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Matrix norms, BPS bounds and marginal stability in $\mathcal{N} = 8$ supergravity

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Abstract: We study the conditions of marginal stability for two-center extremal black holes in $\mathcal{N}$-extended supergravity in four dimensions, with particular emphasis on the $\mathcal{N} = 8$ case.

This is achieved by exploiting triangle inequalities satisfied by matrix norms. Using different norms and relative bounds among them, we establish the existence of marginal stability and split attractor flows both for BPS and some non-BPS solutions.

Our results are in agreement with previous analysis based on explicit construction of multi-center solutions.

Keywords: Black Holes in String Theory, Black Holes, Extended Supersymmetry, Supergravity Models

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1 Introduction

In the present investigation we consider BPS bounds for $\mathcal{N}\,(\geqslant 2)$-extended supergravity theories, in connection with the marginal stability bound of two-center black holes (BHs). Our analysis is mainly devoted to $\mathcal{N} > 2$ theories, since a vast literature and various results are known for the $\mathcal{N} = 2$ case (see e.g. [1]-[12, 13]; for studies on $\mathcal{N} > 2$, see e.g. [14, 21]). Within this latter framework, the most popular application is provided by Calabi-Yau compactifications of (type II) superstrings. This led to the discovery of the phenomenon of split attractor flow for multi-center BHs [1], which are stable BPS solutions, possibly decaying into single center BHs when the scalar flow cross the wall of *marginal stability* (besides refs. cited above, see also e.g. [22–26]).

In this note we first extend the BPS bound to situations in which the central charge is an antisymmetric complex matrix $Z_{AB}\,(\phi, Q)$ rather than a complex function. For BPS configurations, as well as for some non-BPS ones, this is achieved by exploiting Cauchy-Schwarz triangular inequalities for *matrix norms* of various type (see *e.g.* [27, 28]). For instance, in the more familiar case of BPS bound, the so-called *spectral norm* of Z_{AB} is used.

Interestingly, in some $\mathcal{N} = 2$ as well as $\mathcal{N} > 2$ theories, we find double-center non-BPS BH solutions which exhibit a stability region across a wall of marginal stability. This is ultimately due to the fact that non-BPS BHs are supported by different charge orbits; in the case of $\mathcal{N} = 2$ non-BPS solutions with positive quartic G-invariant ($\mathcal{I}_4 > 0$), similar properties to BPS cases can be found. This is actually not surprising, because many non-BPS $\mathcal{N} = 2$ BH solutions may become BPS when embedded in higher $\mathcal{N}$ supergravities.

In fact, our analysis, both for BPS and non-BPS cases, agrees with results on explicit multi-center BPS and non-BPS solutions [6, 8, 9, 11, 14, 15, 19–21].

In order to use the Cauchy-Schwarz inequality, the crucial point is to associate the first order "fake" superpotential W [29–31, 37] to some well-defined matrix norm $\|\mathbf{Z}\|$ of the central charge matrix $\mathbf{Z}$, or of some other charge matrix. Clearly, when this procedure is possible also for non-BPS states, the matrix norm under consideration will be different from the spectral norm $\|\mathbf{Z}\|_s$ which, as mentioned above, pertains to BPS states.

The paper is organized as follows.

In section 2 we discuss the BPS marginal stability in $\mathcal{N}$-extended supergravity by using the spectral norm of Z_{AB}. Then, within $\mathcal{N} = 8$ maximal theory, we derive a manifestly U-duality invariant expression for the marginal stability wall, as well as for the stability equation fixing the relative distance between the two centers of the solution in terms of moduli ϕ and charges Q (with resulting non-vanishing overall angular momentum).

In section 3 we consider several examples in $\mathcal{N} = 2$ and $\mathcal{N} > 2$ supergravity, in which the results derived in section 2 hold for non-BPS BHs, as well. In $\mathcal{N} = 2$, these include special Kähler geometry with $C_{ijk} = 0$, as well as the non-BPS states with $\mathcal{I}_4 > 0$ in theories with homogeneous symmetric vector multiplets' scalar manifolds.

Section 4 is instead devoted to the study of the more intriguing case of non-BPS states with $\mathcal{I}_4 < 0$. Most of the results of our investigation reproduce the findings of [6, 11], namely both the two-center BH and the two one-center BHs produced by the its decay lie on the marginal stability wall, and thus no stable region for multi-center solution exists other than the marginal one. This is related to the fact that, in these examples, the charge vectors Q_1 and Q_2 of the two centers are *mutually local* (namely, their symplectic product vanishes: $\langle Q_1, Q_2 \rangle = 0$).

A non-BPS $\mathcal{I}_4 < 0$ stable double-center BHs can be found, at least in $\mathcal{N} = 8$ supergravity. This is the case in which the Pfaffian $Pf(\mathbf{Z})$ is real, thus with phase $\varphi = \pi$, all along the attractor flow. In fact, under this assumption, the non-BPS "fake" superpotential W_{nBPS} can be associated to the so-called *trace norm* of $\mathbf{Z}$ itself. On the other hand, as recently shown in [38, 39], multi-center non-BPS BHs with constrained positions of the centers and $\mathcal{I}_4 < 0$ (and therefore non-BPS also when uplifted to $\mathcal{N} = 8$) have been explicitly constructed. It would be interesting to investigate the occurrence of the split attractor flow in this framework.

2 BPS bounds and matrix norms

We here consider the generalization of the BPS bound as well as of the Cauchy-Schwarz triangle inequality, which is at the basis of the concept of *marginal stability*. In order to study this problem, we make a small prelude on matrix norms (see e.g. [27, 28] for further details).

2.1 Matrix norms

Given a complex rectangular $n \times m$ matrix $\mathbf{Z}$, its matrix norm $\|\mathbf{Z}\|$ is a consistent generalization of the concept of vector norm, satisfying by definition the following properties:

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$\|\mathbf{Z}\| \geqslant 0$ ($= 0$ *iff* $\mathbf{Z} = 0$); $\|\alpha\mathbf{Z}\| = |\alpha| \, \|\mathbf{Z}\| \; \forall \alpha \in \mathbb{C}$; and

$$\|\mathbf{Z}_1 + \mathbf{Z}_2\| \leqslant \|\mathbf{Z}_1\| + \|\mathbf{Z}_2\| \,. \tag{2.1}$$

In our treatment, we will be mainly concerned of three types of norms, which are particular cases of the so-called *Schatten* p-norms. Such matrix norms are defined as the norms of the real vector σ of the *singular values* of a square $n \times n$ matrix $\mathbf{Z}$ (which are nothing but the absolute values of the eigenvalues of $\mathbf{Z}$ itself: $\sigma \equiv \{\sigma_i\}_{i=1,..,n}$):

$$\|\mathbf{Z}\|_p \equiv \left(\sum_i \sigma_i^p \right)^{1/p} . \tag{2.2}$$

Namely, we will consider the following norms:

1. *Spectral norm.* Starting from the square matrix $\mathbf{Z}$, one can define the positive semi-definite matrix $\mathbf{Z}\mathbf{Z}^\dagger$, whose real positive eigenvalues λ_i's ($i = 1, ..., m$) are the squared *singular values* of $\mathbf{Z}$ itself: $\lambda_i = \sigma_i^2$. The spectral norm $\|\mathbf{Z}\|_s$ of $\mathbf{Z}$ is defined as the *maximum norm* of the vector σ:

$$\|\mathbf{Z}\|_s \equiv \|\sigma\|_\infty \equiv \max\{\sigma_i\} \equiv \sqrt{\lambda_h}, \tag{2.3}$$

where λ_h is the highest eigenvalue of the matrix $\mathbf{Z}\mathbf{Z}^\dagger$. The spectral norm is formally obtained as the $p \to \infty$ limit of the Schatten matrix p-norm (2.2).

