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Relationships between fixed points and eigenvectors in the group $GL(2,\mathbb{R})$

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Abstract

$PSL(2,\mathbb{R})$ is the most frequently studied subgroup of the Möbius transformations. By adding anti-automorphisms

$$G' = \left\{ \frac{a'z + b'}{c'z + d'} : a', b', c', d' \in \mathbb{R}, a'd' - b'c' = -1 \right\}$$

to the group $PSL(2,\mathbb{R})$, the group $G = PSL(2,\mathbb{R}) \cup G'$ is obtained. The elements of this group correspond to matrices of $GL(2,\mathbb{R})$. In this study, we consider the relationships between fixed points of the elements of the group G and eigenvectors of matrices corresponding to the elements of this group.

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1 Introduction

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the extended complex plane. A Möbius transformation is a function f of the form

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Each Möbius transformation is a meromorphic bijection of $\mathbb{C}_\infty$ onto itself and is called an automorphism of $\mathbb{C}_\infty$.

Möbius transformations form a group with respect to composition. If T is a Möbius transformation, then the composition $T \circ R$ is called an anti-automorphism of $\mathbb{C}_\infty$, where $R(z) = -\bar{z}$. The union of automorphisms and anti-automorphisms also form a group under the composition of functions.

If coefficients of Möbius transformations are taken as real numbers, we obtain the most frequently studied subgroup of this group:

$$PSL(2,\mathbb{R}) = \left\{ \frac{az + b}{cz + d} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

By adding anti-automorphisms $G' = \{\frac{a'\bar{z}+b'}{c'\bar{z}+d'} : a', b', c', d' \in \mathbb{R}, a'd' - b'c' = -1\}$ to the group $PSL(2,\mathbb{R})$, the group $G = PSL(2,\mathbb{R}) \cup G'$ is obtained. The elements of this group correspond to the matrices of $GL(2,\mathbb{R})$. If we take $T(z) \in G$, then $T(z)$ has the matrix presentation $T = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$.

The fixed points of automorphisms and anti-automorphisms of the extended complex plane have especially been of great interest in many fields of mathematics, for example, in number theory, functional analysis, theory of complex functions, geometry and group theory (see [1–5] and references therein).

In [6], Beardon gave some relationships between the fixed points of Möbius maps and the lines of the eigenvectors of their corresponding matrices. So, these studies include the transformations of $PSL(2,\mathbb{R})$. In this study, we investigate similar relationships for transformations of G'. Thus we complete the problem for the group G.

2 Preliminaries

In this section we give brief information about complex lines and fixed points of the transformations of G.

Definition 1 [6] A complex line is a one-dimensional subspace of the vector space $\mathbb{C}^{2,t}$ of complex column vectors $(z_1,z_2)^t$. A complex line L is the set of complex scalar multiples of some non-zero point in $\mathbb{C}^{2,t}$, and so it is of the form

$$L=\left\{r\begin{pmatrix}z_1\\z_2\end{pmatrix}:r\in\mathbb{C}\right\}.$$

If $z_2\neq 0$, we can form the quotients $\frac{rz_1}{rz_2}$ of the coordinates of the non-zero points on the line L in Definition 1 and the common value of all of these quotients is the slope $\frac{z_1}{z_2}$ of L. The single complex line whose slope is not defined is

$$L(\infty)=\left\{r\begin{pmatrix}1\\0\end{pmatrix}:r\in\mathbb{C}\right\}\tag{2.1}$$

and, by convention, we say that this line has slope ∞. Given a complex number w, there is a unique complex line $L(w)$ with slope w, namely

$$L(w)=\left\{r\begin{pmatrix}w\\1\end{pmatrix}:r\in\mathbb{C}\right\}.\tag{2.2}$$

Theorem 1 [6] *Let f be a Möbius map with corresponding matrix A. Then $f(w)=w$ if and only if $L(w)$ is a line of eigenvectors of A.*

Here we mention types of the elements in the group G briefly. For each $T\in G$, the point $z\in\mathbb{C}_\infty$ is called a fixed point of T if $T(z)=z$, and the trace of $T(z)$ is defined by $\operatorname{tr}(T)=a+d$. There is a relation between the fixed points and the trace of a transformation of G. Thus we can determine fixed points location in $\mathbb{C}_\infty$ with the trace.

If $T(z)\in PSL(2,\mathbb{R})$, then the number of fixed points of $T(z)$ is at most two. Also, if

(i) $|\operatorname{tr}(T)|>2$, then there are two fixed points in $\mathbb{R}\cup\{\infty\}$ and $T(z)$ is called a hyperbolic element.
(ii) $|\operatorname{tr}(T)|=2$, then there is one fixed point in $\mathbb{R}\cup\{\infty\}$ and $T(z)$ is called a parabolic element.
(iii) $|\operatorname{tr}(T)|<2$, then there are two conjugate fixed points in $\mathbb{C}\cup\{\infty\}$ and $T(z)$ is called an elliptic element.

