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Disconnectedness of the subgraph F^3 for the group Γ^3

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Dedicated to Professor Hari M Srivastava

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Abstract
In this paper we show that the subgraph F^3 is disconnected and that for all integers m, we find all integers a and b such that $(9m^2-4)a^2+4$ and $5b^2\pm 4$ are square. It turns out that the set of numbers b comprises the Fibonacci numbers.

Keywords: modular group; suborbital graph; disconnectedness; Fibonacci numbers

1 Introduction

Let $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ be the extended rationals, let Γ be the modular group acting on $\hat{\mathbb{Q}}$ as with the upper half-plane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z = \frac{x}{y} \to \frac{az+b}{cz+d} = \frac{ax+by}{cx+dy},$$

where a, b, c, and d are rational integers, and let Γ^3 denote the group consisting of the cubes of the elements g of Γ, which is the group $\{g \in \Gamma : ab + cd \equiv 0 (\operatorname{mod} 3)\}$; see [1].

Jones *et al.* [2] used the notion of the imprimitive action [3–5] for a Γ-invariant equivalence relation induced on $\hat{\mathbb{Q}}$ by the congruence subgroup $\Gamma_0(n) = \{g \in \Gamma : c \equiv 0 (\operatorname{mod} n)\}$ to obtain some suborbital graphs and examined their connectedness and forest properties.

In [6], a Γ^3-invariant equivalence relation is introduced by using the subgroup $\Gamma_0^3(n) = \{g \in \Gamma^3 : c \equiv 0 (\operatorname{mod} n)\}$ to obtain suborbital graphs $F^3_{u,n}$. There, the connectivity properties of all subgraphs $F^3_{u,n}$ other than $F^3_{1,1} = F^3$ are examined.

In this paper we show that the subgraph F^3 is disconnected and give some results, which seem important from the point of view of number theory.

2 Preliminaries

Since $\Gamma^3 = \{g^3 : g \in \Gamma\}$, it is easily seen that the elements of Γ^3 are ones of the forms $\begin{pmatrix} 3a & b \\ c & 3d \end{pmatrix}$, $\begin{pmatrix} a & 3b \\ 3c & d \end{pmatrix}$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, and $d \not\equiv 0 (\operatorname{mod} 3)$ in the third matrix. Furthermore, $\Gamma^3_\infty < \Gamma^3_0(n) \le \Gamma^3$ for each positive integer n, where Γ^3_∞ is the stabilizer of ∞ generated by the element $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, and second inclusion is strict if $n > 1$.

Since the group Γ^3 is transitive on $\hat{\mathbb{Q}}$ in [6], any reduced fraction $\frac{r}{s}$ in $\hat{\mathbb{Q}}$ equals $g(\infty)$ for some $g \in \Gamma^3$. Similar to that of [2], we get the following Γ^3-invariant equivalence relation on $\hat{\mathbb{Q}}$ by $\Gamma^3_0(n)$ as $\frac{r}{s} \sim \frac{x}{y}$ if and only if $g^{-1}h \in \Gamma^3_0(n)$, where $g = \begin{pmatrix} r & * \\ s & * \end{pmatrix}$ and h is similar. Furthermore, the above equivalence relation is imprimitive, which means that it is different

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from the identity relation ($a \sim b$ if and only if $a = b$) and the universal relation ($a \sim b$ for all $a, b \in \hat{\mathbb{Q}}$).

From the above defined Γ^3-invariant equivalence relation, we can verify that $\frac{r}{s} \sim \frac{x}{y}$ if and only if $ry - sx \equiv 0 (\bmod n)$. The equivalence classes are called blocks, and a block containing the rational $\frac{x}{y}$ is denoted by $[\frac{x}{y}]$.

Although the equivalence relations are resulting almost the same as in [2], the subgraph $F_{1,1}$ in [2] is easily shown to be connected, but here we will see that the subgraph $F^3_{1,1}$ is disconnected. So, using different subgroups changes the characters of the subgraphs.

