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# $N = 2$ SUGRA BPS multi-center solutions, quadratic prepotentials and Freudenthal transformations

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Abstract: We present a detailed description of $N = 2$ stationary BPS multicenter black hole solutions for quadratic prepotentials with an arbitrary number of centers and scalar fields making a systematic use of the algebraic properties of the matrix of second derivatives of the prepotential, $\mathcal{S}$, which in this case is a scalar-independent matrix. In particular we obtain bounds on the physical parameters of the multicenter solution such as horizon areas and ADM mass. We discuss the possibility and convenience of setting up a basis of the symplectic vector space built from charge eigenvectors of the $\mathcal{S}$, the set of vectors $(\mathrm{P}_\pm q_a)$ with $\mathrm{P}_\pm$ $\mathcal{S}$-eigenspace projectors.

The anti-involution matrix $\mathcal{S}$ can be understood as a Freudenthal duality $\tilde{x} = \mathcal{S}x$. We show that this duality can be generalized to "Freudenthal transformations"

$$x \to \lambda \exp(\theta\mathcal{S})x = ax + b\tilde{x}$$

under which the horizon area, ADM mass and intercenter distances scale up leaving constant the scalars at the fixed points. In the special case $\lambda = 1$, "$\mathcal{S}$-rotations", the transformations leave invariant the solution. The standard Freudenthal duality can be written as $\tilde{x} = \exp\left(\frac{\pi}{2}\mathcal{S}\right)x$. We argue that these generalized transformations leave invariant not only the quadratic prepotential theories but also the general stringy extremal quartic form $\Delta_4$, $\Delta_4(x) = \Delta_4(\cos\theta x + \sin\theta\tilde{x})$ and therefore its entropy at lowest order.

Keywords: Supergravity Models, Supersymmetry and Duality, Black Holes in String Theory, Models of Quantum Gravity

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## Contents



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## 1 Introduction

We present a systematic study of extremal, stationary, multi-center black-hole-type solutions in $N = 2$ $D = 4$ ungauged Einstein-Maxwell supergravity theories minimally coupled to an arbitrary number $n_v$ of vector multiplets, i.e. quadratic prepotentials.

The action of these $4D$ $N = 2$ supergravities can be written, in the framework of special geometry, in terms of a holomorphic section $\Omega$ of the scalar manifold. The corresponding field equations and Bianchi identities remain invariant under the group of symplectic transformations $\mathrm{Sp}(2n_v + 2, \mathbb{R})$. This group acts linearly on the section $\Omega$, which transforms as a symplectic vector when it is parametrized as $\Omega = (X^I, F_I)$, for $I = 0, \ldots, n_v$.

The embedding of the duality group of the moduli space into the symplectic group $\mathrm{Sp}(2n_v + 2, \mathbb{R})$ establishes, in general, a relation between the upper and lower components of $\Omega$, $X^I$ and $F_I = F_I(X^J)$ respectively. In some cases, $F_I$ is the derivative of a single function, the prepotential $F = F(X^J)$. The choice of a particular embedding determines the full Lagrangian of the theory and whether a prepotential exists [1, 2].

In this work, we restrict ourselves to general quadratic prepotentials. These theories[1] include the simplest examples of special Kähler homogeneous manifolds, the axion-dilaton

[1]See [3] for a classification of $N = 2$ SUGRA special Kähler manifolds.

model or the

$$\mathbb{C}P^n \equiv \frac{\mathrm{SU}(1,n)}{\mathrm{U}(1)\times \mathrm{SU}(n)} \tag{1.1}$$

case.

These models correspond to Einstein-Maxwell $N = 2$ supergravities minimally coupled to $n_v$ vector multiplets. They lead to phenomenologically interesting $N = 1$ minimally coupled supergravities [4]. Theories derived from particular examples of these quadratic prepotentials have been studied in detail.[2]

Black hole solutions in $N = 2$ $D = 4$ supergravity have been extensively studied for a long term by now. See, for example, refs. [8–20]. Multicenter black holes have been treated in refs. [21–32]. In this work we show how it is possible a detailed study of stationary multicenter black-hole type solutions with any number of scalar fields and centers, of the properties of the bosonic field solutions and their global and local properties making a systematic and intensive use of the algebraic properties of the matrix of second derivatives of the prepotential, the matrix $\mathcal{S}$ and of the matrix $\mathcal{S}^\dagger$, its adjoint with respect the symplectic product. This matrix is an isometry of the symplectic bilinear form, it connects the real and imaginary parts of symplectic sections of the theory. In this case it is a real scalar-independent $\mathrm{Sp}(2n_v+2,\mathbb{R})$ matrix. Among other results, we obtain bounds for the physical parameter of the multicenter solution such as horizon areas and ADM mass valid for any quadratic prepotentials.

The compatibility of the matrix $\mathcal{S}$ with respect to the symplectic product makes possible the definition of an associated inner product for which these matrices are unitary. We discuss the possibility and convenience of setting up a basis of the $(2n_v + 2)$-dimensional symplectic vector space built from charge eigenvectors of the matrix $\mathcal{S}$. This set of vectors are of the form $(\mathrm{P}_\pm q_a)$, or, alternatively, $(q_a, \mathcal{S}q_a)$, with $\mathrm{P}_\pm$ projectors over the eigenspaces of $\mathcal{S}$ and $q_a$ the center vector charges.

The anti-involution matrix $\mathcal{S}$ can be understood as a Freudenthal duality $\tilde{x} = \mathcal{S}x$ [33, 34]. We will show here that this duality can be generalized to an Abelian group of transformations

$$x \to \lambda \exp(\theta\mathcal{S})x = ax + b\tilde{x}.$$

Under this set of transformations applied to the charge vectors and $\mathcal{I}_\infty = \mathcal{I}(r \to \infty)$, the horizon area, ADM mass and intercenter distances scale up, respectively, as

$$S_h \to \lambda^2 S_h, \quad M_{\mathrm{ADM}} \to \lambda M_{\mathrm{ADM}}, \quad r_{ab} \to \lambda r_{ab},$$

leaving invariant the values of the scalars at the fixed points and at infinity. In the special case $\lambda = 1$, "$\mathcal{S}$-rotations", the transformations leave invariant the solution. The standard Freudenthal duality can be written as the particular rotation

$$\tilde{x} = \exp\left(\frac{\pi}{2}\mathcal{S}\right) x\,.$$

[2]The case $n_v = 1$ corresponds to the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ axion-dilaton black hole [5–7] with prepotential $F = -iX^0X^1$.

We argue at the final section of this work that these generalized Freudenthal transformations leave invariant not only the entropy and other macroscopical quantities of quadratic prepotential theories but also $\Delta_4$, the quartic invariant [33] appearing in the description of more general theories, $4d$ SUGRAs that arise from String and M-theory and therefore the lowest order entropy of these theories.

In section 2, we present some well-known basic aspects of $N = 2$ $D = 4$ supergravity theories and their formulation in terms of special and symplectic geometry. In section 3, we first introduce the matrices $\mathcal{S}_{N,F}$, stressing some of their known properties and deriving new ones. We also construct projective operators (as well as their corresponding symplectic adjoints) based on these matrices. After the consideration of the attractor mechanism in terms of these projectors, we enter in a full explicit description of multicenter black hole solutions, their horizons and their asymptotic properties. This is done in sections 4 and 5. We finally present section 6, which contains a summary and discussion of our work, as well as an outlook on further proposals.

## 2 $N = 2$ $D = 4$ SUGRA and special Kähler geometry

The field content of the $N = 2$ supergravity theory coupled to $n_v$ vector multiplets consists of

$$\left\{e_\mu{}^a\,, A_\mu{}^I\,, z^\alpha\,, \psi_\mu{}^r\,, \lambda_r{}^\alpha\right\}\,, \tag{2.1}$$

with $\alpha = 1, \ldots, n_v$, and $I = 0, \ldots, n_v$. The theory also contains some hypermultiplets, which can be safely taken as constant or neglected (further details can be found in [22], whose notation and concepts we generally adopt). The bosonic $N = 2$ action can be written as

$$S = \int_{M(4d)} R \star 1 + \mathcal{G}_{\alpha\bar{\beta}} dz^\alpha \wedge \star d\bar{z}^{\bar{\beta}} + F^I \wedge G_I\,. \tag{2.2}$$

The fields $F^I, G_I$ are not independent. Whilst $F^I$ is given by $F^I = dA^I$, $G_I$ is a set of combinations of the $F^I$ and their Hodge duals,

$$G_I = a_{IJ} F^I + b_{IJ} \star F^I\,, \tag{2.3}$$

with scalar-dependent coefficients $a_{IJ}$ and $b_{IJ}$.

Abelian charges with respect the $\mathrm{U}(1)^{n_v+1}$ local symmetry of the theory are defined by means of the integrals of the gauge field strengths. The total charges of the geometry are

$$q \equiv (p^I, q_I) \equiv \frac{1}{2\pi^2} \int_{S_\infty} (F^I, G_I)\,. \tag{2.4}$$

Similar charges can be defined for specific finite regions.