2. *Frobenius norm.* The Frobenius norm $\|\mathbf{Z}\|_F$ of the square matrix $\mathbf{Z}$ is defined as the Euclidean norm of the vector σ :

$$\|\mathbf{Z}\|_F \equiv \|\sigma\|_2 \equiv \sqrt{\sum_i \lambda_i} \equiv \sqrt{Tr\left(\mathbf{Z}\mathbf{Z}^\dagger\right)}. \tag{2.4}$$

The Frobenius norm is actually a Schatten matrix 2-norm. As a particular case in which the matrix $\mathbf{Z}$ degenerates to complex vector Z_I, we will also consider the usual Euclidean norm of Z_I ($I = 1, ..., m$) itself, defined as

$$\|Z_I\|_2 \equiv \sqrt{Z_I \overline{Z}^I}. \tag{2.5}$$

3. *Trace (or nuclear) norm.* The trace norm $\|\mathbf{Z}\|_*$ of the square matrix $\mathbf{Z}$ is defined as the 1-norm of the vector σ:

$$\|\mathbf{Z}\|_* \equiv \|\sigma\|_1 \equiv \sum_i \sqrt{\lambda_i} \equiv Tr\left(\sqrt{\mathbf{Z}\mathbf{Z}^\dagger}\right). \tag{2.6}$$

The trace norm is actually a Schatten matrix 1-norm.

The crucial property of all these norms is the Cauchy-Schwarz triangle inequality (2.1), wich we will exploit in order to study the marginal stability of double-center BH configurations, in the case in which the (spatial asymptotical limit of the) relevant matrix norm $\|\mathbf{Z}\|$ is associated to the ADM mass M_{ADM} [40] of the BH solution itself.

The equivalence of the spectral and Frobenius matrix norms is expressed by the following chain of inequalities:

$$\|\mathbf{Z}\|_s \leqslant \|\mathbf{Z}\|_F \leqslant \sqrt{rank\,(\mathbf{Z})}\,\|\mathbf{Z}\|_s\,. \tag{2.7}$$

Let us specify (2.7) for $\mathbf{Z}$ being the central charge matrix of $\mathcal{N}=8$, $d=4$ supergravity. In this case, $rank\,(\mathbf{Z})=8$, and

$$\|\mathbf{Z}\|_F = \sqrt{2\sum_{i=1}^{4}\lambda_i} = \sqrt{2V_{BH}}, \tag{2.8}$$

where V_{BH} is the BH effective potential. Furthermore, due to the antisymmetry of $\mathbf{Z}$ itself, the Bloch-Messiah-Zumino Theorem [41] implies the eigenvalues of $\mathbf{Z}$ and $\mathbf{Z}\mathbf{Z}^\dagger$ to be pairwise; thus, for $\mathbf{Z}$ the chain of inequalities (2.7) can be made more strict:

$$\|\mathbf{Z}\|_s \leqslant \frac{\|\mathbf{Z}\|_F}{\sqrt{2}} \leqslant \sqrt{\frac{rank\,(\mathbf{Z})}{2}}\,\|\mathbf{Z}\|_s\,. \tag{2.9}$$

(2.9) can be rewritten as

$$\sqrt{\lambda_h} \leqslant \sqrt{V_{BH}} \leqslant 2\sqrt{\lambda_h} \Leftrightarrow \lambda_h \leqslant V_{BH} \leqslant 4\lambda_h. \tag{2.10}$$

This can be extended to the non-BPS case, by noticing that the first order "fake" superpotential W_{nBPS} satisfies the bound

$$\|\mathbf{Z}\|_s < W_{nBPS} \leqslant \frac{\|\mathbf{Z}\|_F}{\sqrt{2}} \leqslant 2\,\|\mathbf{Z}\|_s\,, \tag{2.11}$$

where the first upper bound on W_{nBPS} is due to eq. (2.17) further below. If one further applies (2.9) to the quantity $W_{nBPS}\,(\phi, Q_1+Q_2)$ and uses the triangle inequality for $\|\mathbf{Z}\|_s$, the following non-BPS inequality is obtained:

$$\begin{aligned} W_{nBPS}\,(\phi, Q_1+Q_2) &\leqslant 2\,\|\mathbf{Z}\,(\phi, Q_1+Q_2)\|_s \leqslant 2\left[\|\mathbf{Z}\,(\phi, Q_1)\|_s + \|\mathbf{Z}\,(\phi, Q_2)\|_s\right] \\ &< 2\left[W_{nBPS}\,(\phi, Q_1) + W_{nBPS}\,(\phi, Q_2)\right]. \end{aligned} \tag{2.12}$$

In the spatial asymptotical limit, (2.12) is an upper limit for the two-center ADM mass in terms of the ADM masses of the single-center constituents. Note that (2.12) is twice the marginal stability bound, and in some cases it overestimates the actual bound. Indeed, for $\mathcal{N}<8$ non-BPS BHs with $\mathcal{I}_4>0$ (see section 3 further below) the bound satisfied by the corresponding first order "fake" superpotential $W_{\mathcal{I}_4>0}$ is a triangle inequality:

$$W_{\mathcal{I}_4>0}\,(\phi, Q_1+Q_2) \leqslant W_{\mathcal{I}_4>0}\,(\phi, Q_1) + W_{\mathcal{I}_4>0}\,(\phi, Q_2) \tag{2.13}$$

as in the BPS cases, implying that a stability region for double-center solutions exists in this case.

It is also interesting to compare (2.11) with the chain of inequalities obtained in [43]. The lowest bound of (2.11) holds for BPS saturation ($W^2=\lambda_h=\|\mathbf{Z}\|_s^2$), while its highest bound is reached at non-BPS $\mathcal{I}_4<0$ attractor points. Thus, the inequality obtained in [43] is nothing but the equivalence of the spectral and Frobenius norms of the central charge matrix $\mathbf{Z}$ of $\mathcal{N}=8$, $d=4$ supergravity.

