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Supersymmetric gauge theories on squashed five-spheres and their gravity duals

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Abstract: We construct the gravity duals of large N supersymmetric gauge theories defined on squashed five-spheres with $\mathrm{SU}(3)\times\mathrm{U}(1)$ symmetry. These five-sphere backgrounds are continuously connected to the round sphere, and we find a one-parameter family of 3/4 BPS deformations and a two-parameter family of (generically) 1/4 BPS deformations. The gravity duals are constructed in Euclidean Romans $F(4)$ gauged supergravity in six dimensions, and uplift to massive type IIA supergravity. We holographically renormalize the Romans theory, and use our general result to compute the renormalized on-shell actions for the solutions. The results agree perfectly with the large N limit of the dual gauge theory partition function, which we compute using large N matrix model techniques. In addition we compute BPS Wilson loops in these backgrounds, both in supergravity and in the large N matrix model, again finding precise agreement. Finally, we conjecture a general formula for the partition function on any five-sphere background, which for fixed gauge theory depends only on a certain supersymmetric Killing vector.

Keywords: Supersymmetric gauge theory, Gauge-gravity correspondence, Supergravity Models

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1 Introduction

Over the last few years there has been increasing interest in defining and studying supersymmetric gauge theories on curved backgrounds. Such constructions lead to interesting classes of observables that can be computed exactly, which may in turn be used to test and explore conjectured dualities. In this paper we focus on the case of five-dimensional gauge theories. These have been defined on round spheres [1–5], as well as on certain continuous deformations thereof [6, 7], referred to as squashed five-spheres. The main observable that can be computed exactly in these theories is the partition function Z, which depends non-trivially on the background geometry. A particular class of five-dimensional superconformal gauge theories, with gauge group $\mathrm{USp}(2N)$ and arising from a $D4-D8$-system, is expected to have a large N description in terms of massive type IIA supergravity [8–10]. In [5] the large N limit of the partition function of these theories on the *round* sphere was computed and successfully compared to the entanglement entropy of the dual warped $\mathrm{AdS}_6 \times S^4$ supergravity solution.

In this paper we shall present the first construction of gravity duals to gauge theories on non-conformally flat backgrounds (specifically, certain families of squashed five-spheres). As we shall explain, we may effectively work in six-dimensional Romans $F(4)$ supergravity [11], which is a consistent truncation of massive IIA supergravity on S^4 [12]. In particular the computation of [5] effectively determines the six-dimensional Newton constant. Having constructed supergravity solutions that have squashed five-sphere conformal boundaries, we compute the holographic free energy $\mathcal{F} = -\log Z$ by holographically renormalizing the on-shell Euclidean action. More specifically, we construct families of solutions with different numbers of preserved supercharges. Two of these families are shown to be dual to the 1/4 BPS and 3/4 BPS gauge theories defined in [7]. The perturbative partition function for these theories has been computed in [6] and we explicitly show that the large N limit of these partition functions is in precise agreement with the holographic free energies of our supergravity solutions. We also present more general solutions (and in particular a 1/2 BPS solution) which have not previously been considered from the gauge theory side.

From the Killing spinors of a supersymmetric supergravity solution one can always construct a certain Killing vector K. For all solutions found in this paper the free energy is only sensitive to this Killing vector $\mathcal{F} = \mathcal{F}(K)$, and not to other parameters of the solution. It is natural to conjecture that this is also the case for more general solutions, extending what happens in four dimensions [13]. In addition we compute the expectation values of BPS Wilson loops in these backgrounds, both in supergravity and in the large N matrix model, finding precise agreement. Again the expectation value depends only on the Killing vector K.

The rest of this paper is organized as follows. In section 2 we discuss supersymmetric gauge theories defined on squashed five-spheres, their exact partition function and the large N limit. In section 3 we change focus and describe the Romans $F(4)$ supergravity theory

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we will work with. Then in section 4 we present our supergravity solutions dual to the squashed five-sphere backgrounds. In section 5 we apply holographic renormalization to the Romans $F(4)$ supergravity theory and use this to compute the holographic free energy of our solutions. In section 6 we examine the supersymmetry conditions which arise at the conformal boundary for the Romans supergravity theory. Another exact observable that can be computed both in supersymmetric gauge theories and in supergravity are Wilson loops, which are the subject of section 7. Finally, we end in section 8 with some discussion and possible future problems to explore. We also include appendices A, B and C, which expand upon some of the elements in the main body of the paper.

2 Supersymmetric gauge theories on squashed five-spheres

We begin in section 2.1 by describing the squashed five-sphere backgrounds of interest [6]. One can define a supersymmetric gauge theory with general matter content on such a background, and in [7] the perturbative partition function was computed via a twisted reduction of the supersymmetric index in six dimensions,[1] that we summarize in section 2.2. A particular class of five-dimensional gauge theories, with gauge group $\mathrm{USp}(2N)$ and arising from a $D4-D8$ system in massive type IIA string theory, is expected to have a large N limit with a gravity dual. In section 2.3 we compute the large N limit of the partition function for these theories using matrix model techniques.

2.1 SU(3) × U(1) squashed five-sphere

The squashed S^5 backgrounds of interest are homogeneous spaces with symmetry $\mathrm{SU}(3)\times\mathrm{U}(1)$. In particular this is the isometry group of the metric

$$\begin{aligned} \mathrm{d}s_5^2 = &\frac{1}{s^2}(\mathrm{d}\tau + C)^2 + \mathrm{d}\sigma^2 + \frac{1}{4}\sin^2\sigma(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \\ &+\frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi)^2\,, \end{aligned} \tag{2.1}$$

where we have defined the (local) one-form

$$C = -\frac{1}{2}\sin^2\sigma(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi)\ . \tag{2.2}$$

We refer to the parameter s as a squashing parameter, and note that $s=1$ is the round sphere. The coordinates in (2.1) realize the five-sphere as the total space of the Hopf circle bundle over $\mathbb{CP}^2$, where τ is a 2π-period coordinate along the circle fibre. The coordinates $\sigma,\psi,\theta,\varphi$ are then coordinates on the base $\mathbb{CP}^2$, with ψ having period 4π, φ having period 2π, while $\sigma\in[0,\frac{\pi}{2}]$, $\theta\in[0,\pi]$. The local one-form C in (2.2) satisfies

$$\mathrm{d}C \equiv 2\omega \quad = \quad -\sin\sigma\cos\sigma\mathrm{d}\sigma\wedge(\mathrm{d}\psi+\cos\theta\mathrm{d}\varphi) + \frac{1}{2}\sin^2\sigma\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi, \tag{2.3}$$

where ω is the Kähler two-form on $\mathbb{CP}^2$.

[1] See also [14].

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In order to preserve supersymmetry one must also turn on other backgrounds fields. In particular in [6] it was shown that one can define general supersymmetric gauge theories on the above squashed five-sphere, provided one turns on a background $\mathrm{SU}(2)_R$ gauge field

$$\mathcal{A} = \frac{(1+Q\sqrt{1-s^2})\sqrt{1-s^2}}{s^2}(\mathrm{d}\tau + C)\,, \tag{2.4}$$

where we have embedded $\mathrm{U}(1)_R \subset \mathrm{SU}(2)_R$. More precisely, writing the $\mathrm{SU}(2)_R \sim \mathrm{SO}(3)_R$ gauge field as a triplet of one-forms $\mathcal{A}^i$, $i = 1, 2, 3$, we have $\mathcal{A}^1 = \mathcal{A}^2 = 0$, while $\mathcal{A}^3 = \mathcal{A}$ is given by (2.4). For supersymmetric backgrounds the parameter Q takes the values $Q = 1$ and $Q = -3$, which lead to 3/4 BPS and 1/4 BPS solutions, respectively. Notice that the gauge field (2.4) is also invariant under $\mathrm{SU}(3) \times \mathrm{U}(1)$, and is real when $|s| < 1$ but complex for $|s| > 1$.

A supersymmetric background of course admits an appropriate Killing spinor, which then enters the supersymmetry transformations of a supersymmetric gauge theory defined on the background. Recall that a Killing spinor χ on the round S^5 with $s = 1$, solving $\nabla_m \chi = -\frac{\mathrm{i}}{2}\gamma_m \chi$ where γ_m generate the Clifford algebra $\mathrm{Cliff}(5,0)$ in an orthonormal frame, transforms in the $\mathbf{4}$ of the $\mathrm{SU}(4) \sim \mathrm{SO}(6)$ isometry. The squashing breaks this symmetry to $\mathrm{SU}(3) \times \mathrm{U}(1)$, and for $Q = 1$ the resulting Killing spinor transforms as $\mathbf{3}_{\mathbf{+1}}$, while for $Q = -3$ the resulting Killing spinor instead transforms as $\mathbf{1}_{\mathbf{-3}}$. Similarly, solutions to $\nabla_m \chi = \frac{\mathrm{i}}{2}\gamma_m \chi$ transform in the $\bar{\mathbf{4}}$ of $\mathrm{SU}(4)$, which is broken to $\bar{\mathbf{3}}_{\mathbf{-1}}$ and $\mathbf{1}_{\mathbf{+3}}$ in the two cases, respectively.

The corresponding Killing spinor equation for the squashed S^5 was obtained in [6] via a twisted reduction (described in the next subsection) of a standard Killing spinor equation in six dimensions. In order to write this down, we first introduce an orthonormal frame for the metric (2.1)

$$\begin{aligned} e^1_{(5)} &= \frac{1}{s}(\mathrm{d}\tau + C)\,, \qquad e^2_{(5)} = \mathrm{d}\sigma\,, \qquad e^3_{(5)} = \frac{1}{2}\sin\sigma\cos\sigma\tau_3\,, \\ e^4_{(5)} &= \frac{1}{2}\sin\sigma\tau_2\,, \qquad e^5_{(5)} = \frac{1}{2}\sin\sigma\tau_1\,, \end{aligned} \tag{2.5}$$

where τ_i, $i = 1, 2, 3$, are left-invariant one-forms on $\mathrm{SU}(2)$. These are parametrized in terms of the Euler angles as

$$\tau_1 + \mathrm{i}\tau_2 = \mathrm{e}^{-\mathrm{i}\psi}(\mathrm{d}\theta + \mathrm{i}\sin\theta\mathrm{d}\varphi)\,, \qquad \tau_3 = \mathrm{d}\psi + \cos\theta\mathrm{d}\varphi\ . \tag{2.6}$$

The Killing spinor equation then reads

$$\begin{aligned} \nabla_m \chi_I + \frac{\mathrm{i}}{2}\mathcal{A}^i_m (\sigma^i)_I{}^J \chi_J = &-\frac{\mathrm{i}\left(1+Q\sqrt{1-s^2}\right)}{2s}(\sigma^3)_I{}^J \gamma_m \chi_J \\ &+\frac{\sqrt{1-s^2}}{4s}\left(3\gamma_m \not{\varphi} - \not{\varphi}\gamma_m\right)\chi_I\,, \end{aligned} \tag{2.7}$$

which is supplemented by the following algebraic equation

$$Q\sqrt{1-s^2}\chi_I = -\sqrt{1-s^2}\gamma_1 \chi_I - \mathrm{i}\sqrt{1-s^2}(\sigma^3)_I{}^J \not{\varphi}\chi_J\ . \tag{2.8}$$

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Here χ_I, $I=1,2$, form a doublet under the $\mathrm{SU}(2)_R$ symmetry, γ_m generate the Clifford algebra $\mathrm{Cliff}(5,0)$ in the orthonormal frame (2.5), and $(\sigma^i)_I{}^J$ denote the Pauli matrices. Recall also that ω denotes the Kähler form on $\mathbb{CP}^2$, given by (2.3), and if α is a p-form we denote $\not{\alpha} \equiv \frac{1}{p!}\alpha_{m_1\cdots m_p}\gamma^{m_1\cdots m_p}$.

Of course in the case at hand we have that the $\mathrm{SU}(2)_R$ gauge field $\mathcal{A}^i$ is only turned on in the $i=3$ direction, with $\mathcal{A}^3=\mathcal{A}$ given by (2.4), and we may also write (2.7) and (2.8) as

$$\nabla_m\chi_\pm \pm \frac{\mathrm{i}}{2}\mathcal{A}_m\chi_\pm = \mp\frac{\mathrm{i}\left(1+Q\sqrt{1-s^2}\right)}{2s}\gamma_m\chi_\pm + \frac{\sqrt{1-s^2}}{4s}\left(3\gamma_m\not{\omega}-\not{\omega}\gamma_m\right)\chi_\pm\,, \tag{2.9}$$

$$Q\sqrt{1-s^2}\chi_\pm = -\sqrt{1-s^2}\gamma_1\chi_\pm \mp \mathrm{i}\sqrt{1-s^2}\not{\omega}\chi_\pm\,, \tag{2.10}$$

where $\chi_+=\chi_1$, $\chi_-=\chi_2$. Provided the background fields are real, meaning in particular that the metric and $\mathcal{A}$ are real and $|s|<1$, then notice that the equations for χ_- are simply the charge conjugates of the χ_+ equations, where we define the charge conjugate as

$$\chi^c \equiv \mathcal{C}_5\chi^*\,, \tag{2.11}$$

and the charge conjugation matrix $\mathcal{C}_5$ satisfies $\mathcal{C}_5^{-1}\gamma_m\mathcal{C}_5=\gamma_m^*$. In particular it is then consistent to impose the symplectic Majorana condition $\chi_-=\chi_+^c$, or equivalently $\varepsilon_I{}^J\chi_J=\mathcal{C}_5\chi_I^*$, as we shall see below.

Notice that in setting $s=1$ to obtain the round sphere one has that (2.8) is trivially satisfied, while the Killing spinor equation (2.7) implies that χ_1 and χ_2 transform in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of the enhanced $\mathrm{SU}(4)\sim\mathrm{SO}(6)$ symmetry, respectively. In order to present the general solution to (2.7), (2.8) (which is not written in [6]), we first introduce the following basis of $\mathrm{Cliff}(5,0)$

$$\gamma_1=\begin{pmatrix}1_2 & 0\\ 0 & -1_2\end{pmatrix}\,,\qquad \gamma_2=\begin{pmatrix}0 & 1_2\\ 1_2 & 0\end{pmatrix}\,,\qquad \gamma_3=\begin{pmatrix}0 & \mathrm{i}\sigma^3\\ -\mathrm{i}\sigma^3 & 0\end{pmatrix}\,,$$
$$\gamma_4=\begin{pmatrix}0 & \mathrm{i}\sigma^2\\ -\mathrm{i}\sigma^2 & 0\end{pmatrix}\,,\qquad \gamma_5=\begin{pmatrix}0 & \mathrm{i}\sigma^1\\ -\mathrm{i}\sigma^1 & 0\end{pmatrix}\,, \tag{2.12}$$

where as above σ^i, $i=1,2,3$ denote the Pauli matrices, and 1_2 is the 2×2 identity matrix. A choice of the charge conjugation matrix in this basis is

$$\mathcal{C}_5=\begin{pmatrix}-\mathrm{i}\sigma_2 & 0\\ 0 & -\mathrm{i}\sigma_2\end{pmatrix}\,. \tag{2.13}$$

Then for the 1/4 BPS background we find the general solution to (2.7), (2.8) (or equivalently (2.9), (2.10)) is given by

$$\chi_+ = c_+\mathrm{e}^{-\frac{3\mathrm{i}\tau}{2}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\,,\qquad \chi_- = c_-\mathrm{e}^{\frac{3\mathrm{i}\tau}{2}}\begin{pmatrix}-1\\0\\0\\0\end{pmatrix}\,, \tag{2.14}$$

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where $c_\pm$ are integration constants. In particular then notice that the symplectic Majorana condition $\chi_- = \chi_+^c$ simply imposes $c_- = c_+^*$.

For the 3/4 BPS background the solution is a little more complicated. One finds

$$\chi_+ = a_+^{(1)}\, \mathrm{e}^{\mathrm{i}\frac{\tau}{2}} \begin{pmatrix} \cos\sigma + \mathrm{i}\lambda_+(s)\mathrm{e}^{\mathrm{i}\frac{\psi}{2}} S_+^{(1)} \sin\sigma \\ 0 \\ \mathrm{i}\lambda_-(s)\sin\sigma - \mathrm{e}^{\mathrm{i}\frac{\psi}{2}} S_+^{(1)} \cos\sigma \\ -\mathrm{i}\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} S_+^{(2)} \end{pmatrix}, \tag{2.15}$$

where

$$\begin{aligned} S_\pm^{(1)} &= S_\pm^{(1)}(\theta,\varphi) = a_\pm^{(3)} \mathrm{e}^{\pm\mathrm{i}\frac{\varphi}{2}} \cos\frac{\theta}{2} - a_\pm^{(2)} \mathrm{e}^{\mp\mathrm{i}\frac{\varphi}{2}} \sin\frac{\theta}{2}\,, \\ S_\pm^{(2)} &= S_\pm^{(2)}(\theta,\varphi) = a_\pm^{(2)} \mathrm{e}^{\mp\mathrm{i}\frac{\varphi}{2}} \cos\frac{\theta}{2} + a_\pm^{(3)} \mathrm{e}^{\pm\mathrm{i}\frac{\varphi}{2}} \sin\frac{\theta}{2}\,, \end{aligned} \tag{2.16}$$

and where we have introduced $\lambda_\pm(s) \equiv (\pm 1 + \sqrt{1-s^2})/s$. As expected, the solution depends on three integration constants $a_+^{(1)}, a_+^{(2)}, a_+^{(3)}$. Similarly, one finds

$$\chi_- = a_-^{(1)}\, \mathrm{e}^{-\mathrm{i}\frac{\tau}{2}} \begin{pmatrix} 0 \\ \cos\sigma - \mathrm{i}\lambda_+(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} S_-^{(1)} \sin\sigma \\ -\mathrm{i}\mathrm{e}^{\mathrm{i}\frac{\psi}{2}} S_-^{(2)} \\ -\mathrm{i}\lambda_-(s)\sin\sigma - \mathrm{e}^{-\mathrm{i}\frac{\psi}{2}} S_-^{(1)} \cos\sigma \end{pmatrix}, \tag{2.17}$$

where $a_-^{(i)}$ are integration constants. One can once again impose the symplectic Majorana condition, which leads to the relation $(a_-^{(i)})^* = a_+^{(i)}$ for $i = 1, 2, 3$.

2.2 Twisted reduction and the partition function

The backgrounds above may be obtained via a twisted reduction of $\mathbb{R} \times S^5$, starting from the *round* metric on S^5. This is important, as the perturbative partition function on the squashed five-spheres was computed in [7] indirectly, by taking a limit of the supersymmetric index of a corresponding six-dimensional theory on $\mathbb{R} \times S^5$.

We thus begin with the product metric on $\mathbb{R}$ times the round S^5

$$\mathrm{d}s^2_{\mathbb{R}\times S^5} = \mathrm{d}t^2 + \sum_{i=1}^{3} |\mathrm{d}w_i|^2\,, \tag{2.18}$$

where the complex coordinates w_i on $\mathbb{C}^3 \cong \mathbb{R}^6$, $i = 1, 2, 3$, satisfy the constraint $\sum_{i=1}^3 |w_i|^2 = 1$. We then compactify this space by identifying

$$(t, w_i) \sim (t + \beta, \mathrm{e}^{\mathrm{i}\mu_i\beta} w_i)\,, \tag{2.19}$$

where $\beta > 0$ and the μ_i are also sometimes referred to as squashing parameters. Notice that (2.19) is an isometry for $\mu_i \in \mathbb{R}$. We may then change coordinates

$$\rho_i \mathrm{e}^{\mathrm{i}\varphi_i} \equiv \mathrm{e}^{-\mathrm{i}\mu_i t} w_i\,, \tag{2.20}$$

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where $\rho_i \geq 0$ and the φ_i have period 2π. In terms of these new coordinates the identification (2.19) reads $(t, \rho_i, \varphi_i) \sim (t + \beta, \rho_i, \varphi_i)$. We then dimensionally reduce along the t-direction to obtain the five-dimensional metric

$$\mathrm{d}s_5^2 = \sum_{i=1}^{3}(\mathrm{d}\rho_i^2 + \rho_i^2 \mathrm{d}\varphi_i^2) - \frac{1}{1+\sum_{i=1}^{3}\mu_i^2\rho_i^2}\left(\sum_{i=1}^{3}\mu_i\rho_i^2\mathrm{d}\varphi_i\right)^2 . \tag{2.21}$$

Notice that, via the constraint $\sum_{i=1}^{3}\rho_i^2 = 1$, the first term in (2.21) is the round metric on S^5.

One then makes contact with the previous section by choosing

$$\begin{aligned} -\mu_1 &= \mu_2 = \mu_3 = \mathrm{i}\sqrt{1-s^2}\,, && 3/4 \text{ BPS}\,, \\ \mu_1 &= \mu_2 = \mu_3 = -\mathrm{i}\sqrt{1-s^2}\,, && 1/4 \text{ BPS}\,. \end{aligned} \tag{2.22}$$

Notice these are real only if $|s| \geq 1$. The metric (2.21) then agrees with the metric (2.1) on making the standard polar coordinate identifications

$$\rho_1 = \cos\sigma\,, \qquad \rho_2 = \sin\sigma\cos\frac{\theta}{2}\,, \qquad \rho_3 = \sin\sigma\sin\frac{\theta}{2}\,, \tag{2.23}$$

together with

$$\begin{aligned} \varphi_1 &= -\tau\,, \quad \varphi_2 = \tau - \frac{1}{2}(\psi+\varphi)\,, \quad \varphi_3 = \tau - \frac{1}{2}(\psi-\varphi)\,, \quad 3/4 \text{ BPS}\,, \\ \varphi_1 &= \tau\,, \quad \varphi_2 = \tau - \frac{1}{2}(\psi+\varphi)\,, \quad \varphi_3 = \tau - \frac{1}{2}(\psi-\varphi)\,, \quad 1/4 \text{ BPS}\,. \end{aligned} \tag{2.24}$$

The Killing spinor equation (2.7) and algebraic equation (2.8) were then obtained in [6] by dimensionally reducing a standard Killing spinor equation on the $\mathbb{R}\times S^5$ background (2.18).

In practice the perturbative contribution to the squashed S^5 partition function, with more general squashed metric (2.21), was computed in [7] by dimensionally reducing the superconformal index of a corresponding six-dimensional theory on the $\mathbb{R}\times S^5$ background (2.18) with twisted identification (2.19), and then taking the limit $\beta \to 0$, so that the radius of the circle we reduced on to obtain (2.21) is sent to zero. For a gauge theory with gauge group G, prepotential $\mathscr{F}$, which is a cubic polynomial in the scalar σ in the vector multiplet, and matter in the real representation $\mathbf{R}\oplus\bar{\mathbf{R}}$ of G, the result is

$$Z_{\text{pert}} = C(\mathbf{b})\prod_{a=1}^{\text{rank }G}\int_{-\infty}^{\infty}\mathrm{d}\sigma_a\, \mathrm{e}^{-\frac{(2\pi)^3}{b_1b_2b_3}\mathscr{F}(\sigma)}\frac{\prod_\alpha S_3\left(-\mathrm{i}\alpha(\sigma)\mid\mathbf{b}\right)}{\prod_\rho S_3\left(-\mathrm{i}\rho(\sigma)+\frac{1}{2}(b_1+b_2+b_3)\mid\mathbf{b}\right)}\,. \tag{2.25}$$

Here we have introduced

$$\mathbf{b} = (b_1, b_2, b_3)\,, \qquad \text{where} \qquad b_i = 1 + \mathrm{i}\mu_i\,, \tag{2.26}$$

and the prefactor $C(\mathbf{b})$ in (2.25) depends only on (b_1, b_2, b_3), and in particular will not contribute to the large N limit of interest in the next section.[2] The perturbative partition

[2]The precise formula for $C(\mathbf{b})$ may be found in [7].