The theory is defined, in the special geometry formalism, by the introduction of some projective scalar coordinates $X^I$, as for example, 'special' projective coordinates $z^\alpha \equiv X^\alpha / X^0$. By introducing a covariantly holomorphic section of a symplectic bundle, $V$, we are able to arrange $2n_v$ quantities that transform as a vector under symplectic transformations at any point of the manifold. $V$ has the following structure $V = V(z,\bar{z}) \equiv (V^I, V_I)$ and satisfies the following identities:

$$\langle V \mid \bar{V} \rangle \equiv V^t \omega \bar{V} \equiv \bar{V}^I V_I - V^I \bar{V}_I = -i\,, \tag{2.5}$$

where $\omega$ is the symplectic form.[3]

The scalar kinetic term in the action can be written in terms of $V$ as $L_{s,\mathrm{kin}} \sim i \langle D^\mu \bar{V} \mid D_\mu V \rangle$ and the scalar metric is given by

$$\mathcal{G}_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}\,, \tag{2.6}$$

where the Kähler potential $\mathcal{K}$ is defined by the relations $V = \exp(\mathcal{K}/2)\Omega$, being $\Omega \equiv (X^I, F_I)$ a holomorphic section and

$$e^{-\mathcal{K}} = i\left(\bar{X}^I F_I - X^I \bar{F}_I\right) = i \langle \Omega \mid \bar{\Omega} \rangle\,. \tag{2.7}$$

In $N = 2$ theories, the central charge $Z$ can be expressed as a linear function on the charge space:

$$Z(z^\alpha, q) \equiv \langle V \mid q \rangle = e^{\mathcal{K}/2}\left(p^I F_I - q_I X^I\right)\,. \tag{2.8}$$

The embedding of the isometry group of the scalar manifold metric $\mathcal{G}_{\alpha\bar{\beta}}$, into the symplectic group fixes, through the Kähler potential $\mathcal{K}$, a functional relation between the lower and upper parts of $V$ and $\Omega$ [2, 5],

$$F_I = F_I(X^I)\,, \tag{2.9}$$
$$V_I = V_I(V^I)\,. \tag{2.10}$$

There always exists a symplectic frame under which the theory can be described in terms of a single holomorphic function, the *prepotential* $F(X)$. It is a second degree homogeneous function on the projective scalar coordinates $X^I$, such that $F_I(X) = \partial_I F(X)$. For simplicity, we will assume the existence of such prepotential along this study although the results will not depend on such existence. Using the notation $F_{IJ} = \partial_I \partial_J F$, the lower and upper components of $\Omega$ are related by

$$F_I = F_{IJ} X^J\,. \tag{2.11}$$

The lower and upper components of $V$ are related by a field dependent matrix $N_{IJ}$, which is determined by the special geometry relations [9]

$$V_I = N_{IJ} V^J, \tag{2.12}$$
$$D_{\bar{\imath}} \bar{V}_I = N_{IJ} D_{\bar{\imath}} \bar{V}^J. \tag{2.13}$$

[3]We choose a basis such that $\omega = \begin{pmatrix} 0 & -1_{n_v} \\ 1_{n_v} & 0 \end{pmatrix}$.



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The matrix $N$, which also fixes the vector couplings $(a_{IJ}, b_{IJ})$ in the action, can be related to $F_{IJ}$ [35] by

$$N_{IJ} = \bar{F}_{IJ} + T_I T_J \,, \tag{2.14}$$

where the quantities $T_I$ are proportional to the projector of the graviphoton, whose flux defines the $N = 2$ central charge [35]. For our purposes, it is convenient to write the relation between $N_{IJ}$ and $F_{IJ}$ as

$$\begin{aligned} N_{IJ} &\equiv F_{IJ} + N^{\perp}_{IJ} \\ &= F_{IJ} - 2i\text{Im}\,(F_{IJ}) + 2i\frac{\text{Im}\,(F_{IK})\,L^K\text{Im}\,(F_{JQ})\,L^Q}{L^P\text{Im}\,(F_{PQ})\,L^Q}\,, \end{aligned} \tag{2.15}$$

where we have decomposed the matrix $N_{IJ}$ into "longitudinal" (the $F_{IJ}$ themselves) and "transversal" parts ($N^{\perp}_{IJ}$). The perpendicular term (defined by the expression above) annihilates $L^I$, or any multiple of it,

$$N^{\perp}_{IJ}(\alpha L^J) = 0\,. \tag{2.16}$$

From this, (2.12) can be written as

$$\begin{aligned} V_I = N_{IJ}L^J &= \left(F_{IJ} + N^{\perp}_{IJ}\right) L^J \\ &= F_{IJ}L^J\,. \end{aligned} \tag{2.17}$$

Thus, the upper and lower components of $V$ and $\Omega$ are connected by the same matrix $F_{IJ}$.

The existence of functional dependencies among the upper and lower components of the vectors $V$ or $\Omega$ imply further relations between their respective real and imaginary parts. They are related by symplectic matrices $\mathcal{S}(N), \mathcal{S}(F) \in \text{Sp}(2n_v + 2, \mathbb{R})$ which are respectively associated to the quantities $N_{IJ}, F_{IJ}$ as follows:

$$\text{Re}\,(\Omega) = \mathcal{S}(F)\text{Im}\,(\Omega)\,, \tag{2.18}$$

$$\text{Re}\,(V) = \mathcal{S}(N)\text{Im}\,(V) = \mathcal{S}(F)\text{Im}\,(V)\,. \tag{2.19}$$

The last expression is obtained by means of the relation (2.17). These same relations (2.18)–(2.19) are valid for any complex multiple of $\Omega$ or $V$. It is straightforward to show, for example, that for any $\lambda \in \mathbb{C}$, we have

$$\text{Re}\,(\lambda V) = \mathcal{S}(N)\text{Im}\,(\lambda V) = \mathcal{S}(F)\text{Im}\,(\lambda V)\,. \tag{2.20}$$

The matrix $\mathcal{S}(F)$ is, by direct computation (eq. (75) in [35]), of the form

$$\mathcal{S}(F) = \begin{pmatrix} 1 & 0 \\ \text{Re}\,(F_{IJ}) & -\text{Im}\,(F_{IJ}) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \text{Im}\,(F_{IJ}) & \text{Re}\,(F_{IJ})^{-1} \end{pmatrix}^{-1}\,. \tag{2.21}$$

Similarly, the same result applies for $\mathcal{S}(N)$ with $F_{IJ} \to N_{IJ}$.[4]

[4]The matrix $\mathcal{S}_N$ is related to $\mathcal{M}$, the matrix that appears in the black hole effective potential [35] $V_{BH} = -\frac{1}{2}q^t\mathcal{M}q$, by $\omega\mathcal{S}(N) = \mathcal{M}$.

In $N = 2$ theories, $\mathcal{S}(N)$ always exhibits a moduli dependence [4]. However, this is not the case for $\mathcal{S}(F)$. We will focus in this work on the particular case of theories with quadratic prepotentials,

$$F(X) = \frac{1}{2} F_{IJ} X^I X^J \,, \tag{2.22}$$

where $F_{IJ}$ is a complex, constant, symmetric matrix. Then, the corresponding matrix $\mathcal{S}(F)$ is a field-independent, constant matrix. We can assume that Re $(F_{IJ}) = 0$ and Im $(F_{IJ})$ is negative definite. In what follows, we will use the notation $\mathcal{S} \equiv \mathcal{S}(F)$. The condition $e^{-\mathcal{K}} > 0$ and the expression (2.7) implies a restriction on the prepotential. We will write this restriction in a convenient form in section 3 in terms of the positivity of a quadratic form.

### 2.1 General supersymmetric stationary solutions

The most general stationary (time independent) 4-dimensional metric compatible with supersymmetry can be written in the IWP form [36–38],

$$ds^2 = e^{2U}(dt + \omega)^2 - e^{-2U} d\mathbf{x}^2. \tag{2.23}$$

Supersymmetric $N = 2$ supergravity solutions can be constructed systematically following well established methods [20, 22]. In this section we will closely follow the notation of ref. [22]. The 1-form $\omega$ and the function $e^{-2U}$ are related in these theories to the Kähler potential and connection, $\mathcal{K}, Q$ [38]. We demand asymptotic flatness, $e^{-2U} \to 1$ together with $\omega \to 0$ for $|\mathbf{x}| \to \infty$. BPS field equation solutions for the action above (for example, quantities that appear in the metric, as $e^{-2U}$ or $\omega$) can be written in terms of the following real symplectic vectors $\mathcal{R}$ and $\mathcal{I}$

$$\mathcal{R} = \frac{1}{\sqrt{2}} \mathrm{Re} \left( \frac{V}{X} \right) \,, \tag{2.24}$$

$$\mathcal{I} = \frac{1}{\sqrt{2}} \mathrm{Im} \left( \frac{V}{X} \right) \,. \tag{2.25}$$

$X$ is an arbitrary complex function of space coordinates such that $1/X$ is harmonic. The $2n_v + 2$ components of $\mathcal{I}$ and $\mathcal{R}$ are real harmonic functions in $\mathbb{R}^3$. There is an algebraic relation between $\mathcal{R}$ and $\mathcal{I}$ and the solutions can be written in terms only of the vector $\mathcal{I}$. By making use of (2.18)–(2.20), we can write the following *stabilization equation*

$$\mathcal{R} = \mathcal{S} \mathcal{I} \,. \tag{2.26}$$

In practice, specific solutions are determined by giving a particular, suitable, ansatz for the symplectic vector $\mathcal{I}$ as a function of the spacetime coordinates.

Using these symplectic vectors we rewrite the only independent metric component as

$$\begin{aligned} e^{-2U} = e^{-\mathcal{K}} &= \frac{1}{2|X|^2} \\ &= \langle \mathcal{R} \mid \mathcal{I} \rangle = \langle \mathcal{S}\mathcal{I} \mid \mathcal{I} \rangle \,. \end{aligned} \tag{2.27}$$



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Similarly, the time independent 3-dimensional 1-form $\omega = \omega_i dx^i$ satisfies the equation

$$d\omega = 2 \langle \mathcal{I} \mid \star_3 d\mathcal{I} \rangle \ , \tag{2.28}$$

where $\star_3$ is the Hodge dual on flat $\mathbb{R}^3$, together with the integrability condition

$$\langle \mathcal{I} \mid \Delta \mathcal{I} \rangle = 0\,. \tag{2.29}$$

The asymptotic flatness condition implies

$$\langle \mathcal{R}_\infty \mid \mathcal{I}_\infty \rangle = \langle \mathcal{S} \mathcal{I}_\infty \mid \mathcal{I}_\infty \rangle = 1\,. \tag{2.30}$$

The gauge field equations of motion and Bianchi identities can be directly solved in terms of spatially dependent harmonic functions [22]. The modulus of the central charge function defined in (2.8) can be written, taking into account (2.27), as

$$|Z(q)|^2 e^{-2U} = |\langle \mathcal{R} \mid q \rangle|^2 + |\langle \mathcal{I} \mid q \rangle|^2\,. \tag{2.31}$$

At spatial infinity, by assuming the asymptotic flatness condition (2.30), we arrive to

$$|Z_\infty(q)|^2 = |\langle \mathcal{R}_\infty \mid q \rangle|^2 + |\langle \mathcal{I}_\infty \mid q \rangle|^2\,. \tag{2.32}$$

The, assumed time independent, $n_v$ complex scalar fields $z^\alpha$ solutions to the field equations, are given in this formalism by

$$z^\alpha = \frac{\Omega^\alpha}{\Omega^0} = \frac{V^\alpha/X}{V^0/X} = \frac{\mathcal{R}^\alpha + i\mathcal{I}^\alpha}{\mathcal{R}^0 + i\mathcal{I}^0}\,. \tag{2.33}$$

This is, in general, a formal expression as the $\mathcal{I}$ or $\mathcal{R}$ quantities may be scalar dependent.[5]

These scalar fields can, in principle, take any values $(z_\infty)$ at infinity. These values will appear as free parameters in the ansatz that we give for $\mathcal{I}$. Nevertheless, according to the attractor mechanism, the moduli adjust themselves at some fixed points.

