Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 603861, 10 pages
doi:10.1155/2011/603861

Research Article

Q-Functions on Quasimetric Spaces and Fixed Points for Multivalued Maps

J. Marín, S. Romaguera, and P. Tirado

Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n, 46022 Valencia, Spain

Correspondence should be addressed to S. Romaguera, sromague@mat.upv.es

Received 14 December 2010; Revised 26 January 2011; Accepted 31 January 2011

Academic Editor: Qamrul Hasan Ansari

We discuss several properties of Q-functions in the sense of Al-Homidan et al.. In particular, we prove that the partial metric induced by any T_0 weighted quasipseudometric space is a Q-function and show that both the Sorgenfrey line and the Kofner plane provide significant examples of quasimetric spaces for which the associated supremum metric is a Q-function. In this context we also obtain some fixed point results for multivalued maps by using Bianchini-Grandolfi gauge functions.

1. Introduction and Preliminaries

Kada et al. introduced in [1] the concept of w-distance on a metric space and extended the Caristi-Kirk fixed point theorem [2], the Ekeland variation principle [3] and the nonconvex minimization theorem [4], for w-distances. Recently, Al-Homidan et al. introduced in [5] the notion of Q-function on a quasimetric space and then successfully obtained a Caristi-Kirk-type fixed point theorem, a Takahashi minimization theorem, an equilibrium version of Ekeland-type variational principle, and a version of Nadler's fixed point theorem for a Q-function on a complete quasimetric space, generalizing in this way, among others, the main results of [1] because every w-distance is, in fact, a Q-function. This interesting approach has been continued by Hussain et al. [6], and by Latif and Al-Mezel [7], respectively. In particular, the authors of [7] have obtained a nice Rakotch-type theorem for Q-functions on complete quasimetric spaces.

In Section 2 of this paper, we generalize the basic theory of Q-functions to T_0 quasipseudometric spaces. Our approach is motivated, in part, by the fact that in many applications to Domain Theory, Complexity Analysis, Computer Science and Asymmetric Functional Analysis, T_0 quasipseudometric spaces (in particular, weightable T_0

quasipseudometric spaces and their equivalent partial metric spaces) rather than quasimetric spaces, play a crucial role (cf. [8–23], etc.). In particular, we prove that for every weighted T_0 quasipseudometric space the induced partial metric is a Q-function. We also show that the Sorgenfrey line and the Kofner plane provide interesting examples of quasimetric spaces for which the associated supremum metric is a Q-function. Finally, Section 3 is devoted to present a new fixed point theorem for Q-functions and multivalued maps on T_0 quasipseudometric spaces, by using Bianchini-Grandolfi gauge functions in the sense of [24]. Our result generalizes and improves, in several ways, well-known fixed point theorems.

Throughout this paper the letter $\mathbb{N}$ and ω will denote the set of positive integer numbers and the set of nonnegative integer numbers, respectively.

Our basic references for quasimetric spaces are [25, 26].

Next we recall several pertinent concepts.

By a T_0 quasipseudometric on a set X, we mean a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,

(i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$,

(ii) $d(x, z) \leq d(x, y) + d(y, z)$.

A T_0 quasipseudometric d on X that satisfies the stronger condition

(i′) $d(x, y) = 0 \Leftrightarrow x = y$

is called a quasimetric on X.

We remark that in the last years several authors used the term "quasimetric" to refer to a T_0 quasipseudometric and the term "T_1 quasimetric" to refer to a quasimetric in the above sense.

In the following we will simply write T_0 qpm instead of T_0 quasipseudometric if no confusion arises.

A T_0 qpm space is a pair (X, d) such that X is a set and d is a T_0 qpm on X. If d is a quasimetric on X, the pair (X, d) is then called a quasimetric space.

Given a T_0 qpm d on a set X, the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$, is also a T_0 qpm on X, called the conjugate of d, and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d^{-1}(x, y)\}$ is a metric on X, called the supremum metric associated to d.