If $T(z)\in G'$, then it has two fixed points or the set of fixed points is a circle. Also, if

(iv) $\operatorname{tr}(T) \neq 0$, then there are two distinct fixed points on the $\mathbb{R} \cup \{\infty\}$ and $T(z)$ is called a glide reflection.

(v) $\operatorname{tr}(T) = 0$, then the set of the fixed points is a circle and $T(z)$ is called a reflection. For more information, one can consult the references [7] and [8].

Now we find the fixed points of the glide reflections and reflections in the group G. Some straightforward computations show that the fixed points of $T(z)$ are

$$x_{1,2} = \frac{a - d \pm \sqrt{(a + d)^2 + 4}}{2c} \tag{2.3}$$

and these points lie on $\mathbb{R} \cup \{\infty\}$ for any $T(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \in G$ with $\operatorname{tr}(T) \neq 0$. For any $T(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \in G$ with $\operatorname{tr}(T) = 0$, the fixed points of $T(z)$ form a circle centered at $M(\frac{a}{c}, 0)$ and of radius $r = \frac{1}{|c|}$.

3 Eigenvectors of the matrices corresponding to the transformations in the group *G*

If $T(z) \in PSL(2, \mathbb{R})$, then the connection between fixed points of $T(z)$ and lines of eigenvectors for the matrix T corresponding to $T(z)$ is explained by Theorem 1. Now we consider the transformations of the group G which belong to $G^{'}$.

Let $T(z) \in G^{'}$ be any transformation with the corresponding matrix $T = \pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL(2, \mathbb{R})$. The characteristic polynomial for this matrix is

$$\lambda^2 - (\operatorname{tr} T)\lambda - 1 = 0. \tag{3.1}$$

We use the eigenvector representation $\left(\begin{smallmatrix} k_1 \\ k_2 \end{smallmatrix}\right)$ for the matrix T. First we begin with the glide reflections.

3.1 Glide reflections

We will show that the fixed points of a glide reflection $T(z)$ correspond to the two lines of eigenvectors for the matrix T corresponding to $T(z)$. In the following two lemmas, we determine the eigenvalues and eigenvectors of the matrices which correspond to the glide reflections.

Lemma 1 *Let the matrix $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ correspond to any glide reflection. The eigenvalues of T are*

$$\lambda_{1,2} = \frac{a + d \pm \sqrt{(a + d)^2 + 4}}{2}, \tag{3.2}$$

and the eigenvectors of T are

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \frac{a-d\pm\sqrt{(a+d)^2+4}}{2c} r \\ r \end{pmatrix}. \tag{3.3}$$

Proof It is easy to compute the eigenvalues by the condition $a + d \neq 0$ and (3.1). For an eigenvalue λ, we obtain the eigenvector by the following equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \lambda \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

Thus we have $(a-\lambda)k_1 + bk_2 = 0$ and $ck_1 + (d-\lambda)k_2 = 0$. If we choose $k_2 = r$ as a parameter, we find the eigenvector as $\left(\begin{smallmatrix} \frac{\lambda-d}{c}r \\ r \end{smallmatrix}\right)$. Therefore we obtain the eigenvectors as

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \frac{a-d\pm\sqrt{(a+d)^2+4}}{2c} r \\ r \end{pmatrix}. \qquad \square$$

Theorem 2 *Let $T(z)$ be a glide reflection map in the group G with corresponding matrix T. Then $T(w) = w$ if and only if $L(w)$ is a line of eigenvectors of T.*

Proof Let $T(z)$ be a glide reflection map in the group G with a corresponding matrix T. For glide reflections, the lines with slope w, where $w = \frac{a-d\pm\sqrt{(a+d)^2+4}}{2c}$ is a fixed point of $T(z)$, are

$$L(w) = \left\{ r\begin{pmatrix} \frac{a-d\pm\sqrt{(a+d)^2+4}}{2c} \\ 1 \end{pmatrix} : r \in \mathbb{C} \right\}. \tag{3.4}$$

Then T maps $L(w)$ to $L(w')$ if and only if $T(w) = w'$. Thus, w is a fixed point of $T(z)$ if and only if T maps $L(w)$ to itself, and so if and only if each non-zero point on $L(w)$ is an eigenvector of T. $\square$

Example 1 By (2.3) we find the fixed points of the glide reflection $T = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$ as $x = \pm\sqrt{2}$. By (3.2) we find the eigenvalues as $\lambda_{1,2} = 1 \pm \sqrt{2}$. Hence, by Lemma 1, we obtain the following eigenvectors

$$\begin{pmatrix} \sqrt{2}r \\ r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\sqrt{2}r \\ r \end{pmatrix}$$

respectively. We have the slopes $w_1 = \sqrt{2}$ and $w_2 = -\sqrt{2}$.

3.2 Reflections

Recall that we have $\mathrm{tr}(T) = 0$ for any reflection transformation.