We are interested in this work in extremal, single- or multi-center black hole-type solutions determined by an $\mathcal{I}$ ansatz with point-like singularities of the form

$$\mathcal{I} = \mathcal{I}_\infty + \sum_a \frac{q_a}{|\mathbf{x} - \mathbf{x}_a|}\,, \tag{2.34}$$

where $a = 1, \ldots, n_a$ being the number $n_a$ arbitrary and $q_a = ({p_a}^I, q_{aI})$ and $\mathcal{I}_\infty$ real, constant, symplectic vectors.

For this kind of solutions, the quantities $\mathcal{I}_\infty$ are related to the values at infinity of the moduli while the "charge" vectors $q_a$ are related to their values at the fixed points. The fixed values of the scalars, $z(\mathbf{x}) \to z(\mathbf{x}_a) = z_f^a$, are the solutions of the following *attractor equations* [8, 9, 11]:

$$q^a = \mathrm{Re}\left(2i\bar{Z}(z_f^a)V(z_f^a)\right)\,. \tag{2.35}$$

[5] Even for a scalar independent ansatz $\mathcal{I}$, the matrix $\mathcal{S}$ is, in general, scalar dependent.

The prepotential performs its influence throughout $V$ and $Z$ (cf. (2.8)). The scalar attractor values are independent of their asymptotic values and only depend on the discrete charges $z_f^a = z_f^a(q_a)$.

Single center black hole solutions are known to exist for all regions of the moduli scalars at infinity, under very mild conditions on the charge vector. In the multicenter case, for fixed charge vectors, not all the positions $\mathbf{x}_a$ in the ansatz (2.34) are allowed. The integrability condition (2.29) imposes necessary conditions on the relative positions and on the moduli at spatial infinity (through $\mathcal{I}_\infty$) for the existence of a solution. In this framework, a particular black hole solution is completely determined by a triplet of charge vectors, distances and values of the moduli at infinity $(q_a, \mathbf{x}_a, z_\infty^\alpha)$.

## 3 The stabilization matrix, its adjoint and the attractor equations

Let us consider now the attractor equations (2.35) in more detail. We will use the properties of the stabilization matrix $\mathcal{S}$ to solve them in a purely algebraic way to obtain some properties and give some explicit expressions for the scalars at the fixed points.

For this purpose, we first establish some well-known properties of $\mathcal{S}_N, \mathcal{S} \equiv \mathcal{S}_F$ and define new matrices: some projector operators associated to them and their respective symplectic adjoints.

It can be shown by explicit computation that the real symplectic matrices $\mathcal{S}_N, \mathcal{S} \equiv \mathcal{S}_F \in \mathrm{Sp}(2n_v + 2, \mathbb{R})$ defined by (2.18)–(2.19), whose explicit expressions are (2.21), satisfy the relations (see also [35] )

$$\mathcal{S}_N^2 = \mathcal{S}_F^2 = -1 \,. \tag{3.1}$$

From this, it is possible and convenient to define the projector operators[6]

$$\mathrm{P}_\pm = \frac{1 \pm i\mathcal{S}}{2} \,. \tag{3.2}$$

They satisfy the following straightforward properties

$$\begin{aligned} \mathrm{P}_\pm^2 &= \mathrm{P}_\pm \,, \\ \mathcal{S}\mathrm{P}_\pm &= \mp i \mathrm{P}_\pm \,, \\ (\mathrm{P}_\pm)^* &= \mathrm{P}_\mp \,. \end{aligned} \tag{3.3}$$

For $X$, $Y$ arbitrary real vectors, we have

$$\mathrm{P}_\pm X = \mathrm{P}_\pm Y \quad \Rightarrow \quad X = Y \,. \tag{3.4}$$

According to (3.3), $\mathrm{P}_\pm$ are the projectors on the eigenspaces (of equal dimension) of the matrix $\mathcal{S}$. The symplectic space $W$ can be decomposed into eigenspaces of the matrix $\mathcal{S}$:

$$W = W^+ \oplus W^- \,, \tag{3.5}$$

[6] This is done for $\mathcal{S} \equiv \mathcal{S}_F$, but a similar procedure can be done for $\mathcal{S}_N$.

where $W^{\pm} \equiv \mathrm{P}_{\pm} W$. For an arbitrary function of $\mathcal{S}$, $f(\mathcal{S})$, necessarily a linear function of it,

$$f(\mathcal{S}) \equiv a + b\mathcal{S} \equiv \lambda \exp(\theta \mathcal{S}),$$

we have

$$f(\mathcal{S})\mathrm{P}_{\pm} = f(\mp i)\mathrm{P}_{\pm}. \tag{3.6}$$

Complex conjugation interchanges $W^{+}$ and $W^{-}$ subspaces, both subspaces are isomorphic to each other and of dimension $n_v + 1$.

We can rewrite a stabilization relation for the projectors $\mathrm{P}_{\pm}$ analogous to (2.20). For arbitrary $\lambda \in \mathbb{C}$ and $V \in W$, for which there is a relation between its real and imaginary parts of the form $\mathrm{Re}\ (V) = \mathcal{S}\mathrm{Im}\ (V)$, we have

$$\lambda V = \mathrm{Re}\ (\lambda V) + i\mathrm{Im}\ (\lambda V) = 2i\mathrm{P}_{-}\mathrm{Im}\ (\lambda V)\ . \tag{3.7}$$

Thus, the full vector $V$ can be reconstructed applying one of the projectors either from its real or imaginary part. We see that such vectors are fully contained in the subspace $W^{-}$ or, equivalently, they are eigenvectors of $\mathcal{S}$

$$\mathcal{S}V = 2i\mathcal{S}\mathrm{P}_{-}\mathrm{Im}\ (V) = 2\mathrm{P}_{-}\mathrm{Im}\ (V) = iV\,. \tag{3.8}$$

We find it convenient to define the adjoint operator $\mathcal{S}^{\dagger}$ of the matrix $\mathcal{S}$, with respect to the symplectic bilinear product so that, for any vectors $A, B \in W$, we have

$$\langle \mathcal{S}A \mid B \rangle = \left\langle A \,\middle|\, \mathcal{S}^{\dagger} B \right\rangle\ . \tag{3.9}$$

A straightforward computation shows that $\mathcal{S}^{\dagger}$ is given by

$$\mathcal{S}^{\dagger} = -\omega \mathcal{S}^{t} \omega\,. \tag{3.10}$$

Under the assumption of a symmetric $F_{IJ}$ matrix, it is given by

$$\mathcal{S}^{\dagger} = -\mathcal{S}\,. \tag{3.11}$$

In summary, the matrix $\mathcal{S}$ is skew-adjoint with respect to $\omega$ and its square is $\mathcal{S}^2 = -1$. It fulfills an "unitarity" condition $\mathcal{S}^{\dagger}\mathcal{S} = 1$.

$\mathcal{S}$ defines an (almost) complex structure on the symplectic space. This complex structure preserves the symplectic bilinear form, the matrix $\mathcal{S}$ is an isometry of the symplectic space,

$$\langle \mathcal{S}X \mid \mathcal{S}Y \rangle = \langle X \mid Y \rangle\ . \tag{3.12}$$

From (3.11), we see that $\mathcal{S}$ is an element of the symplectic Lie algebra $\mathfrak{sp}(2n_v + 2)$.

Moreover, the bilinear form defined by

$$g(X, Y) \equiv \langle \mathcal{S}X \mid Y \rangle \tag{3.13}$$



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is symmetric.[7] This can be easily seen as (using $\mathcal{S}^\dagger = -\mathcal{S}$):

$$g(X,Y) = \langle \mathcal{S}X \mid Y\rangle = \left\langle X \,\middle|\, \mathcal{S}^\dagger Y\right\rangle = \langle \mathcal{S}Y \mid X\rangle \\ = g(Y,X)\,. \tag{3.14}$$

We will apply these properties to the study of the attractor equations. In general, the matrices $\mathcal{S}_N, \mathcal{S}_F$ are scalar dependent. Only one of them, $\mathcal{S}_F$, is constant, in the case of quadratic prepotentials. Let us write $\mathcal{S}_N^f \equiv \mathcal{S}_N(z_f)$ $\mathcal{S}_F^f \equiv \mathcal{S}_F(z_f)$ for the matrices evaluated at (anyone of) the fixed points. Let us use the sub/superindex $f$ to denote any quantity at the fixed points. For instance, $Z^f \equiv Z(z_f^\alpha)$ or $V^f \equiv V(z_f^\alpha)$. If we multiply both sides of (2.35) by $\mathcal{S}_N^f \equiv \mathcal{S}_N(z_f)$, we arrive to

$$\mathcal{S}_N^f q^a = \mathcal{S}_N^f \mathrm{Re}\left(2i\bar{Z}^f V^f\right) = \mathcal{S}_F^f \mathrm{Re}\left(2i\bar{Z}^f V^f\right) = \mathcal{S}q^a\,, \tag{3.15}$$

where we have used (2.19) and (2.20).[8]

The attractor equations can be written yet in another alternative way. By using (3.7) and (2.35), we can write

$$i\bar{Z}^f V^f = 2\mathrm{P}_- i\bar{Z}^f V^f \\ = \mathrm{P}_- q, \tag{3.16}$$

or its conjugate equation[9]

$$-iZ^f\bar{V}^f = \mathrm{P}_+ q\,. \tag{3.17}$$

That is, the attractor equations simply equalize (a multiple of) the vector $V$ (which, as we have seen above, lies in the subspace $W^-$) with the part of the charge vector which lies in $W^-$.