3 Subgraphs $F^3_{u,n}$

The group Γ^3 acts on $\hat{\mathbb{Q}} \times \hat{\mathbb{Q}}$ through $g : (\alpha, \beta) \to (g(\alpha), g(\beta))$. The orbits are called suborbital. From the suborbital $O^3(\alpha, \beta)$ containing (α, β), we can form the suborbital graph $G^3(\alpha, \beta)$ whose vertices are the elements of $\hat{\mathbb{Q}}$ and the edges are the pairs $(a, b) \in O^3(\alpha, \beta)$, which we denote by $a \to b$ and represent as hyperbolic geodesics in $\mathcal{H}$.

Since Γ^3 acts transitively on $\hat{\mathbb{Q}}$, every suborbital contains a pair $(\infty, \frac{u}{n})$ for some $\frac{u}{n}$ in $\hat{\mathbb{Q}}$, $n \geq 0$, $(u, n) = 1$. In this case, we denote the suborbital graph by $G^3_{u,n}$ for short. From now on, we assume that $n > 0$.

As Γ^3 permutes the blocks transitively, all subgraphs corresponding to the blocks are isomorphic, as in [2]. Therefore, we will only consider the subgraph $F^3_{u,n}$ of $G^3_{u,n}$ whose vertices form the block $[\infty] = \{\frac{x}{y} \in \hat{\mathbb{Q}} \mid y \equiv 0 (\bmod n)\}$. The following two results were proved in [6].

Theorem 1 *$F^3_{u,n} = F^3_{u',n'}$ if and only if $n = n'$ and $u \equiv u' (\bmod 3n)$.*

Theorem 2 *$\frac{r}{s} \to \frac{x}{y}$ is an edge in F^3 $(= F^3_{1,1})$ if and only if*

1. *if $r \equiv 0 (\bmod 3)$, then $y \equiv \pm s (\bmod 3)$ and $ry - sx = \pm 1$, or*
2. *if $s \equiv 0 (\bmod 3)$, then $x \equiv \pm r (\bmod 3)$, and $ry - sx = \pm 1$, or*
3. *if $r, s \not\equiv 0 (\bmod 3)$, then $x \not\equiv \pm r (\bmod 3)$, $y \not\equiv \pm s (\bmod 3)$ and $ry - sx = \pm 1$.*

We can easily get the following lemmas.

Lemma 1 *$\frac{r}{s} \to \frac{x}{y}$ is in F^3 if and only if $\frac{x}{y} \to \frac{r}{s}$ is in F^3.*

Lemma 2 [2] *No edges of F^3 cross in $\mathcal{H}$.*

4 Disconnectedness of F^3

Definition 1 For $m \in \mathbb{N}$ and $m \geq 2$, let $v_1, v_2, \ldots, v_m$ be a finite sequence of vertices of F^3. Then the configuration $v_1 \to v_2 \to \cdots \to v_m$ is called a finite path in F^3. A subgraph $\wedge$ of F^3 is called connected if every two vertices x and y of $\wedge$ are connected by a finite path in F^3. Otherwise, we call $\wedge$ disconnected.

Now we give one of our main theorems.

Theorem 3 *The graph F^3 is disconnected.*

We prove Theorem 3 after giving some theorems, propositions and lemmas as follows.