2.2 BPS Bounds and first order flows

Let us now consider $\mathbf{Z}$ to be the antisymmetric central charge matrix Z_{AB} $(A, B = 1, ..., \mathcal{N})$ centrally extending the local supersymmetry algebra of an $\mathcal{N}$-extended supergravity theory in d space-time dimensions. From the Bloch-Messiah-Zumino Theorem [41, 42], the positive semi-definite matrix $\mathbf{Z}\mathbf{Z}^{\dagger}$ has $[\mathcal{N}/2]$ independent eigenvalues λ_i $(i = 1, ..., [\mathcal{N}/2])$, and the BPS bound reads (at spatial infinity)

$$M_{ADM} \geqslant \lambda_h, \tag{2.14}$$

where M_{ADM} denotes the ADM mass of the considered BH state, whereas, as previously mentioned, $\lambda_h \equiv \max\{\lambda_i\}$. If the BPS bound is saturated by k equal highest eigenvalues of $\mathbf{Z}$, then the corresponding state is called $\frac{k}{\mathcal{N}}$-BPS. In $d = 4$ supergravity, if $k > 1$ the corresponding BH solution has[1] $\mathcal{I}_4 = 0$ and the near-horizon space-time geometry is singular (at least in the Einsteinian approximation). Indeed, it is here worth recalling that the absolute value of the quadratic G-invariant (if any) $\mathcal{I}_2$ or the square root of the absolute value of the quartic G-invariant $\mathcal{I}_4$ is the critical, attractor value of W^2 of the corresponding flow; thus, through the Bekenstein-Hawking entropy-area formula [44, 45], in the Einstein supergravity approximation the entropy of the single-center extremal BH solution reads [46, 47]

$$S_{BH} = \pi \frac{A_H}{4} = \pi \left. W^2 \right|_{\partial W = 0} = \pi \left. V_{BH} \right|_{\partial V_{BH} = 0} = \pi \mathcal{I}, \tag{2.15}$$

where A_H is the area of the BH event horizon, and $\mathcal{I}$ is the G-invariant (G denoting the U-duality group), which does depend on charges, but not on scalar fields. In the theories under consideration in the present paper, $\mathcal{I} = \sqrt{|\mathcal{I}_4|}$, where $\mathcal{I}_4$ is the G-invariant quartic in charges (as in $\mathcal{N} = 8$ supergravity), or $\mathcal{I} = \mathcal{I}_2$, where $\mathcal{I}_2$ is the G-invariant quadratic in charges (as in $\mathcal{N} = 2$ *minimally coupled* $\mathbb{CP}^n$ models and in $\mathcal{N} = 3$ supergravity [48]).

For extremal BHs, the warp factor of the metric and the scalar flow associated with the $\frac{k}{\mathcal{N}}$-BPS solution are determined by the superpotential $W = \sqrt{\lambda_h}$, which satisfies first order flow equations [29]:

$$\dot{U} = -e^U W; \quad \dot{\phi}^{\alpha} = -2 e^U g^{\alpha\beta} \partial_{\beta} W, \tag{2.16}$$

with the effective BH potential given by

$$V_{BH} = W^2 + 2 g^{\alpha\beta} \left(\partial_{\alpha} W \right) \partial_{\beta} W. \tag{2.17}$$

Note that in $\mathcal{N} = 8$, (2.17) can be re-written as a differential relation between the spectral and Frobenius norms of the central charge matrix $\mathbf{Z}$:

$$\|\mathbf{Z}\|_F^2 = 2 \|\mathbf{Z}\|_s^2 + 4 g^{\alpha\beta} \left(\partial_{\alpha} \|\mathbf{Z}\|_s \right) \partial_{\beta} \|\mathbf{Z}\|_s \,. \tag{2.18}$$

The same relations hold true for non-BPS BHs for all $\mathcal{N} \geqslant 3$ theories and for $\mathcal{N} = 2$ models based on symmetric scalar manifolds (for generalizations beyond symmetric spaces,

[1] In $\mathcal{N} = 8$ supergravity for $k = 1$ also a "small" ($\mathcal{I}_4 = 0$) charge orbit exists [50, 57, 58]. This orbit gives both BPS and non-BPS "small" orbits in $\mathcal{N} = 2$ theories. No "small" non-BPS orbits exist in $\mathcal{N} = 8$.

see e.g. [37]), provided one replaces W with the suitable non-supersymmetric first order "fake" superpotential W_{nBPS} [29–36]. For non-BPS BHs supported by generic charge configurations with $\mathcal{I}_4 < 0$, the "fake" superpotential has a complicated expression (see the first, second and fourth of refs. [31–36], and [37]). On the other hand, for all non-BPS BHs with $\mathcal{I}_4 > 0$ the "fake" superpotential can be easily written in terms of a matrix or vector norm of quantities linear in the charges Q. This allows for an analysis of the marginal stability properties also for such a class of non-BPS constituents and non-BPS composites.

2.3 BPS marginal stability for $\mathcal{N} > 2$

We are now going to apply the triangle inequality (2.1) of the matrix norms to the appropriate matrices relevant for the study of extremal BHs in $\mathcal{N}$-extended supergravity theories. As mentioned above, for BPS states, regardless their BPS fraction, the relevant object is the $\mathcal{N} \times \mathcal{N}$ complex antisymmetric central charge matrix $\mathbf{Z} \equiv Z_{[AB]}(\phi, Q)$, which is linear in charges:

$$Z_{AB}(\phi, Q_1 + Q_2) = Z_{AB}(\phi, Q_1) + Z_{AB}(\phi, Q_2)\,. \tag{2.19}$$

Thus, if one assumes the symplectic charge vectors $Q_1 + Q_2$, Q_1 and Q_2 to be all BPS, the triangle inequality for the spectral norm $\|\mathbf{Z}\|_s$ defined by (2.3) yields (in the spatial asymptotical limit) a bound on the ADM masses, as follows (we omit the subscript "ADM" throughout):

$$M(\phi_\infty, Q_1 + Q_2) \leqslant M(\phi_\infty, Q_1) + M(\phi_\infty, Q_2)\,, \tag{2.20}$$

with $M^2 = \lambda_h$, and "ϕ_∞" denoting the spatially asymptotical values of scalar fields. The marginal stability condition corresponds to the saturation of the bound (2.20). Such a saturation defines the marginal stability wall as the (Q_1, Q_2)-dependent *locus* in the (spatially asymptotical) scalar manifold satisfying the equation

$$\sqrt{\lambda_h}(\phi_\infty, Q_1 + Q_2) = \sqrt{\lambda_h}(\phi_\infty, Q_1) + \sqrt{\lambda_h}(\phi_\infty, Q_2)\,. \tag{2.21}$$

By considering $\mathcal{N} = 8$ supergravity, it is worth recalling that the eigenvalues of $\mathbf{Z}\mathbf{Z}^\dagger$ are solutions of the (square root of the) *characteristic equation* [49]

$$\sqrt{\det\left(\mathbf{Z}\mathbf{Z}^\dagger - \lambda\mathbb{I}\right)} = \prod_{i=1}^{4}(\lambda - \lambda_i) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0, \tag{2.22}$$

where the real coefficients a, b, c, d, as well as the explicit expressions of the λ_i's are given, in terms of $Tr\left(\mathbf{Z}\mathbf{Z}^\dagger\right)^K$ $(K = 1, ..., 4)$, in [49] (see also the recent treatment in [50]).