From (3.16)–(3.17), by taking symplectic products, we obtain

$$|Z^f|^2 \left\langle V_f \,\middle|\, \bar{V}_f\right\rangle = \langle \mathrm{P}_- q \mid \mathrm{P}_+ q\rangle = \langle q \mid \mathrm{P}_+ q\rangle \\ = -\frac{i}{2}\langle \mathcal{S}q \mid q\rangle\,. \tag{3.18}$$

If we plug the constraint $\left\langle V \,\middle|\, \bar{V}\right\rangle = -i$, we arrive in a straightforward and purely algebraic way to the well known formula

$$|Z^f|^2 = \frac{1}{2}\langle \mathcal{S}q \mid q\rangle\,, \tag{3.19}$$

[7]The quadratic form $g(X,X)$ is also known as the "$\mathcal{I}_2(X)$" in the literature. The corresponding quartic invariant in this case can be written as $\mathcal{I}_4(X,\ldots) = \langle X \mid X\rangle^2$.

[8]Following [35], we note that $V_{BH} = |Z_i|^2 + |Z|^2 = -\frac{1}{2}q^t\mathcal{S}(N)\omega q$ and $|Z_i|^2 - |Z|^2 = \frac{1}{2}q^t\mathcal{S}(F)\omega q$. At the fixed points, we have $Z_i = 0$, so that $|Z|^2 = -\frac{1}{2}q^t\mathcal{S}_N\omega q = -\frac{1}{2}q^t\mathcal{S}_F\omega q$. This last equation is satisfied by a solution of (2.35). Moreover the symmetric matrix $\omega\mathcal{S}$ is indefinite, it has positive and negative eigenvalues.

[9]These equations are well known in the literature, see for example section (5 ), eq. (319), in [39] and references therein.

which relates the absolute value of the central charge at any fixed point to a quadratic expression on the charge. It is obvious from (3.19) that the positivity of the quadratic form $g(q,q) = \langle \mathcal{S}q \mid q\rangle$ (at least locally at all the fixed points) is a necessary consistency condition for the existence of solutions to the attractor mechanism.

Moreover the consistency condition $e^{-\mathcal{K}} > 0$ can be written as (see (2.7))

$$\begin{aligned} e^{-\mathcal{K}} &= i\,\langle \Omega \mid \bar{\Omega}\rangle = 2\,\langle \mathrm{Re}\;(\Omega) \mid \mathrm{Im}\;(\Omega)\,\rangle \\ &= 2\,\langle \mathcal{S}\mathrm{Im}\;(\Omega) \mid \mathrm{Im}\;(\Omega)\,\rangle > 0. \end{aligned} \tag{3.20}$$

This condition is not automatically satisfied as the symmetric quadratic form $g$ is indefinite.[10] In addition to the symmetric bilinear form $g(X,Y)$, a Hermitian form $h$ of signature $(n_v, 1)$ can be defined from it:

$$h(X,Y) = \langle \mathcal{S}X \mid Y\rangle + i\,\langle X \mid Y\rangle\;, \tag{3.21}$$

which can be written in terms of the projection operators $\mathrm{P}_\pm$ as

$$\begin{aligned} h(X,Y) &= 2i\,\langle \mathrm{P}_- X \mid Y\rangle \\ &= 2i\,\langle \mathrm{P}_- X \mid \mathrm{P}_+ Y\rangle\;. \end{aligned} \tag{3.22}$$

The three defined structures $\{g, \omega, \mathcal{S}\}$ form a compatible triple, each structure can be specified by the two others.

### 3.1 $\mathcal{S}$ transformations and Freudenthal duality

Let us consider "$\mathcal{S}$-transformations" of the type

$$X \to X' = f(\mathcal{S})X,$$

where $f$ is an arbitrary function. $f(\mathcal{S})$ can be written with full generality as a linear expression

$$f(\mathcal{S}) \equiv a + b\mathcal{S}$$

or in "polar form"

$$f(\mathcal{S}) \equiv \lambda \exp(\theta\mathcal{S})$$

where $a, b$ or $\lambda, \theta$ are real parameters. The adjoint is $f(\mathcal{S})^\dagger = f(\mathcal{S}^\dagger) = f(-\mathcal{S})$, $f^\dagger f = a^2 + b^2 = \lambda^2$. Under these transformations $f = f(\mathcal{S})$ the symplectic and $g$ bilinear products (and then the Hermitian product $h$[11]) become scaled:

$$\langle X' \mid Y'\rangle = \langle fX \mid fY\rangle = \left\langle X \,\middle|\, f^\dagger f Y\right\rangle = \lambda^2\,\langle X \mid Y\rangle\,, \tag{3.23}$$

$$\langle \mathcal{S}X' \mid Y'\rangle = \langle \mathcal{S}fX \mid fY\rangle = \left\langle \mathcal{S}X \,\middle|\, f^\dagger f Y\right\rangle = \lambda^2\,\langle \mathcal{S}X \mid Y\rangle\,. \tag{3.24}$$

[10] The matrix $\omega\mathcal{S}$ it has an even number of negative eigenvalues as $\det \omega\mathcal{S} = (-1)^{2n_v+2} = 1$. The signature of $g$ is $(2n_v, 2)$.

[11] Or any other multilinear product built from them

If $\lambda = 1$ both products are invariant under the Abelian U(1)-like group of transformations

$$U_F(\theta) = \exp\theta\mathcal{S},$$

the "$\mathcal{S}$-rotations". Any physical quantity (entropy, ADM mass, scalars at fixed points, intercenter distances, etc.) written in terms of these products (as it will clearly appear in the next sections) will automatically be scaled under the general transformations or invariant under the rotations.

On the other hand it can be easily checked that the "degenerate" Freudenthal duality transformation [33, 34, 40, 41]. is given in our case by the anti-involutive transformation

$$\tilde{X} = \omega\frac{\partial g(X,X)}{\partial X} = \mathcal{S}X\,, \tag{3.25}$$

with $\tilde{\tilde{X}} = -X$.

The Freudenthal duality corresponds to a particular $\mathcal{S}$-transformation, a $\mathcal{S}$-rotation of the type

$$f(\mathcal{S}) = \exp((\pi/2)\mathcal{S}). \tag{3.26}$$

Invariance of quantities as ADM mass and entropy under Freudenthal duality is a special case of a more general behavior of the solutions under the (Abelian) group of general $\mathcal{S}$-transformations. A general $\mathcal{S}$-transformations can be written in terms of Freudenthal duality as

$$X \to \tilde{X}(a,b) = aX + b\tilde{X}, \tag{3.27}$$

or,

$$\tilde{X}(\lambda,\theta) = \lambda\cos\theta X + \lambda\sin\theta\tilde{X}. \tag{3.28}$$

### 3.2 Scalar fields at the fixed points

Let us turn now to the problem of obtaining the values of the moduli at the fixed points and at infinity. The values of the scalar fields at the fixed points can be computed by an explicit expression, which only involves the matrix $\mathcal{S}_F$. The fixed values of the $n_v$ complex scalars $z^\alpha_f(q)$ (at a generic fixed point with charge $q$) are given, using (2.33) and (3.16), by

$$\begin{aligned} z^\alpha_f(q) &= \frac{(\mathcal{S}\mathcal{I})^\alpha + i\mathcal{I}^\alpha}{(\mathcal{S}\mathcal{I})^0 + i\mathcal{I}^0} = \frac{((\mathcal{S}+i)\mathcal{I})^\alpha}{((\mathcal{S}+i)\mathcal{I})^0} \\ &= \frac{(\mathrm{P}_-q)^\alpha}{(\mathrm{P}_-q)^0}\,. \end{aligned} \tag{3.29}$$

That is, the fixed values of the scalars are given in terms of the projection of the charges on the eigenspaces of the matrix $\mathcal{S}$. For quadratic prepotentials, for which $\mathcal{S}$ is a constant, this is a complete and explicit solution of the attractor equations.



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The values of the $n_v$ complex scalars at spatial infinity, $|\mathbf{x}| \to \infty$, are given by[12]

$$z_\infty^\alpha = \lim_{|\mathbf{x}|\to\infty} \frac{(\mathrm{P}_-\mathcal{I})^\alpha}{(\mathrm{P}_-\mathcal{I})^0} = \frac{(\mathrm{P}_-\mathcal{I}_\infty)^\alpha}{(\mathrm{P}_-\mathcal{I}_\infty)^0}\,. \tag{3.30}$$

According to this formula, the 'moduli' $z_\infty^\alpha$ are simple rational functions of the $2n_v + 2$ real constant components of $\mathcal{I}_\infty$. They are thus independent of the fixed attractor values (3.29) (at least for an $\mathcal{I}$ of interest with only point like singularities, as for example the ansatz (2.34)).

We note that (3.30) is formally identical to (3.29), since they both give the values of the scalars at a fixed point in terms of the charges, where the roles of $\mathcal{I}_\infty$ and $q$ are exchanged. It is suggestive then to write an "effective attractor equation" at infinity, where the center charge is replaced by the vector $\mathcal{I}_\infty$. That is, the scalar solutions of the equation

$$\mathcal{I}_\infty = \mathrm{Re}\,\left(2i\bar{Z}V\right)\big|_\infty \tag{3.31}$$

are those ones precisely given by (3.30).

One can extract some algebraic relations for the vectors $\mathcal{I}_\infty$ and $q^a$ and the equations (3.29)–(3.30) in specific cases, for example for solutions with constant scalars. Let us assume $z_f = z_\infty (\neq 0)$. In this case, the equations (3.29)–(3.30) imply the projective equality ($\lambda$ an arbitrary real, non-zero, constant)

$$\mathrm{P}_-\mathcal{I}_\infty = \lambda \mathrm{P}_-q\,. \tag{3.32}$$

which, due to relation (3.4), implies

$$\mathcal{I}_\infty = \lambda q\,. \tag{3.33}$$

The asymptotic flatness condition (2.30) implies, in addition,

$$\lambda^2 = \frac{1}{\langle \mathcal{S}q \mid q\rangle} = \frac{1}{2|Z^f|^2}\,. \tag{3.34}$$

The consistency of the last equation is assured by the positivity of the quadratic form $\langle \mathcal{S}q \mid q\rangle$. Thus, we can finally arrive to a characterization of $\mathcal{I}_\infty$ in the case of constant scalar solutions

$$\mathcal{I}_\infty = \pm\frac{q}{\sqrt{\langle \mathcal{S}q \mid q\rangle}}\,. \tag{3.35}$$

Similar arguments can be stated in the multicenter case.