By Theorem 2, it is easily seen that the graph F^3 is periodic with period 3. That is, if $a \to b$ is in F^3, then $a + 3m \to b + 3m$ is in F^3 for all $m \in \mathbb{Z}$, and therefore, for some m,

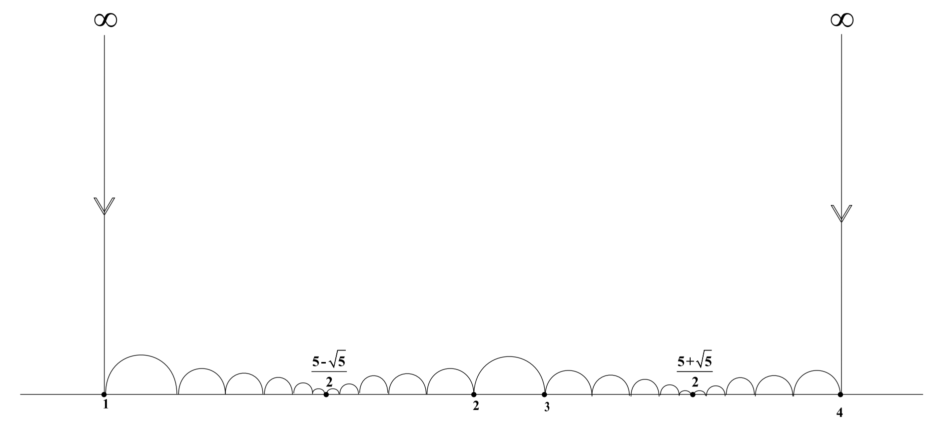

Figure 1 The subgraph F^3.

$a+m$ or $b+m$ (not both) is ∞ or both $a+3m$ and $b+3m$ are in the interval $[1,4]$. Therefore we can only use the interval $[1,4]$ for our calculations as in Figure 1.

It is clear that $T = \begin{pmatrix} 1 & -5 \\ 1 & -4 \end{pmatrix}$ is in Γ^3 and the corresponding transformation $T(z) = \frac{z-5}{z-4}$ is strictly increasing on $[1,4] \cap \mathbb{Q}$. Furthermore, it is easily seen that $T^m(\infty) \to T^m(\frac{1}{1})$ is an edge in F^3 for all non-negative integers m. From this, we get, as an example, a finite path in F^3 as $\infty \to \frac{1}{1} \to \frac{4}{3} \to \frac{11}{8}$.

Lemma 3 *Let T be as above, then the sequence $\{T^m(1)\}$ is strictly monotone increasing and $T(\frac{1}{0}) \to T(\frac{1}{1}) \to T^2(\frac{1}{1}) \to \cdots \to T^m(\frac{1}{1}) \to \cdots$ is an infinite path in F^3 in increasing order.*

Proof The conclusion follows from Theorem 2 and from the fact that $T(z) = \frac{z-5}{z-4}$ is strictly increasing on $[1,4) \cap \mathbb{Q}$. □

Lemma 4 *Let a and b be in $\mathbb{N}$ and let $1 \leq \frac{a}{b} < \frac{5-\sqrt{5}}{2}$, then $\frac{a}{b} < T(\frac{a}{b}) < \frac{5-\sqrt{5}}{2}$.*

Proof From $\frac{a}{b} < \frac{5-\sqrt{5}}{2}$ we get $2a - 5b < -\sqrt{5}b$. Then squaring gives the inequality $-a^2 + 4ab < -ab + 5b^2$. That is, $\frac{a}{b} < \frac{-a+5b}{-a+4b} = T(\frac{a}{b})$. On the other hand, $a^2 - 5ab + 5b^2 > 0$, then it is easily seen that $5(a-4b)^2 < (3a-10b)^2$. As $\frac{a}{b} < 2$, then taking square roots gives $\sqrt{5}(a-4b) > 3a - 10b$. This shows that $T(\frac{a}{b}) < \frac{5-\sqrt{5}}{2}$. □

Proposition 1 *Let T be as above and $1 \leq \frac{a}{b} < \frac{5-\sqrt{5}}{2}$. Then $\frac{a}{b} \to T(\frac{a}{b})$ is an edge in F^3 if and only if there exists a natural number u such that $u^2 = 5b^2 + 4$ and $a = \frac{5b-\sqrt{5b^2+4}}{2}$.*

Proof Let $\frac{a}{b} \to T(\frac{a}{b})$ be an edge in F^3. Then, by using Theorem 2 and Lemma 4, we get $a^2 - 5ab + 5b^2 - 1 = 0$. Since $\frac{a}{b} < \frac{5-\sqrt{5}}{2}$, we have $a = \frac{5b-\sqrt{5b^2+4}}{2}$. This concludes that $\sqrt{5b^2+4}$ is an integer u.