Thus, every T_0 qpm d on X induces, in a natural way, three topologies denoted by τ_d, $\tau_{d^{-1}}$, and τ_{d^s}, respectively, and defined as follows.

(i) τ_d is the T_0 topology on X which has as a base the family of τ_d-open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

(ii) $\tau_{d^{-1}}$ is the T_0 topology on X which has as a base the family of $\tau_{d^{-1}}$-open balls $\{B_{d^{-1}}(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_{d^{-1}}(x, \varepsilon) = \{y \in X : d^{-1}(x, y) < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

(iii) τ_{d^s} is the topology on X induced by the metric d^s.

Note that if d is a quasimetric on X, then d^{-1} is also a quasimetric, and τ_d and $\tau_{d^{-1}}$ are T_1 topologies on X.

Note also that a sequence $(x_n)_{n\in\mathbb{N}}$ in a T_0 qpm space (X, d) is τ_d-convergent (resp., $\tau_{d^{-1}}$-convergent) to $x \in X$ if and only if $\lim_n d(x, x_n) = 0$ (resp., $\lim_n d(x_n, x) = 0$).

It is well known (see, for instance, [26, 27]) that there exists many different notions of completeness for quasimetric spaces. In our context we will use the following notion.

A T_0 qpm space (X,d) is said to be complete if every Cauchy sequence is $\tau_{d^{-1}}$-convergent, where a sequence $(x_n)_{n\in\mathbb{N}}$ is called Cauchy if for each $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $m \geq n \geq n_\varepsilon$.

In this case, we say that d is a complete T_0 qpm on X.

2. Q-Functions on T_0 qpm-Spaces

We start this section by giving the main concept of this paper, which was introduced in [5] for quasimetric spaces.

Definition 2.1. A Q-function on a T_0 qpm space (X,d) is a function $q : X \times X \to [0,\infty)$ satisfying the following conditions:

(Q1) $q(x,z) \leq q(x,y) + q(y,z)$, for all $x,y,z \in X$,

(Q2) if $x \in X$, $M > 0$, and $(y_n)_{n\in\mathbb{N}}$ is a sequence in X that $\tau_{d^{-1}}$-converges to a point $y \in X$ and satisfies $q(x,y_n) \leq M$, for all $n \in \mathbb{N}$, then $q(x,y) \leq M$,

(Q3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d(y,z) \leq \varepsilon$.

If (X,d) is a metric space and $q : X \times X \to [0,\infty)$ satisfies conditions (Q1) and (Q3) above and the following condition:

(Q2′) $q(x,\cdot) : X \to [0,\infty)$ is lower semicontinuous for all $x \in X$, then q is called a w-distance on (X,d) (cf. [1]).

Clearly d is a w-distance on (X,d) whenever d is a metric on X.

However, the situation is very different in the quasimetric case. Indeed, it is obvious that if (X,d) is a T_0 qpm space, then d satisfies conditions (Q1) and (Q2), whereas Example 3.2 of [5] shows that there exists a T_0 qpm space (X,d) such that d does not satisfy condition (Q3), and hence it is not a Q-function on (X,d). In this direction, we next present some positive results.

Lemma 2.2. *Let q be a Q-function on a T_0 qpm space (X,d). Then, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ imply $d^s(y,z) \leq \varepsilon$.*

Proof. By condition (Q3), $d(y,z) \leq \varepsilon$. Interchanging y and z, it follows that $d(z,y) \leq \varepsilon$, so $d^s(y,z) \leq \varepsilon$. □

Proposition 2.3. *Let (X,d) be a T_0 qpm space. If d is a Q-function on (X,d), then $\tau_d = \tau_{d^s}$, and hence, τ_d is a metrizable topology on X.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X which is τ_d-convergent to some $x \in X$. Then, by Lemma 2.2, $\lim_n d^s(x,x_n) = 0$. We conclude that $\tau_d = \tau_{d^s}$. □

Remark 2.4. It follows from Proposition 2.3 that many paradigmatic quasimetrizable topological spaces (X,τ), as the Sorgenfrey line, the Michael line, the Niemytzki plane and the Kofner plane (see [25]), do not admit any compatible quasimetric d which is a Q-function on (X,d).