Let us finish this section with some qualitative remarks. We have arrived to the expressions (3.29)–(3.30) which can be written, in terms of the projective complex, vector $\Omega = (X^I, F_I)$, as

$$\Omega_{\text{fix}} = \mathrm{P}_-q\,, \tag{3.36}$$

$$\Omega_\infty = \mathrm{P}_-\mathcal{I}_\infty\,. \tag{3.37}$$

[12]Let us stress that we have used (2.33) and defined $\mathcal{I}_\infty \equiv \lim_{|\mathbf{x}|\to\infty}\mathcal{I}$, but we have not assumed any particular ansatz for $\mathcal{I}$ up to now.

We could have predicted these expressions a priori:[13] if SUSY solutions are uniquely determined by the symplectic real vectors $q_a$, then the also symplectic but complex vector $\Omega = (X^I, F_I)$ must be related to these vectors in any linear way, respecting symplectic covariance as well. Moreover, the symplectic sections $\Omega$ and $V$ lie in the subspace $W^-$, one eigenspace of the stabilization matrix $\mathcal{S}$. The only possibility for such relation would be the expressions (3.36)–(3.37), where the projections of $q$ or $\mathcal{I}_\infty$ on $W^-$ precisely appear. These expressions, evaluated at the points of maximal symmetry (the horizon and infinity), are equivalent forms of the standard horizon attractor equations and the generalized attractor equation at infinity presented here.

The scalars at fixed points are invariant not only under Freudenthal duality or $\mathcal{S}$-rotations but also under general $\mathcal{S}$-transformations of the corresponding charge vectors. This is clear taking into account equations (3.6) and (3.29). The same conclusion applies to the values of the scalars at infinity (see (3.30)) for transformed vectors $\mathcal{I}_\infty \to \tilde{\mathcal{I}}_\infty(\lambda, \theta)$.

## 4 Complete solutions for quadratic prepotentials

### 4.1 Behavior of the scalar field solutions

In the previous section we have obtained some general results without assuming a specific form for the solutions $\mathcal{I}$. In this section we will make use of the ansatz (2.34) for theories with quadratic prepotentials to obtain a full characterization of the solutions.

Let us insert the ansatz (2.34) into the general expression for the complex scalars, (2.33). The values for the time independent $n_v$ complex scalars, solutions to the field equations, are explicitly given by

$$z^\alpha(\mathbf{x}) = \frac{(\mathrm{P}\_\mathcal{I})^\alpha}{(\mathrm{P}\_\mathcal{I})^0} = \frac{(\mathrm{P}\_\mathcal{I}_\infty)^\alpha + \sum_a \frac{(\mathrm{P}_-q_a)^\alpha}{|\mathbf{x}-\mathbf{x}_a|}}{(\mathrm{P}\_\mathcal{I}_\infty)^0 + \sum_a \frac{(\mathrm{P}_-q_a)^0}{|\mathbf{x}-\mathbf{x}_a|}} \,. \tag{4.1}$$

This equation is a simple rational expression for the value of the scalar fields in the whole space. The fields and their derivatives are regular everywhere, including the fixed points (there could be singularities for special charge configurations which make zero the denominator of (4.1)).

The expression (4.1) interpolates between the values at the fixed points and at infinity. After some simple manipulations, it can be written as

$$z^\alpha(\mathbf{x}) = c^\alpha_\infty(\mathbf{x}) z^\alpha_\infty + \sum_a c^\alpha_a(\mathbf{x}) z^\alpha_{a,f} \,, \tag{4.2}$$

where $c^\alpha_\infty(\mathbf{x})$ and $c^\alpha_a(\mathbf{x})$ are spatial dependent complex functions such that

$$\begin{aligned} c^\alpha_\infty(\mathbf{x}) + c^\alpha_a(\mathbf{x}) &= 1 \,, \\ c^\alpha_\infty(\infty) &= 1 \,, \\ c^\alpha_\infty(\mathbf{x}_a) &= 0 \,, \\ \lim_{\mathbf{x}\to\mathbf{x}_b} c^\alpha_a(\mathbf{x}) &= \delta_{ab} \,. \end{aligned} \tag{4.3}$$

[13] Similar extended arguments are presented in [42] and references therein.

For a single center solution, we note that if $z^\alpha_\infty = z^\alpha_f$ then the scalar fields are constant in all space.

It is straightforward to see that the attractor mechanism is automatically fulfilled by the ansatz (2.34). The value of $z^\alpha$ at any center $\mathbf{x}_a$ is given, taking the corresponding limit in (4.1), by

$$z^\alpha(\mathbf{x}_a) = \frac{(\mathrm{P}_- q_a)^\alpha}{(\mathrm{P}_- q_a)^0} = z^\alpha_f(q_a)\,, \tag{4.4}$$

where, after the second equality, we have used the fixed point expression (3.29), which is a direct consequence of the attractor equations.

On the other hand, the solution at the spatial infinity recovers spherical symmetry. Again, taking limits, we have (with $|\mathbf{x}| \equiv r$)

$$\begin{aligned} z^\alpha_\infty = z^\alpha(r \to \infty) &= \frac{r(\mathrm{P}_-\mathcal{I}_\infty)^\alpha + (\mathrm{P}_- Q)^\alpha}{r(\mathrm{P}_-\mathcal{I}_\infty)^0 + (\mathrm{P}_- Q)^0} \\ &= (1 - c(r))z^\alpha_\infty + c(r) z^\alpha_f(Q)\,, \end{aligned} \tag{4.5}$$

where $z_f(Q)$ is the fixed point scalar value which would correspond, according to the attractor equations, to a total charge $Q \equiv \sum_a q_a$. The asymptotically interpolating function appearing above is unique for all the scalars

$$c(r) = \frac{1}{1 + \frac{r}{r_0}}\,, \tag{4.6}$$

with the (assumed non-zero) scale parameter

$$r_0 = \frac{(\mathrm{P}_- Q)^0}{(\mathrm{P}_-\mathcal{I}_\infty)^0}\,. \tag{4.7}$$

They are such that

$$c(0) = 1\,, \quad c(\infty) = 0\,. \tag{4.8}$$

The *scalar charges* $\Sigma^\alpha$ associated to the scalar fields can be defined by the asymptotic series

$$z^\alpha(r \to \infty) = z^\alpha_\infty + \frac{\Sigma^\alpha}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\,. \tag{4.9}$$

Expanding (4.5), we have

$$z^\alpha(r \to \infty) = z^\alpha_\infty + \frac{r_0(z^\alpha_f(Q) - z^\alpha_\infty)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\,, \tag{4.10}$$

and thus the scalar charges are given by

$$\Sigma^\alpha = r_0\left(z^\alpha_f(Q) - z^\alpha_\infty\right)\,. \tag{4.11}$$

Hence, the scalar charges are fixed in terms of the charge vectors and the asymptotic moduli. In the special case of a single center solution, the expression (4.11) is in agreement with the well known fact that the scalar charges vanish for double extremal black holes. In the multicenter case, from this formula we infer a similar result: the scalar charges vanish if

$$z^{\alpha}_{\infty} = z^{\alpha}_{f}(Q)\,. \tag{4.12}$$

Obviously, this does not mean that the scalar fields are constant in all space. Therefore, the conditions (4.12) could be considered a convenient generalization of double extremal solutions in the multicenter case. Taking into account the considerations of the previous section, (3.35), a possible vector $\mathcal{I}_{\infty}$ corresponding to this solution would be of the form

$$\mathcal{I}_{\infty} = \pm\frac{Q}{\sqrt{\langle \mathcal{S}Q \mid Q\rangle}}\,, \tag{4.13}$$

and the scalar fields would be parametrized at any point of the space by

$$z^{\alpha}(\mathbf{x}) = c^{\alpha}_{\infty}(\mathbf{x})z^{\alpha}_{f}(Q) + \sum_{a} c^{\alpha}_{a}(\mathbf{x})z^{\alpha}_{f}(q_a)\,. \tag{4.14}$$

### 4.2 Intercenter distances and $\mathcal{S}$-transformations

The charge interdistances are restricted, from the 1-form $\omega$ condition of integrability [22], we have (for any charge center $q_b$ )

$$\langle \mathcal{I}_{\infty} \mid q_b\rangle + \sum_{a} \frac{\langle q_a \mid q_b\rangle}{r_{ab}} = 0\,, \tag{4.15}$$

where $r_{ab} = |\mathbf{x}_a - \mathbf{x}_b|$. The solutions for this set of equations give the possible values of the center positions.

Let us study the effect of $\mathcal{S}$-transformations on the intercenter distances for transformed $\mathcal{I}_{\infty}$ and charge symplectic vectors. The vector $\mathcal{I}_{\infty}$ is constrained by the asymptotic flatness condition to a unit fixed $g$-norm, $1 = \langle \mathcal{S}\mathcal{I}_{\infty} \mid \mathcal{I}_{\infty}\rangle$. We consider therefore set of transformations of the type

$$\mathcal{I}_{\infty} \to \tilde{\mathcal{I}}_{\infty}(\theta) = \exp(\theta\mathcal{S})\mathcal{I}_{\infty}, \tag{4.16}$$

$$q_a \to \tilde{q}_a(\lambda,\theta) = \lambda\exp(\theta\mathcal{S})q_a. \tag{4.17}$$

Under these transformations the equations (4.15) become

$$\lambda\,\langle \mathcal{I}_{\infty} \mid q_b\rangle + \lambda^2 \sum_{a} \frac{\langle q_a \mid q_b\rangle}{\tilde{r}_{ab}} = 0. \tag{4.18}$$

Then

$$r_{ab} \to \tilde{r}_{ab} = \lambda r_{ab}, \tag{4.19}$$

the intercenter distances scale (remain invariant) under general $\mathcal{S}$-transformations ($\mathcal{S}$-rotations or Freudenthal dualities) of the charge and $\mathcal{I}_{\infty}$ vectors.



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Let us see the consequence of the integrability equations for a double extremal two center configuration. In this case, if $\mathcal{I}_\infty = \lambda Q$, we have

$$\begin{aligned} 0 &= \lambda \langle Q \mid q_2 \rangle + \frac{\langle q_1 \mid q_2 \rangle}{r_{12}} \\ &= \lambda \langle q_1 \mid q_2 \rangle + \frac{\langle q_1 \mid q_2 \rangle}{r_{12}} \\ &= \langle q_1 \mid q_2 \rangle \left( \lambda + \frac{1}{r_{12}} \right) . \end{aligned} \tag{4.20}$$

If we compare this equation with (4.13), we conclude that, if $\langle q_1 \mid q_2 \rangle \neq 0$, we have to choose the negative sign there and the double extremal intercenter distance is given by

$$r_{12}^2 \big|_{\text{double ext.}} = \langle \mathcal{S} Q \mid Q \rangle \ . \tag{4.21}$$

In the case $\langle q_1 \mid q_2 \rangle = 0$ the intercenter distance is not restristed by the compatibility equation eq. (4.15).