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function thus localizes onto field configurations in which the only non-zero field is a constant mode for the scalar σ in the vector multiplet, and this is then integrated over in (2.25). As usual in such expressions the product over α in the numerator is over roots of G, while the product over ρ in the denominator is over weights in a weight space decomposition of $\mathbf{R}$. Finally, $S_3\left(z \mid \mathbf{b}\right)$ is the triple sine function, which is a special case of the multiple sine functions defined by

$$S_{\mathcal{N}}\left(z \mid \mathbf{b}\right) \equiv \Gamma_{\mathcal{N}}(z \mid \mathbf{b})^{-1}\, \Gamma_{\mathcal{N}}(b_{\rm tot} - z \mid \mathbf{b})^{(-1)^{\mathcal{N}}} \tag{2.27}$$

$$= \prod_{n_1,\ldots,n_{\mathcal{N}}=0}^{\infty} \left[\sum_{i=1}^{\mathcal{N}} n_i b_i + z\right] \prod_{n_1,\ldots,n_{\mathcal{N}}=1}^{\infty} \left[\sum_{i=1}^{\mathcal{N}} n_i b_i - z\right]^{(-1)^{\mathcal{N}-1}} , \tag{2.28}$$

where we have written $\mathbf{b} = (b_1, \ldots, b_{\mathcal{N}})$ and defined $b_{\rm tot} = \sum_{i=1}^{\mathcal{N}} b_i$. The function $\Gamma_{\mathcal{N}}(z \mid \mathbf{b})$ is the so-called Barnes' multiple gamma function

$$\Gamma_{\mathcal{N}}(z \mid \mathbf{b}) \equiv \prod_{n_1,\ldots,n_{\mathcal{N}}=0}^{\infty} \left[\sum_{i=1}^{\mathcal{N}} n_i b_i + z\right]^{-1} . \tag{2.29}$$

We conclude this section by noting from (2.22) and (2.26) that for the $\mathrm{SU}(3) \times \mathrm{U}(1)$ squashed five-spheres in section 2.1

$$\begin{aligned} b_1 &= 1 + \sqrt{1 - s^2}\,, \qquad b_2 = b_3 = 1 - \sqrt{1 - s^2}\,, \qquad && 3/4 \text{ BPS}\,, \\ b_1 &= b_2 = b_3 = 1 + \sqrt{1 - s^2}\,, && 1/4 \text{ BPS}\,. \end{aligned} \tag{2.30}$$

In particular it is straightforward to see [7] that in the 1/4 BPS case the perturbative partition function (2.25) is independent of the squashing parameter s.

It is interesting to note that (2.19) is an isometry of the original six-dimensional $\mathbb{R} \times S^5$ background only for real μ_i, which via (2.22) one sees corresponds to $|s| \geq 1$. On the other hand from (2.30) we see that the parameters b_i are real (and then positive) only if $|s| \leq 1$. The dual six-dimensional supergravity backgrounds we shall construct in section 4 will correspondingly be real for $|s| \leq 1$.

2.3 The large N limit

The result for the perturbative partition function (2.25) in the previous section is valid for a general supersymmetric gauge theory in five dimensions, but we now focus on a particular class of theories with gauge group $G = \mathrm{USp}(2N)$, that arises from a system of N D4-branes and some number of D8-branes and orientifold planes in massive type IIA string theory. These theories are expected to have a large N limit that has a dual description in massive type IIA supergravity [8–10]. Indeed, in [5] the large N limit of the partition function of these theories on the *round* five-sphere was computed and successfully compared to the entanglement entropy of the dual warped $\mathrm{AdS}_6 \times S^4$ supergravity solution. Here the gauge theories flow to a UV superconformal fixed point, and in particular the localization computation in the IR supersymmetric Yang-Mills theory coupled to matter theory successfully reproduces the expected $N^{5/2}$ scaling of the number of degrees of freedom.

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In general one certainly expects non-perturbative contributions to the full partition function Z, in addition to the perturbative result (2.25). In particular in the localization computation of [3] on the round five-sphere one finds that the gauge multiplet localizes onto instanton configurations on $\mathbb{CP}^2$. There is thus a non-perturbative contribution to Z involving a sum over the instanton number. For fixed instanton number $n \neq 0$ and fixed choice of instanton, in addition to the classical instanton action there will also be one-loop determinant contributions around that instanton, plus an integral over the instanton moduli space with fixed n. In general this expression will be very difficult to evaluate. However, in [5] it was argued that in the large N limit these instanton contributions should be suppressed. We shall also assume this to be the case on the squashed five-sphere, although clearly this issue deserves further study. In particular, for general choice of the vector $\mathbf{b} = (b_1, b_2, b_3)$ we expect to find instantons not on $\mathbb{CP}^2$, but rather instantons transverse to the Killing vector $K = \sum_{i=1}^{3} b_i \partial_{\varphi_i}$, as in [15]. These *contact instantons* were discussed in the latter reference in the context of the partition function on Sasaki-Einstein manifolds. In any case, we leave this issue open for future investigation.

Our task thus reduces to computing the large N limit of the perturbative result (2.25), for the USp($2N$) gauge theories of interest. This may be carried out using the matrix model saddle point method originally introduced in [16], and subsequently applied to the round S^5 partition function in [5]. As in the latter reference, we also set the Chern-Simons level for the theory $k = 0$ (thus setting the cubic terms in the prepotential $\mathscr{F}(\sigma)$ to zero). The quadratic and linear terms of $\mathscr{F}(\sigma)$ will only contribute to subleading order in the large N limit. This is because the leading contribution to the free energy arises from the scaling $\sigma = \mathcal{O}(N^{1/2})$. Such a behaviour for σ leads to an $\mathcal{O}(N^2)$ contribution for the classical parts in the perturbative partition function (2.25). Thus in the limit of large N we only have to analyse the behaviour of the two one-loop determinants from the vector and matter multiplets. In particular, for a given theory we will have to find the expansion of the logarithm of the triple sine function entering (2.25).

The USp($2N$) gauge theories have N_f matter fields in the fundamental and a single hypermultiplet in the antisymmetric representation of the gauge group. Let us denote an element in the Cartan subalgebra for USp($2N$) as $\{\lambda_1, \ldots, \lambda_N\}$, so that $\sigma = \mathrm{diag}(\lambda_1, \ldots, \lambda_N, -\lambda_1, \ldots, -\lambda_N)$. The Weyl group acts as $\lambda_i \to -\lambda_i$ for each i, and also permutes the λ_i. If the normalized weights of the fundamental representation are given by $\pm e_i$, where $\{e_1, \ldots, e_N\}$ is a basis of $\mathbb{R}^N$, then the antisymmetric representation has weights $\{e_i \pm e_j\}_{i \neq j}$ and the adjoint representation has weights $\{e_i \pm e_j\}_{i \neq j} \cup \{\pm 2e_i\}_{i=1}^{N}$. Therefore we can write the free energy for this theory as

$$\begin{aligned}\mathcal{F}(\lambda_i) = &\sum_{\substack{i,j=1 \\ i \neq j}}^{N} G_V(\lambda_i + \lambda_j \mid \mathbf{b}) + G_V(\lambda_i - \lambda_j \mid \mathbf{b}) + G_H(\lambda_i + \lambda_j \mid \mathbf{b}) + G_H(\lambda_i - \lambda_j \mid \mathbf{b}) \\ &+ \sum_{i=1}^{N} G_V\left(2\lambda_i \mid \mathbf{b}\right) + G_V\left(-2\lambda_i \mid \mathbf{b}\right) + N_f\left[G_H\left(\lambda_i \mid \mathbf{b}\right) + G_H\left(-\lambda_i \mid \mathbf{b}\right)\right], \end{aligned} \tag{2.31}$$

where G_V and G_H are the logarithms of the triple sine functions in the numerator and denominator of (2.25) for the vector and the hypermultiplets, respectively. We are interested

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in their asymptotics for large λ_i only, because we assume that the eigenvalues scale with N^α for some $\alpha > 0$. These asymptotics are explicitly computed in appendix C, and here we simply quote the results:

$$\begin{aligned} G_V(x \mid \mathbf{b}) + G_V(-x \mid \mathbf{b}) &= -\log S_3\left(-\mathrm{i}x \mid \mathbf{b}\right) - \log S_3\left(\mathrm{i}x \mid \mathbf{b}\right) \\ &\sim \frac{\pi}{3\, b_1 b_2 b_3}|x|^3 - \frac{\pi\left(b_{\mathrm{tot}}^2 + b_1 b_2 + b_1 b_3 + b_2 b_3\right)}{6\, b_1 b_2 b_3}|x|\,, \end{aligned} \tag{2.32}$$

where we have expanded in the limit $|x| \to \infty$. Here we have assumed that $b_i > 0$ for each $i = 1, 2, 3$, as this is the case of interest — see equation (2.30) and the discussion after it. Similarly, for the free energy contribution of the hypermultiplet we obtain

$$G_H(x \mid \mathbf{b}) = \log S_3\left(\frac{1}{2}b_{\mathrm{tot}} - \mathrm{i}x \mid \mathbf{b}\right) \sim -\frac{\pi}{6\, b_1 b_2 b_3}|x|^3 - \frac{\pi\left(b_1^2 + b_2^2 + b_3^2\right)}{24\, b_1 b_2 b_3}|x|\,, \tag{2.33}$$

in the asymptotic limit $|x| \to \infty$.

Using the Weyl symmetry of USp($2N$) we may take $\lambda_i \geq 0$, and we shall furthermore assume that these eigenvalues scale as $\lambda_i = N^\alpha x_i$ to leading order in the large N limit, with $\alpha > 0$. We next introduce the density

$$\rho(x) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - x_i\right)\,, \tag{2.34}$$

which becomes an $\mathcal{L}^1$ function with

$$\int \rho(x)\mathrm{d}x = 1\,, \tag{2.35}$$

once we take $N \to \infty$. In that limit, the discrete sums in (2.31) become Riemann integrals

$$\frac{1}{N}\sum_{i=1}^{N} \longrightarrow \int_0^{x_\star} \rho(x)\mathrm{d}x\ . \tag{2.36}$$

Hence taking the large N limit of (2.31), we obtain to leading order

$$\begin{aligned} \mathcal{F} \approx\ & N^2 \int_0^{x_\star} \rho(x) \int_0^{x_\star} \rho(y)\Big[G_V(\lambda(x) \pm \lambda(y) \mid \mathbf{b}) + G_H(\lambda(x) \pm \lambda(y) \mid \mathbf{b})\Big]\mathrm{d}y\,\mathrm{d}x \\ & + N \int_0^{x_\star} \rho(x)\Big[G_V(\pm 2\lambda(x) \mid \mathbf{b}) + N_f\, G_H(\pm\lambda(x) \mid \mathbf{b})\Big]\mathrm{d}x\,. \end{aligned} \tag{2.37}$$

By assumption we have $\lambda(x) = N^\alpha x$ to leading order in the continuum limit, and hence we may use the above expansions for the vector and hypermultiplet contributions (2.32), (2.33) respectively. Then the leading order term in the first line of (2.37) scales as $N^{2+\alpha}$, because the cubic terms in the asymptotic expansion of G_H and G_V cancel. The leading order term of the second line in (2.37) however does not cancel, and is given by $N^{1+3\alpha}$. In order to obtain a non-trivial saddle point, both terms must contribute and we deduce that $\alpha = 1/2$.

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Putting everything together we obtain

$$\mathcal{F} = -N^{5/2} \int_0^{x_\star} \rho(x) \int_0^{x_\star} \rho(y) \left[\frac{\pi b_{\rm tot}^2}{8 b_1 b_2 b_3} \left(|x+y| + |x-y| \right) - \frac{(8-N_f)\pi}{3\ b_1 b_2 b_3} |x|^3 \right] \mathrm{d}y\ \mathrm{d}x + \mathcal{O}\left(N^{3/2}\right) . \tag{2.38}$$

It thus remains to solve a simple variational problem for $\rho(x)$ extremizing the free energy. We add a Lagrange multiplier term to impose the constraint (2.35), namely $\mu \left(\int_0^{x_\star} \rho(x) \mathrm{d}x - 1 \right)$, and then solve $\frac{\partial F}{\partial \rho} = 0$ for $\rho(x)$. Doing so we find (with $N_f < 8$)

$$\rho(x) = \frac{4(8-N_f)}{b_{\rm tot}^2} |x| \,, \tag{2.39}$$

inside the interval $[0, x_\star]$, with ρ identically zero outside this interval, and where extremizing $\mathcal{F}$ over the end-point $x_\star$ gives

$$x_\star^2 = \frac{b_{\rm tot}^2}{2(8-N_f)} \,. \tag{2.40}$$

We may then evaluate the free energy by substituting these saddle point configurations back into (2.37) to obtain

$$\mathcal{F} = -\frac{\sqrt{2}\pi b_{\rm tot}^3}{15\sqrt{8-N_f}\ b_1 b_2 b_3} N^{5/2} + \mathcal{O}\left(N^{3/2}\right) , \tag{2.41}$$

which may be rewritten as (where recall we have assumed that $b_i > 0$ for each $i = 1, 2, 3$)

$$\mathcal{F} = \frac{(b_1+b_2+b_3)^3}{27 b_1 b_2 b_3} \mathcal{F}_{S^5_{\rm round}} \,, \tag{2.42}$$

where $\mathcal{F}_{S^5_{\rm round}}$ is the large N limit of the free energy on the round five-sphere computed in reference [5]

$$\mathcal{F}_{S^5_{\rm round}} = -\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}} + \mathcal{O}\left(N^{3/2}\right) . \tag{2.43}$$

We note that the above result has a very similar structure to that obtained in three dimensions [17]. Also notice that we get the same result, (2.42), for the orbifold theories discussed in [5, 10].

We conclude this section by noting that for the SU(3) × U(1) squashed five-spheres, with the vector $\mathbf{b} = (b_1, b_2, b_3)$ given by (2.30), we obtain the large N free energies

$$\mathcal{F} = \begin{cases} \dfrac{1}{27 s^2} \dfrac{(3-\sqrt{1-s^2})^3}{1-\sqrt{1-s^2}} \mathcal{F}_{S^5_{\rm round}} \,, & 3/4 \text{ BPS} \,, \\ \mathcal{F}_{S^5_{\rm round}} \,, & 1/4 \text{ BPS} \,. \end{cases} \tag{2.44}$$

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3 Romans $F(4)$ supergravity

When the USp$(2N)$ superconformal theories discussed in section 2 are put on the round S^5, they are conjectured to be dual in the large N limit to the $\mathrm{AdS}_6 \times S^4$ solution of massive type IIA supergravity [8–10]. In order to find gravity duals to the same superconformal theories put on different background five-manifolds, it is then natural to work in the six-dimenional Romans $F(4)$ supergravity theory [11]. The key here is that, as shown in [12], the Romans theory is a consistent truncation of massive type IIA supergravity on S^4. In the next subsection we shall review this uplift to ten dimensions, and then present the Romans theory in Euclidean signature in section 3.2.

3.1 Uplift to massive type IIA

The Romans theory [11] is a six-dimensional gauged supergravity that admits an AdS_6 vacuum. The bosonic fields consist of the metric, a dilaton ϕ, a two-form potential B, a one-form potential A, together with an $\mathrm{SU}(2) \sim \mathrm{SO}(3)$ gauge field A^i, $i = 1, 2, 3$. It is convenient to introduce the scalar field $X \equiv \exp(-\phi/2\sqrt{2})$, and we define the field strengths as $H = \mathrm{d}B$, $F = \mathrm{d}A + \frac{2}{3}gB$, $F^i = \mathrm{d}A^i - \frac{1}{2}g\varepsilon_{ijk}A^j \wedge A^k$. Here g denotes the gauge coupling constant. Notice that B appears in the field strength for A.

As shown in [12], this Romans theory is a consistent truncation of massive type IIA supergravity on S^4. This means that any solution to the Romans theory automatically uplifts, via the non-linear Kaluza-Klein ansatz of [12] presented in (3.1) below, to a solution of massive type IIA. Moreover, the $\mathrm{AdS}_6 \times S^4$ solution of the latter is the uplift of the AdS_6 vacuum of the Romans theory.

We shall later need some details of how the six-dimensional solutions uplift to ten dimensions. The gauge coupling constant g is related to the ten-dimensional mass parameter by $m_{\mathrm{IIA}} = \frac{\sqrt{2}}{3}g$, while the remaining fields uplift via

$$
\begin{aligned}
\mathrm{d}s_{10}^2 &= (\sin\xi)^{\frac{1}{12}} X^{\frac{1}{8}} \left[\Delta^{\frac{3}{8}} \mathrm{d}s_6^2 + 2g^{-2}\Delta^{\frac{3}{8}} X^2 \mathrm{d}\xi^2 + \frac{1}{2}g^{-2}\Delta^{-\frac{5}{8}} X^{-1} \cos^2\xi \sum_{i=1}^{3} (\hat{\tau}^i - gA^i)^2 \right] , \\
F_{(4)} &= -\frac{\sqrt{2}}{6} g^{-3} s^{1/3} c^3 \Delta^{-2} U \, \mathrm{d}\xi \wedge \mathrm{vol}_3 - \sqrt{2} g^{-3} s^{4/3} c^4 \Delta^{-2} X^{-3} \, \mathrm{d}X \wedge \mathrm{vol}_3 \\
&\quad + \sqrt{2} g^{-1} s^{1/3} c X^4 \, *H \wedge \mathrm{d}\xi - \frac{1}{\sqrt{2}} s^{4/3} X^{-2} * F + \frac{1}{\sqrt{2}} g^{-2} s^{1/3} c \, F^i h^i \wedge \mathrm{d}\xi \\
&\quad - \frac{1}{4\sqrt{2}} g^{-2} s^{4/3} c^2 \Delta^{-1} X^{-3} F^i \wedge h^j \wedge h^k \, \varepsilon_{ijk} \, , \\
F_{(3)} &= s^{2/3} H + g^{-1} s^{-1/3} c \, F \wedge \mathrm{d}\xi \, , \\
F_{(2)} &= \frac{1}{\sqrt{2}} s^{2/3} F \, , \qquad \mathrm{e}^{\Phi} = s^{-5/6} \Delta^{1/4} X^{-5/4} \, ,
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
\Delta &\equiv X \cos^2\xi + X^{-3} \sin^2\xi \, , \\
U &\equiv X^{-6} s^2 - 3X^2 c^2 + 4X^{-2} c^2 - 6X^{-2} \, .
\end{aligned}
\tag{3.2}
$$

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Here $\mathrm{d}s^2_{10}$ is the ten-dimensional metric in Einstein frame, Φ is the ten-dimensional dilaton, $F_{(3)}$ is the NS-NS three-form field strength, while $F_{(4)}$ and $F_{(2)}$ are the RR four-form and two-form field strengths, respectively. The $\hat{\tau}^i$, $i = 1, 2, 3$, are left-invariant one-forms on a copy of $\mathrm{SU}(2) \cong S^3$. These are defined precisely as in (2.6), except here this S^3 is in the internal space (hence the hats). We have also defined $h^i \equiv \hat{\tau}^i - gA^i$, $\mathrm{vol}_3 \equiv h^1 \wedge h^2 \wedge h^3$, and $s = \sin\xi$ and $c = \cos\xi$. The Hodge duals in (3.1) are computed with respect to the six-dimensional metric $\mathrm{d}s^2_6$. This is defined on some six-manifold M_6, and the ten-dimensional metric in (3.1) then describes a warped product $M_6 \times S^4$. More precisely, the solution only describes "half" of a four-sphere, where the coordinate $\xi \in (0, \frac{\pi}{2}]$ is a polar coordinate for which constant $\xi \in (0, \frac{\pi}{2})$ slices are three-spheres, parametrized by Euler angles on S^3 as in (2.6). The solution is smooth at the north pole $\xi = \frac{\pi}{2}$, where the S^3 slices of S^4 collapse to zero size, but singular on the equator $\xi = 0$. Nevertheless, it is argued in [9, 10] that the supergravity solution (3.1) can be trusted away from this singularity.

3.2 Euclidean theory

The equations of motion and action for the Romans theory in Lorentz signature appear in [11, 12]. However, the gravity duals to the large N field theories on the squashed five-sphere of section 2 will be constructed in Euclidean signature. The corresponding Wick rotation is not entirely straightforward because the Romans theory contains Chern-Simons-type couplings, that become purely imaginary in Euclidean signature in order that the theory is gauge invariant. The associated factors of i are also crucial for supersymmetry in Euclidean signature. The Euclidean equations of motion for the Romans supergravity fields are

$$\begin{aligned}
\mathrm{d}\left(X^4 * H\right) &= \frac{\mathrm{i}}{2} F \wedge F + \frac{\mathrm{i}}{2} F^i \wedge F^i + \frac{2}{3} g X^{-2} * F \,, \\
\mathrm{d}(X^{-2} * F) &= -\mathrm{i} F \wedge H \,, \\
D(X^{-2} * F^i) &= -\mathrm{i} F^i \wedge H \,, \\
\mathrm{d}\left(X^{-1} * \mathrm{d}X\right) &= -g^2 \left(\frac{1}{6} X^{-6} - \frac{2}{3} X^{-2} + \frac{1}{2} X^2\right) * 1 \\
&\quad - \frac{1}{8} X^{-2} \left(F \wedge *F + F^i \wedge *F^i\right) + \frac{1}{4} X^4 H \wedge *H \,.
\end{aligned} \tag{3.3}$$

Here $D\omega^i = \mathrm{d}\omega^i - g\varepsilon_{ijk} A^j \wedge \omega^k$ is the SO(3) covariant derivative, and our convention for the Hodge duality operator is fixed via

$$\alpha \wedge *\beta = \frac{1}{p!} \alpha_{\mu_1 \cdots \mu_p} \beta^{\mu_1 \cdots \mu_p} * 1 \,, \tag{3.4}$$

where α and β are p-forms.[3] The Einstein equation is

$$\begin{aligned}
R_{\mu\nu} &= 4X^{-2} \partial_\mu X \partial_\nu X + g^2 \left(\frac{1}{18} X^{-6} - \frac{2}{3} X^{-2} - \frac{1}{2} X^2\right) g_{\mu\nu} + \frac{1}{4} X^4 \left(H^2_{\mu\nu} - \frac{1}{6} H^2 g_{\mu\nu}\right) \\
&\quad + \frac{1}{2} X^{-2} \left(F^2_{\mu\nu} - \frac{1}{8} F^2 g_{\mu\nu}\right) + \frac{1}{2} X^{-2} \left((F^i)^2_{\mu\nu} - \frac{1}{8} (F^i)^2 g_{\mu\nu}\right) \,,
\end{aligned} \tag{3.5}$$

where $F^2_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho$, $H^2_{\mu\nu} = H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma}$.

[3] In particular this convention differs from that in [12].

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The Euclidean action which gives rise to these field equations is

$$I_E = -\frac{1}{16\pi G_N}\int\Big[R*1 - 4X^{-2}\mathrm{d}X\wedge *\mathrm{d}X - g^2\left(\frac{2}{9}X^{-6} - \frac{8}{3}X^{-2} - 2X^2\right)*1 \\ - \frac{1}{2}X^{-2}\left(F\wedge *F + F^i\wedge *F^i\right) - \frac{1}{2}X^4 H\wedge *H \\ - \mathrm{i}B\wedge\left(\frac{1}{2}\mathrm{d}A\wedge\mathrm{d}A + \frac{1}{3}B\wedge\mathrm{d}A + \frac{2}{27}g^2 B\wedge B + \frac{1}{2}F^i\wedge F^i\right)\Big]\,. \tag{3.6}$$

In particular notice that the final term is a Chern-Simons-type coupling, and is accompanied by a factor of i. This is required for gauge-invariance in the path integral with Euclidean measure $\exp(-I_E)$. It is also implied by supersymmetry. Indeed, a solution to the above equations of motion is supersymmetric provided the following Killing spinor equation and dilatino equation hold:

$$D_\mu\epsilon_I = \frac{\mathrm{i}}{4\sqrt{2}}g(X + \frac{1}{3}X^{-3})\Gamma_\mu\Gamma_7\epsilon_I - \frac{\mathrm{i}}{16\sqrt{2}}X^{-1}F_{\nu\rho}(\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu\Gamma^\rho)\epsilon_I \\ -\frac{1}{48}X^2 H_{\nu\rho\sigma}\Gamma^{\nu\rho\sigma}\Gamma_\mu\Gamma_7\epsilon_I + \frac{1}{16\sqrt{2}}X^{-1}F^i_{\nu\rho}(\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu\Gamma^\rho)\Gamma_7(\sigma^i)_I{}^J\epsilon_J\,, \tag{3.7}$$

$$0 = -\mathrm{i}X^{-1}\partial_\mu X\Gamma^\mu\epsilon_I + \frac{1}{2\sqrt{2}}g\left(X - X^{-3}\right)\Gamma_7\epsilon_I + \frac{\mathrm{i}}{24}X^2 H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_7\epsilon_I \\ -\frac{1}{8\sqrt{2}}X^{-1}F_{\mu\nu}\Gamma^{\mu\nu}\epsilon_I - \frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma_7(\sigma^i)_I{}^J\epsilon_J\,. \tag{3.8}$$

Here ϵ_I, $I = 1, 2$, are two Dirac spinors, Γ_μ generate the Clifford algebra $\mathrm{Cliff}(6,0)$ in an orthonormal frame, and we have defined the chirality operator $\Gamma_7 = \mathrm{i}\Gamma_{012345}$, which satisfies $\Gamma_7^2 = 1$. The $\mathrm{SO}(3)\sim\mathrm{SU}(2)$ gauge field A^i is an R-symmetry gauge field, with the spinor ϵ_I transforming in the two-dimensional representation via the Pauli matrices $(\sigma^i)_I{}^J$. Thus the covariant derivative acting on the spinor is $D_\mu\epsilon_I = \nabla_\mu\epsilon_I + \frac{\mathrm{i}}{2}gA^i_\mu(\sigma^i)_I{}^J\epsilon_J$.