In the sequel, we show that, nevertheless, it is possible to construct an easy but, in several cases, useful Q-function on any quasimetric space, as well as a suitable Q-functions on any weightable T_0 qpm space.

Recall that the discrete metric on a set X is the metric d_{01} on X defined as $d_{01}(x,x) = 0$, for all $x \in X$, and $d_{01}(x,y) = 1$, for all $x, y \in X$ with $x \neq y$.

Proposition 2.5. *Let (X,d)be a quasimetric space. Then, the discrete metric on $X$is a Q-function on (X,d).*

Proof. Since d_{01} is a metric it obviously satisfies condition (Q1) of Definition 2.1.

Now suppose that $(y_n)_{n\in\mathbb{N}}$ is a sequence in X that $\tau_{d^{-1}}$-converges to some $y \in X$, and let $x \in X$ and $M > 0$ such that $d_{01}(x, y_n) \leq M$, for all $n \in \mathbb{N}$. If $M \geq 1$, then $d_{01}(x,y) \leq M$. If $M < 1$, we deduce that $x = y_n$, for all $n \in \mathbb{N}$. Since $\lim_n d(y_n, y) = 0$, it follows that $d(x,y) = 0$, so $x = y$, and thus $d_{01}(x,y) = 0 < M$. Hence, condition (Q2) is also satisfied.

Finally, d_{01} satisfies condition (Q3) taking $\delta = 1/2$ for every $\varepsilon > 0$. □

Example 2.6. On the set $\mathbb{R}$ of real numbers define $d : \mathbb{R}\times\mathbb{R} \to [0,1]$ as $d(x,y) = 1$ if $x > y$, and $d(x,y) = \min\{y - x, 1\}$ if $x \leq y$. Then, d is a quasimetric on $\mathbb{R}$ and the topological space $(\mathbb{R}, \tau_d)$ is the celebrated Sorgenfrey line. Since d^s is the discrete metric on $\mathbb{R}$, it follows from Proposition 2.5 that d^s is a Q-function on $(\mathbb{R}, d)$.

Example 2.7. The quasimetric d on the plane $\mathbb{R}^2$, constructed in Example 7.7 of [25], verifies that $(\mathbb{R}^2, \tau_d)$ is the so-called Kofner plane and that d^s is the discrete metric on $\mathbb{R}^2$, so, by Proposition 2.5, d^s is a Q-function on $(\mathbb{R}^2, d)$.

Matthews introduced in [14] the notion of a weightable T_0 qpm space (under the name of a "weightable quasimetric space"), and its equivalent partial metric space, as a part of the study of denotational semantics of dataflow networks.

A T_0 qpm space (X,d) is called weightable if there exists a function $w : X \to [0,\infty)$ such that for all $x, y \in X, d(x,y) + w(x) = d(y,x) + w(y)$. In this case, we say that d is a weightable T_0 qpm on X. The function w is said to be a weighting function for (X,d) and the triple (X,d,w) is called a weighted T_0 qpm space.

A partial metric on a set X is a function $p : X\times X \to [0,\infty)$ such that, for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(ii) $p(x,x) \leq p(x,y)$,

(iii) $p(x,y) = p(y,x)$,

(iv) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.

A partial metric space is a pair (X,p) such that X is a set and p is a partial metric on X.

Each partial metric p on X induces a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < \varepsilon + p(x,x)\}$, for all $x \in X$ and $\varepsilon > 0$.

The precise relationship between partial metric spaces and weightable T_0 qpm spaces is provided in the next result.