### 4.3 Near horizon and infinity geometry

Let us now study the gravitational field. The metric has the form given by (2.23), with the asymptotic flatness conditions $-g_{rr} = \langle \mathcal{R}_\infty \mid \mathcal{I}_\infty \rangle = 1$ and $\omega(|\mathbf{x}| \to \infty) \to 0$. For point-like sources, as those represented by the ansatz (2.34), the compatibility equation (2.29) takes the form [22]

$$N \equiv \sum_a \langle \mathcal{I}_\infty \mid q_a \rangle = \langle \mathcal{I}_\infty \mid Q \rangle = 0 \, . \tag{4.22}$$

An explicit computation of the total field strength shows that (4.22) is equivalent to the requirement of absence of NUT charges: only after imposing the condition $N = 0$, the overall integral of the $(F^I, G_I)$ field strengths at infinity is equal to $Q = \sum q_a$. Another consequence of the condition $N = 0$, which can be checked by direct computation from (2.28), is that the 1-form $\omega$ takes the same value at any of the horizons of the centers that make up the multicenter black hole. This value is also equal to its value at spacial infinity, which can be taken to be zero.

Let us write a more explicit expression for the $g_{rr}$ component at any space point. We can write, using the 'stabilization equation' (2.26) and the ansatz (2.34), the expression

$$\begin{aligned} \langle \mathcal{R} \mid \mathcal{I} \rangle &= \left\langle \mathcal{S}\mathcal{I}_\infty + \sum_a \frac{\mathcal{S} q_a}{|\mathbf{x} - \mathbf{x}_a|} \,\middle|\, \mathcal{I}_\infty + \sum_b \frac{q_b}{|\mathbf{x} - \mathbf{x}_b|} \right\rangle \\ &= 1 + \sum_b \frac{1}{|\mathbf{x} - \mathbf{x}_b|} \left( \langle \mathcal{S}\mathcal{I}_\infty \mid q_b \rangle + \langle \mathcal{S} q_b \mid \mathcal{I}_\infty \rangle \right) + \sum_{a,b} \frac{\langle \mathcal{S} q_a \mid q_b \rangle}{|\mathbf{x} - \mathbf{x}_a||\mathbf{x} - \mathbf{x}_b|} \\ &= 1 + 2 \sum_b \frac{\langle \mathcal{S}\mathcal{I}_\infty \mid q_b \rangle}{|\mathbf{x} - \mathbf{x}_b|} + \sum_{a,b} \frac{\langle \mathcal{S} q_a \mid q_b \rangle}{|\mathbf{x} - \mathbf{x}_a||\mathbf{x} - \mathbf{x}_b|} \, , \end{aligned} \tag{4.23}$$

where we have used the property $\mathcal{S}^\dagger = -\mathcal{S}$ and the asymptotic flatness condition $\langle \mathcal{S}\mathcal{I}_\infty \mid \mathcal{I}_\infty\rangle = 1$. We introduce now the quantities

$$M_a \equiv \langle \mathcal{S}\mathcal{I}_\infty \mid q_a\rangle \ , \tag{4.24}$$
$$A_{ab} \equiv \langle \mathcal{S} q_a \mid q_b\rangle \ , \tag{4.25}$$

where $A_{ab}$ is symmetric in its indices due to the property (3.14).

With these definitions, we finally write the expression for the metric element as

$$\begin{aligned} -g_{rr} &= \langle \mathcal{R} \mid \mathcal{I}\rangle \\ &= 1 + 2\sum_a \frac{M_a}{|\mathbf{x}-\mathbf{x}_a|} + \sum_{a,b} \frac{A_{ab}}{|\mathbf{x}-\mathbf{x}_a||\mathbf{x}-\mathbf{x}_b|} \,. \end{aligned} \tag{4.26}$$

If the metric element (4.26) describes a black hole, then the right part should be kept always positive and finite for any finite $|\mathbf{x}|$.[14] A sufficient condition for its positivity is, for example, that the mass-like $M_a$ and area-like $A_{ab}$ parameters are all positive.

**Behavior at fixed points and at infinity.** We will define new quantities from the behavior of the metric at infinity: the mass $M_{\text{ADM}}$ and $A_\infty$. At spatial infinity $|\mathbf{x}| \to \infty$, $1/|\mathbf{x}-\mathbf{x}_a| \to 1/r$, the metric element (4.26) becomes spherically symmetric:

$$\begin{aligned} -g_{rr} &= 1 + \frac{2\sum_a M_a}{r} + \frac{\sum_{ab} A_{ab}}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \\ &\equiv 1 + \frac{2M_{\text{ADM}}}{r} + \frac{A_\infty}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \,. \end{aligned} \tag{4.27}$$

The second equation defines $M_{\text{ADM}}$ and $A_\infty$. Comparing both expressions and using (4.24), (4.25), we have

$$M_{\text{ADM}} = \sum_a M_a = \langle \mathcal{S} I_\infty \mid Q\rangle \ , \tag{4.28}$$
$$A_\infty = \sum_{ab} A_{ab} = \langle \mathcal{S} Q \mid Q\rangle \ . \tag{4.29}$$

The expression for the central charge at infinity, (2.32), becomes then

$$|Z_\infty|^2 = |\ \langle \mathrm{P}_+ \mathcal{I}_\infty \mid Q\rangle\ |^2 = |\ N + iM\ |^2 \tag{4.30}$$
$$= M_{\text{ADM}}^2 + N^2 \tag{4.31}$$

where $N$ is defined by (4.22). The compatibility condition $N = 0$ is equivalent to the saturation of a BPS condition

$$|Z_\infty|^2 = M_{\text{ADM}}^2 = |\ \langle \mathcal{S}\mathcal{I}_\infty \mid Q\rangle\ |^2 \,. \tag{4.32}$$

Unlike $A_\infty$, the $M_{\text{ADM}}$ quantity depends on the values of the scalars at infinity through the implicit dependence of $\mathcal{I}_\infty$ on them. These can take arbitrary values or, at least, can

[14] Consider, for example, that $-g_{rr} \sim e^{-\mathcal{K}} > 0$.

be chosen in a continuous range. In the single center case, for any given charge vector, one can obtain a certain particular solution by setting the scalar fields to constant values ($z_f^\alpha = z_\infty^\alpha$), giving this the minimal possible $M_{\rm ADM}$ mass [10]. For multicenter solutions and generic non-trivial charge vectors, it is not possible to have constant scalar fields. Nevertheless, we can still proceed to the extremization of $M_{\rm ADM}(z_\infty^\alpha)$, with respect to the scalar fields at infinity for a given configuration.

Let us suppose a configuration with null scalar charges. In this case $\mathcal{I}_\infty = \pm\lambda Q$, $\lambda = 1/\sqrt{\langle \mathcal{S}Q \mid Q\rangle}$. We have $M_{\rm ADM} = \pm 1/\lambda$, the positivity of $M_{\rm ADM}$ obliges us to choose the positive sign. For a two center case this in turn implies $\langle q_1 \mid q_2\rangle = 0$ and $r_{12}$ unrestricted (see section (4.2)). Such $\mathcal{I}_\infty$ trivially satisfies the absence of NUT charge ($N = 0$) condition, and for it $z_\infty^\alpha = z_f(Q)$. This configuration can be considered a multicenter generalization of the double extremal solutions.

Let us proceed now with the study of the geometry near the centers. For $\mathbf{x} \to \mathbf{x}_a$ the metric element given by (4.26) becomes spherically symmetric. Moreover, it can be shown that, by fixing additive integration constants, we can take $\omega_a = \omega(\mathbf{x} \to \mathbf{x}_a) = 0$ at the same time that $\omega_\infty = \omega(\mathbf{x} \to \infty) = 0$. As a consequence, the metric at any of the horizon components with charge $q_a$ approaches an $AdS_2 \times S^2$ metric of the form

$$ds^2 = \frac{r^2}{\langle \mathcal{S}q_a \mid q_a\rangle} dt^2 - \frac{\langle \mathcal{S}q_a \mid q_a\rangle}{r^2} d\mathbf{x}^2 \,. \tag{4.33}$$

This is a Robinson-Bertotti-like metric. The Robertson-Bertotti-like mass parameter $M_{RB}$ is given by

$$M^2_{RB,a} = \langle \mathcal{S}q_a \mid q_a\rangle \,, \tag{4.34}$$

this is a charge extremal condition impliying the positivity of the charge products: $\langle \mathcal{S}q_a \mid q_a\rangle > 0$.[15]

Then, the near horizon geometry is completely determined in terms of the individual horizon areas $S_{h,a} = \langle \mathcal{S}q_a \mid q_a\rangle$. The horizon area $S_h$ is the sum of the areas of its disconnected parts

$$S_h = \sum_a S_{h,a} = \sum_a \langle \mathcal{S}q_a \mid q_a\rangle = 2\sum_a |Z_{f,a}|^2 \,. \tag{4.35}$$

This expression can be compared with the area corresponding to a single center black hole with the same total charge $Q = \sum_a q_a$, which is given by $S_h(q = Q) = \langle \mathcal{S}Q \mid Q\rangle$.

The relation between the asymptotic "area" $A_\infty$ and the multicenter horizon area, or horizon entropy $S_h$, is simply

$$\begin{aligned} A_\infty &= \langle \mathcal{S}Q \mid Q\rangle = \sum_{a,b} \langle \mathcal{S}q_a \mid q_b\rangle \\ &= S_h + 2\sum_{a<b} \langle \mathcal{S}q_a \mid q_b\rangle \,. \end{aligned} \tag{4.36}$$

[15]The positivity of these quantities implies diverse restrictions as the quadratic form $g(X,Y) = \langle \mathcal{S}X \mid Y\rangle$ is undefinite with a signature including an even number of negative signs.