The marginal decay of a "large" $(\mathcal{I}_4 > 0)$ $\frac{1}{8}$-BPS two-center BH state into two single-center BPS states $(k_1, k_2 = 1, 2, 4)$

$$\frac{1}{8}\text{-BPS "large"} \longrightarrow \frac{k_1}{8}\text{-BPS} + \frac{k_2}{8}\text{-BPS} \tag{2.23}$$

can be studied by using (2.21) and the aforementioned expressions of λ_i's. Examples of (2.23) with k_1 and/or $k_2 > 1$ have been considered e.g. in [3, 9].

We note that, since $\sqrt{|\mathcal{I}_4(Q_1+Q_2)|} \neq \sqrt{|\mathcal{I}_4(Q_1)|}+\sqrt{|\mathcal{I}_4(Q_2)|}$, the two-center solution can have less or more entropy than the single-center solution with the same charge vector Q_1+Q_2. While the BPS single-center BH does not exist if $\mathcal{I}_4(Q_1+Q_2)<0$, in the cases discussed e.g. in [3, 9] the BPS multi-center solution has $\mathcal{I}_4\left(\sum_i Q_i\right) \gtrless 0$, but its entropy is always given by $\sum_i \sqrt{\mathcal{I}_4(Q_i)}$, with $\mathcal{I}_4(Q_i) \geqslant 0\ \forall i$.

When at least one of the final two single-center BH states is non-BPS, namely for cases

$$\frac{1}{8}\text{-BPS } \textit{“large”} \longrightarrow \frac{k_1}{8}\text{-BPS } + \text{ nBPS}; \tag{2.24}$$

$$\frac{1}{8}\text{-BPS } \textit{“large”} \longrightarrow \text{nBPS } + \text{ nBPS}, \tag{2.25}$$

there is no marginal decay. Indeed, due to the non-saturation of the BPS bound by one center or both centers, it respectively holds that (at spatial infinity)

$$\|\mathbf{Z}_1+\mathbf{Z}_2\|_s \leqslant \|\mathbf{Z}_1\|_s+\|\mathbf{Z}_2\|_s < \begin{cases} M_1+\|\mathbf{Z}_2\|_s\,; \\ M_1+M_2, \end{cases} \tag{2.26}$$

where we use the short-hand notation $\mathbf{Z}_\alpha \equiv \mathbf{Z}(Q_\alpha)$ and $M_\alpha \equiv M(Q_\alpha)$ $(\alpha = 1,2)$ throughout.

2.4 $\mathcal{N}=8$ BPS stability conditions

Given a two-center BH solution, let us now turn to consider the formula of the relative distance $|\overrightarrow{x_1}-\overrightarrow{x_2}|$ of the two single-center BH constituents with *mutually non-local* charges $\langle Q_1, Q_2\rangle \neq 0$.

In the $\mathcal{N}=2$ theory (in which $Z_{AB}=Z\epsilon_{AB}$, $A,B=1,2$) such a distance is [3]

$$|\overrightarrow{x_1}-\overrightarrow{x_2}| = \frac{1}{2}\frac{\langle Q_1,Q_2\rangle\,|Z_1+Z_2|}{\mathrm{Im}\left(Z_1\overline{Z_2}\right)}, \tag{2.27}$$

where $Z_i \equiv Z(\phi_\infty, Q_i)$ $(i=1,2)$, and[2]

$$2\left|\mathrm{Im}\left(Z_1\overline{Z_2}\right)\right| = \sqrt{4\,|Z_1|^2\,|Z_2|^2-\left(|Z_1+Z_2|^2-|Z_1|^2-|Z_2|^2\right)^2}. \tag{2.28}$$

Eq. (2.27) implies the stability region for the double-center BH solution to occur for $\langle Q_1,Q_2\rangle \mathrm{Im}\left(Z_1\overline{Z_2}\right) > 0$, while it is forbidden for $\langle Q_1,Q_2\rangle \mathrm{Im}\left(Z_1\overline{Z_2}\right) < 0$. Note that the quantity $\langle Q_1,Q_2\rangle \mathrm{Im}\left(Z_1\overline{Z_2}\right)$ is even under the center exchange $1 \leftrightarrow 2$. The scalar flow is directed from the stability region towards the instability region, crossing the wall of marginal stability at $\langle Q_1,Q_2\rangle \mathrm{Im}\left(Z_1\overline{Z_2}\right)=0$. This implies that the stability region is placed *beyond* the marginal stability wall, and *on the opposite side* of the split attractor flows.

[2] Note that $\mathrm{Im}\left(Z_1\overline{Z_2}\right)=0$ both describes marginal and *anti-marginal* stability [12, 13]. *Marginal stability* (at which $\mathrm{Re}\left(Z_1\overline{Z_2}\right)>0$) further requires $|Z_1+Z_2|^2 > |Z_1|^2+|Z_2|^2$. In the other branch $|Z_1+Z_2|^2 < |Z_1|^2+|Z_2|^2$, *anti-marginal stability* (at which $\mathrm{Re}\left(Z_1\overline{Z_2}\right)<0$) corresponds to $|Z_1+Z_2| = ||Z_1|-|Z_2||$.
All these bounds can be reformulated for $\mathcal{N}>2$ BPS states by replacing $|Z|$ with $\sqrt{\lambda_h}=\|\mathbf{Z}\|_s$.