Returning to the equations of motion (3.3), notice that the exterior derivative of the first equation (the equation of motion for B) implies the second equation on using the Bianchi identities for F and F^i, where note that $\mathrm{d}F = \frac{2}{3}gH$. This is related to the fact that the theory possesses a gauge invariance $A\to A+\frac{2}{3}g\lambda$, $B\to B-\mathrm{d}\lambda$, where λ is an arbitrary one-form. Using this freedom one can then gauge away $A = 0$, leaving $F = \frac{2}{3}gB$. The kinetic term for F in the action (3.6) then becomes a mass term for the B-field; that is, the B-field "eats" the U(1) gauge field A in a Higgs-like mechanism. Notice that there is also a cubic Chern-Simons coupling for B in (3.6), making it a somewhat exotic field. We may also make a simple rescaling of the fields via $g_{\mu\nu}\to\frac{1}{g^2}g_{\mu\nu}$, $B\to\frac{1}{g^2}B$, $A\to\frac{1}{g}A$, $A^i\to\frac{1}{g}A^i$, after which one sees that the coupling constant g only appears in the action as an overall constant $1/g^4$ factor. Thus we may without loss of generality set $g = 1$, which we henceforth will do.

In appendix A we compute the integrability conditions for the Killing spinor equation (3.7) and dilatino equation (3.8), and show that these are compatible with the equations of motion (3.3), (3.5).

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3.3 Killing vector bilinear

Given a supersymmetric solution to the Euclidean Romans theory, one can verify that the bilinear

$$K_\mu \equiv \varepsilon^{IJ}\epsilon_I^T \mathcal{C}\Gamma_\mu \epsilon_J \,, \tag{3.9}$$

is a Killing one-form. Here $\mathcal{C}$ is the charge conjugation matrix, satisfying $\Gamma_\mu^T = \mathcal{C}^{-1}\Gamma_\mu\mathcal{C}$ and in our conventions is antisymmetric satisfying $\mathcal{C}^2 = -1$. If we also impose a symplectic Majorana condition

$$\mathcal{C}\epsilon_I^* = \varepsilon_I{}^J \epsilon_J \,, \tag{3.10}$$

then this Killing one-form may be rewritten as

$$K_\mu = \epsilon_I^\dagger \Gamma_\mu \epsilon_I \,, \tag{3.11}$$

which is then manifestly real. In particular we will be able to impose this symplectic Majorana condition for the solutions we construct in section 4. In this "real" case the Killing spinors ϵ_I define an SU(2) structure on M_6. One could similarly analyse the differential conditions on the corresponding SU(2) structure bilinears, but we shall leave this for the future.

4 Supergravity solutions

In this section we present supergravity duals to the $\mathrm{SU}(3)\times\mathrm{U}(1)$ squashed five-sphere backgrounds of section 2. Via the consistent truncation to the Romans theory in the previous section, this effectively becomes a filling problem in six-dimensional gauged supergravity: one seeks a smooth, asymptotically locally Euclidean AdS_6 supersymmetric supergravity solution, with conformal boundary data given by the squashed five-sphere background in section 2. In particular this means the bulk supergravity solution is equipped with an $\mathrm{SU}(2)_R$ doublet of Killing spinors ϵ_I, $I = 1, 2$, solving (3.7) and (3.8), which should suitably approach the boundary Killing spinors in section 2.1. We shall indeed find such fillings for both the 3/4 BPS and 1/4 BPS solutions. In the process shall extend the 1/4 BPS solution to a two-parameter family of solutions, containing a one-parameter 1/2 BPS subfamily of new solutions.

4.1 SU(3) × U(1) invariant ansatz

The squashed five-sphere backgrounds of section 2.1 have SU(3) × U(1) symmetry, and one expects this symmetry to be preserved by the bulk supergravity filling. Indeed, for asymptotically locally Euclidean AdS solutions of the *vacuum* Einstein equations this is a theorem [18]. This leads to the following ansatz for the Romans supergravity fields

$$\begin{aligned}
\mathrm{d}s_6^2 &= \alpha^2(r)\mathrm{d}r^2 + \gamma^2(r)(\mathrm{d}\tau + C)^2 + \beta^2(r)\Big[\mathrm{d}\sigma^2 + \frac{1}{4}\sin^2\sigma(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \\
&\qquad + \frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\psi + \cos\theta\mathrm{d}\varphi)^2\Big] \,, \\
B &= p(r)\mathrm{d}r\wedge(\mathrm{d}\tau + C) + \frac{1}{2}q(r)\mathrm{d}C \,, \\
A^i &= f^i(r)(\mathrm{d}\tau + C) \,, \quad i = 1,2,3 \,,
\end{aligned} \tag{4.1}$$

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together with $X = X(r)$. Recall here that we have used the gauge freedom to set the U(1) gauge field (which is really a Stueckelberg field) to $A = 0$. The additional coordinate r is a radial coordinate, and we shall choose a parametrization in which the conformal boundary is at $r = \infty$. For fixed r, provided $\gamma(r)$ and $\beta(r)$ are non-zero the constant r surfaces in (4.1) are squashed five-spheres. We shall seek solutions with the topology of a ball, so that $r \in [r_0, \infty)$ with $r = r_0$ being the origin. At this point the squashed five-spheres must become *round* in order that the metric extends smoothly to the origin of the ball. Similarly, in order for the gauge fields B, A^i in (4.1) to be non-singular at the origin they must tend to zero sufficiently quickly at $r = r_0$. In writing the ansatz (4.1) we have used the fact that the only $\mathrm{SU}(3) \times \mathrm{U}(1)$ invariant one-form on the squashed five-sphere is the global angular form $\mathrm{d}\tau + C$ for the Hopf fibration $S^1 \hookrightarrow S^5 \to \mathbb{CP}^2$, while the only invariant two-form is the pull-back $\frac{1}{2}\mathrm{d}C = \omega$ of the Kähler form on $\mathbb{CP}^2$.

Substituting the cohomogeneity one ansatz (4.1) into the equations of motion (3.3) and Einstein equation (3.5) leads to a rather complicated coupled system of ODEs. The equations of motion for the background $\mathrm{SU}(2)_R$ gauge field imply $f^i(r) = \kappa_i f(r)$, $i = 1, 2, 3$. The equations for the other fields then depend only on the $\mathrm{SU}(2) \sim \mathrm{SO}(3)$ invariant $\kappa_1^2 + \kappa_2^2 + \kappa_3^2$, which we can set to one by rescaling $f(r)$. The equations of motion then result in the coupled ODEs for the functions $\alpha(r)$, $\beta(r)$, $\gamma(r)$, $p(r)$, $q(r)$, $f(r)$, $X(r)$, which can be found in appendix B.1.

Since the solutions we find are continuously connected to Euclidean AdS_6, we first present the latter in these coordinates:

$$\begin{aligned} \alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}}\,, & \beta(r) = \gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\,, \\ p(r) = q(r) = f(r) &= 0\,, & X(r) &= 1\,. \end{aligned} \tag{4.2}$$

Here only the metric is non-trivial, and (4.2) realizes Euclidean AdS_6 as a hyperbolic ball with radial coordinate $r \in [\frac{1}{\sqrt{6}}, \infty)$, with the conformal boundary at infinity $r = \infty$. Thus the origin is at $r_0 = \frac{1}{\sqrt{6}}$. Notice in particular that the conformal boundary at $r = \infty$ is equipped with a *round* metric on S^5, which is conformally flat. We would like to find families of solutions that generalize (4.2) by allowing for a squashed five-sphere boundary, keeping the metric asymptotically locally Euclidean AdS near $r = \infty$. That is, near $r = \infty$ the metric should approach

$$\mathrm{d}s_6^2 \simeq \frac{9\mathrm{d}r^2}{2r^2} + 27r^2\mathrm{d}s_5^2\,, \tag{4.3}$$

where $\mathrm{d}s_5^2$ is the squashed five-sphere (2.1). For such solutions we may thus define the squashing parameter by

$$\lim_{r\to\infty} \frac{\gamma(r)}{r} \;=\; 3\sqrt{3}\,\frac{1}{s}\,, \tag{4.4}$$

so that $s = 1$ for the round sphere. Even though we did not manage to find supersymmetric solutions in closed form, the solutions can nevertheless be given as expansions around different limits. In general notice that we can use reparametrization invariance to set

$$\beta(r) \;=\; \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\,, \tag{4.5}$$

which we assume henceforth. In particular this fixes the origin of the ball to be at $r_0 = \frac{1}{\sqrt{6}}$.

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In the following we summarize the various families of supersymmetric solutions we have constructed with the ansatz (4.1). Details of the computations may be found in appendix B.

4.2 3/4 BPS solutions

There is a one-parameter family of 3/4 BPS solutions parametrized by the squashing parameter s. The solution expanded around the conformal boundary is given by

$$
\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} + \frac{8+s^2}{36\sqrt{2}s^2}\frac{1}{r^3} + \dots\,, \\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{-16+7s^2}{12\sqrt{3}s^3}\frac{1}{r} - \frac{-1280+1120s^2+241s^4}{2592\sqrt{3}s^5}\frac{1}{r^3} + \dots\,, \\
X(r) &= 1 + \frac{1-s^2-3\sqrt{1-s^2}}{54s^2}\frac{1}{r^2} + \frac{s^2\sqrt{1-s^2}\kappa}{12\left(1-s^2+\sqrt{1-s^2}\right)}\frac{1}{r^3} + \dots\,, \\
p(r) &= -\frac{\mathrm{i}\sqrt{\frac{2}{3}}\left(s^2+3\sqrt{1-s^2}-1\right)}{s^3}\frac{1}{r^2} + \dots\,, \\
q(r) &= -\frac{3\mathrm{i}\left(\sqrt{6}\sqrt{1-s^2}\right)}{s}r + \frac{\sqrt{\frac{2}{3}}\mathrm{i}\sqrt{1-s^2}\left(5s^2+9\sqrt{1-s^2}-5\right)}{3s^3}\frac{1}{r} + \dots\,, \\
f(r) &= \frac{1-s^2+\sqrt{1-s^2}}{s^2} + \frac{2\left(-2+2s^2-(2+s^2)\sqrt{1-s^2}\right)}{9s^4}\frac{1}{r^2} + \frac{\kappa}{r^3} + \dots\,,
\end{aligned}
\tag{4.6}
$$

where we have computed this expansion up to $\mathcal{O}(1/r^9)$. The extra parameter κ is fixed by requiring regularity at the origin $r = \frac{1}{\sqrt{6}}$ (see (4.8) below). Notice that the $\mathrm{SU}(2)_R$ gauge field at the conformal boundary agrees with the gauge field (2.4) with $Q = 1$. We may also expand the solution around Euclidean AdS_6, which has $s = 1$:

$$
\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \frac{\Big(-5\sqrt{6}+330\sqrt{6}r^2-3744r^3+1620\sqrt{6}r^4}{9\sqrt{2}r^2\left(6r^2-1\right)^{9/2}} \\
&\qquad \frac{+8640r^5-7560\sqrt{6}r^6+5184\sqrt{6}r^8\Big)}{9\sqrt{2}r^2\left(6r^2-1\right)^{9/2}}(1-s) + \dots\,, \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} - \frac{\Big(55\sqrt{2}-384\sqrt{3}r+1080\sqrt{2}r^2+768\sqrt{3}r^3}{6\left(6r^2-1\right)^{7/2}} \\
&\qquad \frac{-5400\sqrt{2}r^4+11232\sqrt{2}r^6-11664\sqrt{2}r^8\Big)}{6\left(6r^2-1\right)^{7/2}}(1-s) + \dots\,, \\
X(r) &= 1 - \frac{\left(\sqrt{2}\left(1-2\sqrt{6}r+6r^2\right)\right)}{3\left(6r^2-1\right)^2}\sqrt{1-s} + \dots\,, \\
p(r) &= \frac{18\mathrm{i}\sqrt{2}\left(\sqrt{6}-16r+12\sqrt{6}r^2-12\sqrt{6}r^4\right)}{\left(6r^2-1\right)^3}\sqrt{1-s} + \dots\,,
\end{aligned}
$$

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$$
\begin{aligned}
q(r) &= -\frac{3\mathrm{i}\sqrt{2}\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)}{\left(6r^2-1\right)^2}\sqrt{1-s}+\dots\,,\\
f(r) &= \frac{\sqrt{2}\left(-3+8\sqrt{6}r-36r^2+36r^4\right)}{\left(6r^2-1\right)^2}\sqrt{1-s}+\dots\,.
\end{aligned}
\tag{4.7}
$$

In particular one can check that these functions lead to a regular solution at the origin $r=\frac{1}{\sqrt{6}}$, although this is not manifest in the formulas presented above. Indeed, we have computed this expansion up to sixth order, and by comparing the two expansions we find that regularity at the origin fixes the parameter κ in (4.6) via

$$
\frac{3\sqrt{3}}{4}\kappa=\delta+\frac{\sqrt{2}}{3}\delta^2+\frac{113}{36}\delta^3+\frac{25}{9\sqrt{2}}\delta^4+\frac{1127}{288}\delta^5+\frac{35}{9\sqrt{2}}\delta^6+\dots\,,
\tag{4.8}
$$

where we have introduced

$$
\delta^2\equiv\frac{1}{s}-1\,.
\tag{4.9}
$$

The explicit solution ϵ_I to the Killing spinor (3.7) and dilatino equation (3.8) for this solution may be found in appendix B. In particular there are three independent constants of integration after imposing the symplectic Majorana condition (3.10). Using this solution one can compute the Killing vector bilinear (3.9). Requiring that this Killing vector lies in the Lie algebra of the maximal torus $\mathrm{U}(1)^3\subset\mathrm{SU}(3)\times\mathrm{U}(1)$ fixes the constants of integration, up to an overall irrelevant scaling. In this case we obtain

$$
K=b_1\partial_{\varphi_1}+b_2\partial_{\varphi_2}+b_3\partial_{\varphi_3}\,,
\tag{4.10}
$$

where $b_1=1+\sqrt{1-s^2}$, $b_2=b_3=1-\sqrt{1-s^2}$ and the coordinates φ_i are related to τ, ψ and φ via (2.24).

4.3 1/4 BPS solutions

We also find a two-parameter family of 1/4 BPS solutions, parametrized by the squashing parameter s and the background $\mathrm{SU}(2)_R$ field at the conformal boundary, which is parametrized by f_0. The solution expanded around the conformal boundary is given by

$$
\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r}-\frac{f_0^2s^2+9\left(-2+s^2\right)-6f_0\left(-1+s^2\right)}{36\sqrt{2}}\frac{1}{r^3}+\dots\,,\\
\gamma(r) &= \frac{3\sqrt{3}}{s}r+\frac{2f_0^2s^2-12f_0\left(-1+s^2\right)+9\left(-3+2s^2\right)}{12\sqrt{3}s}\frac{1}{r}+\dots\,,\\
X(r) &= 1+\frac{18-3f_0-18s^2+12f_0s^2-2f_0^2s^2}{54}\frac{1}{r^2}+\dots\,,\\
p(r) &= \frac{\mathrm{i}\sqrt{\frac{2}{3}}(-3+f_0)\left(3+(-3+f_0)s^2\right)}{s}\frac{1}{r^2}+\dots\,,\\
q(r) &= -\frac{3\mathrm{i}\sqrt{6}\left(3+(-3+f_0)s^2\right)}{s}r\\
&\quad+\frac{\mathrm{i}\left(3+(-3+f_0)s^2\right)\left(f_0^2s^2+9\left(-1+s^2\right)-6f_0\left(1+s^2\right)\right)}{6\sqrt{6}s}\frac{1}{r}+\frac{\xi_1}{r^2}+\dots\,,
\end{aligned}
$$

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$$f(r) = f_0 + \frac{2(-3+f_0)f_0}{9}\frac{1}{r^2} + \frac{\xi_2}{r^3} + \dots \ . \tag{4.11}$$

Again, we have found this solution up to $\mathcal{O}(1/r^9)$. The constants ξ_1 and ξ_2 are again fixed by requiring regularity at the origin.

There are a number of interesting special cases. First, we obtain the one-parameter family of 1/4 BPS squashed five-spheres of section 2.1 by choosing the constant f_0 so as to reproduce (2.4) with $Q = -3$. That is, $f_0 = (1 - 3\sqrt{1-s^2})\sqrt{1-s^2}/s^2$. We show explicitly in appendix B that the supergravity Killing spinor matches onto the five-dimensional spinors in section 2.1. Another interesting case is $f_0 = 0$. In this case the $\mathrm{SU}(2)_R$ background gauge field is completely switched off, but the solution is still supersymmetric with a squashed five-sphere at the conformal boundary. This solution has enhanced supersymmetry — as we show in appendix B it is 1/2 BPS. On the other hand we may also set $s = 1$, so that the conformal boundary is the *round* five-sphere, but keep the parameter f_0. This shows that one can define non-trivial Killing spinors on the *round* S^5 by turning on other fields.

We may also expand the solution around Euclidean AdS_6 with $s = 1$:

$$\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \frac{\sqrt{3}\left(1 - 54r^2 + 96\sqrt{6}r^3 - 324r^4 + 216r^6\right)}{2r^2\left(6r^2-1\right)^{7/2}}(1-s) + \dots \ , \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} + \frac{\left(15 - 48\sqrt{6}r + 270r^2 - 540r^4 + 648r^6\right)}{\sqrt{2}\left(6r^2-1\right)^{5/2}}(1-s) + \dots \ , \\
X(r) &= 1 + \frac{\left(1 - 2\sqrt{6}r + 6r^2\right)(4+\omega)}{\left(6r^2-1\right)^2}(1-s) + \dots \ , \\
p(r) &= -\frac{18\mathrm{i}\sqrt{2}\left(-\sqrt{3} + 8\sqrt{2}r - 12\sqrt{3}r^2 + 12\sqrt{3}r^4\right)(6+\omega)}{\left(6r^2-1\right)^3}(1-s) + \dots \ , \\
q(r) &= -\frac{3\mathrm{i}\left(-4 + 9\sqrt{6}r - 24r^2 - 12\sqrt{6}r^3 + 36\sqrt{6}r^5\right)(6+\omega)}{\left(6r^2-1\right)^2}(1-s) + \dots \ , \\
f(r) &= \frac{\left(-3 + 8\sqrt{6}r - 36r^2 + 36r^4\right)\omega}{\left(6r^2-1\right)^2}(1-s) + \dots \ ,
\end{aligned} \tag{4.12}$$

where we have introduced the parameter ω via $(1-s)\omega = f_0$. As before it can be checked explicitly that the solution is regular at $r = \frac{1}{\sqrt{6}}$, and we have checked this up to fourth order in the expansion variable

$$\delta \equiv \frac{1}{s} - 1 \ . \tag{4.13}$$

Comparing this expansion with the expansion around the conformal boundary we deduce

$$\begin{aligned}
\xi_1 = {} & 2\mathrm{i}(6+\omega)\delta - \frac{\mathrm{i}\left(144 + 98\omega + 13\omega^2\right)}{5}\delta^2 \\
& + \frac{\mathrm{i}\left(307719 + 209547\omega + 41094\omega^2 + 1282\omega^3\right)}{9450}\delta^3 \\
& - \frac{\mathrm{i}\left(26693550 + 21683700\omega + 6126111\omega^2 + 771474\omega^3 + 51568\omega^4\right)}{623700}\delta^4 + \dots \ ,
\end{aligned}$$

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$$\xi_2 = \frac{2}{3}\sqrt{\frac{2}{3}}\omega\delta - \frac{2\left(-\sqrt{6}\omega + 2\sqrt{6}\omega^2\right)}{45}\delta^2 + \frac{\left(-999\sqrt{6}\omega - 594\sqrt{6}\omega^2 + 244\sqrt{6}\omega^3\right)}{42525}\delta^3 + \frac{\left(32724\sqrt{6}\omega + 26082\sqrt{6}\omega^2 + 6105\sqrt{6}\omega^3 + 935\sqrt{6}\omega^4\right)}{1403325}\delta^4 + \dots \,. \tag{4.14}$$

The explicit solution ϵ_I to the dilatino and Killing spinor equation (3.8), (3.7) for this solution may also be found in appendix B. In this case there is a single integration constant (for generic f_0, or equivalently ω). The Killing vector automatically lies in the Lie algebra of the torus $\mathrm{U}(1)^3 \subset \mathrm{SU}(3) \times \mathrm{U}(1)$, and with an appropriate scaling we obtain

$$K = \partial_\tau \;=\; b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} + b_3\partial_{\varphi_3}\,, \tag{4.15}$$

where $b_1 = b_2 = b_3 = 1$ and the coordinates φ_i are related to τ, ψ and φ via (2.24).

5 Holographic free energy

In this section we describe how the on-shell action for the Euclidean Romans theory detailed in section 3 can be computed, and for asymptotically locally Euclidean AdS solutions holographically renormalized by adding boundary counterterms [19–21]. For the supersymmetric solutions presented in section 4 we evaluate the renormalized on-shell action and determine the holographic free energies.

5.1 On-shell action

We will work in the gauge $A = 0$. Starting from the Euclidean action (3.6) and using the equations of motion (3.3) together with the Einstein equation (3.5) and its trace, we find the following for the on-shell action defined on a manifold M_6 with boundary ∂M_6

$$I_{\text{on-shell}} = I_{\text{bulk}} + I_{\text{boundary}}\,, \tag{5.1}$$

where

$$I_{\text{bulk}} = \frac{1}{16\pi G_N}\int_{M_6}\frac{4}{9}X^{-2}\left(2+3X^4\right)*1 + \frac{1}{3}X^{-2}F^i\wedge *F^i + \frac{\mathrm{i}}{3}B\wedge F^i\wedge F^i\,, \tag{5.2}$$

$$I_{\text{boundary}} = \frac{1}{16\pi G_N}\int_{\partial M_6}\frac{2}{3}\left(X^{-1}*\mathrm{d}X\right) + \frac{1}{3}(B\wedge X^4 * H)\,. \tag{5.3}$$

Here we have used Stokes' theorem to write a total derivative as a boundary integral. In particular this assumes that the potentials B and A^i are globally defined, which is the case for our supergravity solutions. The Hodge duals in (5.3) are defined on M_6, and then restricted to the boundary. The on-shell action is divergent due to the infinite volume of M_6 and ∂M_6, and from divergences in the supergravity fields as the conformal boundary $r \to \infty$ is approached. Consequently, I_{bulk} should be understood as integrated up to a finite cut-off which is then sent to infinity only after adding counterterms which regularize the divergences. In addition, because of the presence of boundary terms in the on-shell action, one should add a Gibbons-Hawking term [22]

$$I_{\text{GH}} = -\frac{1}{8\pi G_N}\int_{\partial M_6}\mathcal{K}\sqrt{\det h}\,\mathrm{d}^5x\,. \tag{5.4}$$

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This involves the trace $\mathcal{K}$ of the extrinsic curvature of the boundary, and where h_{mn} is the induced boundary metric, and also leads to divergences. Hence the finite on-shell action is

$$I_{\text{renormalized}} = I_{\text{on-shell}} + I_{\text{GH}} + I_{\text{counterterms}} \,. \tag{5.5}$$

In the next subsection we determine the precise form of the counterterms.

5.2 Boundary counterterms

The counterterms needed to regularize the action of the Euclidean Romans $F(4)$ theory were stated without derivation in [23]. Here we provide a full account of their construction. We assume a general expansion of the fields for an asymptotically locally Euclidean AdS_6 solution. In particular, we take the metric to be given in Fefferman-Graham form [24, 25]

$$\mathrm{d}s_6^2 = \frac{\ell^2}{z^2}\mathrm{d}z^2 + \frac{1}{z^2}\gamma_{mn}(z,x)\mathrm{d}x^m\mathrm{d}x^n \,, \tag{5.6}$$

where $\ell = 3/\sqrt{2}$ is the AdS_6 radius, and in turn

$$\gamma_{mn}(z,x) = \gamma^0_{mn} + z^2\gamma^2_{mn} + z^4\gamma^4_{mn} + \mathcal{O}(z^5) \,. \tag{5.7}$$

Here $\gamma^0_{mn}(x)$ is the metric induced on the conformal boundary which, due to the radial coordinate transformation $r \to \frac{1}{z}$, is now at $z = 0$. The Gibbons-Hawking term is then

$$I_{\text{GH}} = \frac{1}{8\pi G_N}\int_{\partial M_6}\frac{z}{\ell}\partial_z\sqrt{\det h}\;\mathrm{d}^5x \,, \tag{5.8}$$

and $h_{mn} = \frac{1}{z^2}\gamma_{mn}$ is the induced metric on the boundary.