Theorem 2.8 (Matthews [14]). (a) *Let (X,d) be a weightable T_0 qpm space with weighting function. Then, the function $p_d : X\times X \to [0,\infty)$ defined by $p_d(x,y) = d(x,y) + w(x)$, for all $x, y \in X$, is a partial metric on X. Furthermore $\tau_d = \tau_{p_d}$.*

(b) *Conversely, let (X,p) be a partial metric space. Then, the function $d_p : X\times X \to [0,\infty)$ defined by $d_p(x,y) = p(x,y) - p(x,x)$, for all $x, y \in X$ is a weightable T_0 qpm on X with weighting function w given by $w(x) = p(x,x)$ for all $x \in X$. Furthermore $\tau_p = \tau_{d_p}$.*

Remark 2.9. The domain of words, the interval domain, and the complexity quasimetric space provide distinguished examples of theoretical computer science that admit a structure of a weightable T_0 qpm space and, thus, of a partial metric space (see, e.g., [14, 20, 21]).

Proposition 2.10. *Let* (X, d, w) *be a weighted* T_0 *qpm space. Then, the induced partial metric* p_d *is a* Q*-function on* (X, d).

Proof. We will show that p_d satisfies conditions (Q1), (Q2), and Q(3) of Definition 2.1.

(Q1) Let $x, y, z \in X$, then

$$p_d(x,z) \leq p_d(x,y) + p_d(y,z) - p_d(y,y) \leq p_d(x,y) + p_d(y,z). \tag{2.1}$$

(Q2) Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in X which is $\tau_{d^{-1}}$-convergent to some $y \in X$. Let $x \in X$ and $M > 0$ such that $p_d(x, y_n) \leq M$, for all $n \in \mathbb{N}$.

Choose $\varepsilon > 0$. Then, there exists $n_\varepsilon \in \mathbb{N}$ such that $d(y_n, y) < \varepsilon$, for all $n \geq n_\varepsilon$. Therefore,

$$\begin{aligned} p_d(x,y) &= d(x,y) + w(x) \leq d(x, y_{n_\varepsilon}) + d(y_{n_\varepsilon}, y) + w(x) \\ &= p_d(x, y_{n_\varepsilon}) + d(y_{n_\varepsilon}, y) < M + \varepsilon. \end{aligned} \tag{2.2}$$

Since ε is arbitrary, we conclude that $p_d(x, y) \leq M$.

(Q3) Given $\varepsilon > 0$, put $\delta = \varepsilon/2$. If $p_d(x, y) \leq \delta$ and $p_d(x, z) \leq \delta$, it follows

$$\begin{aligned} d(y,z) &= p_d(y,z) - w(y) \leq p_d(y,z) \\ &\leq p_d(y,x) + p_d(x,z) \leq 2\delta = \varepsilon. \end{aligned} \tag{2.3}$$

□

3. Fixed Point Results

Given a T_0 qpm space (X, d), we denote by 2^X the collection of all nonempty subsets of X, by $Cl_{d^{-1}}(X)$ the collection of all nonempty $\tau_{d^{-1}}$-closed subsets of X, and by $Cl_{d^s}(X)$ the collection of all nonempty τ_{d^s}-closed subsets of X.

Following Al-Homidan et al. [5, Definition 6.1] if (X, d) is a quasimetric space, we say that a multivalued map $T : X \to 2^X$ is q-contractive if there exists a Q-function q on (X, d) and $r \in [0, 1)$ such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying $q(u, v) \leq rq(x, y)$.

Latif and Al-Mezel (see [7]) generalized this notion as follows.

If (X, d) is a quasimetric space, we say that a multivalued map $T : X \to 2^X$ is generalized q-contractive if there exists a Q-function q on (X, d) such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying

$$q(u,v) \leq k(q(x,y))q(x,y), \tag{3.1}$$

where $k : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{r\to t^+} k(r) < 1$ for all $t \geq 0$.

Then, they proved the following improvement of the celebrated Rakotch fixed point theorem (see [28]).