Conversely, it is clear to see $M = \begin{pmatrix} \frac{-5b+\sqrt{5b^2+4}}{2} & 5b \\ -b & \frac{5b+\sqrt{5b^2+4}}{2} \end{pmatrix}$ is in Γ^3 and that $M(\frac{1}{0}) = \frac{a}{b}$ and $M(\frac{1}{1}) = T(\frac{a}{b})$. Therefore, by the definition of edges of F^3, the configuration $\frac{a}{b} \to T(\frac{a}{b})$ is an edge in F^3. □

Theorem 4 *The positive rational number $\frac{x}{y}$ is in $\{T^m(\frac{1}{0}) : m \in \mathbb{N}\} = A$ if and only if there exists a natural number u such that $5y^2 + 4 = u^2$ and $x = \frac{5y-\sqrt{5y^2+4}}{2}$.*

Proof From Proposition 1, the 'if' part is clear.

Conversely, we show that under the hypothesis, $\frac{x}{y}$ is in the set A. Since $T^m(\frac{1}{0}) = \frac{\frac{5y-\sqrt{5y^2+4}}{2}}{y}$ for any $m \in \mathbb{N}$, $\lim_{m\to\infty} T^m(\frac{1}{0}) = \frac{5-\sqrt{5}}{2}$. As $1 \leq \frac{x}{y} < \frac{5-\sqrt{5}}{2}$, if $\frac{x}{y}$ is not in A, there exists $k \in \mathbb{N}$ such that $T^k(\frac{1}{0}) < \frac{x}{y} < T^{k+1}(\frac{1}{0})$. From this, we get

$$T^k\begin{pmatrix}1\\0\end{pmatrix} = \frac{\frac{5a-\sqrt{5a^2+4}}{2}}{a} < \frac{\frac{5y-\sqrt{5y^2+4}}{2}}{y} < 1 + \frac{a}{3a - \frac{3a-\sqrt{5a^2+4}}{2}} = \frac{\frac{5a-v}{2}}{a} < \frac{\frac{5y-u}{2}}{y} < \frac{\frac{5a+v}{2}}{\frac{3a+v}{2}}.$$

We know that $\frac{\frac{5y-u}{2}}{y} \to T(\frac{\frac{5y-u}{2}}{y}) = 1 + \frac{y}{3y-\frac{3y-u}{2}}$ is an edge in F^3. From Lemma 2 it must be that $T(\frac{\frac{5y-u}{2}}{y}) = \frac{\frac{5y+u}{2}}{\frac{3y+u}{2}} < \frac{\frac{5a+v}{2}}{\frac{3a+v}{2}}$. This gives the inequality $vy < au$. But, from $\frac{\frac{5a-v}{2}}{a} < \frac{\frac{5y-u}{2}}{y}$, we arrive at the inequality $vy > au$, a contradiction. This concludes the proof of the theorem. □

Corollary 1 $\frac{1}{0} \to 0 + \frac{1}{1} \to 1 + \frac{1}{3} \to 1 + \frac{3}{8} \xrightarrow{T} \cdots \xrightarrow{T} 1 + \frac{a_n}{b_n} \xrightarrow{T} 1 + \frac{b_n}{3b_n - a_n} \xrightarrow{T} \cdots$ *is an infinite path in* F^3, *and all vertices of the path are smaller than* $\frac{5-\sqrt{5}}{2}$, *and the natural numbers* x *and* y *in the vertex* $1 + \frac{x}{y}$ *are such that* $5x^2 + 4$ *and* $5y^2 + 4$ *are square. Furthermore,* $a_n = \frac{3b_n - \sqrt{5b_n^2+4}}{2}$ *and* $\lim_{n\to\infty}(1 + \frac{a_n}{b_n}) = \frac{5-\sqrt{5}}{2}$, *where* T *is as above.*

Proof Lemma 3 and Theorem 4 conclude the proof. □

If we follow the way of the above, we arrive at the following two results without proofs.