The Ricci tensor of the six-dimensional metric (5.6) is

$$\begin{aligned}
R_{zz} &= -\frac{5}{z^2} - \frac{1}{2}\left[\mathrm{Tr}\left(\gamma^{-1}\partial_z^2\gamma\right) - \frac{1}{z}\mathrm{Tr}\left(\gamma^{-1}\partial_z\gamma\right) - \frac{1}{2}\mathrm{Tr}\left(\gamma^{-1}\partial_z\gamma\right)^2\right] \,, \\
R_{mn} &= -\frac{5}{\ell^2 z^2}\gamma_{mn} - \frac{1}{\ell^2}\left[\frac{1}{2}\partial_z^2\gamma - \frac{2}{z}\partial_z\gamma - \frac{1}{2}(\partial_z\gamma)\gamma^{-1}(\partial_z\gamma) + \frac{1}{4}(\partial_z\gamma)\mathrm{Tr}\left(\gamma^{-1}\partial_z\gamma\right)\right. \\
&\qquad\qquad \left. -\ell^2 R(\gamma) - \frac{1}{2z}\gamma\mathrm{Tr}\left(\gamma^{-1}\partial_z\gamma\right)\right]_{mn} \,, \\
R_{zm} &= \frac{1}{2}(\gamma^{-1})^{np}\left[\nabla_m\gamma_{np,z} - \nabla_p\gamma_{mn,z}\right] \,,
\end{aligned} \tag{5.9}$$

with ∇ being the covariant derivative for $\gamma(z,x)$. We also assume an asymptotic expansion for bulk scalar and gauge fields, namely

$$\begin{aligned}
X &= 1 + zX_1 + z^2X_2 + \cdots \,, \\
B &= \frac{1}{z}b + \mathrm{d}z\wedge A_0 + B_0 + z\mathrm{d}z\wedge A_1 + zB_1 + \cdots \,, \\
H = \mathrm{d}B &= -\frac{1}{z^2}\mathrm{d}z\wedge b + \frac{1}{z}\mathrm{d}b - \mathrm{d}z\wedge\mathrm{d}A_0 + \mathrm{d}B_0 + \mathrm{d}z\wedge B_1 - z\mathrm{d}z\wedge\mathrm{d}A_1\cdots \,, \\
F^i &= f^i + \mathrm{d}z\wedge A_0^i + z\mathrm{d}z\wedge A_1^i + zF_1^i + \cdots \,.
\end{aligned} \tag{5.10}$$

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The $1/z$ term appearing in the B-field expansion is non-standard but is justified by being compatible with the equations of motion as we will see below.

It is useful to establish some formulas. We write (in general)

$$\alpha \wedge *\alpha = \|\alpha\|^2 \mathrm{vol}\,, \tag{5.11}$$

to define the norm $\|\cdot\|$ of a p-form. The inner product of two p-forms α, β is denoted $\langle \alpha, \beta \rangle$. First we compute

$$\begin{aligned} *\alpha_p &= -\ell z^{2p-6} \left(*_\gamma \alpha_p\right) \wedge \mathrm{d}z\,, \\ *(\mathrm{d}z \wedge \alpha_{p-1}) &= \frac{1}{\ell} z^{2p-6} *_\gamma \alpha_{p-1}\,, \end{aligned} \tag{5.12}$$

where α_p represents a general p-form that is orthogonal to ∂_z. Here the volume forms are related as

$$\mathrm{vol}_6 = \frac{\ell}{z^6} \mathrm{d}z \wedge \mathrm{vol}_\gamma \;=\; \frac{\ell}{z^6} \mathrm{d}z \wedge \sqrt{\det \gamma}\; \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^5\,. \tag{5.13}$$

We will need the expansion of the determinant and Hodge dual for γ_{mn}. The former is

$$\begin{aligned} \sqrt{\det \gamma} = \sqrt{\det \gamma^0} \Big[1 + \frac{z^2}{2} \mathrm{Tr}\left[\gamma^2 (\gamma^0)^{-1}\right] + \frac{z^4}{2} \mathrm{Tr}\left[\gamma^4 (\gamma^0)^{-1}\right] \\ - \frac{z^4}{4} \mathrm{Tr}\left[\gamma^2 (\gamma^0)^{-1}\right]^2 + \frac{z^4}{8} \left(\mathrm{Tr}\left[\gamma^2 (\gamma^0)^{-1}\right]\right)^2 + \mathcal{O}(z^5) \Big]\,, \end{aligned} \tag{5.14}$$

whilst the latter may be computed similarly as

$$*_\gamma\, \alpha_p = *_{\gamma^0} \alpha_p + z^2 \left[\frac{1}{2} \mathrm{Tr}\left[\gamma^2 (\gamma^0)^{-1}\right] *_{\gamma^0} \alpha_p - p *_{\gamma^0} (\gamma^2 \circ \alpha_p) \right] + \mathcal{O}(z^4)\,. \tag{5.15}$$

Here we have defined the p-form

$$(\gamma^2 \circ \alpha_p)_{m_1 \cdots m_p} \equiv (\gamma^2)_{[m_1}{}^n (\alpha_p)_{|n| m_2 \cdots m_p]}\,, \tag{5.16}$$

and indices are always raised with γ^0, so $(\gamma^2)_m{}^n \equiv (\gamma^2)_{mp} (\gamma^0)^{pn}$.

The idea now is to substitute these expansions into the Romans field equations and then on-shell action. We first look at the lowest order term in z in each of the X, B and Einstein equations. The leading order term in the X equation of motion dictates

$$X_1 = 0\,. \tag{5.17}$$

Specifically, the term $\frac{1}{z^5} \mathrm{d}z \wedge \mathrm{vol}_{\gamma^0}$ has a coefficient proportional to X_1 times a non-zero number, thus forcing $X_1 = 0$. Next one finds that the leading order term in the B equation of motion, which is proportional to $\frac{1}{z^3} \mathrm{d}z \wedge *_{\gamma^0} b$, has a coefficient that is zero if and only if $\ell^2 = 9/2$. Similarly, the leading order term in the mn component of the Einstein equation, which is $\mathcal{O}(1/z^2)$, is satisfied if and only if $\ell^2 = 9/2$. We will substitute $\ell = 3/\sqrt{2}$ from now on.

The first divergence we encounter, which is at order $\mathcal{O}(1/\epsilon^5)$ where $z = \epsilon$ is the finite cut-off, comes from expanding the $\frac{4}{9} X^{-2} \left(2 + 3X^4\right) * 1$ integrand in I_{bulk} and the Gibbons-Hawking term. It is

$$I^{\mathrm{div}}_{\mathcal{O}(1/\epsilon^5)} = \frac{1}{8\pi G_N} \frac{1}{\epsilon^5} \int_{\partial M_6} -\frac{4\sqrt{2}}{3} \sqrt{\det \gamma^0}\; \mathrm{d}^5 x\,, \tag{5.18}$$

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and is simply cancelled by adding the counterterm

$$I_5^{\text{counterterm}} = \frac{1}{8\pi G_N} \cdot \frac{4\sqrt{2}}{3} \int_{\partial M_6} \sqrt{\det h}\ \mathrm{d}^5x\ . \tag{5.19}$$

We write the counterterm action in terms of the induced boundary metric h_{mn} as the divergences most naturally appear in this form [26]. There is no divergence at $\mathcal{O}(1/\epsilon^4)$ as a consequence of $X_1 = 0$. The divergence at $\mathcal{O}(1/\epsilon^3)$ has contributions from each of I_{bulk}, I_{boundary}, I_{GH} and the expansion of $I_5^{\text{counterterm}}$, and is

$$I^{\text{div}}_{\mathcal{O}(1/\epsilon^3)} = \frac{1}{8\pi G_N}\frac{1}{\epsilon^3}\int_{\partial M_6}\left[\frac{4\sqrt{2}}{9}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] + \frac{1}{9\sqrt{2}}\|b\|^2_{\gamma^0}\right]\sqrt{\det\gamma^0}\ \mathrm{d}^5x\,. \tag{5.20}$$

Clearly we will need some control on γ^2, and this comes from the $\mathcal{O}(1)$ term in the mn direction of the Einstein equation. Carefully expanding we find this fixes

$$\gamma^2_{mn} = -\frac{3}{2}\left[R(\gamma^0)_{mn} - \frac{1}{8}R(\gamma^0)\gamma^0_{mn}\right] + \frac{1}{2}b^2_{mn} - \frac{3}{16}\|b\|^2_{\gamma^0}\gamma^0_{mn}\ . \tag{5.21}$$

Here $R(g)_{mn} = \mathrm{Ric}(g)_{mn}$ denotes the Ricci tensor of a metric g_{mn}, with $R(g)$ the Ricci scalar. The curvature terms in γ^2_{mn} are standard [19], while the terms involving b are specific to the Romans theory and boundary conditions we are considering. Taking the trace of (5.21), or alternatively examining the zz component of the Einstein equation at order $\mathcal{O}(1)$, gives

$$\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] = -\frac{9}{16}R(\gamma^0) + \frac{1}{16}\|b\|^2_{\gamma^0}\ . \tag{5.22}$$

This expression will need to be used extensively due to its appearance in the Hodge dual and metric determinant. Substituting $\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]$ into the right hand side of $I^{\text{div}}_{\mathcal{O}(1/\epsilon^3)}$ leads to

$$I^{\text{div}}_{\mathcal{O}(1/\epsilon^3)} = \frac{1}{8\pi G_N}\frac{1}{\epsilon^3}\int_{\partial M_6}\left[-\frac{1}{2\sqrt{2}}R(\gamma^0) + \frac{1}{6\sqrt{2}}\|b\|^2_{\gamma^0}\right]\sqrt{\det\gamma^0}\ \mathrm{d}^5x\,, \tag{5.23}$$

and the appropriate counterterm is therefore

$$I_3^{\text{counterterm}} = \frac{1}{8\pi G_N}\int_{\partial M_6}\left[\frac{1}{2\sqrt{2}}R(h) - \frac{1}{6\sqrt{2}}\|B\|^2_h\right]\sqrt{\det h}\ \mathrm{d}^5x\ . \tag{5.24}$$

A priori there is also an $\mathcal{O}(1/\epsilon^2)$ divergence, but one easily sees from the various expansions that only the scalar field contributes to it. This term (temporarily reinstating the AdS length scale) is

$$I^{\text{div}}_{\mathcal{O}(1/\epsilon^2)} = \frac{1}{8\pi G_N}\left(\frac{4\ell}{9}\cdot\frac{1}{2} - \frac{1}{\ell}\right)\frac{1}{\epsilon^2}\int_{\partial M_6} X_3\sqrt{\det\gamma^0}\ \mathrm{d}^5x\ =\ 0\,, \tag{5.25}$$

where the first term comes from expanding the bulk integral (5.2), while the second (which cancels it) comes from the boundary $X^{-1}{*}\mathrm{d}X$ term in (5.3). Thus this potential divergence is zero, without needing a counterterm or indeed even needing to use any of the equations of motion.

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Continuing we find there are many terms that contribute at $\mathcal{O}(1/\epsilon)$ including A_1 and B_0 from the asymptotic expansion of the B-field. It is prudent to look at higher orders of z in the equations of motion for simplifications along the lines of $X_1 = 0$. Indeed by looking at the $z^{-2}\mathrm{d}z \wedge \alpha_3$ coefficient of the B-field equation of motion we find

$$B_0 = 0\,. \tag{5.26}$$

The $z^{-1}\alpha_4$ coefficient similarly implies

$$A_1 = 0\,. \tag{5.27}$$

With these simplifications the $\mathcal{O}(1/\epsilon)$ divergence becomes

$$\begin{aligned} I^{\mathrm{div}}_{\mathcal{O}(1/\epsilon)} = \frac{1}{8\pi G_N}\frac{1}{\epsilon}\int_{\partial M_6}\Bigg[&\frac{29\sqrt{2}}{9}(X_2)^2 + \frac{2\sqrt{2}}{9}X_4 + \frac{2\sqrt{2}}{9}X_2\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] + \frac{\sqrt{2}}{4}\|f^i\|^2_{\gamma^0} \\ &-\frac{\sqrt{2}}{72}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]\|b\|^2_{\gamma^0} + \frac{\sqrt{2}}{18}\langle b, \gamma^2\circ b\rangle + \frac{2\sqrt{2}}{9}X_2\|b\|^2_{\gamma^0} + \frac{\sqrt{2}}{18}\langle b, \mathrm{d}A_0\rangle \\ &+\frac{4\sqrt{2}}{3}\mathrm{Tr}\left[\gamma^4(\gamma^0)^{-1}\right] - \frac{2\sqrt{2}}{3}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]^2 + \frac{\sqrt{2}}{3}\left(\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]\right)^2 \\ &-\frac{\sqrt{2}}{4}R(\gamma^0)_{ij}(\gamma^2)^{ij} + \frac{\sqrt{2}}{8}R(\gamma^0)\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]\Bigg]\sqrt{\det\gamma^0}\;\mathrm{d}^5x\,. \end{aligned} \tag{5.28}$$

We now seek to determine A_0, X_4 and γ^4 in terms of lower order boundary quantities such as b. Examination of the $z^{-2}\alpha_4$ coefficient of the B-field equation of motion gives

$$\mathrm{d}*_{\gamma^0} b = -\frac{\mathrm{i}\sqrt{2}}{3}b\wedge b - \frac{4}{9}*_{\gamma^0} A_0\,, \tag{5.29}$$

which we should regard as fixing A_0 in terms of the boundary field b. Specifically, since $*_5^2 = 1$ on any form, we solve this as

$$A_0 = -\frac{9}{4}*_{\gamma^0}\left(\mathrm{d}*_{\gamma^0} b + \frac{\mathrm{i}\sqrt{2}}{3}b\wedge b\right)\,. \tag{5.30}$$

Note we may also write $*_{\gamma^0}\mathrm{d}*_{\gamma^0} b = \delta_{\gamma^0} b$ in terms of the adjoint δ_{γ^0} of d with respect to γ^0. The $z^{-1}\mathrm{d}z\wedge\alpha_3$ coefficient determines B_1 to be

$$B_1 = *_{\gamma^0}\left(\frac{9}{4}\mathrm{d}*_{\gamma^0}\mathrm{d}b - \frac{\mathrm{i}\sqrt{2}}{3}b\wedge A_0\right) + 2bX_2 - \frac{1}{2}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]b + 2\gamma^2\circ b\,, \tag{5.31}$$

which may be rewritten as

$$\begin{aligned} B_1 = \frac{9}{4}*_{\gamma^0}&\left[\mathrm{d}*_{\gamma^0}\mathrm{d}b + \frac{\mathrm{i}\sqrt{2}}{3}b\wedge\delta_{\gamma^0}b - \frac{2}{9}b\wedge *_{\gamma^0}(b\wedge b)\right] + 2bX_2 \\ &-\frac{1}{2}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]b + 2\gamma^2\circ b\,. \end{aligned} \tag{5.32}$$

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The next coefficient we need is X_4, the coefficient of z^4 in the expansion of $X(z, x^m)$ and is found from the $z^{-2}\mathrm{d}z \wedge \mathrm{vol}_{\gamma^0}$ terms in the X field equation

$$\begin{aligned} X_4 = &-\frac{9}{4}\Delta_{\gamma^0} X_2 - X_2 \mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right] - \frac{11}{2}(X_2)^2 + \frac{3}{4}X_2\|b\|^2_{\gamma^0} \\ &+\frac{9}{16}\|\mathrm{d}b\|^2_{\gamma^0} - \frac{1}{36}\|A_0\|^2_{\gamma^0} - \frac{1}{2}\langle B_1, b\rangle + \frac{1}{4}\langle b, \mathrm{d}A_0\rangle - \frac{9}{32}\|f^i\|^2_{\gamma^0}\,. \end{aligned} \tag{5.33}$$

Here $\Delta_{\gamma^0} = \delta_{\gamma^0}\mathrm{d}$ acting on functions but will not contribute for a compact boundary (after integrating by parts).

We also need γ^4_{mn}, which comes from expanding the zz component of the Einstein equation at $\mathcal{O}(z^2)$:

$$\begin{aligned} \mathrm{Tr}\left[\gamma^4(\gamma^0)^{-1}\right] = &+\frac{1}{4}\mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]^2 - \frac{5}{2}(X_2)^2 - \frac{1}{24}\|A_0\|^2_{\gamma^0} + \frac{9}{32}\|\mathrm{d}b\|^2_{\gamma^0} - \frac{3}{8}X_2\|b\|^2_{\gamma^0} \\ &+\frac{1}{4}\langle b, B_1\rangle - \frac{1}{8}\langle b, \mathrm{d}A_0\rangle + \frac{9}{64}\|f^i\|^2_{\gamma^0}\,. \end{aligned} \tag{5.34}$$

Next we record some intermediate formulae which follow from the expression for γ^2_{mn} in (5.21):

$$\begin{aligned} \mathrm{Tr}\left[\gamma^2(\gamma^0)^{-1}\right]^2 = &\frac{9}{4}\left[R(\gamma^0)_{mn}R(\gamma^0)^{mn} - \frac{11}{64}R(\gamma^0)^2\right] + \frac{1}{4}\mathrm{Tr}_{\gamma^0}b^4 \\ &-3\langle \mathrm{Ric}(\gamma^0)\circ b, b\rangle_{\gamma^0} + \frac{75}{128}R(\gamma^0)\|b\|^2_{\gamma^0} - \frac{51}{256}\|b\|^4_{\gamma^0}\,, \\ R(\gamma^0)_{mn}(\gamma^2)^{mn} = &-\frac{3}{2}R(\gamma^0)_{mn}R(\gamma^0)^{mn} + \frac{3}{16}R(\gamma^0)^2 + \langle \mathrm{Ric}(\gamma^0)\circ b, b\rangle_{\gamma^0} \\ &-\frac{3}{16}R(\gamma^0)\|b\|^2_{\gamma^0}\,, \\ \langle \gamma^2\circ b, b\rangle = &-\frac{3}{2}\langle \mathrm{Ric}(\gamma^0)\circ b, b\rangle_{\gamma^0} + \frac{1}{4}\mathrm{Tr}_{\gamma^0}b^4 + \frac{3}{16}R(\gamma^0)\|b\|^2_{\gamma^0} - \frac{3}{16}\|b\|^4_{\gamma^0}\,. \end{aligned} \tag{5.35}$$

Here we have defined $\mathrm{Tr}_{\gamma^0}b^4 \equiv b_m{}^n b_n{}^p b_p{}^q b_q{}^m$. Notice that $\mathrm{Tr}_{\gamma^0}b^2 = -2\|b\|^2_{\gamma^0}$, with this notation.

We now have all that we need to compute the $\mathcal{O}(1/\epsilon)$ counterterm. Inserting all our intermediate results along with the newfound expressions for X_4 etc. into $I^{\mathrm{div}}_{\mathcal{O}(1/\epsilon)}$ in (5.28) leads to

$$\begin{aligned} I^{\mathrm{div}}_{\mathcal{O}(1/\epsilon)} = &\frac{1}{8\pi G_N}\frac{1}{\epsilon}\int_{\partial M_6}\Bigg\{\Bigg[-\frac{3}{4\sqrt{2}}R(\gamma^0)_{mn}R(\gamma^0)^{mn} + \frac{15}{64\sqrt{2}}R(\gamma^0)^2 \\ &+\frac{3}{4\sqrt{2}}\|f^i\|^2_{\gamma^0} - \frac{1}{12\sqrt{2}}\mathrm{Tr}_{\gamma^0}b^4 + \frac{13}{192\sqrt{2}}\|b\|^4_{\gamma^0} + \frac{1}{\sqrt{2}}\|\mathrm{d}b\|^2_{\gamma^0} \\ &-\frac{5}{8\sqrt{2}}\|\mathrm{d}*_{\gamma^0}b + \frac{\mathrm{i}\sqrt{2}}{3}b\wedge b\|^2_{\gamma^0} + \frac{1}{4\sqrt{2}}\langle b, \mathrm{d}\delta_{\gamma^0}b + \frac{\mathrm{i}\sqrt{2}}{3}\mathrm{d}[*_{\gamma^0}b\wedge b]\rangle \\ &-\frac{4\sqrt{2}}{3}(X_2)^2 + \frac{1}{\sqrt{2}}\langle \mathrm{Ric}(\gamma^0)\circ b, b\rangle_{\gamma^0} - \frac{9}{32\sqrt{2}}R(\gamma^0)\|b\|^2_{\gamma^0}\Bigg]\sqrt{\det\gamma^0}\;\mathrm{d}^5x \\ &+\frac{1}{4\sqrt{2}}\langle b, *_{\gamma^0}\left[\mathrm{d}*_{\gamma^0}\mathrm{d}b + \frac{\mathrm{i}\sqrt{2}}{3}b\wedge\delta b - \frac{2}{9}b\wedge *_{\gamma^0}(b\wedge b)\right]\rangle\Bigg\}\,. \end{aligned} \tag{5.36}$$

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The corresponding counterterm is hence

$$
\begin{aligned}
I_1^{\text{counterterm}} = \frac{1}{8\pi G_N}\int_{\partial M_6}\Bigg\{\Bigg[&\frac{3}{4\sqrt{2}}R(h)_{mn}R(h)^{mn}-\frac{15}{64\sqrt{2}}R(h)^2\\
&-\frac{3}{4\sqrt{2}}\|F^i\|_h^2+\frac{1}{12\sqrt{2}}\mathrm{Tr}_h B^4-\frac{13}{192\sqrt{2}}\|B\|_h^4-\frac{1}{\sqrt{2}}\|\mathrm{d}B\|_h^2\\
&+\frac{5}{8\sqrt{2}}\|\mathrm{d}*_h B+\frac{\mathrm{i}\sqrt{2}}{3}B\wedge B\|_h^2-\frac{1}{4\sqrt{2}}\langle B,\mathrm{d}\delta_h B+\frac{\mathrm{i}\sqrt{2}}{3}\mathrm{d}*_h B\wedge B\rangle_h\\
&+\frac{4\sqrt{2}}{3}(1-X)^2-\frac{1}{\sqrt{2}}\langle\mathrm{Ric}(h)\circ B,B\rangle_h+\frac{9}{32\sqrt{2}}R(h)\|B\|_h^2\Bigg]\sqrt{\det h}\,\mathrm{d}^5x\\
&-\frac{1}{4\sqrt{2}}B\wedge\left[\mathrm{d}*_h\mathrm{d}B+\frac{\sqrt{2}\mathrm{i}}{3}B\wedge\delta_h B-\frac{2}{9}B\wedge*_h(B\wedge B)\right]\Bigg\}\,.
\end{aligned}
\tag{5.37}
$$

Once again the pure gravity terms found in the first line agree with the literature [19].

A priori the bulk integral in (5.2) is logarithmically divergent. Of course a log divergence should not appear, as the boundary is odd-dimensional and on general grounds one does not expect local anomalies. In keeping with this argument the equations of motion at even higher order in z constrain the fields such that the potential log divergence cancels without the need for a counterterm.