Theorem 3.1 (Lafit and Al-Mezel [7, Theorem 2.3]). *Let (X, d) be a complete quasimetric space. Then, for each generalized q-contractive multivalued map $T : X \to Cl_{d^{-1}}(X)$ there exists $z \in X$ such that $z \in T(z)$.*

On the other hand, Bianchini and Grandolfi proved in [29] the following fixed point theorem.

Theorem 3.2 (Bianchini and Grandolfi [29]). *Let (X, d) be a complete metric space and let $T : X \to X$ be a map such that for each $x, y \in X$*

$$d(T(x), T(y)) \leq \varphi(d(x, y)), \tag{3.2}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$, for all $t > 0$ (φ^n denotes the nth iterate of φ). Then, T has a unique fixed point.

A function $\varphi : [0, \infty) \to [0, \infty)$ satisfying the conditions of the preceding theorem is called a Bianchini-Grandolfi gauge function (cf [24, 30]).

It is easy to check (see [30, Page 8]) that if φ is a Bianchini-Grandolfi gauge function, then $\varphi(t) < t$, for all $t > 0$, and hence $\varphi(0) = 0$.

Our next result generalizes Bianchini-Grandolfi's theorem for Q-functions on complete T_0 qpm spaces.

Theorem 3.3. *Let (X, d) be a complete T_0 qpm space, q a Q-function on X, and $T : X \to Cl_{d^s}(X)$ a multivalued map such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying*

$$q(u, v) \leq \varphi(q(x, y)), \tag{3.3}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Bianchini-Grandolfi gauge function. Then, there exists $z \in X$ such that $z \in T(z)$ and $q(z, z) = 0$.

Proof. Fix $x_0 \in X$ and let $x_1 \in T(x_0)$. By hypothesis, there exists $x_2 \in T(x_1)$ such that $q(x_1, x_2) \leq \varphi(q(x_0, x_1))$. Following this process, we obtain a sequence $(x_n)_{n\in\omega}$ with $x_n \in T(x_{n-1})$ and $q(x_n, x_{n+1}) \leq \varphi(q(x_{n-1}, x_n))$, for all $n \in \mathbb{N}$. Therefore

$$q(x_n, x_{n+1}) \leq \varphi^n(q(x_0, x_1)), \tag{3.4}$$

for all $n \in \mathbb{N}$.

Now, choose $\varepsilon > 0$. Let $\delta = \delta(\varepsilon) \in (0, \varepsilon)$ for which condition (Q3) is satisfied. We will show that there is $n_\delta \in \mathbb{N}$ such that $q(x_n, x_m) < \delta$ whenever $m > n \geq n_\delta$.

Indeed, if $q(x_0, x_1) = 0$, then $\varphi(q(x_0, x_1)) = 0$ and thus $q(x_n, x_{n+1}) = 0$, for all $n \in \mathbb{N}$, so, by condition (Q1), $q(x_n, x_m) = 0$ whenever $m > n$.

If $q(x_0,x_1)>0$, $\sum_{n=0}^{\infty}\varphi^n(q(x_0,x_1))<\infty$, so there is $n_\delta\in\mathbb{N}$ such that

$$\sum_{n=n_\delta}^{\infty}\varphi^n(q(x_0,x_1))<\delta. \tag{3.5}$$

Then, for $m>n\geq n_\delta$, we have

$$\begin{aligned}q(x_n,x_m)&\leq q(x_n,x_{n+1})+q(x_{n+1},x_{n+2})+\cdots+q(x_{m-1},x_m)\\&\leq\varphi^n(q(x_0,x_1))+\varphi^{n+1}(q(x_0,x_1))+\cdots+\varphi^{m-1}(q(x_0,x_1))\\&\leq\sum_{j=n_\delta}^{\infty}\varphi^j(q(x_0,x_1))<\delta.\end{aligned} \tag{3.6}$$

In particular, $q(x_{n_\delta},q_n)\leq\delta$ and $q(x_{n_\delta},q_m)\leq\delta$ whenever $n,m>n_\delta$, so, by Lemma 2.2, $d^s(x_n,x_m)\leq\varepsilon$ whenever $n,m>n_\delta$.