Theorem 5 *Let* $T = \begin{pmatrix}-1 & 5\\-1 & 4\end{pmatrix}$ *and* $\frac{5-\sqrt{5}}{2} < \frac{a}{b} \leq 2$. *Then* $\frac{5-\sqrt{5}}{2} < T(\frac{a}{b}) < \frac{a}{b}$ *and* $\frac{a}{b} \to T(\frac{a}{b})$ *is an edge in* F^3 *if and only if* $5b^2 - 4$ *is a square and* $a = \frac{5b-\sqrt{5b^2-4}}{2}$.

Corollary 2 $3 \xrightarrow{T} 2 = 1 + \frac{1}{1} \xrightarrow{T} 1 + \frac{1}{2} \xrightarrow{T} 1 + \frac{2}{5} \xrightarrow{T} \cdots \xrightarrow{T} 1 + \frac{a_n}{b_n} \xrightarrow{T} 1 + \frac{b_n}{3b_n - a_n} \xrightarrow{T} \cdots$ *is an infinite path in* F^3 *in decreasing order such that all vertices of the path are greater than* $\frac{5-\sqrt{5}}{2}$, *and the natural numbers* x *and* y *in the vertex* $1 + \frac{x}{y}$ *are such that* $5x^2 - 4$ *and* $5y^2 - 4$ *are squares. Furthermore,* $a_n = \frac{3b_n - \sqrt{5b_n^2-4}}{2}$ *and* $\lim_{n\to\infty}(1 + \frac{a_n}{b_n}) = \frac{5-\sqrt{5}}{2}$, *where* T *is as above.*

Theorem 6 *Let* k *be a natural number and let the vertices* $v_1, v_2, \ldots, v_k$ *in* $[1,3]$ *of* F^3 *be such that at least one is smaller and one is greater than* $\frac{5-\sqrt{5}}{2}$. *Then the path* $\infty \to v_1 \to v_2 \to \cdots \to v_k$ *does not occur in* F^3.

Proof If the above situation occurs, since $\frac{5-\sqrt{5}}{2}$ is not a vertex in F^3, there exists $1 \leq m < k$ such that $v_m < \frac{5-\sqrt{5}}{2} < v_{m+1}$ and $v_m \to v_{m+1}$ is an edge in F^3.

Since the sequences of the vertices in Corollary 1 and Corollary 2 converge to $\frac{5-\sqrt{5}}{2}$, Lemma 2 gives that there exist naturals m and n such that

$$T^m\begin{pmatrix}1\\0\end{pmatrix} = v_m \to v_{m+1} = T^n\begin{pmatrix}3\\1\end{pmatrix}.$$

Suppose first that $n > m$. Then, multiplying by T^{-m} and using Theorem 5, we have $\frac{1}{0} \to T^{n-m}(\frac{3}{1}) = \frac{\frac{5b-\sqrt{5b^2-4}}{2}}{b}$ for some $b \in \mathbb{N}$. Therefore, from Theorem 2, we get $b = 1$. That is, $\frac{1}{0} \to \frac{3}{1}$ is an edge in F^3, a contradiction.

