t}{\lambda}}\right)^n \\ &= \sum_{n=0}^{\infty} B_{n,q}(x) \left(\frac{\log(1+\lambda)}{\lambda}\right)^n \frac{t^n}{n!}. \end{aligned} \tag{2.13}$$

For $r \in \mathbb{N}$, we define the *modified degenerate q-Bernoulli polynomials of order r* as follows:

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)} (1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda} t} \, du_q(x_1) \, du_q(x_2) \cdots du_q(x_r) \\ &= \sum_{n=0}^{\infty} \widetilde{B}_{n,q,\lambda}^{(r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.14}$$

When $x = 0$, $\widetilde{B}^{(r)}_{n,q,\lambda}(0) = \widetilde{B}^{(r)}_{n,q,\lambda}$ are called the *modified degenerate q-Bernoulli numbers of order r*.

We observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}(1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda}t}\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}\sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n\\
&\qquad\times[x_1+x_2+\cdots+x_r+x]_q^n\frac{t^n}{n!}\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}[x_1+x_2+\cdots+x_r+x]_q^n\\
&\qquad\times d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\frac{t^n}{n!}\\
&\quad=\sum_{n=0}^{\infty}\left(\left(\frac{\log(1+\lambda)}{\lambda}\right)^n B^{(r)}_{n,q}(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.15}$$

Therefore, we are able to derive the following theorem.

Theorem 2.6 *For $n \geq 0$, $\widetilde{B}^{(r)}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}^{(r)}_{n,q,\lambda}(x) = \left(\frac{\log(1+\lambda)}{\lambda}\right)^n B^{(r)}_{n,q}(x).\tag{2.16}$$

Now, we consider

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}(1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda}t}\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}\sum_{l=0}^{\infty}\binom{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda}t}{l}\lambda^l\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\sum_{l=0}^{\infty}\sum_{n=0}^{l}\frac{S_1(l,n)}{l!}\lambda^{l-n}t^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)}\\
&\qquad\times[x_1+x_2+\cdots+x_r+x]_q^n\,d\mu_q(x_1)\,d\mu_q(x_2)\cdots d\mu_q(x_r)\\
&\quad=\sum_{l=0}^{\infty}\sum_{n=0}^{l}\frac{S_1(l,n)}{l!}\lambda^{l-n}t^n B^{(r)}_{n,q}(x)\\
&\quad=\sum_{n=0}^{\infty}\left(\sum_{l=n}^{\infty}\frac{S_1(l,n)}{l!}\lambda^{l-n}n!B^{(r)}_{n,q}(x)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.17}$$

Now, (2.17) yields the following theorem.

Theorem 2.7 *For $n \geq 0$, $\widetilde{B}^{(r)}_{n,q,\lambda}(x)$ can be written as*

$$\widetilde{B}^{(r)}_{n,q,\lambda}(x) = \sum_{l=n}^{\infty}\frac{S_1(l,n)}{l!}\lambda^{l-n}n!B^{(r)}_{n,q}(x).\tag{2.18}$$

Now, we observe that, for $d \in \mathbb{N}$,

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-(x_1+x_2+\cdots+x_r)} (1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda} t} \, du_q(x_1)\, du_q(x_2) \cdots du_q(x_r) \\
&= \lim_{N\to\infty} \frac{1}{[dp^N]_q^r} \sum_{x_1=0}^{dp^N-1} \cdots \sum_{x_r=0}^{dp^N-1} (1+\lambda)^{\frac{[x_1+x_2+\cdots+x_r+x]_q}{\lambda} t} \\
&= \lim_{N\to\infty} \frac{1}{[dp^N]_q^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \sum_{x_1=0}^{dp^N-1} \cdots \sum_{x_r=0}^{dp^N-1} (1+\lambda)^{\frac{[a_1+\cdots+a_r+x+dx_1+dx_2+\cdots+dx_r]_q}{\lambda} t} \\
&= \lim_{N\to\infty} \frac{1}{[d]_q^r [p^N]_{q^d}^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \sum_{x_1=0}^{p^N-1} \cdots \sum_{x_r=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[d]_q [\frac{a_1+\cdots+a_r+x}{d}+x_1+x_2+\cdots+x_r]_{q^d} t} \\
&= \frac{1}{[d]_q^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \lim_{N\to\infty} \frac{1}{[p^N]_{q^d}^r} \sum_{x_1=0}^{p^N-1} \cdots \sum_{x_r=0}^{p^N-1} (1+\lambda)^{\frac{1}{\lambda}[\frac{a_1+\cdots+a_r+x}{d}+x_1+x_2+\cdots+x_r]_{q^d} [d]_q t} \\
&= \frac{1}{[d]_q^r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{-d(x_1+x_2+\cdots+x_r)} \\
&\quad \times (1+\lambda)^{\frac{1}{\lambda}[\frac{a_1+\cdots+a_r+x}{d}+x_1+x_2+\cdots+x_r]_{q^d} [d]_q t} \, du_{q^d}(x_1)\, du_{q^d}(x_2) \cdots du_{q^d}(x_r) \\
&= \sum_{n=0}^{\infty} \left([d]_q^{n-r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \widetilde{B}_{n,q^d,\lambda}^{(r)} \left(\frac{a_1+\cdots+a_r+x}{d} \right) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.19}$$

Finally, by comparing the coefficients on both sides of (2.19) we get the following theorem.

Theorem 2.8 *For $n \geq 0$ and $d \in \mathbb{N}$, $\widetilde{B}_{n,q}^{(r)}(x)$ can be written as*

$$\widetilde{B}_{n,q,\lambda}^{(r)}(x) = [d]_q^{n-r} \sum_{a_1=0}^{d-1} \cdots \sum_{a_r=0}^{d-1} \widetilde{B}_{n,q^d,\lambda}^{(r)} \left(\frac{a_1+\cdots+a_r+x}{d} \right). \tag{2.20}$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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