Collating all the expressions for the counterterms we finally arrive at [23]

$$
\begin{aligned}
I_{\text{counterterms}} = \frac{1}{8\pi G_N}\int_{\partial M_6}\Bigg\{\Bigg[&\frac{4\sqrt{2}}{3}+\frac{1}{2\sqrt{2}}R(h)-\frac{1}{6\sqrt{2}}\|B\|_h^2\\
&+\frac{3}{4\sqrt{2}}R(h)_{mn}R(h)^{mn}-\frac{15}{64\sqrt{2}}R(h)^2\\
&-\frac{3}{4\sqrt{2}}\|F^i\|_h^2+\frac{1}{12\sqrt{2}}\mathrm{Tr}_h B^4-\frac{13}{192\sqrt{2}}\|B\|_h^4-\frac{1}{\sqrt{2}}\|\mathrm{d}B\|_h^2\\
&+\frac{5}{8\sqrt{2}}\|\mathrm{d}*_h B+\frac{\mathrm{i}\sqrt{2}}{3}B\wedge B\|_h^2-\frac{1}{4\sqrt{2}}\langle B,\mathrm{d}\delta_h B+\frac{\mathrm{i}\sqrt{2}}{3}\mathrm{d}*_h B\wedge B\rangle_h\\
&+\frac{4\sqrt{2}}{3}(1-X)^2-\frac{1}{\sqrt{2}}\langle\mathrm{Ric}(h)\circ B,B\rangle_h+\frac{9}{32\sqrt{2}}R(h)\|B\|_h^2\Bigg]\sqrt{\det h}\,\mathrm{d}^5x\\
&-\frac{1}{4\sqrt{2}}B\wedge\left[\mathrm{d}*_h\mathrm{d}B+\frac{\sqrt{2}\mathrm{i}}{3}B\wedge\delta_h B-\frac{2}{9}B\wedge*_h(B\wedge B)\right]\Bigg\}\,.
\end{aligned}
\tag{5.38}
$$

5.3 Free energy of the solutions

The renormalized on-shell action determined in the previous subsection holds for all Romans supergravity solutions which are asymptotically locally AdS. In particular we may use these results to compute the holographic free energy for the supersymmetric solutions of section 4. In order to present the results, we first split the renormalized action as

$$
I_{\text{renormalized}} = I_{\text{bulk}} + I_{\text{non-bulk}}\,,
\tag{5.39}
$$

where I_{bulk} is the bulk integral given by (5.2), while

$$
I_{\text{non-bulk}} = I_{\text{boundary}} + I_{\text{GH}} + I_{\text{counterterms}}\,,
\tag{5.40}
$$

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where I_{boundary} is the boundary contribution to the on-shell action (5.3), I_{GH} is the Gibbons-Hawking term, while $I_{\text{counterterms}}$ is the full counterterm (5.38). For our SU(3) × U(1) ansatz (4.1), with $f^1(r) \equiv f^2(r) \equiv 0$ and $f^3(r) = f(r)$, we have in particular

$$I_{\text{bulk}} = \frac{\pi^2}{36G_N}\int_{r=\frac{1}{\sqrt{6}}}^{\Lambda}\left[3X^2(r)\alpha(r)\beta^4(r)\gamma(r) + 6\mathrm{i}f(r)\left[f(r)p(r) + q(r)f'(r)\right]\right.$$
$$\left. + \frac{24f^2(r)\alpha^2(r)\gamma^2(r) + 8\alpha^2(r)\beta^4(r)\gamma^2(r) + 3\beta^4(r)(f'(r))^2}{4X^2(r)\alpha(r)\gamma(r)}\right]\mathrm{d}r\,, \quad (5.41)$$

where Λ is the cut-off for the r coordinate.

3/4 BPS solution. For the one-parameter family of 3/4 BPS solutions in section 4.2 we obtain

$$\begin{aligned} I_{\text{bulk}} = \frac{\pi^2}{36G_N}\Bigg[&\frac{6561\sqrt{\frac{3}{2}}}{s}\Lambda^5 - \frac{243\sqrt{\frac{3}{2}}\left(3+12s^2+\sqrt{1-s^2}\right)}{s^3}\Lambda^3 \\ &- \frac{2187\sqrt{6}\kappa\left(-1+\sqrt{1-s^2}\right)}{8s}\Lambda^2 \\ &+ \frac{27\left[\sqrt{\frac{3}{2}}\left(74+66s^4-14\sqrt{1-s^2}-s^2\left(5+4\sqrt{1-s^2}\right)\right)\right]}{4s^5}\Lambda \\ &-243 + \frac{81\delta}{2\sqrt{2}} - 1377\delta^2 - \frac{1467\delta^3}{8\sqrt{2}} - \frac{6693\delta^4}{2} - \frac{44073\delta^5}{64\sqrt{2}} - 4482\delta^6 + \mathcal{O}(\delta^7)\Bigg] \\ &+\mathcal{O}\left(\frac{1}{\Lambda}\right), \end{aligned} \quad (5.42)$$

together with

$$\begin{aligned} I_{\text{non-bulk}} = \frac{\pi^2}{36G_N}\Bigg[&-\frac{6561\sqrt{\frac{3}{2}}}{s}\Lambda^5 + \frac{243\sqrt{\frac{3}{2}}\left(3+12s^2+\sqrt{1-s^2}\right)}{s^3}\Lambda^3 \\ &+ \frac{2187\sqrt{6}\kappa\left(-1+\sqrt{1-s^2}\right)}{8s}\Lambda^2 \\ &- \frac{27\left[\sqrt{\frac{3}{2}}\left(74+66s^4-14\sqrt{1-s^2}-s^2\left(5+4\sqrt{1-s^2}\right)\right)\right]}{4s^5}\Lambda \\ &+ \frac{81\sqrt{\frac{3}{2}}\left(-16+16\sqrt{1-s^2}+13s^2\left(1+3\sqrt{1-s^2}\right)\right)\kappa}{8s^3}\Bigg] + \mathcal{O}\left(\frac{1}{\Lambda}\right), \end{aligned} \quad (5.43)$$

where recall that κ is given as a series in δ in (4.8). Adding the two contributions and taking the cut-off $\Lambda \to \infty$, the divergences cancel and we are left with the following finite result

$$I_{\text{renormalized}} = -\frac{27\pi^2}{4G_N}\left(1 + \frac{8}{3}\delta^2 + \frac{16\sqrt{2}}{27}\delta^3 + \frac{68}{27}\delta^4 + \frac{28\sqrt{2}}{27}\delta^5 + \frac{32}{27}\delta^6 + \dots\right), \quad (5.44)$$

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where the six-dimensional Newton constant is given by[4]

$$G_N = \frac{15\pi\sqrt{8-N_f}}{4\sqrt{2}N^{5/2}}. \tag{5.45}$$

The holographic free energy is identified with $I_{\text{renormalized}}$ and agrees precisely with the series expansion of the large N field theory result (2.44)

$$\mathcal{F} = \frac{1}{27s^2}\frac{(3-\sqrt{1-s^2})^3}{1-\sqrt{1-s^2}}\mathcal{F}_{\text{round } S^5}\,, \tag{5.46}$$

where recall that $s = 1/(1+\delta^2)$.

1/4 BPS solution. We may similarly compute the holographic free energy of the two-parameter family of 1/4 BPS solutions in section 4.3. Again we obtain two divergent contributions whose divergences cancel. The finite piece may be computed as an expansion in $\delta = \frac{1}{s} - 1$ using the series expansions of the parameters ξ_1, ξ_2 in (4.14). Putting everything together we obtain

$$I_{\text{renormalized}} = -\frac{27\pi^2}{4G_N}\left(1+\mathcal{O}(\delta^5)\right)\,. \tag{5.47}$$

This again agrees with large N field theory result (2.44). Of course the latter field theory result was computed for a one-parameter subfamily of boundary conditions in section 2, while here we have a more general two-parameter family. We shall elaborate on this in section 8.

6 Boundary supersymmetry conditions

In this section we determine the form of the Euclidean Romans supersymmetry conditions, given in section 3, near the five-dimensional conformal boundary. Closely related work has appeared in [27]. Our conventions are the following: we use $x^\mu = (r, x^m)$ to denote six-dimensional coordinates, so that the indices $\mu, \nu, \ldots \in \{0,1,2,3,4,5\}$. Six-dimensional frame indices are indexed by $A, B, \ldots \in \{0,1,2,3,4,5\}$ and five-dimensional frame indices by early Roman letters a, b etc.

We continue to use the Fefferman-Graham coordinates outlined in subsection 5.2, although compared to that section we change coordinates $z \to 1/r$ so that the conformal boundary is now at $r = \infty$. We can then scale the r coordinate $r \to \lambda r$ without changing the position of the conformal boundary or modifying the five-dimensional boundary metric γ^0. After this scaling the asymptotic six-dimensional metric is now

$$\mathrm{d}s_6^2 = \frac{\ell^2}{r^2}\mathrm{d}r^2 + \lambda^2 r^2 \gamma_{mn}\mathrm{d}x^m\mathrm{d}x^n\,, \tag{6.1}$$

where

$$\gamma_{mn} = \gamma^0_{mn} + \frac{1}{\lambda^2 r^2}\gamma^2_{mn} + \frac{1}{\lambda^4 r^4}\gamma^4_{mn} + \frac{1}{\lambda^5 r^5}\gamma^5_{mn} + \mathcal{O}\left(\frac{1}{r^6}\right)\,. \tag{6.2}$$

[4]This was effectively calculated in [5] by identifying the holographic free energy of Euclidean AdS_6 with an entanglement entropy. The $N^{5/2}$ scaling of the free energy had previously been predicted in [9].

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We introduce a six-dimensional vielbein e^A such that

$$\mathrm{d}s_6^2 = e^A e^A \;=\; e^0 e^0 + e^a e^a \,. \tag{6.3}$$

If we denote by $e^a_{(5)}$ the vielbein for γ^0, then the six-dimensional frame components may be written as

$$e^0_r \;=\; \frac{\ell}{r}\,, \qquad e^0_m \;=\; 0\,, \qquad e^a_r \;=\; 0\,, \qquad e^a_m(r,x) \;=\; \lambda r e^a_{(5)m}(x) + \cdots\,, \tag{6.4}$$

where the ellipsis denotes subleading powers of r which will not play a part in what follows. The inverse frame is

$$e^r_0 = \frac{r}{\ell}\,, \qquad e^m_0 \;=\; 0\,, \qquad e^r_a \;=\; 0\,, \qquad (e^a_m)^{-1} \;=\; e^m_a \;=\; \frac{1}{\lambda r} e^m_{(5)a} + \ldots\,. \tag{6.5}$$

The six-dimensional spin connection is given by $\omega_\mu{}^{AB} \;=\; e^{\nu[A}\partial_\mu e_\nu{}^{B]} - e^{\nu[A}\partial_\nu e_\mu{}^{B]} - e^{[A}_\nu e^{B]}_\sigma e^C_\mu \partial^\nu e^\sigma_C$ and from this expression it is easy to show that

$$\omega_r{}^{bc} = 0 = \omega_r{}^{0b}\,, \qquad \omega_a{}^{0c} = -\frac{1}{\ell}\delta^c_a + \ldots\,, \qquad \omega_a{}^{bc} = \frac{1}{\lambda r}\omega^{(5d)bc}_a + \ldots\,, \tag{6.6}$$

where $\omega^{(5d)bc}_a$ is the spin connection associated with the 5d boundary metric γ^0.

Incorporating some of the results from the holographic renormalization in subsection 5.2, the asymptotic bulk field expansions in the local six-dimensional coordinates are[5]

$$\begin{aligned} X &= 1 + \frac{1}{r^2}X_2 + \cdots\,, \\ F = \frac{2}{3}B &= \frac{2r}{3}b - \frac{2}{3r^2}\mathrm{d}r\wedge A_0 + \cdots\,, \\ H = \mathrm{d}B &= \mathrm{d}r\wedge b + r\mathrm{d}b + \cdots\,, \\ F^i &= f^i + \cdots\,, \\ \mathcal{A}^i &= a^i + \cdots\,. \end{aligned} \tag{6.7}$$

Note that not all the fields appearing on the right hand side are independent. For example $f^i = \mathrm{d}a^i - \frac{1}{2}\varepsilon^{ijk}a^j\wedge a^k$ and A_0 was found in subsection 5.2 to be given by

$$A_0 = -\frac{9}{4}*_{\gamma^0}\left(\mathrm{d}*_{\gamma^0} b + \frac{\mathrm{i}\sqrt{2}}{3}b\wedge b\right)\,. \tag{6.8}$$

However, for simplicity we keep A_0 and substitute in terms of b only at the end of our computation. Converting the bulk field expansions first into the six-dimensional frame and then into the 5d frame using (6.5) we can read off the following components for the asymptotic fields

$$\begin{aligned} H_{0ab} &= \frac{r}{\ell(\lambda r)^2}b_{ab} + \mathcal{O}\left(\frac{1}{r^3}\right)\,, & H_{abc} &= \frac{r}{(\lambda r)^3}(\mathrm{d}b)_{abc} + \mathcal{O}\left(\frac{1}{r^4}\right)\,, \\ F_{0a} &= -\frac{2}{3\ell\lambda r^2}(A_0)_a + \mathcal{O}\left(\frac{1}{r^3}\right)\,, & F_{ab} &= \frac{2r}{3(\lambda r)^2}b_{ab} + \mathcal{O}\left(\frac{1}{r^3}\right)\,, \end{aligned}$$

[5]In this section we use a calligraphic font $\mathcal{A}^i$ to denote the SU(2) gauge field so that there is no confusion with other notation.

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$$
\begin{aligned}
X + \frac{1}{3}X^{-3} &= \frac{4}{3} + \mathcal{O}\left(\frac{1}{r^4}\right), & X - X^{-3} &= \frac{4}{r^2}X_2 + \mathcal{O}\left(\frac{1}{r^3}\right), \\
X^{-1}\partial_0 X &= -\frac{2}{\ell r^2}X_2 + \mathcal{O}\left(\frac{1}{r^3}\right), & X^{-1}\partial_a X &= \mathcal{O}\left(\frac{1}{r^3}\right), \\
F^i_{ab} &= \frac{1}{(\lambda r)^2}f^i_{ab} + \mathcal{O}\left(\frac{1}{r^3}\right), & F^i_{0a} &= \mathcal{O}\left(\frac{1}{r^3}\right), \\
\mathcal{A}^i_a &= \frac{1}{\lambda r}a^i_a + \mathcal{O}\left(\frac{1}{r^3}\right), & \mathcal{A}^i_0 &= \mathcal{O}\left(\frac{1}{r^3}\right).
\end{aligned} \tag{6.9}
$$

The full six-dimensional Killing spinor equation for the Euclidean Romans theory, where all indices are orthonormal frame indices, is

$$
\begin{aligned}
D_A\epsilon_I = {} & \frac{\mathrm{i}}{4\sqrt{2}}\left(X + \frac{1}{3}X^{-3}\right)\Gamma_A\Gamma_7\epsilon_I - \frac{1}{48}X^2 H_{BCD}\Gamma^{BCD}\Gamma_A\Gamma_7\epsilon_I \\
& - \frac{\mathrm{i}}{16\sqrt{2}}X^{-1}F_{BC}(\Gamma_A{}^{BC} - 6\delta_A{}^B\Gamma^C)\epsilon_I \\
& + \frac{1}{16\sqrt{2}}X^{-1}F^i_{BC}(\Gamma_A{}^{BC} - 6\delta_A{}^B\Gamma^C)\Gamma_7(\sigma^i)_I{}^J\epsilon_J\,,
\end{aligned} \tag{6.10}
$$

where $D_A\epsilon_I = \partial_A\epsilon_I + \frac{1}{4}\omega_A{}^{BC}\Gamma_{BC}\epsilon_I + \frac{\mathrm{i}}{2}\mathcal{A}^i_A(\sigma^i)_I{}^J\epsilon_J$. Taking the free index to be $A = 0$ and substituting the field components (6.9) leads to

$$
\partial_r\epsilon_I = +\frac{\mathrm{i}}{2r}\Gamma_0\Gamma_7\epsilon_I + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{6.11}
$$

Similarly, if we take the free index in the Killing spinor equation to be $A = a$ then we find

$$
\begin{aligned}
\nabla_a\epsilon_I = {} & \frac{\lambda}{3\sqrt{2}}r\Gamma_a(\mathrm{i}\Gamma_7 - \Gamma_0)\epsilon_I - \frac{\mathrm{i}}{2}a^i_a(\sigma^i)_I{}^J\epsilon_J \\
& - \frac{\mathrm{i}}{24\lambda\sqrt{2}}b_{bc}\Gamma_a{}^{bc}(1 + \mathrm{i}\Gamma_0\Gamma_7)\epsilon_I + \frac{\mathrm{i}}{4\lambda\sqrt{2}}b_{ab}\Gamma^b\left(1 + \frac{\mathrm{i}}{3}\Gamma_0\Gamma_7\right)\epsilon_I + \mathcal{O}\left(\frac{1}{r}\right),
\end{aligned} \tag{6.12}
$$

with ∇_a being the covariant derivative with respect to the 5d spin connection.

Now we decompose the six-dimensional gamma matrices and spinors. We take our coordinate independent Cliff$(6,0)$ gamma matrices to be

$$
\Gamma_0 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix}, \qquad \Gamma_a = \begin{pmatrix} 0 & \mathrm{i}\gamma_a \\ -\mathrm{i}\gamma_a & 0 \end{pmatrix}, \qquad \Gamma_7 = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix}, \tag{6.13}
$$

where γ_a are a Hermitian basis of Cliff$(5,0)$. The six-dimensional spinor ϵ_I is decomposed as

$$
\epsilon_I = \begin{pmatrix} \epsilon^+_I \\ \epsilon^-_I \end{pmatrix}, \tag{6.14}
$$

where $\epsilon^\pm_I$ are 4-component spinors.

With this basis of gamma matrices and splitting of the spinors, the r direction of the Killing spinor equation (6.11), to lowest order in r, is

$$
\begin{pmatrix} \partial_r\epsilon^+_I \\ \partial_r\epsilon^-_I \end{pmatrix} = \frac{\mathrm{i}}{2r}\begin{pmatrix} \epsilon^-_I \\ -\epsilon^+_I \end{pmatrix}. \tag{6.15}
$$

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The general solution determines the asymptotic dependence on r:

$$\epsilon_I \;=\; \begin{pmatrix} \epsilon_I^+ \\ \epsilon_I^- \end{pmatrix} \;=\; \sqrt{r}\begin{pmatrix} \chi_I \\ -\mathrm{i}\chi_I \end{pmatrix} + \frac{1}{\sqrt{r}}\begin{pmatrix} \varphi_I \\ \mathrm{i}\varphi_I \end{pmatrix} + \cdots , \tag{6.16}$$

where χ_I, φ_I depend only on the boundary coordinates x^m. Having found the asymptotic dependence on r for the spinors ϵ_I we can then substitute into the remaining components of the Killing spinor equation (6.12). Taking only the lowest terms in r gives two copies of

$$\nabla_a \chi_I \;=\; -\frac{\lambda\sqrt{2}\mathrm{i}}{3}\gamma_a\varphi_I - \frac{\mathrm{i}}{2}a_a^i(\sigma^i)_I{}^J\chi_J - \frac{\mathrm{i}}{12\lambda\sqrt{2}}b_{bc}\gamma_a{}^{bc}\chi_I + \frac{\mathrm{i}}{3\lambda\sqrt{2}}b_{ab}\gamma^b\chi_I\,. \tag{6.17}$$

This is the five-dimensional boundary Killing spinor equation.

Now recall that the six-dimensional dilatino condition in the frame reads

$$\begin{aligned} 0 = {} & -\mathrm{i}X^{-1}\partial_A X\Gamma^A\epsilon_I + \frac{1}{2\sqrt{2}}\left(X - X^{-3}\right)\Gamma_7\epsilon_I + \frac{\mathrm{i}}{24}X^2 H_{ABC}\Gamma^{ABC}\Gamma_7\epsilon_I \\ & - \frac{1}{8\sqrt{2}}X^{-1}F_{AB}\Gamma^{AB}\epsilon_I - \frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^i_{AB}\Gamma^{AB}\Gamma_7(\sigma^i)_I{}^J\epsilon_J\,. \end{aligned} \tag{6.18}$$

We may follow precisely the same steps as for the Killing spinor equation to determine the asymptotic form of the dilatino equation. Doing so we find the five-dimensional constraint

$$\begin{aligned} 0 = {} & -\frac{1}{6\sqrt{2}}b_{ab}\gamma^{ab}\varphi_I - \frac{\sqrt{2}}{3}\lambda^2 X_2\chi_I + \frac{\mathrm{i}}{24\lambda}(\mathrm{d}b)_{abc}\gamma^{abc}\chi_I + \frac{\lambda\mathrm{i}}{8}\nabla^b b_{ab}\gamma^a\chi_I \\ & + \frac{\lambda}{48\sqrt{2}}b_{ab}b_{cd}\gamma^{abcd}\chi_I + \frac{\mathrm{i}}{8\sqrt{2}}f^i_{ab}\gamma^{ab}(\sigma^i)_I{}^J\chi_J\,. \end{aligned} \tag{6.19}$$

We would prefer to have five-dimensional supersymmetry conditions which are homogeneous in the spinor χ_I instead of the current dependence on both χ_I and φ_I. To remove φ_I we contract (6.17) with γ^a. This gives

$$\begin{aligned} \varphi_I &= \frac{\mathrm{i}}{5}\frac{3}{\lambda\sqrt{2}}\left[\gamma^a\left(\delta_I^J\nabla_a + \frac{\mathrm{i}}{2}a_a^i(\sigma^i)_I{}^J + \frac{\mathrm{i}}{12\lambda\sqrt{2}}b_{bc}(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c)\delta_I^J\right)\right]\chi_J \\ &\equiv \frac{\mathrm{i}}{5}\frac{3}{\lambda\sqrt{2}}D_I{}^J\chi_J\,. \end{aligned} \tag{6.20}$$

We may then write the boundary Killing spinor equation in the form

$$\left(\tilde{\nabla}_I{}^J{}_a - \frac{1}{5}\gamma_a D_I{}^J\right)\chi_J \;=\; 0\,, \tag{6.21}$$

where $\tilde{\nabla}_I{}^J{}_a = \delta_I^J\nabla_a + \frac{\mathrm{i}}{2}a_a^i(\sigma^i)_I{}^J + \frac{\mathrm{i}}{12\lambda\sqrt{2}}b_{bc}(\gamma_a{}^{bc} - 4\delta_a^b\gamma^c)\delta_I^J$. The boundary dilatino constraint reads

$$\begin{aligned} 0 = {} & -\frac{\mathrm{i}}{20\lambda}b_{ab}\gamma^{ab}D_I{}^J\chi_J - \frac{\sqrt{2}}{3}\lambda^2 X_2\chi_I + \frac{\mathrm{i}}{24\lambda}(\mathrm{d}b)_{abc}\gamma^{abc}\chi_I + \frac{\lambda\mathrm{i}}{8}\nabla^b b_{ab}\gamma^a\chi_I \\ & + \frac{\lambda}{48\sqrt{2}}b_{ab}b_{cd}\gamma^{abcd}\chi_I + \frac{\mathrm{i}}{8\sqrt{2}}f^i_{ab}\gamma^{ab}(\sigma^i)_I{}^J\chi_J\,. \end{aligned} \tag{6.22}$$

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For vanishing b-field, solutions of (6.21) are known as charged conformal Killing spinors (CCKS), or twistor spinors. Within the current context of gauge/gravity duality, CCKS have been classified for 3-manifolds and 4-manifolds in both Euclidean and Lorentzian signature in [28–31]. More recently, solutions in five dimensions (with arbitrary signature) have been studied in [32]. To our knowledge the more general charged conformal Killing spinor equation, where the charge is with respect to both the triplet of one-forms a^i and the two-form b, has not been studied in the literature. It would be interesting to understand the relationship between the five-dimensional conditions found here from the Romans supergravity theory and the rigid limit of five-dimensional $\mathcal{N}=1$ Poincaré supergravity [33, 34] studied in [35, 36].

Finally, whilst we do not yet understand the general properties of a solution to (6.21), we are able to state the precise relation between the spinors φ_I and χ_I for our supersymmetric solutions (for which $\lambda = 3\sqrt{3}$). For the 3/4 BPS solution we find

$$\varphi_I \;=\; (-1)^I \frac{3-\sqrt{1-s^2}}{6\sqrt{6}s}\chi_I - (-1)^I \frac{4\sqrt{1-s^2}}{6\sqrt{6}s}\gamma_1\chi_I\,, \tag{6.23}$$

and for the two-parameter family of 1/4 BPS solutions

$$\varphi_I \;=\; \frac{(f_0-3)s}{6\sqrt{6}}\chi_I\,. \tag{6.24}$$

In appendix B we give further details of the explicit six-dimensional Killing spinors and their relation to the five-dimensional spinors of section 2.

7 Wilson loops

In this section we compute the expectation values of certain BPS Wilson loops, both in the large N matrix model of section 2.3 and also in the supergravity dual solutions of section 4. More precisely it will be important to uplift these solutions to massive type IIA supergravity, where the Wilson loop in the fundamental representation is dual to a fundamental string. Minus the action of this string precisely matches the logarithm of the Wilson loop VEV in the large N limit, as a function of the parameters of the solutions.