By using the fundamental identities of $\mathcal{N}=2$ special Kähler geometry in presence of two (mutually non-local) symplectic charge vectors Q_1 and Q_2 (see e.g. [1, 52, 53]), one can compute that at BPS attractor points of the centers 1 *or* 2:

$$\mathcal{N}=2:\langle Q_1,Q_2\rangle=-2\mathrm{Im}\left(Z_1\overline{Z_2}\right)\Rightarrow 2\langle Q_1,Q_2\rangle\,\mathrm{Im}\left(Z_1\overline{Z_2}\right)=-\langle Q_1,Q_2\rangle^2<0. \tag{2.29}$$

By using (2.27) and (2.29), one obtains $|\vec{x_1}-\vec{x_2}|<0$: this means that, as expected, the BPS attractor points of the centers 1 *or* 2 do not belong to the stability region of the two-center BH solution. Furthermore, the result (2.29) also consistently implies:

$$\text{stability region}:\quad \begin{array}{l}\langle Q_1,Q_2\rangle\,\mathrm{Im}\left(Z_1\overline{Z_2}\right)\\ =|\langle Q_1,Q_2\rangle|\sqrt{4|Z_1|^2|Z_2|^2-\left(|Z_1+Z_2|^2-|Z_1|^2-|Z_2|^2\right)^2}>0;\end{array} \tag{2.30}$$

$$\text{instability region}:\quad \begin{array}{l}\langle Q_1,Q_2\rangle\,\mathrm{Im}\left(Z_1\overline{Z_2}\right)\\ =-|\langle Q_1,Q_2\rangle|\sqrt{4|Z_1|^2|Z_2|^2-\left(|Z_1+Z_2|^2-|Z_1|^2-|Z_2|^2\right)^2}<0,\end{array} \tag{2.31}$$

where a particular case of (2.31), holding at the attractor points, is given by (2.29).

By replacing $|Z|$ with $\sqrt{\lambda_h}$ in (2.28), the generalization of (2.27) to $\mathcal{N}=8$ maximal supergravity reads

$$|\vec{x_1}-\vec{x_2}|=\frac{|\langle Q_1,Q_2\rangle|\sqrt{\lambda_{1+2,h}}}{\sqrt{4\lambda_{1,h}\lambda_{2,h}-\left(\lambda_{1+2,h}-\lambda_{1,h}-\lambda_{2,h}\right)^2}}, \tag{2.32}$$

where $\lambda_{1+2,h}\equiv\lambda_h\left(\phi_\infty,Q_1+Q_2\right)$ and $\lambda_{i,h}\equiv\lambda_h\left(\phi_\infty,Q_i\right)$ $(i=1,2)$. Note that eq. (2.32) is manifestly $\mathcal{N}=8$ U-duality invariant (written in terms of $Tr\left(\mathbf{Z}\mathbf{Z}^\dagger\right)^K$ $(K=1,...,4)$), and it reduces to (2.27) in the $\mathcal{N}=2$ case. It is here worth remarking that $\mathcal{I}_4$ of the $\mathcal{N}=8$ theory is a (moduli independent) $G=E_{7(7)}$-invariant constructed with the (moduli dependent) $H=\mathrm{SU}(8)$-invariants $Tr\left(\mathbf{Z}\mathbf{Z}^\dagger\right)^K$ $(K=1,2)$ and $Pf\left(\mathbf{Z}\right)$ [46, 48, 51].

Moreover, a result similar to (2.29) holds for $\mathcal{N}=8$ supergravity, as well. Indeed, by exploiting the $\mathcal{N}=8$ generalization of the $\mathcal{N}=2$ special geometry identities [53]

$$\langle Q_1,Q_2\rangle=-\mathrm{Im}\left(Tr\left(\mathbf{Z}_1\mathbf{Z}_2^\dagger\right)\right), \tag{2.33}$$

one can compute that at the $\frac{1}{8}$-BPS attractor points of the centers 1 *or* 2:

$$\mathcal{N}=8:|\langle Q_1,Q_2\rangle|=\sqrt{4\lambda_{h,1}\lambda_{h,2}-\left(\lambda_{1,h}+\lambda_{2,h}-\lambda_{1+2,h}\right)^2}. \tag{2.34}$$

However, note that $\frac{1}{8}$-BPS attractor points of the centers 1 *or* 2 do not belong to the stability region of the two-center BH solution, but instead they are placed, with respect to the stability region, on the *opposite* side of the marginal stability wall.

It is worth concluding the present section by remarking that the results (2.29) and (2.34) are consistent with situations in which the ADM masses are always on the marginal stability wall (for a given set of charges, and within a suitable subspace of the scalar manifold, such as vanishing axions), and then also $\langle Q_1, Q_2 \rangle = 0$ (mutually local charges), thus not constraining $|\vec{x_1} - \vec{x_2}|$ in any way (with resulting vanishing overall angular momentum). For instance, this holds for the limit (scalarless) case of Reissner-Nördstrom double-center BH solutions in $\mathcal{N} = 2$ pure supergravity. Some other cases are discussed in section 4.

3 Marginal stability for non-BPS $\mathcal{I}_2 < 0$ and $\mathcal{I}_4 > 0$ black holes

We now consider particular non-BPS double-center BH solutions for which marginal stability walls can be discussed in full generality.

Let us start with the $\mathcal{N} = 2$ theories with $\mathbb{CP}^n$ vector multiplets' scalar manifolds, namely the models in which the n vector multiplets are *minimally coupled* to the gravity multiplet [54] (see also e.g. [55]). Such models all have $C_{ijk} = 0$, and only one type of non-BPS attractors, namely the ones with vanishing central charge at the horizon ($Z_H = 0$) and $\mathcal{I}_2 < 0$ ($\mathcal{I}_2$ denoting the quadratic G-invariant of these theories). The first order "fake" superpotential for non-BPS $Z_H = 0$ is nothing but the Euclidean norm (2.5) of the complex vector of matter charges $Z_a \equiv D_a Z$ ($a = 1, ..., n$) in local flat indices [30] ($(z, \overline{z})$ denotes the $\mathcal{N} = 2$ - or $\mathcal{N} = 6$ - , $d = 4$ complex scalars throughout):

$$W(z, \overline{z}; Q) = \sqrt{g^{i\overline{j}} Z_i \overline{Z}_{\overline{j}}} = \sqrt{\sum_a |Z_a|^2} = \|D_a Z\|_2 \,. \tag{3.1}$$

Thus, due to the linearity of $D_a Z$ in the charges Q:

$$D_a Z(z, \overline{z}; Q_1 + Q_2) = D_a Z(z, \overline{z}; Q_1) + D_a Z(z, \overline{z}; Q_2) \,, \tag{3.2}$$

the non-BPS $Z_H = 0$ "fake" superpotential given by (3.1) satisfies the triangle inequality:

$$W(z, \overline{z}; Q_1 + Q_2) \leqslant W(z, \overline{z}; Q_1) + W(z, \overline{z}; Q_2) \,. \tag{3.3}$$

Since the spatial asymptotical limit of W is nothing but the ADM mass (namely $M \equiv W(\phi_\infty, Q)$), it follows that the saturation of the spatial asymptotical limit of (3.3) yields the marginal stability condition for the decay

$$nBPS \longrightarrow nBPS + nBPS, \tag{3.4}$$

with $\mathcal{I}_2(Q_1 + Q_2) < 0$, $\mathcal{I}_2(Q_1) < 0$ and $\mathcal{I}_2(Q_2) < 0$.