For one center solution we always have $A_\infty = S_h$. For example, in the case of two centers with charges $q_1$, $q_2$, the difference is[16]

$$A_\infty - S_h = 2\,\langle \mathcal{S} q_1 \mid q_2 \rangle > 0\,. \tag{4.37}$$

Let us finally remark that under $\mathcal{S}$-transformations $\mathcal{I}_\infty \to \tilde{\mathcal{I}}_\infty(\theta)$, $q_a \to \tilde{q}_a(\lambda,\theta)$ the $ADM$ mass and the horizon areas scale as

$$\tilde{M}_{\rm ADM} = \lambda M_{\rm ADM}\,, \tag{4.38}$$

$$\tilde{S}_h = \lambda^2 S_h\,. \tag{4.39}$$

Under the same transformations, the scalars at the fixed points and at infinity remain invariant whereas the intercenter distances $r_{ab}$ scale as (4.19).

## 5 Freudenthal duals and charge vector expansions

It is well known the utility of the use of the section $V$, its derivatives $D_\alpha V$ and their complex conjugates as a basis for the symplectic space. Any real symplectic vector $X$ can be expanded as

$$X = 2{\rm Im}\left( Z(X)\bar{V} + g^{\alpha\bar{\beta}} D_\alpha Z(X) \bar{D}_{\bar{\beta}} \bar{V}\right)$$

with $Z(X) = \langle V \mid X\rangle$. The existence and properties of such expansions are based on the symplectic properties of $V$ and its derivatives as well as on the existence of an anti-involution $\mathcal{S}(N)$ for which $\mathcal{S}(N)V = iV$ and $\mathcal{S}(N)D_\alpha V = iD_\alpha V$.[17]

We will define here alternative expansions using the properties of the matrix $\mathcal{S} \equiv \mathcal{S}(F)$. As we have seen before, the projectors $\mathrm{P}_\pm$ split the $(2n_v+2)$-dimensional space $W$ into two $(n_v+1)$-dimensional eigenspaces

$$W = W^+ \oplus W^-\,,$$

in which the eigenvectors of $\mathcal{S}$ (therefore eigenvectors of general $\mathcal{S}$-transformations) with eigenvalues $\pm i$, respectively, lie (cf. section 3).

Given a set of generic real charge vectors $(q_1,\ldots,q_n)$, the sets $(\mathrm{P}_+ q_a)$, respectively $(\mathrm{P}_- q_a)$, possibly completed with additional suitable vectors, can be considered a basis for

[16] In ref. [7] it has been shown that, for quadratic prepotentials, the single center BPS extremal black hole area with charge $Q = q_1 + q_2$ is always larger than the corresponding two-center area

$$S_h(Q = q_1 + q_2) \geq S_{h,1} + S_{h,2}\,.$$

Or equivalently, taking into account that $A_\infty$ is also the area of the corresponding single center black hole with the same total charge $A_{\infty\, q1,q2} = S_h(Q = q_1 + q_2)$, we have

$$A_\infty - S_h = 2\,\langle \mathcal{S} q_1 \mid q_2 \rangle \geq 0\,.$$

[17] See for example section 2.2.2 in [43] and references therein.

the eigenspaces $W^+$, respectively $W^-$. Let us consider the $W$ subspace $B(q_n)$ generated by eigenvectors of the matrix $\mathcal{S}$ associated to center charges, directly of the complex form

$$B(q_n) \equiv \mathrm{Span}(\mathrm{P}_\pm q_1, \ldots, \mathrm{P}_\pm q_n)\,, \tag{5.1}$$

or, equivalently, in the real basis formed by charge vectors and their Freudenthal duals $\tilde{q}_i = \mathcal{S} q_i$

$$B(q_n) \equiv \mathrm{Span}(q_1, \ldots, q_n, \mathcal{S}q_1, \ldots, \mathcal{S}q_n)\,. \tag{5.2}$$

In particular, we can consider the subspace $B(q_{na})$ generated by the $n_a$ pairs $(q_a,\, \mathcal{S}q_a)$ of center charges, whose dimension is, in general, $\dim B(q_{na}) \leq 2n_a$. The dimension of the orthogonal complement to this space, $B(q_{na})^\perp$, i.e. those vectors $s$ such that $\langle q \mid s\rangle = \langle \mathcal{S}q \mid s\rangle = 0$ is, generically, $\dim B(q_{na})^\perp = 2(n_v - n_a) + 2$.[18] This dimension is zero for one scalar, one center black holes ($n_v = 0, n_a = 1$). The set of vectors $(q_a,\, \mathcal{S}q_a)$ may form themselves a (maybe overcomplete) basis for the $(2n_v + 2)$ symplectic space. Otherwise, they can be extended with as many other vectors $(s_i)$ as necessary to complete such basis. Any real symplectic vector of interest (e.g. $\mathcal{I}_\infty$) can be conveniently expanded as

$$X \,=\, 2\mathrm{Re}\,\alpha^a \mathrm{P}_+ q_a + 2\mathrm{Re}\,\gamma^i \mathrm{P}_+ s_i\,, \tag{5.3}$$

where $\alpha^a, \gamma^i$ are complex parameters or, equivalently, as

$$X \,=\, \alpha^a q_a + \tilde{\alpha}^a \mathcal{S} q_a + \gamma^i s_i + \tilde{\gamma}^i \mathcal{S} s_i\,, \tag{5.4}$$

where $\alpha^a, \tilde{\alpha}^a, \gamma^i, \tilde{\gamma}^i$ are in this case real parameters.[19] Let us note that under this same expansion the dual vector $\tilde{X} = \mathcal{S}X$ has respectively complex components $(-i\alpha, \ldots)$ or real ones $(-\tilde{\alpha}^a, \alpha^a, \ldots)$.

We can use expansions of different quantities in such a basis formed by charge and extra vectors, to get different results. In a simple illustrative case, by decomposition of the $\mathcal{I}_\infty$ vector, we will study different properties. In particular, we will see how the extremality of the solutions imposes strong conditions on such extra vectors.

### 5.1 Decomposition of $\mathcal{I}_\infty$ and double extremality

We will decompose now the vector $\mathcal{I}$ into a basis of charge and extra vectors. For the sake of simplicity we will discuss here the case of a single center solution and one complex scalar. The dimension of the symplectic space is $2n_v + 2 = 4$. We will see, in particular, how the extremality of the solutions imposes strong conditions on such extra vectors. In addition, we will show, using this decomposition, the double extremality of the black hole solutions for quadratic prepotentials.

Let us decompose the vector $\mathcal{I}_\infty$ in the following way (with $\langle \mathcal{S}q \mid q\rangle \neq 0$):

$$\mathcal{I}_\infty \,=\, \alpha q + \beta \mathcal{S} q + \gamma s + \epsilon \mathcal{S} s\,, \tag{5.5}$$

[18] Or equivalently, $B(q_{na})^\perp$ is defined as the set of vectors $s$ such that $h(s, q) = 0$ for all $q \in (q_{na})$, where $h$ is the Hermitian inner product defined in section 3.

[19] Naturally, other bases are possible or convenient, as for example bases including linear combinations of the charge vectors, the total charge vector $Q$, $\mathcal{I}_\infty$, etc.

where $\alpha, \beta, \gamma, \epsilon \in \mathbb{R}$ and $s \in B(q_{na}, \mathcal{S}q_{na})^{\perp}$ is arbitrary. The bilinear form $g(X,Y) = \langle X \mid Y \rangle$ is indefinite, it has a signature with an even number of minus signs. A physical requirement is that $A_\infty = \langle \mathcal{S}q \mid q \rangle > 0$ that implies that, if we choose

$$\langle s \mid q \rangle = \langle s \mid \mathcal{S}q \rangle = 0 \tag{5.6}$$

then[20] we are obliged to choose

$$\langle \mathcal{S}s \mid s \rangle = -1. \tag{5.7}$$

This vector $s$ can be always determined by a modified Gram-Schmidt procedure for a given pair of vectors $(q, \mathcal{S}q)$. By projecting the relation (5.5) over any of the individual vectors $(q, \mathcal{S}q)$, we get

$$\langle \mathcal{I}_\infty \mid q \rangle = \beta \langle \mathcal{S}q \mid q \rangle \ , \tag{5.8}$$

$$\langle \mathcal{I}_\infty \mid \mathcal{S}q \rangle = -\alpha \langle \mathcal{S}q \mid q \rangle \ . \tag{5.9}$$

Using (4.22), (4.28) and (4.29), we can rewrite these last two expressions respectively as

$$N = \beta A_\infty \,, \tag{5.10}$$

$$M_{\text{ADM}} = -\alpha A_\infty \,, \tag{5.11}$$

from which we read the coefficients $\alpha$, $\beta$ in terms of some other, more physical, parameters. The condition $N = 0$ implies that $\beta = 0$, hence the $\mathcal{I}_\infty$ vector does not contain any component in the "$\mathcal{S}q$" direction.

Let us consider now the asymptotic flatness condition and apply the ansatz (5.5) for $\mathcal{I}_\infty$, but without imposing at this moment the $N = 0$ condition. If we define $\Delta^2 \equiv \gamma^2 + \epsilon^2$ and make use of (4.31) and the values for $\alpha$, $\beta$, we have

$$\begin{aligned} 1 &= \langle \mathcal{S}\mathcal{I}_\infty \mid \mathcal{I}_\infty \rangle \\ &= \left(\alpha^2 + \beta^2\right) \langle \mathcal{S}q \mid q \rangle + \left(\gamma^2 + \epsilon^2\right) \langle \mathcal{S}s \mid s \rangle \\ &= \frac{M_{\text{ADM}}^2 + N^2}{A_\infty{}^2} \langle \mathcal{S}q \mid q \rangle - \Delta^2 \,, \end{aligned} \tag{5.12}$$

or equivalently,

$$|Z_\infty|^2 = M_{\text{ADM}}^2 + N^2 = \langle \mathcal{S}q \mid q \rangle \left(1 + \Delta^2\right). \tag{5.13}$$

The BPS condition $|Z_\infty| = M_{\text{ADM}} = \langle \mathcal{S}q \mid q \rangle$ is only fulfilled if $N = 0$ (in agreement with (4.32)) and the additional condition $\Delta = 0$.

The vanishing of these quantities can be directly seen by imposing extremality in the metric elements, by requesting extremal RN black hole type metric or, $-g_{rr} \sim f^2$ with $f$ an spatially harmonic function. The metric component $g_{rr}$ is

$$\begin{aligned} -g_{rr} &= 1 + \frac{2M_{\text{ADM}}}{r} + \frac{\langle \mathcal{S}q \mid q \rangle}{r^2} \\ &= 1 + \frac{2M_{\text{ADM}}}{r} + \frac{(M_{\text{ADM}}^2 + N^2)/(1+\Delta^2)}{r^2} \\ &= \left(1 + \frac{M_{\text{ADM}}}{r}\right)^2 + \frac{1}{r^2}\frac{1}{1+\Delta^2}\left(M_{\text{ADM}}^2\Delta^2 + N^2\right) . \end{aligned} \tag{5.14}$$

[20]By a simple application of the Silverster inertia theorem.