7.1 Large N field theory

An interesting observable to consider is the VEV of the Wilson loop in a representation $\mathbf{R}$ of the gauge group G:

$$\langle\, W_{\mathbf{R}}\,\rangle = \frac{1}{\dim \mathbf{R}}\left\langle \mathrm{Tr}_{\mathbf{R}}\,\mathcal{P}\exp\int (\mathscr{A}_m\dot{x}^m + \sigma|\dot{x}|)\,\mathrm{d}t\right\rangle\,. \tag{7.1}$$

Here $\mathscr{A}$ denotes the dynamical gauge field for the gauge group G, σ is the scalar in the corresponding vector multipet, and the worldline is parametrized by $x^m(t)$. It is straightforward to see that (7.1) is invariant under the supersymmetry transformations for the

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squashed five-sphere (2.21) appearing in section 3.3 of [6] provided the Wilson loop wraps an orbit of the Killing vector bilinear[6]

$$K_m = \varepsilon^{IJ}\chi_I{}^T\mathcal{C}_{(5)}\gamma_m\chi_J \ . \tag{7.2}$$

That is, we take $x^m(t)$ to be an integral curve of K. The supersymmetry variations of the two terms in (7.1) then cancel each other.

The large N limit of (7.1) for the USp($2N$) gauge theories described in section 2.3 was computed for the round five-sphere in [37]. It is straightforward to extend this to the more general squashed sphere matrix model in section 2.3. The key point is that the insertion of the Wilson loop into the path integral does not affect the leading order saddle point configuration because its logarithm scales as $N^{1/2}$, while the free energy instead scales as $N^{5/2}$. The dynamical gauge field $\mathscr{A}$ localizes to zero, so only the constant scalar σ contributes to the Wilson loop (7.1) in the localization computation. Thus the VEV (7.1), for the fundamental representation of USp($2N$), is effectively computed in the large N matrix model as

$$\langle\, W_{\text{fund}}\,\rangle = \int_0^{x_\star} \mathrm{e}^{2\pi\mathscr{L}\lambda(x)}\,\rho(x)\mathrm{d}x\,, \tag{7.3}$$

where $\rho(x)$ is the saddle point eigenvalue density (2.39), with the eigenvalues supported on $[0, x_\star]$ with $x_\star$ given by (2.40). We have also denoted by $2\pi\mathscr{L} = \int |\dot{x}|\,\mathrm{d}t$ the length of the integral curve of K that is wrapped by the Wilson loop, and recall that $\lambda(x) = N^{1/2}x$ to leading order. Thus we find the large N result

$$\log\langle\, W_{\text{fund}}\,\rangle = \frac{(b_1+b_2+b_3)\sqrt{2}\pi\mathscr{L}}{\sqrt{8-N_f}}N^{1/2} + o(N^{1/2})\ . \tag{7.4}$$

Relative to the round sphere result we thus have

$$\log\langle\, W_{\text{fund}}\,\rangle = \frac{(b_1+b_2+b_3)\mathscr{L}}{3}\log\langle\, W_{\text{fund}}\,\rangle_{\text{round}}\ . \tag{7.5}$$

Indeed, recalling that

$$K = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} + b_3\partial_{\varphi_3}\,, \tag{7.6}$$

in terms of the standard U(1)^3 action on $S^5 \subset \mathbb{R}^2\oplus\mathbb{R}^2\oplus\mathbb{R}^2$, then the orbits of K are always closed circles at the origins of any two copies of $\mathbb{R}^2$. If we call these U(1)^3 invariant circles S^1_i, $i=1,2,3$, then $\mathscr{L} = 1/b_i$ and we may write

$$\log\langle\, W_{\text{fund},\,S^1_i}\,\rangle = \frac{(b_1+b_2+b_3)}{3b_i}\log\langle\, W_{\text{fund}}\,\rangle_{\text{round}}\ . \tag{7.7}$$

Notice that this formula is invariant under a constant rescaling $K \to c\cdot K$. We now explain how to reproduce this large N result from the dual supergravity solutions.

[6] Of course we have similarly defined a Killing vector K in the six-dimensional bulk as (3.9). The latter restricts to (7.2) on the conformal boundary, so this is only a slight abuse of notation.

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7.2 Dual fundamental strings

The supergravity dual of the Wilson loop W_{fund} was studied in [37] for the round five-sphere. The supergravity background is in this case the massive type IIA uplift $\text{AdS}_6 \times S^4$ of the AdS_6 vacuum of the Romans theory of section 3. The Wilson loop maps to a fundamental string sitting at the north pole $\xi = \frac{\pi}{2}$ of the internal S^4, in the notation of section 3.1. The string then wraps a copy of $\mathbb{R}^2 \subset \text{AdS}_6$ parametrized by the radial direction r in AdS together with the Wilson loop curve $S^1 \subset S^5$.

We now generalize this to our supergravity backgrounds in section 4. Here the type IIA background is a warped and fibred product $M_6 \times S^4$, together with various non-trivial background fluxes. However, M_6 still has the topology of a ball, with a natural radial direction r. Thus the candidate dual of the Wilson loops computed in the previous section is a fundamental string sitting at $\xi = \frac{\pi}{2}$ in the internal S^4 of (3.1), together with the Wilson loop curve $S^1 \subset S^5_{\text{squashed}}$ and the radial direction r. This is then a copy of $\Sigma_2 \cong \mathbb{R}^2 \subset M_6$, and we would like to compute the regularized action of a fundamental string wrapping this submanifold.

In order to compute the string action we must first convert to the string frame metric in (3.1), which introduces a factor of $\mathrm{e}^{\Phi/2}$, where Φ is the ten-dimensional dilaton. The induced string frame metric on M_6 at the north pole $\xi = \frac{\pi}{2}$ of S^4 is then

$$\mathrm{d}s^2_{M_6} \,|_{\xi=\frac{\pi}{2},\,\text{string}} = X^{-2}\mathrm{d}s^2_6\,, \tag{7.8}$$

where $\mathrm{d}s^2_6$ is the Romans supergravity metric. The B-field then uplifts to the type IIA B-field with curvature $F_{(3)} = H = \mathrm{d}B$ via (3.1) at the north pole $\xi = \frac{\pi}{2}$. In section 3 we have set most of the physical scaling parameters to specific numerical values — for example the Romans mass is set to $m_{\text{IIA}} = \frac{\sqrt{2}}{3}$, while the correctly normalized value for the supergravity dual to the $\text{USp}(2N)$ gauge theories is $(8 - N_f)/(2\pi\ell_s)$ where ℓ_s is the string length. In particular restoring the AdS radius to its physical value

$$L^4 = \frac{8\pi^2 N}{9(8-N_f)}\ell_s^4\,, \tag{7.9}$$

(as in [37]) the string frame action is

$$S = \frac{N^{1/2}\sqrt{2}}{3\sqrt{(8-N_f)}}\int_{\Sigma_2} X^{-2}\sqrt{\det\gamma}\,\mathrm{d}^2x + \mathrm{i}B\,, \tag{7.10}$$

where γ_{ab} is the metric induced on Σ_2 via its embedding into the Romans metric $\mathrm{d}s^2_6$ on M_6, and we have included the usual Wess-Zumino coupling to the ten-dimensional B-field. More precisely, (7.10) is divergent, and as usual one may regularize it by cutting off the r integral at some $r = \Lambda$, and including a boundary counterterm given by the length of the boundary $S^1 \subset S^5$ at $r = \Lambda$. Thus the regularized action reads

$$S_{\text{string}} = \frac{N^{1/2}\sqrt{2}}{3\sqrt{(8-N_f)}}\left[\int_{\Sigma_2}\left(X^{-2}\sqrt{\det\gamma}\,\mathrm{d}^2x + \mathrm{i}B\right) - \frac{3}{\sqrt{2}}\text{length}(\partial\Sigma_2)\right]\,, \tag{7.11}$$

where this is understood to mean the limit as one takes the cut-off $\Lambda \to \infty$. We now compute this for our various solutions.

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1/4 BPS background. We begin with the 1/4 BPS background, as in this case the supersymmetric Killing vector bilinear is simply $K = \partial_\tau$ (up to an irrelevant constant rescaling). Via the SU(3) symmetry of the background all orbits of K are equivalent, and thus there is effectively only one Wilson loop to compute. This wraps the τ and r directions at, say, $\sigma = 0$ (which is a point on the base $\mathbb{CP}^2$ of $S^1_{\text{Hopf}} \hookrightarrow S^5 \to \mathbb{CP}^2$, all points being equivalent under SU(3)). The regularized string action (7.11) is

$$S_{\text{string}} = \lim_{\Lambda\to\infty} \frac{N^{1/2}2\sqrt{2}\pi}{3\sqrt{(8-N_f)}} \left[\int_{r=\frac{1}{\sqrt{6}}}^{\Lambda} \left[X^{-2}(r)\alpha(r)\gamma(r) + \mathrm{i}\, p(r) \right] \mathrm{d}r - \frac{3}{\sqrt{2}}\gamma(\Lambda) \right] , \tag{7.12}$$

where we have used that τ has period 2π. Evaluating this for the two-parameter family of 1/4 BPS solutions, as a series in the parameter δ, we find

$$-S_{\text{string}} = \frac{3\sqrt{2}\pi}{\sqrt{8-N_f}} N^{1/2} + \mathcal{O}(\delta^5) , \tag{7.13}$$

which agrees precisely with the large N field theory result (7.4) since $K = \partial_\tau = \partial_{\varphi_1} + \partial_{\varphi_2} + \partial_{\varphi_3}$ so that $b_1 = b_2 = b_3 = 1$.

3/4 BPS background. For the 3/4 BPS solution recall that the supersymmetric Killing vector K has $b_1 = 1 + \sqrt{1-s^2}$, $b_2 = b_3 = 1 - \sqrt{1-s^2}$. For generic values of the squashing parameter s the generic orbit of K will be open. However, the orbits always close over the circles S^1_i defined in section 7.1, which have lengths $\mathscr{L} = 2\pi/b_i$. Since $b_2 = b_3$ these circles give rise to two distinct Wilson loop VEVs:

$$\frac{\log \langle\, W_{\text{fund},\, S^1_i} \,\rangle}{\log \langle\, W_{\text{fund}} \,\rangle_{\text{round}}} = \begin{cases} \dfrac{3-\sqrt{1-s^2}}{3(1+\sqrt{1-s^2})} , & i \;=\; 1 , \\[2ex] \dfrac{3-\sqrt{1-s^2}}{3(1-\sqrt{1-s^2})} , & i \;=\; 2, 3 . \end{cases} \tag{7.14}$$

We may then compare these results to the regularized string action (7.11), where for S^1_i the fundamental string wraps the circle φ_i together with the r direction. More precisely, S^1_1 is located at $\sigma = 0$ in the coordinates (2.1), while S^1_2 is located at $\{\sigma = \frac{\pi}{2},\, \theta = 0\}$, as one sees from (2.23). The result for S^1_3 is the same as that for S^1_2 due to the SU(2) $\subset$ SU(3) symmetry preserved by the bosonic solution and supersymmetric Killing vector. On the other hand, due to the signs in (2.24) the relevant string actions to compute are then

$$\frac{N^{1/2}2\sqrt{2}\pi}{3\sqrt{(8-N_f)}} \left[\int_{r=\frac{1}{\sqrt{6}}}^{\Lambda} \left[X^{-2}(r)\alpha(r)\gamma(r) \pm \mathrm{i}\, p(r) \right] \mathrm{d}r - \frac{3}{\sqrt{2}}\gamma(\Lambda) \right] , \tag{7.15}$$

respectively. Evaluating this for the one-parameter family of 3/4 BPS solutions, as a series in the parameter δ up to sixth order where $\delta^2 = \frac{1}{s} - 1$, we find

$$\frac{S_{\text{string},S^1_1}}{S_{\text{string}}\,|_{\delta=0}} = 1 - \frac{4\sqrt{2}}{3}\delta + \frac{8}{3}\delta^2 - \frac{5\sqrt{2}}{3}\delta^3 + \frac{4}{3}\delta^4 - \frac{7}{12\sqrt{2}}\delta^5 + 0\cdot\delta^6 + \dots , \tag{7.16}$$

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while

$$\frac{S_{\text{string},S_2^1}}{S_{\text{string}}\,|_{\delta=0}} = 1 + \frac{2\sqrt{2}}{3}\delta + \frac{4}{3}\delta^2 + \frac{5}{3\sqrt{2}}\delta^3 + \frac{2}{3}\delta^4 + \frac{7}{24\sqrt{2}}\delta^5 + 0\cdot\delta^6 + \dots \ . \tag{7.17}$$

These agree precisely with the series expansions of (7.14) computed in field theory.

8 Discussion and conjectures

In this paper we have constructed supergravity duals to the USp($2N$) superconformal gauge theories on SU(3) × U(1) squashed five-spheres. These constitute a one-parameter family of 3/4 BPS solutions, and a two-parameter family of generically 1/4 BPS. The latter include new supersymmetric squashed five-sphere geometries with the background SU(2)$_R$ gauge field turned off, and moreover these have enhanced 1/2 BPS supersymmetry. By holographically renormalizing the Euclidean Romans supergravity theory, we have computed the holographic free energy for our solutions. We then compared this to the large N limit of the partition function of the gauge theories, and found perfect agreement. Given a supersymmetric supergravity solution one can construct the Killing vector $K^\mu = \varepsilon^{IJ}\epsilon_I^T\mathcal{C}\gamma^\mu\epsilon_J$, where ϵ_I, $I = 1, 2$, is the SU(2)$_R$ doublet of Killing spinors. For our solutions the free energy takes the form

$$\mathcal{F} = \frac{(|b_1| + |b_2| + |b_3|)^3}{27|b_1 b_2 b_3|}\mathcal{F}_{\text{AdS}_6}\,, \tag{8.1}$$

where we write the supersymmetric Killing vector as $K = \sum_{i=1}^3 b_i\partial_{\varphi_i}$, and ∂_{φ_i} are standard generators of U(1)^3 ⊂ SU(3) × U(1) acting on $S^5 \subset \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$. Given the corresponding $4d/3d$ results of [13, 38], it is then natural to conjecture that (8.1) holds for *any* supersymmetric supergravity solution with the topology of a six-ball and for which the supersymmetric Killing vector K may be written as $K = \sum_{i=1}^3 b_i\partial_{\varphi_i}$. In the present paper we chose orientation conventions so that $b_i > 0$ for $i = 1, 2, 3$. More generally we expect the orientations of ∂_{φ_i} to be fixed as in [13], leading to the modulus signs in (8.1). We shall comment further on this below. We also conjecture that any supersymmetric gauge theory, with finite N, defined on the conformal boundary of such a supergravity solution depends only on b_1, b_2, b_3.

We have also computed certain BPS Wilson loops, both in supergravity and in the large N gauge theories, again finding agreement. In this case we find that one can write the Wilson loop VEV as

$$\log\langle W\rangle = \frac{|b_1| + |b_2| + |b_3|}{3|b_i|}\log\langle W\rangle_{\text{AdS}_6}\,, \tag{8.2}$$

where the Wilson loop wraps the φ_i circle. Again, it is natural to conjecture that (8.2) holds for general supergravity backgrounds with U(1)^3 symmetry and the topology of a six-ball. A general proof of the analogous formula to (8.2) for the Wilson loop VEV in four dimensions appears in [39].

There are many natural directions which one could follow up. Firstly, it would be interesting to study supersymmetric gauge theories on a general class of supersymmetric background five-manifolds, generalizing the work done in lower dimensions in [28, 38, 40, 41].

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One should then be able to prove (or disprove) the conjectures made above. In particular it would be interesting to study five-manifolds with different topology. Some work in this direction appears in [15], where the authors studied the case where the boundary is a Sasaki-Einstein manifold. It would also be very interesting to study systematically the geometry of Euclidean Romans supergravity backgrounds, as alluded to in section 3.3. Here it is natural to expect that general supersymmetric solutions on the six-ball have a canonical complex structure, so that $M_6 \cong \mathbb{C}^3$. If this is the case, then introducing standard complex coordinates $z_i = \rho_i e^{i\varphi_i}$, $i = 1, 2, 3$, fixes the relative orientations of ∂_{φ_i}. In analysing the asymptotic expansion of the bulk Killing spinor equation, we have obtained a boundary charged conformal Killing spinor equation, where the charge is with respect to both a one-form and also a two-form. To our knowledge, this type of equation has not been studied in the literature. In particular, it is an open problem to relate this equation to a more standard Killing spinor equation, of the type (6.17), in general.

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A Integrability conditions

Here we compute the integrability conditions for the Killing spinor equation (3.7) and dilatino equation (3.8) of the Euclidean Romans theory.

Recall that a supersymmetric solution must satisfy

$$\begin{aligned} D_\mu \epsilon_I = {} & \frac{i}{4\sqrt{2}} g (X + \frac{1}{3} X^{-3}) \Gamma_\mu \Gamma_7 \epsilon_I - \frac{1}{48} X^2 H_{\nu\rho\sigma} \Gamma^{\nu\rho\sigma} \Gamma_\mu \Gamma_7 \epsilon_I \\ & - \frac{i}{16\sqrt{2}} X^{-1} F_{\nu\rho} (\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu \Gamma^\rho) \epsilon_I + \frac{1}{16\sqrt{2}} X^{-1} F^i_{\nu\rho} (\Gamma_\mu{}^{\nu\rho} - 6\delta_\mu{}^\nu \Gamma^\rho) \Gamma_7 (\sigma^i)_I{}^J \epsilon_J \,, \end{aligned} \tag{A.1}$$

$$\begin{aligned} \delta\lambda_I = 0 = {} & -i X^{-1} \partial_\mu X \Gamma^\mu \epsilon_I + \frac{1}{2\sqrt{2}} g \left(X - X^{-3} \right) \Gamma_7 \epsilon_I + \frac{i}{24} X^2 H_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \Gamma_7 \epsilon_I \\ & - \frac{1}{8\sqrt{2}} X^{-1} F_{\mu\nu} \Gamma^{\mu\nu} \epsilon_I - \frac{i}{8\sqrt{2}} X^{-1} F^i_{\mu\nu} \Gamma^{\mu\nu} \Gamma_7 (\sigma^i)_I{}^J \epsilon_J \,, \end{aligned} \tag{A.2}$$

where λ_I is the dilatino field. Let us also record the component form of the Romans field equations in (3.3) and (3.5)

$$\begin{aligned} (E_g)_{\mu\nu} \equiv {} & R_{\mu\nu} - 4X^{-2} \partial_\mu X \partial_\nu X - g^2 \left(\frac{1}{18} X^{-6} - \frac{1}{2} X^2 - \frac{2}{3} X^{-2} \right) g_{\mu\nu} \\ & - \frac{1}{4} X^4 (H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} - \frac{1}{6} g_{\mu\nu} H^{\rho\sigma\tau} H_{\rho\sigma\tau}) - \frac{1}{2} X^{-2} (F_\mu{}^\rho F_{\nu\rho} - \frac{1}{8} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}) \\ & - \frac{1}{2} X^{-2} (F^i_\mu{}^\rho F^i_{\nu\rho} - \frac{1}{8} g_{\mu\nu} F^{i\rho\sigma} F^i_{\rho\sigma}) \,, \end{aligned}$$

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$$
\begin{aligned}
(E_X) &\equiv \nabla^\mu(X^{-1}\partial_\mu X) + g^2\left(\frac{1}{2}X^2 - \frac{2}{3}X^{-2} + \frac{1}{6}X^{-6}\right) - \frac{1}{24}X^4 H^{\mu\nu\rho}H_{\mu\nu\rho} \\
&\quad + \frac{1}{16}X^{-2}(F^{\mu\nu}F_{\mu\nu} + F^{i\mu\nu}F^i_{\mu\nu})\,, \\
(E_A)^\mu &\equiv \nabla_\nu(X^{-2}F^{\nu\mu}) - \frac{\mathrm{i}}{12}\varepsilon^{\mu\nu\rho\sigma\tau\kappa}F_{\nu\rho}H_{\sigma\tau\kappa}\,, \\
(E_{A^i})^\mu &\equiv D_\nu(X^{-2}F^{i\nu\mu}) - \frac{\mathrm{i}}{12}\varepsilon^{\mu\nu\rho\sigma\tau\kappa}F^i_{\nu\rho}H_{\sigma\tau\kappa}\,, \\
(E_B)^{\mu\nu} &\equiv \nabla_\rho(X^4 H^{\rho\mu\nu}) - \frac{2}{3}gX^{-2}F^{\mu\nu} - \frac{\mathrm{i}}{8}\varepsilon^{\mu\nu\rho\sigma\tau\kappa}(F_{\rho\sigma}F_{\tau\kappa} + F^i_{\rho\sigma}F^i_{\tau\kappa})\,.
\end{aligned}
\tag{A.3}
$$

The equations of motion are then $E_{\text{field}} = 0$. In addition, the gauge fields satisfy Bianchi identities $B_{\text{field}} = 0$, where we define

$$
\begin{aligned}
(B_F)_{\mu\nu\rho} &\equiv \nabla_{[\mu}F_{\nu\rho]} - \frac{2}{9}gH_{\mu\nu\rho}\,, \\
(B_{F^i})_{\mu\nu\rho} &\equiv D_{[\mu}F^i_{\nu\rho]}\,, \\
(B_H)_{\mu\nu\rho\sigma} &\equiv \nabla_{[\mu}H_{\nu\rho\sigma]}\,.
\end{aligned}
\tag{A.4}
$$

Taking the commutator of the Killing spinor equation (A.1) we find the integrability condition to be

$$
\mathcal{I}_{\mu\nu I}{}^J\epsilon_J = 0\,,
\tag{A.5}
$$

where

$$
\begin{aligned}
\mathcal{I}_{\mu\nu I}{}^J\epsilon_J = {}& \frac{1}{4}R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\epsilon_I + \frac{\mathrm{i}}{2}gF^i_{\mu\nu}(\sigma^i)_I{}^J\epsilon_J + \Big[-\frac{\mathrm{i}}{4\sqrt{2}}g(1-X^{-4})\partial_\mu X\Gamma_\nu\Gamma_7\epsilon_I \\
&+\frac{1}{24}X\partial_\mu X H^{\rho\sigma\tau}\Gamma_{\rho\sigma\tau}\Gamma_\nu\Gamma_7\epsilon_I + \frac{1}{48}X^2\nabla_\mu H^{\rho\sigma\tau}\Gamma_{\rho\sigma\tau}\Gamma_\nu\Gamma_7\epsilon_I \\
&-\frac{\mathrm{i}}{16\sqrt{2}}X^{-2}\partial_\mu X F_{\rho\sigma}J_\nu{}^{\rho\sigma}\epsilon_I + \frac{\mathrm{i}}{16\sqrt{2}}X^{-1}\nabla_\mu F_{\rho\sigma}J_\nu{}^{\rho\sigma}\epsilon_I \\
&+\frac{1}{16\sqrt{2}}X^{-2}\partial_\mu X F^i_{\rho\sigma}J_\nu{}^{\rho\sigma}\Gamma_7(\sigma^i)_I{}^J\epsilon_J - \frac{1}{16\sqrt{2}}X^{-1}\nabla_\mu F^i_{\rho\sigma}J_\nu{}^{\rho\sigma}\Gamma_7(\sigma^i)_I{}^J\epsilon_J \\
&-\frac{1}{32}g^2(\frac{1}{9}X^{-6} + \frac{2}{3}X^{-2} + X^2)\Gamma_\nu\Gamma_\mu\epsilon_I - \frac{1}{2304}X^4 H^{\lambda\omega\theta}H^{\rho\sigma\tau}\Gamma_{\lambda\omega\theta}\Gamma_\nu\Gamma_{\rho\sigma\tau}\Gamma_\mu\epsilon_I \\
&+\frac{1}{512}X^{-2}F_{\omega\theta}F_{\rho\sigma}J_\nu{}^{\omega\theta}J_\mu{}^{\rho\sigma}\epsilon_I + \frac{1}{512}X^{-2}F^i_{\omega\theta}F^i_{\rho\sigma}J_\nu{}^{\omega\theta}J_\mu{}^{\rho\sigma}\epsilon_I \\
&+\frac{\mathrm{i}}{512}X^{-2}\varepsilon_{ijk}F^i_{\omega\theta}F^j_{\rho\sigma}J_\nu{}^{\omega\theta}J_\mu{}^{\rho\sigma}(\sigma^k)_I{}^J\epsilon_J \\
&+\frac{\mathrm{i}}{192\sqrt{2}}g(X^3 + \frac{1}{3}X^{-1})H^{\rho\sigma\tau}\Big(\Gamma_\nu\Gamma_{\rho\sigma\tau}\Gamma_\mu - \Gamma_{\rho\sigma\tau}\Gamma_\nu\Gamma_\mu\Big)\epsilon_I \\
&+\frac{1}{128}gX^{-1}(X + \frac{1}{3}X^{-3})F_{\rho\sigma}\Big(\Gamma_\nu J_\mu{}^{\rho\sigma} - J_\nu{}^{\rho\sigma}\Gamma_\mu\Big)\Gamma_7\epsilon_I \\
&+\frac{\mathrm{i}}{128}gX^{-1}(X + \frac{1}{3}X^{-3})F^i_{\rho\sigma}\Big(\Gamma_\nu J_\mu{}^{\rho\sigma} + J_\nu{}^{\rho\sigma}\Gamma_\mu\Big)(\sigma^i)_I{}^J\epsilon_J \\
&+\frac{\mathrm{i}}{768\sqrt{2}}XF_{\rho\sigma}H^{\lambda\omega\theta}\Big(\Gamma_{\lambda\omega\theta}\Gamma_\nu J_\mu{}^{\rho\sigma} - J_\nu{}^{\rho\sigma}\Gamma_{\lambda\omega\theta}\Gamma_\mu\Big)\Gamma_7\epsilon_I
\end{aligned}
$$