The same holds true for the unique non-BPS ("large") charge orbit of $\mathcal{N} = 3$ supergravity [56] (see also e.g. [55]). This theory also has a quadratic G-invariant $\mathcal{I}_2$, and a first order non-BPS "fake" superpotential which is the Euclidean norm (2.5) of the complex vector of matter charges Z_I ($I = 1, ..., n_V$, n_V denoting the number of vector multiplets) [30]:

$$W(z, \overline{z}; Q) = \|Z_I\|_2 \equiv \sqrt{Z_I \overline{Z}^I}. \tag{3.5}$$

Thus, due to the linearity of Z_I in the charges Q:

$$Z_I(z,\overline{z};Q_1+Q_2) = Z_I(z,\overline{z};Q_1) + Z_I(z,\overline{z};Q_2), \tag{3.6}$$

the non-BPS $Z_H = 0$ "fake" superpotential given by (3.5) satisfies the triangle inequality (3.3), whose spatial asymptotical limit yields an analogue bound for the ADM masses. The saturation of such a bound is the marginal stability condition for the decay (3.4).

For theories with a quartic G-invariant $\mathcal{I}_4$, the non-BPS charge orbit with $\mathcal{I}_4 > 0$ can also be discussed in a fairly general way. The crucial observation is that this orbit is non-BPS for lower $\mathcal{N}$'s, but it becomes BPS when the model is embedded in maximal ($\mathcal{N} = 8$) supergravity. Indeed, it is worth noticing that in $\mathcal{N} = 8$ supergravity, unlike lower-$\mathcal{N}$ theories, the unique non-BPS charge orbit is "large" with $\mathcal{I}_4 < 0$ [57, 58]. Thus, since the marginal bounds on moduli and charges are insensitive to the value of $\mathcal{N}$, the treatment of double-center BH solutions can be performed (for studies of this issue within a $d = 3$ approach, see [19]).

As an illustrative example, let us consider the $\mathcal{N} = 6$ theory, characterized by the central charge matrix Z_{AB} and a complex singlet charge X. This theory shares the very same bosonic sector with the $\mathcal{N} = 2$ "magic" model based on the degree-3 Euclidean Jordan algebra over the quaternions ($J_3^{\mathbb{H}}$), with central charge $Z \equiv X$ [48, 59, 60]. After the analysis of [61], the $\mathcal{N} = 6$ $\frac{1}{6}$-BPS "large" orbit becomes the $\mathcal{N} = 2$ non-BPS $Z_H = 0$. Thus, the non-BPS $Z_H = 0$ of the $\mathcal{N} = 2$ $J_3^{\mathbb{H}}$ "magic" supergravity has $W = \sqrt{\lambda_h} = \left\|\mathbf{Z}_{\mathcal{N}=6}\right\|_s > |X|$, and it satisfies the marginal stability bound because of the triangle inequality on $\left\|\mathbf{Z}_{\mathcal{N}=6}\right\|_s$ itself. In this case, the formula (2.32), clearly with λ_h denoting the maximal eigenvalue of the semi-positive definite matrix $\mathbf{Z}_{\mathcal{N}=6}\mathbf{Z}^{\dagger}_{\mathcal{N}=6}$. On the other hand, the $\mathcal{N} = 6$ non-BPS $\mathbf{Z}_H \neq 0$ "large" orbit corresponds to the $\mathcal{N} = 2$ ($\frac{1}{2}$-)BPS "large" orbit [61], with first order superpotential $|Z| = |X| > \sqrt{\lambda_h} = \left\|\mathbf{Z}_{\mathcal{N}=6}\right\|_s$. Thus, due to the linearity of $X(z,\overline{z};Q)$ in the charges Q, the triangle inequality (which here is a mere consequence of the Cauchy-Schwarz inequality on complex numbers)

$$|X(z,\overline{z};Q_1+Q_2)| \leqslant |X(z,\overline{z};Q_1)| + |X(z,\overline{z};Q_2)| \tag{3.7}$$

applies. The relative distance of the two centers $|\overrightarrow{x_1} - \overrightarrow{x_2}|$ can be computed simply by taking eq. (2.27) and replacing Z with X.

By exploiting the fact that the complex matter charges' vector $D_a Z$ in local flat indices ($a = 1, ..., n_V$) can be re-arranged in terms of an antisymmetric complex matrix embedded in the central charge matrix Z_{AB} of $\mathcal{N} = 8$ supergravity, one can show the above analysis to hold true for the non-BPS $\mathcal{I}_4 > 0$ charge orbits of the remaining $\mathcal{N} = 2$ "magic" models (based on $J_3^{\mathbb{A}}$, with $\mathbb{A} = \mathbb{C}, \mathbb{R}$), which are consistent truncation of the quaternionic model. The "magic" octonionic model, based on $J_3^{\mathbb{O}}$, cannot be obtained through consistent truncation of $\mathcal{N} = 8$ theory, but the above analysis can be still shown to hold, since the matter charges of the $n_V = 27$ vector multiplets re-arrange (in an $USp(8)$-irreducible way) as a skew-traceless 8×8 complex antisymmetric matrix Z^0_{AB}, whose skew-trace is the $\mathcal{N} = 2$ central charge Z.

Therefore, we conclude that the non-BPS $\mathcal{I}_4 > 0$ composites and constituents may satisfy the marginal stability condition, with a region of stable double-center BH solutions.

Note that this situation is different from the one discussed in [6, 11] in which no stable multi-center configurations were found for non-BPS composites. However, it confirms the analysis of explicit multi-center non-BPS solutions with $\mathcal{I}_4 > 0$ performed in [19–21].

4 Marginal stability for non-BPS $\mathcal{I}_4 < 0$ black holes

In the present section, we discuss the condition of marginal stability for non-BPS states with $\mathcal{I}_4 < 0$. In this case, the above reasoning ascribing the non-BPS lower-$\mathcal{N}$ BH states to BPS orbits in higher-$\mathcal{N}$ theories cannot be repeated, because non-BPS BH states with $\mathcal{I}_4 < 0$ are all uplifted to non-BPS $\mathcal{I}_4 < 0$ in maximal supergravity (for investigations within $\mathcal{N} = 8$ and $\mathcal{N} = 4$ theories, see e.g. the first and second refs. of [19]).

We actually find that this occurs, in all known examples, in the rather trivial situation in which the charge vectors Q_1 and Q_2 of the two centers are *mutually local* (i.e. $\langle Q_1, Q_2\rangle = 0$). A non-trivial case is discussed at the end of the present section.