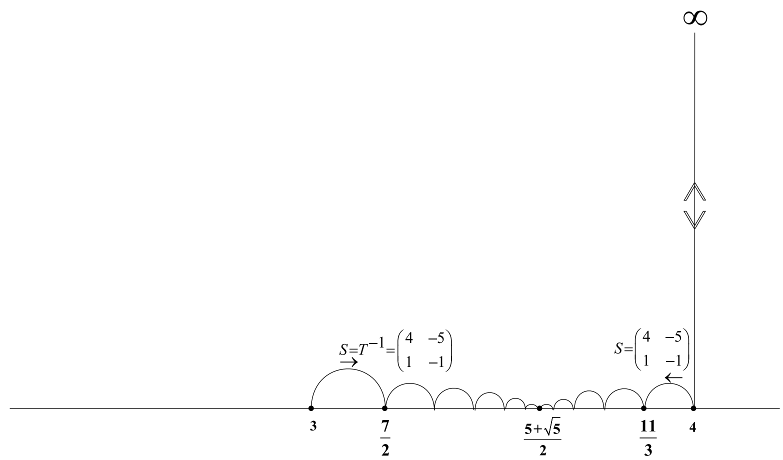

Figure 2 A path in F^3.

Suppose now that $m > n$. Then, again multiplying by T^{-n}, we have $T^{m-n}(\frac{1}{0}) \to \frac{3}{1}$. In this case, $T^{m-n}(\frac{1}{0}) \geq 2$. But, in any case, $T^{m-n}(\frac{1}{0}) < \frac{5-\sqrt{5}}{2}$, a contradiction.

Finally, let $m = n$. Then $\frac{1}{0} \to \frac{3}{1}$ must be an edge in F^3, a contradiction. These complete the proof. □

Let $S = T^{-1} = \begin{pmatrix} 4 & -5 \\ 1 & -1 \end{pmatrix}$. Then likewise we do before we give the following five results without a proof.

Lemma 5 *Let* $3 \leq \frac{a}{b} < \frac{5+\sqrt{5}}{2}$. *Then* $\frac{a}{b} < S(\frac{a}{b}) < \frac{5+\sqrt{5}}{2}$. *Furthermore,* $\frac{a}{b} \to S(\frac{a}{b})$ *is an edge in* F^3 *if and only if* $\sqrt{5b^2-4}$ *is a natural number and* $a = \frac{5b+\sqrt{5b^2-4}}{2}$.

Corollary 3 $3 = 4 - \frac{1}{1} \xrightarrow{S} 4 - \frac{1}{2} \xrightarrow{S} 4 - \frac{2}{5} \xrightarrow{S} \cdots \xrightarrow{S} 4 - \frac{\frac{3b-\sqrt{5b^2-4}}{2}}{b} \xrightarrow{S} \cdots$ *is an infinite path in* F^3 *in increasing order, as seen Figure* 2, *and the limit of the sequence of vertices is* $\frac{5+\sqrt{5}}{2}$.

Lemma 6 *Let* $\frac{5+\sqrt{5}}{2} < \frac{a}{b} \leq 4$. *Then* $\frac{5+\sqrt{5}}{2} < S(\frac{a}{b}) < \frac{a}{b}$. *Furthermore,* $\frac{a}{b} \to S(\frac{a}{b})$ *is an edge in* F^3 *if and only if* $5b^2+4$ *is a square and* $a = \frac{5b+\sqrt{5b^2+4}}{2}$.

Lemma 7 $4 - \frac{0}{1} \xrightarrow{S} 4 - \frac{1}{3} \xrightarrow{S} 4 - \frac{3}{8} \xrightarrow{S} \cdots \xrightarrow{S} 4 - \frac{\frac{3b-\sqrt{5b^2+4}}{2}}{b} \xrightarrow{S} \cdots$ *is an infinite path in* F^3 *in decreasing order, and the limit of the sequence of vertices is* $\frac{5+\sqrt{5}}{2}$.

Theorem 7 *Let k be a natural number and let the vertices* $v_1, v_2, \ldots, v_k$, *in* $[3,4]$, *of* F^3 *be such that at least one is smaller and one is greater than* $\frac{5+\sqrt{5}}{2}$. *Then* $v_1 \to v_2 \to \cdots \to v_k \to \infty$ *does not occur in* F^3.