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$$-\frac{1}{768\sqrt{2}}XF^i_{\rho\sigma}H^{\lambda\omega\theta}\Big(\Gamma_{\lambda\omega\theta}\Gamma_\nu J_\mu{}^{\rho\sigma}-J_\nu{}^{\rho\sigma}\Gamma_{\lambda\omega\theta}\Gamma_\mu\Big)(\sigma^i)_I{}^J\epsilon_J$$
$$+\frac{\mathrm{i}}{512}X^{-2}(F_{\rho\sigma}F^i_{\omega\theta}-F_{\omega\theta}F^i_{\rho\sigma})J_\nu{}^{\rho\sigma}J_\mu{}^{\omega\theta}\Gamma_7(\sigma^i)_I{}^J\epsilon_J-(\mu\leftrightarrow\nu)\Big]\,, \tag{A.6}$$

and we have defined the Clifford algebra element

$$J_\mu{}^{\rho\sigma}\equiv\Gamma_\mu{}^{\rho\sigma}-6\delta_\mu{}^\rho\Gamma^\sigma\ . \tag{A.7}$$

Taking the covariant derivative of the dilatino equation (A.2) and contracting with Γ^μ leads to

$$\begin{aligned}
&\Gamma^\mu D_\mu(\delta\lambda_I)-\frac{\mathrm{i}}{2\sqrt{2}}g\Big(X-\frac{7}{3}X^{-3}\Big)\Gamma_7\delta\lambda_I+\frac{1}{24}X^2H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\Gamma_7\delta\lambda_I \\
&+\frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F_{\mu\nu}\Gamma^{\mu\nu}\delta\lambda_I+\frac{1}{8\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma_7(\sigma^i)_I{}^J\delta\lambda_J \\
&=\mathrm{i}\,(E_X)\,\epsilon_I-\frac{1}{4\sqrt{2}}X\,(E_A)_\mu\,\Gamma^\mu\epsilon_I-\frac{\mathrm{i}}{4\sqrt{2}}X\,(E_{A^i})_\mu\,\Gamma^\mu\Gamma_7(\sigma^i)_I{}^J\epsilon_J+\frac{\mathrm{i}}{8}X^{-2}\,(E_B)_{\mu\nu}\,\Gamma^{\mu\nu}\Gamma_7\epsilon_I \\
&\quad-\frac{1}{8\sqrt{2}}X^{-1}\,(B_F)_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}\epsilon_I-\frac{\mathrm{i}}{8\sqrt{2}}X^{-1}\,(B_{F^i})_{\mu\nu\rho}\,\Gamma^{\mu\nu\rho}\Gamma_7(\sigma^i)_I{}^J\epsilon_J \\
&\quad+\frac{\mathrm{i}}{24}X^2\,(B_H)_{\mu\nu\rho\sigma}\,\Gamma^{\mu\nu\rho\sigma}\Gamma_7\epsilon_I\ .
\end{aligned} \tag{A.8}$$

We may similarly contract $\mathcal{I}_{\mu\nu I}{}^J\epsilon_J$ with Γ^ν. After a very lengthy calculation we find

$$\begin{aligned}
&\Gamma^\nu\mathcal{I}_{\mu\nu I}{}^J\epsilon_J+\frac{\mathrm{i}}{2}\Gamma_\mu\Gamma_\nu D^\nu(\delta\lambda_I)+2\mathrm{i}X^{-1}\partial_\mu X\delta\lambda_I+\frac{1}{2\sqrt{2}}g\Big(X-\frac{5}{3}X^{-3}\Big)\Gamma_\mu\Gamma_7\delta\lambda_I \\
&-\frac{\mathrm{i}}{16}X^2H_{\mu\nu\rho}\Gamma^{\nu\rho}\Gamma_7\delta\lambda_I+\frac{\mathrm{i}}{16}X^2H^{\nu\rho\sigma}\Gamma_{\mu\nu\rho\sigma}\Gamma_7\delta\lambda_I-\frac{1}{8\sqrt{2}}X^{-1}F^{\nu\rho}\Gamma_{\mu\nu\rho}\delta\lambda_I \\
&+\frac{1}{4\sqrt{2}}X^{-1}F_{\mu\nu}\Gamma^\nu\delta\lambda_I-\frac{\mathrm{i}}{4\sqrt{2}}X^{-1}F^i_{\mu\nu}\Gamma^\nu\Gamma_7(\sigma^i)_I{}^J\delta\lambda_J+\frac{\mathrm{i}}{8\sqrt{2}}X^{-1}F^{i\nu\rho}\Gamma_{\mu\nu\rho}\Gamma_7(\sigma^i)_I{}^J\delta\lambda_J \\
&=\frac{1}{2}\,(E_X)\,\Gamma_\mu\epsilon_I-\frac{1}{2}\,(E_g)_{\mu\nu}\,\Gamma^\nu\epsilon_I-\frac{1}{8}X^{-2}\,(E_B)^{\nu\rho}\,\Gamma_{\mu\nu\rho}\Gamma_7\epsilon_I \\
&\quad-\frac{\mathrm{i}}{2\sqrt{2}}X\,(E_A)_\mu\,\epsilon_I+\frac{1}{2\sqrt{2}}X\,(E_{A^i})_\mu\,\Gamma_7(\sigma^i)_I{}^J\epsilon_J-\frac{1}{24}X^2\,(B_H)^{\nu\rho\sigma\tau}\,\Gamma_{\mu\nu\rho\sigma\tau}\Gamma_7\epsilon_I \\
&\quad-\frac{3\mathrm{i}}{4\sqrt{2}}X^{-1}\,(B_F)_{\mu\nu\rho}\,\Gamma^{\nu\rho}\epsilon_I+\frac{3}{4\sqrt{2}}X^{-1}\,(B_{F^i})_{\mu\nu\rho}\,\Gamma^{\nu\rho}\Gamma_7(\sigma^i)_I{}^J\epsilon_J\ .
\end{aligned} \tag{A.9}$$

B Supersymmetric supergravity solutions

B.1 The equations

The solutions found in this paper arise from the following SU(3) × U(1) symmetric ansatz for the supergravity fields

$$\begin{aligned}
\mathrm{d}s_6^2=\alpha^2(r)\mathrm{d}r^2+\gamma^2(r)(\mathrm{d}\tau+C)^2+\beta^2(r)\Big[&\mathrm{d}\sigma^2+\frac{1}{4}\sin^2\sigma(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2) \\
&+\frac{1}{4}\cos^2\sigma\sin^2\sigma(\mathrm{d}\psi+\cos\theta\mathrm{d}\varphi)^2\Big]\,,
\end{aligned}$$

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$$
\begin{aligned}
B &= p(r)\mathrm{d}r \wedge (\mathrm{d}\tau + C) + \frac{1}{2} q(r)\mathrm{d}C\,, \\
A^i &= f^i(r)(\mathrm{d}\tau + C)\,,
\end{aligned} \tag{B.1}
$$

together with $X = X(r)$. The equations of motion for the background $\mathrm{SU}(2)_R$ gauge field imply

$$
f^i(r) \;=\; \kappa_i f(r)\;. \tag{B.2}
$$

The equations for the other fields then depend only on the $\mathrm{SU}(2) \sim \mathrm{SO}(3)$ invariant $\kappa_1^2+\kappa_2^2+\kappa_3^2$, which we can set to one by rescaling $f(r)$. Explicitly, one finds that substituting the ansatz (B.1) into the equations of motion (3.3) and Einstein equation (3.5) leads to following coupled system of ODEs:

$$
\frac{\lambda\gamma X^4}{\alpha} = \mathrm{i}f^2 + \mathrm{i}\frac{q^2}{9} + \frac{p\beta^4}{9\alpha\gamma X^2}\,, \tag{B.3}
$$

$$
\left(\frac{\lambda\gamma X^4}{\alpha}\right)' = 2\mathrm{i}ff' + \mathrm{i}\left(\frac{2}{3}\right)^2 pq + \left(\frac{2}{3}\right)^2 \frac{q\alpha\gamma}{X^2}\,, \tag{B.4}
$$

$$
\left(\frac{\beta^4 f'}{2\alpha\gamma X^2}\right)' - \frac{4\alpha\gamma f}{X^2} \;=\; -2\mathrm{i}f\lambda\,, \tag{B.5}
$$

$$
\begin{aligned}
\frac{\alpha}{\gamma\beta^4}\left(\frac{\gamma\beta^4 X'}{\alpha X}\right)' = &-\frac{1}{8X^2}\left(\frac{f'^2}{\gamma^2} + \frac{8\alpha^2 f^2}{\beta^4}\right) - \left(\frac{2}{3}\right)^2 \frac{1}{8X^2}\left(\frac{p^2}{\gamma^2} + 2\frac{\alpha^2 q^2}{\beta^4}\right) \\
&+ \frac{X^4\lambda^2}{2\beta^4} - \frac{\alpha^2}{6X^6} + \frac{2\alpha^2}{3X^2} - \frac{\alpha^2 X^2}{2}\,,
\end{aligned} \tag{B.6}
$$

$$
-\frac{\beta''}{\beta} + \frac{\beta'}{\beta}\frac{(\alpha\gamma)'}{\alpha\gamma} - \frac{(\alpha\gamma)^2}{\beta^4} \;=\; \left(\frac{X'}{X}\right)^2 + \frac{X^4\lambda^2}{4\beta^4}\,, \tag{B.7}
$$

$$
\begin{aligned}
&-\frac{\gamma''}{\gamma} + \frac{\beta''}{\beta} + \frac{\alpha'}{\alpha}\left(\frac{\gamma'}{\gamma} - \frac{\beta'}{\beta}\right) - 3\frac{\beta'}{\beta}\left(\frac{\gamma'}{\gamma} - \frac{\beta'}{\beta}\right) + \frac{6\alpha^2}{\beta^4}(\gamma^2 - \beta^2) \\
&= \;-\frac{X^4\lambda^2}{2\beta^4} + \frac{1}{2X^2}\left(\frac{f'^2}{\gamma^2} - \frac{4\alpha^2 f^2}{\beta^4}\right) + \left(\frac{2}{3}\right)^2 \frac{1}{2X^2}\left(\frac{p^2}{\gamma^2} - \frac{\alpha^2 q^2}{\beta^4}\right)\,,
\end{aligned} \tag{B.8}
$$

$$
\begin{aligned}
-\frac{\gamma''}{\gamma} + \frac{\alpha'}{\alpha}\frac{\gamma'}{\gamma} - 4\frac{\beta'}{\beta}\frac{\gamma'}{\gamma} + 4\frac{(\alpha\gamma)^2}{\beta^4} = &\frac{\alpha^2}{18X^6} - \frac{2\alpha^2}{3X^2} - \frac{\alpha^2 X^2}{2} - \frac{X^4\lambda^2}{2\beta^4} \\
&+ \frac{1}{2X^2}\left[\frac{f'^2}{\gamma^2} - \frac{1}{4}\left(\frac{f'^2}{\gamma^2} + \frac{8\alpha^2 f^2}{\beta^4}\right)\right] \\
&+ \left(\frac{2}{3}\right)^2 \frac{1}{2X^2}\left[\frac{p^2}{\gamma^2} - \frac{1}{4}\left(\frac{p^2}{\gamma^2} + \frac{2\alpha^2 q^2}{\beta^4}\right)\right]\,.
\end{aligned} \tag{B.9}
$$

where we have introduced $\lambda = q' - 2p$. These are seven equations for seven functions. In addition one can explicitly check that the equations are invariant under changes in the parametrization $r \to \rho(r)$.

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B.2 General solutions

Before writing the general series solutions to the above coupled system of ODEs, let us present the solution for Euclidean AdS_6 in these coordinates:

$$\alpha(r) = \frac{3\sqrt{3}}{\sqrt{6r^2-1}}\,, \qquad \beta(r) = \gamma(r) = \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\,,$$
$$p(r) = q(r) = f(r) = 0\,, \qquad X(r) = 1\,. \tag{B.10}$$

Here only the metric is non-trivial, and the above realizes Euclidean AdS_6 as a hyperbolic ball with radial coordinate $r \in [\frac{1}{\sqrt{6}}, \infty)$, with the conformal boundary at infinity $r = \infty$. The point $r = \frac{1}{\sqrt{6}}$ is the origin of the ball, where the transverse copies of S^5 collapse smoothly to zero. Notice in particular that the conformal boundary at $r = \infty$ is equipped with a *round* metric on S^5, which is conformally flat. We would like to find families of solutions that generalize (B.10) by allowing for a squashed five-sphere boundary, keeping the metric asymptotically locally Euclidean AdS near $r = \infty$. We define the squashing parameter by:

$$\lim_{r\to\infty} \frac{\gamma(r)}{r} = 3\sqrt{3}\,\frac{1}{s}\,, \tag{B.11}$$

so that $s = 1$ for the round sphere. Even though we did not manage to find solutions in closed form, the solutions can nevertheless be given as expansions around different limits. In general notice that we can use reparametrization invariance to set

$$\beta(r) = \frac{3\sqrt{6r^2-1}}{\sqrt{2}}\,, \tag{B.12}$$

which we assume henceforth. In particular we shall only seek solutions with the topology of a ball, so that from (B.12) necessarily $r = \frac{1}{\sqrt{6}}$ is the origin of the ball. Correspondingly, the fields must satisfy certain boundary conditions at this point in order that the full solution is smooth at the origin.

B.2.1 Expansion around the conformal boundary

When finding gravity duals to a given boundary theory, it is natural to perform an expansion around the conformal boundary at $r = \infty$. This also has the advantage that the squashing parameter can be explicitly seen in the solution. Starting from a general expansion and imposing the equations of motion in section B.1 we find

$$\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} + \frac{486 + q_0^2 s^2}{1944\sqrt{2}s^2}\frac{1}{r^3} + \dots\,,\\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{-486 + \left(243 - q_0^2\right)s^2}{324\sqrt{3}s^3}\frac{1}{r} + \dots\,,\\
X(r) &= 1 + \frac{-486q_0 + 72\mathrm{i}\sqrt{6}q_0^2 s + 486q_0 s^2 + 7q_0^3 s^2 + 5832 s^2 q_2}{11664 q_0 s^2}\frac{1}{r^2} + \frac{x_3}{r^3} + \dots\,,\\
p(r) &= \frac{q_0\left(54 - \sqrt{6}\mathrm{i}q_0 s\right)}{162 s^2}\frac{1}{r^2} + \dots\,,
\end{aligned}$$

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$$
\begin{aligned}
q(r) &= q_0 r + \frac{q_2}{r} + \frac{q_3}{r^2} + \dots \,, \\
f(r) &= f_0 - \frac{f_0\left(54 - \sqrt{6}\mathrm{i}q_0 s\right)}{81 s^2}\frac{1}{r^2} + \frac{f_3}{r^3} + \dots \,.
\end{aligned}
\tag{B.13}
$$

In addition to the squashing parameter s, the solution depends on $q_0, f_0, f_3, q_2, q_3, x_3$ and an extra parameter α_5, which appears at higher order in the expansion for $\alpha(r)$. All other coefficients in the expansion are fixed in terms of these constants. Of course, some of these parameters will be fixed in the full solution by requiring the correct boundary conditions at the origin $r = \frac{1}{\sqrt{6}}$, but at this point they are arbitrary.

B.2.2 Expansion around Euclidean AdS

The family of solutions we seek should approach Euclidean AdS_6 (B.10) as we take the squashing parameter $s \to 1$. Hence it should be possible to expand the solutions around this limit in terms of a perturbation parameter δ. Thus we make the ansatz

$$
\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \delta\ \alpha^{(1)}(r) + \delta^2\ \alpha^{(2)}(r) + \dots \,, \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} + \delta\ \gamma^{(1)}(r) + \delta^2\ \gamma^{(2)}(r) + \dots \,, \\
X(r) &= 1 + \delta\ X^{(1)}(r) + \delta^2\ X^{(2)}(r) + \dots \,, \\
p(r) &= \delta\ p^{(1)}(r) + \delta^2\ p^{(2)}(r) + \dots \,, \\
q(r) &= \delta\ q^{(1)}(r) + \delta^2\ q^{(2)}(r) + \dots \,, \\
f(r) &= \delta\ f^{(1)}(r) + \delta^2\ f^{(2)}(r) + \dots \,.
\end{aligned}
\tag{B.14}
$$

Substituting this expansion into the equations of motion and expanding in powers of δ, at each order we obtain a system of linear differential equations which can be solved in closed form with some effort. For instance, at first order we find

$$
\begin{aligned}
\alpha^{(1)}(r) &= -c_\gamma \frac{\left(1 - 54r^2 + 96\sqrt{6}r^3 - 324r^4 + 216r^6\right)}{\sqrt{6}r^2\left(6r^2-1\right)^{7/2}} \,, \\
\gamma^{(1)}(r) &= c_\gamma \frac{\left(-5 + 16\sqrt{6}r - 90r^2 + 180r^4 - 216r^6\right)}{\left(6r^2-1\right)^{5/2}} \,, \\
X^{(1)}(r) &= c_x \frac{\left(1 - 2\sqrt{6}r + 6r^2\right)}{\left(6r^2-1\right)^2} \,, \\
p^{(1)}(r) &= c_q \frac{\left(\sqrt{6} - 16r + 12\sqrt{6}r^2 - 12\sqrt{6}r^4\right)}{3\left(6r^2-1\right)^3} \,, \\
q^{(1)}(r) &= -c_q \frac{\left(-4 + 9\sqrt{6}r - 24r^2 - 12\sqrt{6}r^3 + 36\sqrt{6}r^5\right)}{18\left(6r^2-1\right)^2} \,, \\
f^{(1)}(r) &= c_f \frac{\left(-3 + 8\sqrt{6}r - 36r^2 + 36r^4\right)}{\left(6r^2-1\right)^2} \,.
\end{aligned}
\tag{B.15}
$$

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The constants of integration have been partially fixed by requiring regularity at the origin $r = \frac{1}{\sqrt{6}}$. In particular we have

$$\alpha^{(1)}(r) \sim \left(r - \frac{1}{\sqrt{6}}\right)^{1/2}, \qquad \gamma^{(1)}(r) \sim \left(r - \frac{1}{\sqrt{6}}\right)^{3/2},$$
$$p^{(1)}(r) \sim 1 \sim X^{(1)}(r), \qquad q^{(1)}(r) \sim \left(r - \frac{1}{\sqrt{6}}\right) \sim f^{(1)}(r). \tag{B.16}$$

Here $\rho \sim (r - \frac{1}{\sqrt{6}})^{1/2}$ is geodesic distance from the origin at $\rho = 0$. We can furthermore fix an extra constant of integration by fixing a relation between δ and the squashing parameter s (such that $\delta \to 0$ as $s \to 1$). As seen in the next section it will be convenient not to do this uniformly.

B.3 Imposing supersymmetry

We are interested in solutions that preserve some supersymmetry. In order for this to happen, there should exist non-trivial eight-component Killing spinors ϵ_1, ϵ_2 solving the Killing spinor equation (3.7) and dilatino equation (3.8). We choose the frame

$$e^0 = \alpha(r)\mathrm{d}r\,, \qquad e^1 = \gamma(r)(\mathrm{d}\tau + C)\,, \qquad e^2 = \beta(r)\mathrm{d}\sigma\,, \tag{B.17}$$
$$e^3 = \frac{1}{2}\beta(r)\sin\sigma\cos\sigma\tau_3\,, \qquad e^4 = \frac{1}{2}\beta(r)\sin\sigma\tau_2\,, \qquad e^5 = \frac{1}{2}\beta(r)\sin\sigma\tau_1\,,$$

and the following basis for six-dimensional gamma matrices

$$\Gamma_0 = \begin{pmatrix} 0 & 1_4 \\ 1_4 & 0 \end{pmatrix}, \qquad \Gamma_m = \begin{pmatrix} 0 & \mathrm{i}\gamma_m \\ -\mathrm{i}\gamma_m & 0 \end{pmatrix}, \qquad m = 1, \ldots, 5\,,$$
$$\Gamma_7 = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix}, \tag{B.18}$$

where 1_4 is the 4×4 unit matrix and γ_m are the five-dimensional gamma matrices given explicitly in section 2.1.

The vanishing of the dilatino variation as well as each component of the integrability condition (A.6) for the Killing spinor equation have the following general structure

$$P\epsilon_1 + Q\epsilon_2 = 0\,,$$
$$R\epsilon_1 + S\epsilon_2 = 0\,, \tag{B.19}$$

where P, Q, R, S are 8×8 matrices, whose components are in general complicated functions of the fields. After setting $f_i(r) = \kappa_i f(r)$ we observe the following $\mathrm{SU}(2)_R$ structure

$$\begin{pmatrix} A + \kappa_3 B & (\kappa_1 - \mathrm{i}\kappa_2)B \\ (\kappa_1 + \mathrm{i}\kappa_2)B & A - \kappa_3 B \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0\,, \tag{B.20}$$

in terms of 8×8 matrices A, B. We can then diagonalize the block matrix and consider the equivalent problem

$$\begin{pmatrix} A + B & 0 \\ 0 & A - B \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = 0\,, \tag{B.21}$$

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where we have without loss of generality set $\kappa_1^2+\kappa_2^2+\kappa_3^2=1$. There are four independent conditions. One of these arises from the dilatino variation, whose matrices we denote by A_0, B_0, and the other three conditions arise from integrability of the Killing spinor equation, whose matrices we denote by A_M, B_M with $M \in \{12, 13, 34\}$ (all other components of the integrability condition (A.6) are equivalent to one of these). The dilatino condition as well as $M = 12$ and $M = 34$ have the following structure:

$$A \pm B = \begin{pmatrix} * & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & * \\ * & 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & * \end{pmatrix} . \tag{B.22}$$

The existence of a non-trivial solution requires, for instance, $\det(A+B)=0$. The above structure implies the determinant factorizes into four factors

$$\det(A+B) \;=\; F_1 F_2 F_3 F_4 \;=\; 0\,, \tag{B.23}$$

where the factors F_i are complicated functions of the supergravity fields $\alpha(r)$, $\beta(r)$, $\gamma(r)$, $p(r)$, $q(r)$, $f(r)$, $X(r)$. F_1 and F_3 differ only by a change of sign in $f(r)$, and the same happens for F_2 and F_4. We find two distinct classes of solutions which we describe in the following.