Lemma 2 *Let the matrix $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ correspond to any reflection. The eigenvalues of T are*

$$\lambda_{1,2} = \pm 1.$$

Proof By (3.1), if we use the condition $\mathrm{tr}(T) = 0$, the result is obtained. $\square$

First we begin the case $c = 0$. For this case, the set of fixed points is a circle with radius ∞ (that is, a line on the complex plane).

Lemma 3 *Let the matrix $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ correspond to any reflection with $c = 0$. We have*

$$T = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad T = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}.$$

Proof The proof is easy by the facts that $ad - bc = -1$ $(a, b, c, d \in \mathbb{R})$ and $\mathrm{tr}(T) = 0$. $\square$

Lemma 4

(i) *For the matrix* $T = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$, *we have the eigenvalues* $\lambda = 1$ *and* $\lambda = -1$ *and the eigenvectors*

$$\begin{pmatrix} r \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{b}{2}r \\ r \end{pmatrix}, \tag{3.5}$$

respectively.

(ii) *For the matrix* $T = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$, *we have the eigenvalues* $\lambda = 1$ *and* $\lambda = -1$ *and the eigenvectors*

$$\begin{pmatrix} \frac{b}{2}r \\ r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r \\ 0 \end{pmatrix}, \tag{3.6}$$

respectively.

Proof It is easy to compute the eigenvalues $\lambda = 1$ and $\lambda = -1$ by the condition $a + d = 0$ and (3.1). For these eigenvalues, we obtain the eigenvectors by the following equation

$$\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \lambda \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

If we choose $k_2 = r$ as a parameter, we find the eigenvectors as $\begin{pmatrix} r \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -\frac{b}{2}r \\ r \end{pmatrix}$. The second part of the proof can be obtained similarly. □

In the first part of Lemma 4, we have the slopes as $w_1 = \infty$ and $w_2 = -\frac{b}{2}$. In the second part, we have $w_1 = \frac{b}{2}$ and $w_2 = \infty$.

Lemma 5

(i) *The matrix* $T = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$ *represents the reflection* $T(z) = \frac{\overline{z}+b}{-1}$. *The set of the fixed points of this reflection is a circle with radius* ∞, *that is, the line* $x = -\frac{b}{2}$.

(ii) *The matrix* $T = \begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$ *represents the reflection* $T(z) = -\overline{z} + b$. *The set of the fixed points of this reflection is the circle* $x = \frac{b}{2}$.

Proof The proof follows by straightforward computations. □

In the following theorem, we explain the relationship between fixed points of the reflections with $c = 0$ and eigenvectors of the matrices corresponding to those reflections.

Theorem 3 *Let* $T(z)$ *be a reflection map in the group* G *with* $c = 0$ *and let* T *be the matrix corresponding to* $T(z)$. *Then* $L(\infty)$ *and* $L(\pm\frac{b}{2})$ *are the lines of the eigenvectors of the matrix* T *and the set of the fixed points of the reflection* $T(z)$ *is the line* $x = \pm\frac{b}{2}$.

Proof The proof follows by Lemma 3, Lemma 4 and Lemma 5. □

Finally, we consider the reflections with $c \neq 0$. Lemma 6 can be proven in a similar way as Lemma 4.

Lemma 6 *Let the matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ correspond to any reflection with $c \neq 0$. We have the following eigenvectors for the eigenvalues $\lambda = 1$ and $\lambda = -1$*

$$\begin{pmatrix} \frac{1-d}{c}r \\ r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\frac{d+1}{c}r \\ r \end{pmatrix}, \tag{3.7}$$

respectively.

In Lemma 6, we have the slopes as $w_1 = \frac{1-d}{c}$ and $w_2 = -\frac{d+1}{c}$. In the following theorem, we explain the relationship between the set of fixed points of the reflections with $c \neq 0$ and eigenvectors of matrices corresponding to those reflections.

Theorem 4 *Let $T(z)$ be a reflection map in the group G with $c \neq 0$ and let T be the corresponding matrix of $T(z)$. If $L(w_1)$ and $L(w_2)$ are the lines of the eigenvectors of the matrix T, then the set of the fixed points of the reflection $T(z)$ is the circle centered at $M(\frac{w_1+w_2}{2}, 0)$ and of radius $\frac{|w_1-w_2|}{2}$.*

Proof For the slopes $w_1 = \frac{1-d}{c}$ and $w_2 = -\frac{d+1}{c}$, we have

$$\frac{w_1 + w_2}{2} = \frac{a}{c}$$

and

$$\frac{|w_1 - w_2|}{2} = \frac{1}{|c|}.$$

Then the proof follows by Lemma 6. □

Example 2 The fixed point set of the reflection $T = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$ is a circle. By Theorem 4, we find the equation of this circle. By Lemma 6, eigenvectors of the matrix T are

$$\begin{pmatrix} 3r \\ r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} r \\ r \end{pmatrix}.$$

Then we have $w_1 = 3$ and $w_2 = 1$. Thus the fixed point set is a circle centered at $M(\frac{w_1+w_2}{2}, 0) = M(2, 0)$ and of radius $\frac{|w_1-w_2|}{2} = \frac{|3-1|}{2} = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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