Proof We conclude the proof as in Theorem 6. □

Proof of Theorem 3 Theorems 6 and 7 conclude that the vertices of F^3 in $(\frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2})$ are not connected to the vertex ∞. That is, the graph F^3 is disconnected. □

We finally give one of our main results as follows.

Theorem 8 *For all natural numbers m, the natural numbers b that make the number $(9m^2-4)b^2+4$ square are* $0, 1, 3m, 9m^2-1, 3m(9m^2-1)-3m, \dots, a, b, 3mb-a, \dots$.

Proof For the proof, we only use, as above, the interval $[1,4]$. It is clear that the matrix $M = \begin{pmatrix} -1 & 3m+2 \\ -1 & 3m+1 \end{pmatrix}$ is in Γ^3. Theorem 2 gives that $\frac{1}{1} \to M(\frac{1}{1}) = \frac{3m+1}{3m}$ is an edge in F^3. Since the transformation $M(x) = \frac{-x+3m+2}{-x+3m+1}$ is an increasing function on $[1,4]$ and $\frac{1}{1} < M(\frac{1}{1}) = \frac{3m+1}{3m}$, we can easily see that, for all $k \in \mathbb{N}$, $M^k(\frac{1}{1}) < M^{k+1}(\frac{1}{1})$. That is, the sequence $\{M^k(1)\}$ is an increasing sequence. Furthermore, $M^k(1) = [1; \underbrace{3m, 3m, 3m, \dots, 3m}_{k \text{ times}}]$, or

$$M^k(1) = 1 + \cfrac{1}{3m - \cfrac{1}{3m - \cfrac{1}{3m - \cdots - \cfrac{1}{3m}}}}.$$

And more, $\lim_{k\to\infty} M^k(1) = \frac{3m+2-\sqrt{9m^2-4}}{2}$. So, for all $k \in \mathbb{N}$, $M^k(1) < \frac{3m+2-\sqrt{9m^2-4}}{2}$. If $\frac{a}{b} < \frac{3m+2-\sqrt{9m^2-4}}{2}$ we can easily see that $T(\frac{a}{b}) < \frac{3m+2-\sqrt{9m^2-4}}{2}$, and furthermore, if $\frac{a}{b} \to T(\frac{a}{b}) = \frac{-a+(3m+2)b}{-a+(3m+1)b}$ is an edge in F^3, then, by Theorem 2, $a^2-(3m+2)b+(3m+2)b^2-1=0$. Solving the equation, we have $a = \frac{(3m+2)b-\sqrt{(9m^2-4)b^2+4}}{2}$, where we get the sign '–' since for all $k \in \mathbb{N}$, $M^k(1) < \frac{3m+2-\sqrt{9m^2-4}}{2}$. Because a is an integer, $\sqrt{(9m^2-4)b^2+4}$ must be an integer. According to Theorem 2, for all $k \in \mathbb{N}$, $M^k(\frac{1}{1}) \to M^{k+1}(\frac{1}{1})$ is an edge in F^3. Therefore

$$\frac{1}{1} \xrightarrow{M} M\left(\frac{1}{1}\right) \xrightarrow{M} M^2\left(\frac{1}{1}\right) \xrightarrow{M} \cdots \xrightarrow{M} \left(\frac{a}{b}\right) \xrightarrow{M} M\left(\frac{a}{b}\right) \xrightarrow{M} \cdots$$

is an infinite path γ in F^3. All denominators of vertices $\frac{a}{b}$ of γ make $(9m^2-4)b^2+4$ square. We can rewrite γ as

$$1+\frac{0}{1} \to 1+\frac{1}{3m} \to 1+\frac{3m}{9m^2-1} \to 1+\frac{9m^2-1}{3m(9m^2-1)-3m} \to \cdots \to 1+\frac{a'}{b'} \\ \to 1+\frac{b'}{3mb'-a'} \to \cdots.$$

And we conclude that the numbers $0, 1, 3m, 9m^2-1, 3m(9m^2-1)-3m, \dots, a_0, b_0, 3mb_0-a_0, \dots$ make $(9m^2-4)b^2+4$ square.