B.3.1 3/4 BPS solutions

There is a class of solutions that satisfies

$$F_1 \;=\; F_2 \;=\; F_3 \;=\; 0\,, \qquad F_4 \;\neq\; 0\,. \tag{B.24}$$

These are a one-parameter family of solutions parametrized by the squashing parameter s. The solution expanded around the conformal boundary is given by

$$\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} + \frac{8+s^2}{36\sqrt{2}s^2}\frac{1}{r^3} + \dots\,, \\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{-16+7s^2}{12\sqrt{3}s^3}\frac{1}{r} - \frac{-1280+1120s^2+241s^4}{2592\sqrt{3}s^5}\frac{1}{r^3} + \dots\,, \\
X(r) &= 1 + \frac{1-s^2-3\sqrt{1-s^2}}{54s^2}\frac{1}{r^2} + \frac{s^2\sqrt{1-s^2}\kappa}{12\left(1-s^2+\sqrt{1-s^2}\right)}\frac{1}{r^3} + \dots\,, \\
p(r) &= -\frac{\mathrm{i}\sqrt{\frac{2}{3}}\left(s^2+3\sqrt{1-s^2}-1\right)}{s^3}\frac{1}{r^2} + \dots\,,
\end{aligned} \tag{B.25}$$

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$$q(r) = -\frac{3\mathrm{i}\left(\sqrt{6}\sqrt{1-s^2}\right)}{s}r + \frac{\sqrt{\frac{2}{3}}\mathrm{i}\sqrt{1-s^2}\left(5s^2+9\sqrt{1-s^2}-5\right)}{3s^3}\frac{1}{r} + \dots ,$$
$$f(r) = \frac{1-s^2+\sqrt{1-s^2}}{s^2} + \frac{2\left(-2+2s^2-(2+s^2)\sqrt{1-s^2}\right)}{9s^4}\frac{1}{r^2} + \frac{\kappa}{r^3} + \dots .$$

The extra parameter κ is fixed by requiring regularity at the origin. The solution expanded around Euclidean AdS_6 has $c_\gamma = 0$, hence it is convenient to set the relation between the expansion parameter and the squashing parameter to be

$$\frac{1}{s} \;=\; 1+\delta^2 \;. \tag{B.26}$$

With this choice the solution is given by

$$\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \frac{\Big(-5\sqrt{6}+330\sqrt{6}r^2-3744r^3+1620\sqrt{6}r^4+8640r^5}{9\sqrt{2}r^2\left(6r^2-1\right)^{9/2}} \\
&\qquad \frac{-7560\sqrt{6}r^6+5184\sqrt{6}r^8\Big)}{9\sqrt{2}r^2\left(6r^2-1\right)^{9/2}}\delta^2 + \dots , \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} - \frac{\Big(55\sqrt{2}-384\sqrt{3}r+1080\sqrt{2}r^2+768\sqrt{3}r^3}{6\left(6r^2-1\right)^{7/2}} \\
&\qquad \frac{-5400\sqrt{2}r^4+11232\sqrt{2}r^6-11664\sqrt{2}r^8\Big)}{6\left(6r^2-1\right)^{7/2}}\delta^2 + \dots , \\
X(r) &= 1 - \frac{\left(\sqrt{2}\left(1-2\sqrt{6}r+6r^2\right)\right)}{3\left(6r^2-1\right)^2}\delta + \dots , \\
p(r) &= \frac{18\mathrm{i}\sqrt{2}\left(\sqrt{6}-16r+12\sqrt{6}r^2-12\sqrt{6}r^4\right)}{\left(6r^2-1\right)^3}\delta + \dots , \\
q(r) &= -\frac{3\mathrm{i}\sqrt{2}\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)}{\left(6r^2-1\right)^2}\delta + \dots , \\
f(r) &= \frac{\sqrt{2}\left(-3+8\sqrt{6}r-36r^2+36r^4\right)}{\left(6r^2-1\right)^2}\delta + \dots .
\end{aligned} \tag{B.27}$$

We have computed the solution up to sixth order in δ. Comparing this expansion with the expansion around the conformal boundary we can compute the coefficient κ as a series expansion in δ. We obtain

$$\frac{3\sqrt{3}}{4}\kappa \;=\; \delta + \frac{\sqrt{2}}{3}\delta^2 + \frac{113}{36}\delta^3 + \frac{25}{9\sqrt{2}}\delta^4 + \frac{1127}{288}\delta^5 + \frac{35}{9\sqrt{2}}\delta^6 + \dots . \tag{B.28}$$

B.3.2 1/4 BPS solutions

There is another class of supersymmetric solutions that satisfies

$$F_1\,,\, F_2\,,\, F_3 \;\neq\; 0\,, \qquad F_4 \;=\; 0\,. \tag{B.29}$$

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These are a two-parameter family of solutions and are parametrized by the squashing parameter s and the background $\mathrm{SU}(2)_R$ field at the conformal boundary, which is parametrized by f_0. The solution expanded around the conformal boundary is given by

$$\begin{aligned}
\alpha(r) &= \frac{3}{\sqrt{2}}\frac{1}{r} - \frac{f_0^2 s^2 + 9\left(-2+s^2\right) - 6f_0\left(-1+s^2\right)}{36\sqrt{2}}\frac{1}{r^3} + \dots\,, \\
\gamma(r) &= \frac{3\sqrt{3}}{s}r + \frac{2f_0^2 s^2 - 12f_0\left(-1+s^2\right) + 9\left(-3+2s^2\right)}{12\sqrt{3}s}\frac{1}{r} + \dots\,, \\
X(r) &= 1 + \frac{18 - 3f_0 - 18s^2 + 12f_0 s^2 - 2f_0^2 s^2}{54}\frac{1}{r^2} + \dots\,, \\
p(r) &= \frac{\mathrm{i}\sqrt{\frac{2}{3}}(-3+f_0)\left(3+(-3+f_0)s^2\right)}{s}\frac{1}{r^2} + \dots\,, \\
q(r) &= -\frac{3\mathrm{i}\sqrt{6}\left(3+(-3+f_0)s^2\right)}{s}r \\
&\quad + \frac{\mathrm{i}\left(3+(-3+f_0)s^2\right)\left(f_0^2 s^2 + 9\left(-1+s^2\right) - 6f_0\left(1+s^2\right)\right)}{6\sqrt{6}s}\frac{1}{r} + \frac{\xi_1}{r^2} + \dots\,, \\
f(r) &= f_0 + \frac{2(-3+f_0)f_0}{9}\frac{1}{r^2} + \frac{\xi_2}{r^3} + \dots\,.
\end{aligned} \tag{B.30}$$

The constants ξ_1 and ξ_2 are fixed by requiring regularity at the origin. Note that a particular case corresponds to $f_0 = 0$. In this case the $\mathrm{SU}(2)_R$ background field is turned off, but the solution is still supersymmetric with a squashed five-sphere at the conformal boundary. In this case $F_4 = F_2 = 0$, so we have enhanced supersymmetry; that is, this one-parameter family of solutions with $f_0 = 0$ is 1/2 BPS.

As an expansion around Euclidean AdS we parametrize the solution in terms of the expansion parameter δ and an extra parameter ω, related to s and f_0 above by

$$\frac{1}{s} \;=\; 1+\delta\,, \qquad f_0 \;=\; \delta\,\omega\,. \tag{B.31}$$

With this choice the solution is given by

$$\begin{aligned}
\alpha(r) &= \frac{3\sqrt{3}}{\sqrt{6r^2-1}} + \frac{\sqrt{3}\left(1-54r^2+96\sqrt{6}r^3-324r^4+216r^6\right)}{2r^2\left(6r^2-1\right)^{7/2}}\delta + \dots\,, \\
\gamma(r) &= \frac{3\sqrt{6r^2-1}}{\sqrt{2}} + \frac{\left(15-48\sqrt{6}r+270r^2-540r^4+648r^6\right)}{\sqrt{2}\left(6r^2-1\right)^{5/2}}\delta + \dots\,, \\
X(r) &= 1 + \frac{\left(1-2\sqrt{6}r+6r^2\right)(4+\omega)}{\left(6r^2-1\right)^2}\delta + \dots\,, \\
p(r) &= -\frac{18\mathrm{i}\sqrt{2}\left(-\sqrt{3}+8\sqrt{2}r-12\sqrt{3}r^2+12\sqrt{3}r^4\right)(6+\omega)}{\left(6r^2-1\right)^3}\delta + \dots\,, \\
q(r) &= -\frac{3\mathrm{i}\left(-4+9\sqrt{6}r-24r^2-12\sqrt{6}r^3+36\sqrt{6}r^5\right)(6+\omega)}{\left(6r^2-1\right)^2}\delta + \dots\,, \\
f(r) &= \frac{\left(-3+8\sqrt{6}r-36r^2+36r^4\right)\omega}{\left(6r^2-1\right)^2}\delta + \dots\,.
\end{aligned} \tag{B.32}$$

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As before it can be checked explicitly that the solution is regular at $r = \frac{1}{\sqrt{6}}$. We have computed this solution explicitly up to fourth order in δ. Comparing this expansion with the expansion around the conformal boundary we deduce

$$\begin{aligned}\xi_1 &= 2\mathrm{i}(6+\omega)\delta - \frac{1}{5}\mathrm{i}\left(144+98\omega+13\omega^2\right)\delta^2 \\ &\quad + \frac{\mathrm{i}\left(307719+209547\omega+41094\omega^2+1282\omega^3\right)}{9450}\delta^3 \\ &\quad - \frac{\mathrm{i}\left(26693550+21683700\omega+6126111\omega^2+771474\omega^3+51568\omega^4\right)}{623700}\delta^4+\dots\,, \end{aligned} \tag{B.33}$$

$$\begin{aligned}\xi_2 &= \frac{2}{3}\sqrt{\frac{2}{3}}\omega\delta - \frac{2}{45}\left(-\sqrt{6}\omega+2\sqrt{6}\omega^2\right)\delta^2 + \frac{\left(-999\sqrt{6}\omega-594\sqrt{6}\omega^2+244\sqrt{6}\omega^3\right)}{42525}\delta^3 \\ &\quad + \frac{\left(32724\sqrt{6}\omega+26082\sqrt{6}\omega^2+6105\sqrt{6}\omega^3+935\sqrt{6}\omega^4\right)}{1403325}\delta^4+\dots\,. \end{aligned} \tag{B.34}$$

B.4 Killing spinors

Having found the above supersymmetric solutions we now proceed to solve the dilatino equation (3.8) and Killing spinor equation (3.7) for the Killing spinors ϵ_I, $I = 1, 2$.

3/4 BPS solution. For the 3/4 BPS solution we find

$$\epsilon_1 = a_+^{(1)}\,\mathrm{e}^{\mathrm{i}\frac{\tau}{2}}\begin{pmatrix} k_2(r)\left[\cos\sigma+\mathrm{i}\lambda_+(s)\mathrm{e}^{\mathrm{i}\frac{\psi}{2}}S_+^{(1)}\sin\sigma\right] \\ 0 \\ \mathrm{i}k_3(r)\left[\sin\sigma-\mathrm{i}\lambda_+(s)\mathrm{e}^{\mathrm{i}\frac{\psi}{2}}S_+^{(1)}\cos\sigma\right] \\ \mathrm{i}k_3(r)\lambda_+(s)\,\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_+^{(2)} \\ -\mathrm{i}k_4(r)\left[\cos\sigma+\mathrm{i}\lambda_+(s)\mathrm{e}^{\mathrm{i}\frac{\psi}{2}}S_+^{(1)}\sin\sigma\right] \\ 0 \\ k_1(r)\left[\sin\sigma-\mathrm{i}\lambda_+(s)\mathrm{e}^{\mathrm{i}\frac{\psi}{2}}S_+^{(1)}\cos\sigma\right] \\ k_1(r)\,\lambda_+(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_+^{(2)} \end{pmatrix}, \tag{B.35}$$

$$\epsilon_2 = a_-^{(1)}\,\mathrm{e}^{-\mathrm{i}\frac{\tau}{2}}\begin{pmatrix} 0 \\ \mathrm{i}k_4(r)\left[\cos\sigma-\mathrm{i}\lambda_-(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_-^{(1)}\sin\sigma\right] \\ -k_1(r)\lambda_-(s)\,\mathrm{e}^{\mathrm{i}\frac{\psi}{2}}S_-^{(2)} \\ k_1(r)\left[\sin\sigma+\mathrm{i}\lambda_-(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_-^{(1)}\cos\sigma\right] \\ 0 \\ k_2(r)\left[\cos\sigma-\mathrm{i}\lambda_(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_-^{(1)}\sin\sigma\right] \\ \mathrm{i}k_3(r)\,\lambda_-(s)\mathrm{e}^{\mathrm{i}\frac{\psi}{2}}S_-^{(2)} \\ -\mathrm{i}k_3(r)\left[\sin\sigma+\mathrm{i}\lambda_-(s)\mathrm{e}^{-\mathrm{i}\frac{\psi}{2}}S_-^{(1)}\cos\sigma\right] \end{pmatrix}, \tag{B.36}$$

where we have introduced

$$S_\pm^{(1)} = S_\pm^{(1)}(\theta,\varphi) = a_\pm^{(3)}\mathrm{e}^{\pm\mathrm{i}\frac{\varphi}{2}}\cos\frac{\theta}{2} - a_\pm^{(2)}\mathrm{e}^{\mp\mathrm{i}\frac{\varphi}{2}}\sin\frac{\theta}{2}\,,$$

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$$S^{(2)}_{\pm} = S^{(2)}_{\pm}(\theta,\varphi) = a^{(2)}_{\pm} \mathrm{e}^{\mp \mathrm{i}\frac{\varphi}{2}} \cos\frac{\theta}{2} + a^{(3)}_{\pm} \mathrm{e}^{\pm \mathrm{i}\frac{\varphi}{2}} \sin\frac{\theta}{2}\,,$$
$$\lambda_{\pm}(s) = \frac{\pm 1 + \sqrt{1-s^2}}{s}\,. \tag{B.37}$$

The Killing spinors contain in total six constants of integration $a^{(i)}_{\pm}$, $i = 1,2,3$. These constants of integration are generically complex, but imposing the symplectic Majorana condition $\mathcal{C}\epsilon^*_I = \varepsilon_I{}^J \epsilon_J$ enforces certain reality conditions. The functions $k_i(r)$ are functions of the radial coordinate only and can be expanded either around Euclidean AdS or around the boundary. For instance, expanding around the conformal boundary we obtain

$$\begin{aligned}
k_1(r) &= \frac{-1+\sqrt{1-s^2}}{s}\sqrt{r} + \frac{1}{2\sqrt{6}}\frac{1}{\sqrt{r}} + \dots\,,\\
k_2(r) &= \sqrt{r} - \frac{5\sqrt{1-s^2}-3}{6\sqrt{6}s}\frac{1}{\sqrt{r}} + \dots\,,\\
k_3(r) &= \frac{-1+\sqrt{1-s^2}}{s}\sqrt{r} - \frac{1}{2\sqrt{6}}\frac{1}{\sqrt{r}} + \dots\,,\\
k_4(r) &= \sqrt{r} + \frac{5\sqrt{1-s^2}-3}{6\sqrt{6}s}\frac{1}{\sqrt{r}} + \dots\,,
\end{aligned} \tag{B.38}$$

Notice that the expansion of the Killing spinor around the boundary is precisely of the form

$$\epsilon_I = \begin{pmatrix} \epsilon^+_I \\ \epsilon^-_I \end{pmatrix} = \sqrt{r}\begin{pmatrix} \chi_I \\ -\mathrm{i}\chi_I \end{pmatrix} + \frac{1}{\sqrt{r}}\begin{pmatrix} \varphi_I \\ \mathrm{i}\varphi_I \end{pmatrix} + \cdots\,, \tag{B.39}$$

which arises from the general analysis of section 6 and should of course hold for our particular solution. This allows us to immediately identify the boundary five-dimensional Killing spinor χ_I corresponding to our bulk solution. Note that this precisely agrees with (2.15).

1/4 BPS solution. For the 1/4 BPS solution we find

$$\epsilon_1 = c_+\,\mathrm{e}^{-\frac{3\mathrm{i}\tau}{2}}\begin{pmatrix} 0 \\ k_2(r) \\ 0 \\ 0 \\ 0 \\ -\mathrm{i}\,k_1(r) \\ 0 \\ 0 \end{pmatrix}, \qquad \epsilon_2 = -c_-\,\mathrm{e}^{\frac{3\mathrm{i}\tau}{2}}\begin{pmatrix} k_1(r) \\ 0 \\ 0 \\ 0 \\ -\mathrm{i}\,k_2(r) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{B.40}$$

The solution depends now on two constants of integration $c_\pm$. The functions of the radial coordinate admit the following expansion around the conformal boundary

$$k_1(r) = \sqrt{r} + \frac{(f_0-3)s}{6\sqrt{6}}\frac{1}{\sqrt{r}} + \frac{5(f_0-3)^2s^2 + 6(4f_0-9)}{432}\left(\frac{1}{r}\right)^{3/2} + \dots\,,$$

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$$k_2(r) = \sqrt{r} - \frac{(f_0-3)s}{6\sqrt{6}}\frac{1}{\sqrt{r}} + \frac{5(f_0-3)^2s^2+6(4f_0-9)}{432}\left(\frac{1}{r}\right)^{3/2} + \dots \ . \tag{B.41}$$

As before, the corresponding Killing spinors at the boundary can be identified. In this case they are indeed of the form (2.14), as expected. Finally, let us mention that the supersymmetry gets enhanced for the case $f_0 = 0$ (or equivalently $\omega = 0$). In this limit the gauge field vanishes and so the two Killing spinors ϵ_I for $I = 1, 2$ decouple and have the same structure. They read

$$\epsilon_I = \begin{pmatrix} c_I^{(2)}\, k_1(r) \mathrm{e}^{\frac{3\mathrm{i}\tau}{2}} \\ c_I^{(1)}\, k_2(r) \mathrm{e}^{-\frac{3\mathrm{i}\tau}{2}} \\ 0 \\ 0 \\ -\mathrm{i}\, c_I^{(2)}\, k_2(r) \mathrm{e}^{\frac{3\mathrm{i}\tau}{2}} \\ -\mathrm{i}\, c_I^{(1)}\, k_1(r) \mathrm{e}^{-\frac{3\mathrm{i}\tau}{2}} \\ 0 \\ 0 \end{pmatrix} , \tag{B.42}$$

where $c_I^{(j)}$ for $j = 1, 2$ are the integration constants and where the r-dependent functions $k_i(r)$ are the same as in the 1/4 BPS case, with $f_0 = 0$. This solution may thus be referred to as a 1/2 BPS solution.

C Asymptotics of multiple sine functions

Let us start by defining Barnes' multiple zeta function,

$$\zeta_{\mathcal{N}}(s, w \mid \mathbf{a}) \equiv \sum_{m_1,\dots,m_{\mathcal{N}}=0}^{\infty} (w + m_1 a_1 + \cdots m_{\mathcal{N}} a_{\mathcal{N}})^{-s} \ , \tag{C.1}$$

where $\mathbf{a} = (a_1, \dots, a_{\mathcal{N}})$, $\mathrm{Re}\, w > 0$, $\mathrm{Re}\, s > \mathcal{N}$ and $a_1, \dots, a_{\mathcal{N}} > 0$. This function is meromorphic in s, with simple poles at $s = 1, \dots, \mathcal{N}$. One can then define the Barnes multiple gamma function $\Gamma_{\mathcal{N}}(w \mid \mathbf{a}) \equiv \exp\left[\Psi_{\mathcal{N}}(w \mid \mathbf{a})\right]$, where

$$\Psi_{\mathcal{N}}(w \mid \mathbf{a}) \equiv \frac{\mathrm{d}}{\mathrm{d}s}\zeta_{\mathcal{N}}(s, w \mid \mathbf{a})\,|_{s=0} \ . \tag{C.2}$$

In order to compute the asymptotics of the multiple gamma function, and the closely related multiple sine function, we have to express this function in a more convenient way. In [42], it was observed that there is an expansion of $\Psi_{\mathcal{N}}(w)$ of the form

$$\begin{aligned} \Psi_{\mathcal{N}}(w \mid \mathbf{a}) = {} & \frac{(-1)^{\mathcal{N}+1}}{\mathcal{N}!} B_{\mathcal{N},\mathcal{N}}(w) \log w + (-1)^{\mathcal{N}} \sum_{k=0}^{\mathcal{N}-1} \frac{B_{\mathcal{N},k}(0) w^{\mathcal{N}-k}}{k!(\mathcal{N}-k)!} \sum_{\ell=1}^{\mathcal{N}-k} \frac{1}{\ell} \\ & + \sum_{k=\mathcal{N}+1}^{\mathcal{M}} \frac{(-1)^k}{k!} B_{\mathcal{N},k}(0)\, w^{\mathcal{N}-k} (k-\mathcal{N}-1)! + \mathcal{R}_{\mathcal{N},\mathcal{M}}(w) \, , \end{aligned} \tag{C.3}$$

where

$$\mathcal{R}_{\mathcal{N},\mathcal{M}}(w) \equiv \int_0^\infty \frac{\mathrm{d}t}{t} \mathrm{e}^{-wt} \left(\prod_{j=1}^{\mathcal{N}} \left(1 - \mathrm{e}^{-a_j t}\right)^{-1} - \sum_{k=0}^{\mathcal{M}} \frac{(-1)^k}{k!} B_{\mathcal{N},k}(0) t^{k-\mathcal{N}} \right) , \tag{C.4}$$

and $\mathcal{M} \geq \mathcal{N}$ as well as $\mathrm{Re}\, w > 0$. The functions $B_{\mathcal{N},\mathcal{M}}(w)$ are the so-called multiple Bernoulli polynomials and can be determined by expanding and solving the following relation

$$\frac{t^{\mathcal{N}} \mathrm{e}^{xt}}{\prod_{j=1}^{\mathcal{N}} \left(\mathrm{e}^{a_j t} - 1\right)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{\mathcal{N},n}(x) , \tag{C.5}$$

for $B_{\mathcal{N},\mathcal{M}}(w)$. It was further shown in [42] that in the asymptotic limit $|w| \to \infty$ and $|\arg w| < \pi$ the remainder $\mathcal{R}_{\mathcal{N},\mathcal{M}}(w)$ behaves as

$$\mathcal{R}_{\mathcal{N},\mathcal{M}}(w) = \mathcal{O}\left(w^{\mathcal{N}-\mathcal{M}-1}\right) , \tag{C.6}$$

and hence in the asymptotic limit is suppressed by the first three terms in (C.3). Similarly, the third term in (C.3) behaves as

$$\sum_{k=\mathcal{N}+1}^{\mathcal{M}} \frac{(-1)^k}{k!} B_{\mathcal{N},k}(0)\, w^{\mathcal{N}-k} (k - \mathcal{N} - 1)! = \mathcal{O}\left(w^{-1}\right) , \tag{C.7}$$

in the asymptotic limit $|w| \to \infty$. Hence for our purposes we shall only focus on the asymptotics of the first two contributions to $\Psi_{\mathcal{N}}$.

We are interested in the asymptotic expansion of the so-called multiple sine function, which is defined in terms of the Gamma function as

$$S_{\mathcal{N}}(w \mid \mathbf{a}) \equiv \Gamma_{\mathcal{N}}(w \mid \mathbf{a})^{-1}\, \Gamma_{\mathcal{N}}(a_{\mathrm{tot}} - w \mid \mathbf{a})^{(-1)^{\mathcal{N}}} , \tag{C.8}$$

where $a_{\mathrm{tot}} = \sum_{i=1}^{\mathcal{N}} a_i$. To compute the large N limit of the free energy, we are interested in the asymptotics of the logarithm of these functions

$$\log S_{\mathcal{N}}(w \mid \mathbf{a}) = -\Psi_{\mathcal{N}}(w \mid \mathbf{a}) - \Psi_{\mathcal{N}}(a_{\mathrm{tot}} - w \mid \mathbf{a})^{(-1)^{\mathcal{N}}} . \tag{C.9}$$

Focusing on the case $\mathcal{N} = 3$, we find the following Bernoulli polynomials

$$\begin{aligned}
B_{3,0}(x) &= \frac{1}{a_1 a_2 a_3} , \\
B_{3,1}(x) &= \frac{x}{a_1 a_2 a_3} - \frac{a_{\mathrm{tot}}}{2 a_1 a_2 a_3} , \\
B_{3,2}(x) &= \frac{x^2}{a_1 a_2 a_3} - \frac{a_{\mathrm{tot}}}{a_1 a_2 a_3} x + \frac{a_{\mathrm{tot}}^2 + (a_1 a_2 + a_1 a_3 + a_2 a_3)}{6 a_1 a_2 a_3} , \\
B_{3,3}(x) &= \frac{x^3}{a_1 a_2 a_3} - \frac{3 a_{\mathrm{tot}}}{2 a_1 a_2 a_3} x^2 + \frac{a_{\mathrm{tot}}^2 + (a_1 a_2 + a_1 a_3 + a_2 a_3)}{6 a_1 a_2 a_3} x \\
&\quad - \frac{a_{\mathrm{tot}} (a_1 a_2 + a_1 a_3 + a_2 a_3)}{4 a_1 a_2 a_3} .
\end{aligned} \tag{C.10}$$

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We can then compute (C.3) and take the asymptotic limit of the logarithm of the triple sine function to obtain

$$\log S_3(w \mid \mathbf{a}) = \operatorname{sign} \operatorname{Re} w \left[\frac{\mathrm{i}\pi}{6a_1a_2a_3} w^3 - \frac{\mathrm{i}\pi a_{\mathrm{tot}}}{4a_1a_2a_3} w^2 + \frac{\mathrm{i}\pi \left(a_{\mathrm{tot}}^2 + a_1a_2 + a_1a_3 + a_2a_3\right)}{12a_1a_2a_3} \right.$$
$$\left. - \frac{\mathrm{i}\pi a_{\mathrm{tot}} \left(a_1a_2 + a_1a_3 + a_2a_3\right)}{24a_1a_2a_3} + \mathcal{O}\left(w^{-1}\right) \right] . \tag{C.11}$$

This procedure generalizes to any choice of $\mathcal{N}$, and gives a straightforward method to obtain the asymptotics of these functions.

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