The metric element is of the form $-g_{rr} \sim f^2$ with $f$ an spatially harmonic function if and only if the second term of the previous expression is zero, that is, if and only if

$$M_{\mathrm{ADM}}^2 \Delta^2 + N^2 = 0\,. \tag{5.15}$$

Thus, the conditions $N = 0$ and $\Delta = 0$ (which is equivalent to $\gamma = \epsilon = 0$ in (5.5)) are necessary conditions to recover an extremal RN black hole type metric. In this case, the central charge at infinity is

$$|Z_\infty|^2 = M_{\mathrm{ADM}}^2 = \langle \mathcal{S}q \mid q \rangle\ . \tag{5.16}$$

We see that the vanishing of the non-extremality parameter $\Delta$ is equivalent to require that $\mathcal{I}_\infty$ is fully contained in the subspace $\mathrm{Span}(q, \mathcal{S}q)$, whereas the condition $N = 0$ further restricts it to be proportional to the vector charge $\mathcal{I}_\infty = q/M_{\mathrm{ADM}}$. In this case, after imposing the conditions $N = \Delta = 0$, we can finally write

$$\mathcal{I} = \frac{q}{M_{\mathrm{ADM}}} \left(1 + \frac{M_{\mathrm{ADM}}}{r}\right)\ . \tag{5.17}$$

As a consequence of having $\mathcal{I}_\infty = q/M_{\mathrm{ADM}}$ the scalar fields $z^\alpha$ are constant everywhere and equal to their values at the fixed point (see (3.30) and the discussion in section 4). It might be interesting to remark that in this expression the "unphysical" vector $\mathcal{I}$ appears written in terms of the physical quantities $q$ and $M_{\mathrm{ADM}}$ which can be chosen by hand from the beginning.

## 6 Summary and concluding remarks

We have presented a systematic study of general, stationary, multicenter black hole solutions in $N = 2$ $D = 4$ Einstein-Maxwell supergravity theories minimally coupled to scalars, i.e. theories with quadratic prepotentials. We have assumed a generic multicenter ansatz (2.34), which depends on $q_a$ and $\mathcal{I}_\infty$.

This analysis is heavily based on the use of the algebraic properties of the anti-involutive matrix $\mathcal{S}$, the constant matrix of second derivatives of the prepotential of the theory and of the, defined in this work, symplectic adjoint $\mathcal{S}^\dagger$. They are "unitary", $\mathcal{S}\mathcal{S}^\dagger = 1$, with respect to the symplectic product. The matrix $\mathcal{S}$ defines a complex structure on the $(2n_v + 2)$-dimensional symplectic space.

By defining suitable projector operators $\mathrm{P}_\pm$, the symplectic $(2n_v{+}2)$-dimensional space is decomposed into eigenspaces of $\mathcal{S}$, $W = W^+ \oplus W^-$. We show that any symplectic section, whose real and imaginary parts are related $\mathrm{Re}\,(X) = \mathcal{S}\mathrm{Im}\,(X)$, lies in the subspace $W^-$. With the help of these projector operators, we write a purely algebraic expression for the attractor equations, which equalizes the symplectic section $V \in W^-$ to the projection of the corresponding charge vector on that subspace, $\mathrm{P}_- q^a \in W^-$ (3.16)–(3.17). The modulus of the central charge function is given in terms of the norm of a charge vector, which is written in terms of the inner product $g$, (3.19).

We obtain expressions for the scalar fields evaluated at the fixed points (3.29) and at infinity (3.30). They are given, in a similar way, in terms of the projections of the center



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charges vectors $q_a$ and $\mathcal{I}_\infty$ on $W^-$, respectively. The values of the $n_v$ complex scalars at spatial infinity are given by (3.30)

$$z^\alpha_\infty = \lim_{|\mathbf{x}|\to\infty} \frac{(\mathrm{P}\_\mathcal{I})^\alpha}{(\mathrm{P}\_\mathcal{I})^0} = \frac{(\mathrm{P}\_\mathcal{I}_\infty)^\alpha}{(\mathrm{P}\_\mathcal{I}_\infty)^0} \,. \tag{6.1}$$

This is an explicit formula where the moduli $z^\alpha_\infty$ are simple rational functions of the $2n_v+2$ real constant components of $\mathcal{I}_\infty$.

We write expressions for the scalar field solutions at any space point (4.2) in terms of $q_a$ and $\mathcal{I}_\infty$ (4.7). They are interpolating expressions between the fixed point values and moduli values at infinity. In particular, the formalism allows us to easily study a configuration analogous to the *double extremal* case in a multicenter scenario: configurations such that $z^\alpha_\infty = z^\alpha_f(Q)$, with $Q$ the total charge. The vanishing of the scalar charges is shown to be equivalent to this condition. This is in close analogy with the single center case, in which the vanishing of the scalar charges is a necessary and sufficient condition for the double extremality of the black hole [10].

We have written the metric element $-g_{rr}$ in terms of area-like $A_{ab}$ and mass-like quantities $M_a$ (4.26) involving the bilinear product $g$. The study of the near horizon and infinity geometry of the solution lead us to the consideration of the area-like quantities $A_{ab} = \langle \mathcal{S}q_a \mid q_b\rangle$ and $A_\infty = \sum_{ab} A_{ab} = \langle \mathcal{S}Q \mid Q\rangle$, in addition to the horizon areas $S_{h,a} = \langle \mathcal{S}q_a \mid q_a\rangle$.

In section 5 we have proposed a decomposition of the $(2n_v+2)$-dimensional symplectic vector space in a basis of eigenvectors of the matrix $\mathcal{S}$. This set of vectors are of the form $(\mathrm{P}_\pm q_a)$, or, alternatively, $(q_a, \mathcal{S}q_a)$, with $\mathrm{P}_\pm$ projectors over the eigenspaces of $\mathcal{S}$ and $q_a$ the center vector charges. Any real symplectic vector of interest (e.g. $\mathcal{I}_\infty$) can be conveniently expanded as ($\alpha^a, \gamma^i$ are complex parameters )

$$X \;= 2\mathrm{Re}\, \alpha^a \mathrm{P}_+ q_a + 2\mathrm{Re}\, \gamma^i \mathrm{P}_+ s_i \,, \tag{6.2}$$

or as ( real $\alpha^a, \tilde\alpha^a, \gamma^i, \tilde\gamma^i$) $X = \alpha^a q_a + \tilde\alpha^a \mathcal{S} q_a + \gamma^i s_i + \tilde\gamma^i \mathcal{S} s_i$. Some simple properties of the solutions are studied using this decomposition. The decomposition can be seen as an alternative to the well known expansions in terms of the section $V$, its derivatives $D_\alpha V$ and their complex conjugates as a basis for the symplectic space. A formalism which allows that any real symplectic vector $X$ can be expanded as $X = 2\mathrm{Im}\left(Z(X)\bar V + g^{\alpha\bar\beta} D_\alpha Z(X) \bar D_{\bar\beta} \bar V\right)$

The anti-involution matrix $\mathcal{S}$ can be understood as a Freudenthal duality $\tilde q = \mathcal{S}q$ [33, 34]. Under this transformation of the charges the horizon area, ADM mass and other properties of the solutions remain invariant. We have shown, for the quadratic prepotential theories studied here, that this duality can be generalized to an Abelian group of transformations ("Freudenthal transformations") of the form

$$x \to \lambda \exp(\theta \mathcal{S}) x = ax + b\tilde x.$$

Under this set of transformations applied to the charge vectors and $\mathcal{I}_\infty$, the horizon area, ADM mass and intercenter distances scale up, respectively, as

$$S_h \to \lambda^2 S_h, \quad M_{\mathrm{ADM}} \to \lambda M_{\mathrm{ADM}}, \quad r_{ab} \to \lambda r_{ab}, \tag{6.3}$$



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leaving invariant the values of the scalars at the fixed points and at infinity. In the special case $\lambda = 1$, "$\mathcal{S}$-rotations", the transformations leave invariant the solution. The standard Freudenthal duality can be written as the particular rotation

$$\tilde{x} = \exp(\pi/2\mathcal{S})x.$$

It is immediate to ask the question whether such transformations can be generalized to 4$d$ theories with general prepotentials, not associated to "degenerate" U-duality groups, including stringy black holes. We can see that this is indeed the case using a simple argument as follows (a more detailed investigation is presented in [44]). The U-duality quartic invariant defined ([33], using here a slightly adapted notation ) as

$$2\Delta_4(x) \equiv \langle T(x) \mid x \rangle$$

can be written also, using the definition of Freudenthal duality, as

$$\Delta_4(x) = \frac{1}{4} \langle \tilde{x} \mid x \rangle^2 .$$

Let us note then that, for a general transformation this quantity scale as

$$\begin{aligned} 2\Delta_4(ax + b\tilde{x})^{1/2} &= \left\langle \widetilde{ax + b\tilde{x}} \,\middle|\, ax + b\tilde{x} \right\rangle = \langle a\tilde{x} - bx \mid ax + b\tilde{x} \rangle & (6.4)\\ &= (a^2 + b^2) \langle \tilde{x} \mid x \rangle & (6.5)\\ &= 2(a^2 + b^2)\Delta_4(x)^{1/2}. & (6.6) \end{aligned}$$

For $a^2 + b^2 = 1$, a $\mathcal{S}$-rotation, the quantity $\Delta_4$ for any $U$-duality group, and then the lowest order entropy of any extremal stringy black hole, is invariant under these transformations.

Moreover the invariance of $\Delta_4$ is shown [44] to be equivalent to the conditions

$$\begin{aligned} \Delta_4(x, x, \tilde{x}, \tilde{x}) &= \frac{1}{3}\Delta_4(x), & (6.7)\\ \Delta_4(x, \tilde{x}, \tilde{x}, \tilde{x}) &= \Delta_4(x, x, x, \tilde{x}) = 0. & (6.8) \end{aligned}$$

For the special case of $D = 4$ theories with U-duality groups of "degenerate type $E_7$" such conditions (6.7)–(6.8) can be easily checked by an explicit computation.

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