Let us now show the only non-negative integers b such that $(9m^2-4)b^2+4$ is square.

Conversely, suppose that there is a natural number t such that $(9m^2-4)t^2+4$ is square. Then $\frac{a_1}{b_1} = \frac{\frac{(3m+2)t-\sqrt{(9m^2-4)t^2+4}}{2}}{t}$ is smaller than $\frac{3m+2-\sqrt{9m^2-4}}{2}$ and, from Theorem 2, we get that $\frac{a_1}{b_1} \to T(\frac{a_1}{b_1})$ is an edge in F^3. Suppose, for some $k \in \mathbb{N}$, $M^k(\frac{1}{1}) < \frac{a_1}{b_1} < M^{k+1}(\frac{1}{1})$. Lemma 2 says that $T^m(\frac{1}{1}) < \frac{a_1}{b_1} < T^{m+1}(\frac{a_1}{b_1}) < M^{k+1}(\frac{1}{1})$. Therefore, for some y,

$$\begin{aligned} M^k\left(\frac{1}{1}\right) &= 1+\frac{\frac{3my-\sqrt{(9m^2-4)y^2+4}}{2}}{y} < \frac{a_1}{b_1} = 1+\frac{\frac{3mt-\sqrt{(9m^2-4)t^2+4}}{2}}{t} \\ &\to T\left(\frac{a_1}{b_1}\right) = 1+\frac{t}{3mt-\frac{3mt-\sqrt{(9m^2-4)t^2+4}}{2}} < M^{k+1}\left(\frac{1}{1}\right) \\ &= 1+\frac{y}{3my-\frac{3my-\sqrt{(9m^2-4)y^2+4}}{2}}. \end{aligned}$$

From the first inequality, we conclude that $t\sqrt{(9m^2-4)y^2+4} > y\sqrt{(9m^2-4)y^2+4}$. From the second inequality, we just have $t\sqrt{(9m^2-4)y^2+4} < y\sqrt{(9m^2-4)y^2+4}$, a contradiction. Consequently, $\frac{a_1}{b_1}$ must be in the set $\{M^k(\frac{1}{1}) : k \in \mathbb{N}\}$. This completes the proof of the theorem. □

From Corollaries 1 and 2, we get the following without a proof.

Corollary 4 *The non-negative integers b such that* $5b^2+4$ *is square are* $0, 1, 3, 8, 21, 55, 144, \dots, a, b, 3b-a, \dots$.

Corollary 5 *The non-negative integers b making* $5b^2-4$ *square are* $1, 2, 5, 13, 34, 89, \dots, a, b, 3b-a, \dots$.

From Corollaries 4 and 5, we conclude the following important corollary.

Corollary 6 *Let* $\{a_n\}$ *and* $\{b_n\}$ *be the sequences* $(0, 1, 3, \dots, a, b, 3b-a, \dots)$ *and* $(1, 2, 5, \dots, c, d, 3d-c, \dots)$, *respectively. Then the sequence* $(a_1, b_1, a_2, b_2, \dots, a_n, b_n, \dots)$ *is the Fibonacci sequence*.

Proof Let us see that $a_n + b_n = a_{n+1}$ and $b_n + a_{n+1} = b_{n+1}$ for all n in $\mathbb{N}$ by induction. Suppose that the assertion is true up to the natural number k. Let us show that $a_{k+1} + b_{k+1} = a_{k+2}$. Since $a_{k+1} = 3a_k - a_{k-1}$ and $b_{k+1} = 3b_k - b_{k-1}$, $a_{k+1} + b_{k+1} = 3(a_k + b_k) - (a_{k-1} + b_{k-1}) = 3a_{k+1} - a_k = a_{k+2}$. The other is similar. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together, read and approved the final manuscript.

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