It is here worth commenting on the $\mathcal{N} = 2$ cases discussed in [29], which are characterized by a"twisted" ("fake") central charge. Let us consider for instance the case discussed, in the "electric" configuration $\left(p^0, q_1\right)$ of the so-called 1-modulus t^3 model, in section 5 therein. In the ($\frac{1}{2}$-)BPS branch ($p^0 q_1 < 0$), the first order superpotential reads

$$W_{BPS} = |Z|\,;\;\; Z = \frac{tq_1 + p^0 t^3}{\sqrt{-i\left(t-\overline{t}\right)^3}}, \tag{4.1}$$

while in the non-BPS $Z_H \neq 0$ branch ($p^0 q_1 > 0$) the first order superpotential reads

$$W_{nBPS} = |Z_{twist}|\,;\;\; Z_{twist} = \frac{tq_1 + p^0 t^2\overline{t}}{\sqrt{-i\left(t-\overline{t}\right)^3}} = t\frac{\left(q_1 + p^0\left|t\right|^2\right)}{\sqrt{-i\left(t-\overline{t}\right)^3}}; \tag{4.2}$$

$$\Downarrow$$

$$W_{nBPS} = \pm\left|t\right|\frac{\left(q_1 + p^0\left|t\right|^2\right)}{\sqrt{-i\left(t-\overline{t}\right)^3}} \;\text{ for } p^0, q_1 \gtrless 0. \tag{4.3}$$

Thus, W_{nBPS} given by (4.3) is linear in charges, whereas W_{BPS} given by (4.1) is not:

$$\left|Z_{twist}\left(Q_1+Q_2\right)\right| = \left|Z_{twist}\left(Q_1\right)\right| + \left|Z_{twist}\left(Q_2\right)\right|; \tag{4.4}$$

$$\left|Z\left(Q_1+Q_2\right)\right| \leqslant \left|Z\left(Q_1\right)\right| + \left|Z\left(Q_2\right)\right|. \tag{4.5}$$

Thus, the twist $t^3 \to t^2\overline{t}$ determining $Z \to Z_{twist}$ makes the stability region for the two-center non-BPS configuration empty. The multi-center solutions discussed in [6, 11] are of this kind. Note that $\left(p^0, q_1\right)$ is a closed subspace with respect to charge addition, as in general "electric" and "magnetic" configurations (discussed further below) are, as well.

In the particular $\mathcal{N} = 2$ cases discussed in [29], it is observed that $W_{\mathcal{I}_4<0}\left(z,\overline{z};Q\right)$, also in presence of non-vanishing axions, is *linear* in charges (we omit the subscript "$\mathcal{I}_4 < 0$" throughout):

$$W\left(z,\overline{z};Q_1+Q_2\right) = W\left(z,\overline{z};Q_1\right) + W\left(z,\overline{z};Q_2\right). \tag{4.6}$$

The property (4.6) has an obvious consequence, namely that non-BPS double-center configurations always occur at the marginal stability wall in the moduli space, since the spatial asymptotical limit of (4.6) reads

$$M_{1+2}\left(z_{\infty},\overline{z}_{\infty};Q_1+Q_2\right)=M_1\left(z_{\infty},\overline{z}_{\infty};Q_1\right)+M_2\left(z_{\infty},\overline{z}_{\infty};Q_2\right). \tag{4.7}$$

Therefore, for these charge configurations the non-BPS BH bound states are never stable but rather only marginally stable, thus producing two single-center BH solutions with mutually local charges ($\langle Q_1,Q_2\rangle=0$) and no constraints on the relative distance $|\overrightarrow{x_1}-\overrightarrow{x_2}|$ between the two centers (and therefore vanishing overall angular momentum). A further example is provided by eq. (4.1) of the first ref. of [31–36].

A particular subset of such configurations are the "electric" (p^0,q_i) and "magnetic" (q_0,p^i) ones, which may be axion-free. By plugging them into the explicit general expression of Z computed in [62], one finds that such configurations support a real or purely imaginary central charge: $Z=\pm\overline{Z}$. As a consequence, both BPS and non-BPS constituents do not form a stable composite, and the moduli are always on the marginal stability wall. Notice that the situation is different for the $\left(p^0,q_0\right)$ charge configuration (corresponding to the presence of only $D0$ and $D6$ branes [9]). This configuration may (but does not necessarily) support axion-free solutions but, as already evident in the 1-modulus t^3 model (see e.g. eq. (3.5) of the first ref. of [31–36]), $W_{\mathcal{I}_4<0}$ is not linear in charges nor the absolute value of a complex quantity linear in charges.

Thus, apart from the $\left(p^0,q_0\right)$ case, it seems that many known simple non-BPS $\mathcal{I}_4<0$ configurations are exactly marginal. This situation agrees with the conclusions of the analysis of [6, 11].

By using norm inequalities, the only non-BPS $\mathcal{I}_4<0$ configurations which may exhibit a stability region for double-center BH solutions (and a corresponding wall of marginal stability for their decay into two single-center BH solutions) seem to be the ones which can be uplifted to a very particular non-BPS configurations of $\mathcal{N}=8$ supergravity, namely one with constant phase. In such a case, one of the duality ($SU(8)$-) invariants of the theory, namely the Pfaffian $Pf(\mathbf{Z})$ of the central charge matrix $\mathbf{Z}$, is constrained to be real all along the corresponding scalar flow; this corresponds to the phase φ of $\mathbf{Z}$ to be set to its non-BPS critical value $\varphi_H=\pi$ all along the flow. For this configurations, the first order non-BPS "fake" superpotential can be computed to be nothing but (one quarter of) the trace norm (2.6) of $\mathbf{Z}$ itself [30] (see also the second and third refs. of [31–36]):

$$W_{\mathcal{I}_4<0,\varphi=\pi}(\phi,Q)=\frac{1}{2}\sum_{i=1}^{4}\sqrt{\lambda_i}=\frac{1}{4}Tr\left(\sqrt{\mathbf{Z}\mathbf{Z}^{\dagger}}\right)=\frac{1}{4}\left\|\mathbf{Z}\right\|_{*}. \tag{4.8}$$

Consequently, $W_{\mathcal{I}_4<0,\varphi=\pi}$ satisfies the triangle inequality

$$W_{\mathcal{I}_4<0,\varphi=\pi}(\phi,Q_1+Q_2)\leqslant W_{\mathcal{I}_4<0,\varphi=\pi}(\phi,Q_1)+W_{\mathcal{I}_4<0,\varphi=\pi}(\phi,Q_2), \tag{4.9}$$

provided that (recall (2.33))

$$\langle Q_1,Q_2\rangle=-\mathrm{Im}\left(Tr\left(\mathbf{Z}_1\mathbf{Z}_2^{\dagger}\right)\right)\neq 0, \tag{4.10}$$

and that $Pf(\mathbf{Z}_1 + \mathbf{Z}_2)$, $Pf(\mathbf{Z}_1)$ and $Pf(\mathbf{Z}_2)$ are all real; this latter condition can equivalently be recast as

$$\varphi(\phi, Q_1 + Q_2) = \varphi(\phi, Q_1) = \varphi(\phi, Q_2) = \pi, \tag{4.11}$$

all along the attractor flow. The marginal stability condition would correspond to the saturation of the bound (4.9), within the conditions (4.10) and (4.11).

By performing the supersymmetry reduction $\mathcal{N} = 8 \to \mathcal{N} = 2$ and using the $\mathcal{N} = 2$ formalism introduced in the first ref. of [31–36] and in [37], the constancy of the phase φ along the non-BPS $\mathcal{I}_4 < 0$ attractor flow corresponds to the vanishing of the H-invariant i_3 (and to $i_4 < 0$). Thus, the $\mathcal{N} = 2$ analogues of conditions (4.10) and (4.11) respectively read as follows (for the equality in the l.h.s. of (4.12), see e.g. [1, 52]):

$$\langle Q_1, Q_2\rangle = 2\mathrm{Im}\left[-Z_1\overline{Z_2} + g^{i\bar{j}}(D_iZ_1)\,\overline{D_{\bar{j}}Z_2}\right] \neq 0; \tag{4.12}$$

$$i_3(z,\overline{z};Q_1+Q_2) = i_3(z,\overline{z};Q_1) = i_3(z,\overline{z};Q_2) = 0,\;\; i_4(z,\overline{z};Q_1+Q_2) < 0. \tag{4.13}$$

The moduli dependence of (4.11) and (4.13) yields a co-dimension three subspace of scalar manifold. Thus, in the $\mathcal{N} = 8 \to \mathcal{N} = 2$ supersymmetry reduction, if the three real conditions entailed by (4.13) are all independent, they admit consistent solutions in presence of mutually non-local charges $\langle Q_1, Q_2\rangle \neq 0$ only with at least two (complex) scalar fields.

5 Concluding remarks

In the present investigation, we have analyzed the marginal stability bound for BPS extremal (two-center composite) BHs in $\mathcal{N} > 2$ supergravity, as well as whether this bound can be extended to non-BPS configurations.

By denoting the central charge matrix with $\mathbf{Z}$, for BPS BHs we found that the Cauchy-Schwarz triangle inequality applies to the ADM mass $M = \lim_{r\to\infty}\sqrt{\lambda_h} = \lim_{r\to\infty}\|\mathbf{Z}\|_s$, where λ_h is the highest eigenvalue of the semi-positive definite matrix $\mathbf{Z}\mathbf{Z}^\dagger$, and $\|\cdot\|_s$ stands for the *spectral* matrix norm. This generalization of the marginal stability bound uses the property of matrix norm as well as the linearity of $\mathbf{Z}$ in charges Q:

$$\|\mathbf{Z}(\phi, Q_1+Q_2)\|_s = \|\mathbf{Z}(\phi,Q_1) + \mathbf{Z}(\phi,Q_2)\|_s \leqslant \|\mathbf{Z}(\phi,Q_1)\|_s + \|\mathbf{Z}(\phi,Q_2)\|_s\,. \tag{5.1}$$

Furthermore, we found that all non-BPS BHs of the $\mathcal{N} = 2$ *minimal coupling* $\mathbb{CP}^n$ sequence (characterized by $C_{ijk} = 0$) and of $\mathcal{N} = 3$ supergravity, satisfy a marginal stability bound identical to the one of their BPS counterparts. These theories share the properties that they cannot be uplifted to $d = 5$ space-time dimensions, they have a G-invariant $\mathcal{I}_2$ which is quadratic in charges, which defines the supersymmetry preserving features of the charge orbits as follows:

$$\mathrm{BPS}: \mathcal{I}_2 \geqslant 0;\;\; \mathrm{nBPS}: \mathcal{I}_2 < 0. \tag{5.2}$$

For theories with a G-invariant $\mathcal{I}_4$ quartic in charges and $\mathcal{N} < 8$, two types of "large" attractor non-BPS solutions exist, depending on whether $\mathcal{I}_4 \gtrless 0$.

For $\mathcal{I}_4 > 0$ non-BPS BHs, the marginal stability bound as for the BPS BHs applies. An obvious example is provided by the $\mathcal{N} = 6$ theory, which shares the same bosonic sector of the $\mathcal{N} = 2$ "magic" quaternionic ($J_3^{\mathbb{H}}$-based) supergravity, but with the role of BPS and non-BPS (both with $\mathcal{I}_4 > 0$) interchanged [61]. This example actually extends to the $\mathcal{I}_4 > 0$ non-BPS BHs of all $\mathcal{N} \geqslant 2$-extended supergravities with symmetric (vector multiplets') scalar manifolds. The $\mathcal{N} = 5$ case is particularly simple, because such a theory has only two orbits, both BPS: one "large" ($\frac{1}{5}$-BPS) and one "small" ($\frac{2}{5}$-BPS). At least for "magic" $\mathcal{N} = 2$ models (with the exclusion of the octonionic one), this result for $\mathcal{I}_4 > 0$ non-BPS BHs is not surprising, because such theories can be seen as sub-theories of the maximal $\mathcal{N} = 8$ supergravity, in which in fact the constraint $\mathcal{I}_4 > 0$ defines a unique ($\frac{1}{8}$-BPS) orbit [57, 58].

For $\mathcal{I}_4 < 0$ non-BPS BHs, we found that most examples (characterized by particular charge configurations and moduli dependence) saturate the marginal stability bound, and thus they cannot admit stable double-center composite solutions. It would be interesting to determine under which circumstances, for a generic charge configuration belonging to the non-BPS "large" orbit $\frac{E_{7(7)}}{E_{6(6)}}$, the $\mathcal{N} = 8$ non-BPS first order "fake" superpotential, which in the asymptotical spatial limit yields the ADM mass, satisfies the marginal stability bound. It should be recalled that the *Ansatz* of flat $d = 3$ spatial slices of the BH geometry, made in [6, 11], has been removed in [38, 39], in which a general solution for non-BPS multi-center BHs, with constrained centers and non-vanishing overall angular momentum, has been explicitly obtained.

For stable configurations with a wall of marginal stability, the split attractor flow will occur not only for BPS cases, but also for non-BPS cases for which a stability region in the moduli space exists. In this paper we have shown that, at least in the supergravity approximation, this is not limited to BPS solutions, but it extends to a broad class of non-BPS solutions.

Finally, it would be interesting to investigate, in the case of $\mathcal{N} \geqslant 2$ non-BPS and also $\mathcal{N} > 2$ BPS configurations, the fate of the *"moduli spaces"* [63, 64] of scalar flows across the split occurring at the marginal stability wall, which may be thus reduce or remove the "flat directions" spanning the corresponding *"moduli space"*. For $\mathcal{N} = 8$ $\frac{1}{8}$-BPS ("large") attractor flow, the "flat directions" have an $\mathcal{N} = 2$ interpretation in terms of hypermultiplets' scalar degrees of freedom [48, 63, 65]. Therefore, the double-center solutions removing the "flat directions" would be genuine $\mathcal{N} = 8$ solutions with no $\mathcal{N} = 2$ interpretation (concerning this, see [19, 21]).

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