We have proved that $(x_n)_{n\in\omega}$ is a Cauchy sequence in (X,d) (in fact, it is a Cauchy sequence in the metric space (X,d^s)). Since (X,d) is complete there exists $z\in X$ such that $\lim_n d(x_n,z)=0$.

Next, we show that $z\in T(z)$.

To this end, we first prove that $\lim_n q(x_n,z)=0$. Indeed, choose $\varepsilon>0$. Fix $n\geq n_\delta$. Since $q(x_n,x_m)\leq\delta$ whenever $m>n$, it follows from condition (Q2) that $q(x_n,z)\leq\delta<\varepsilon$ whenever $n\geq n_\delta$.

Now for each $n\in\mathbb{N}$ take $y_n\in T(z)$ such that

$$q(x_n,y_n)\leq\varphi(q(x_{n-1},z)). \tag{3.7}$$

If $q(x_{n-1},z)=0$, it follows that $q(x_n,y_n)=0$. Otherwise we obtain $q(x_n,y_n)<q(x_{n-1},z)$. Hence, $\lim_n q(x_n,y_n)=0$, and by Lemma 2.2,

$$\lim_n d^s(z,y_n)=0. \tag{3.8}$$

Therefore, $z\in Cl_{d^s}(T(z))=T(z)$.

It remains to prove that $q(z,z)=0$.

Since $z\in T(z)$, we can construct a sequence $(z_n)_{n\in\mathbb{N}}$ in X such that $z_1\in T(z)$, $z_{n+1}\in T(z_n)$ and

$$q(z,z_n)\leq\varphi^n(q(z,z)),\quad\forall n\in\mathbb{N}. \tag{3.9}$$

Since $\sum_{n=0}^{\infty}\varphi^n(q(z,z))<\infty$, it follows that $\lim_n\varphi^n(q(z,z))=0$, and thus $\lim_n q(z,z_n)=0$. So, by Lemma 2.2, $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X,d) (in fact, it is a Cauchy sequence in (X,d^s)). Let $u\in X$ such that $\lim_n d(z_n,u)=0$. Given $\varepsilon>0$, there is $n_\varepsilon\in\mathbb{N}$ such that $q(z,z_n)\leq\varepsilon$, for all $n\geq n_\varepsilon$. By applying condition (Q2), we deduce that $q(z,u)\leq\varepsilon$, so $q(z,u)=0$. Since $\lim_n q(x_n,z)=0$, it follows from condition (Q1) that $\lim_n q(x_n,u)=0$. Therefore, $d^s(z,u)\leq\varepsilon$, for all $\varepsilon>0$, by condition (Q3). We conclude that $z=u$, and thus $q(z,z)=0$. □

The next example illustrates Theorem 3.3.

Example 3.4. Let $X = [0,\pi]$ and let d be the T_0 qpm on X given by $d(x,y) = \max\{y-x,0\}$. It is well known that d is weightable with weighting function w given by $w(x) = x$, for all $x \in X$. Let q be partial metric induced by d. Then, q is a Q-function on (X,d) by Proposition 2.10. Note also that, by Theorem 2.8 (a),

$$q(x,y) = \max\{y-x,0\} + x = \max\{x,y\}, \tag{3.10}$$

for all $x,y \in X$. Moreover (X,d) is clearly complete because d^s is the Euclidean metric on X and thus (X,d^s) is a compact metric space.

Now define $T : X \to Cl_{d^s}(X)$ by

$$T(x) = \{0\} \cup \left\{\sin\frac{x}{2n} : n \in \mathbb{N}\right\}, \tag{3.11}$$

for all $x \in X$. Note that $T(x) \notin Cl_{d^{-1}}(X)$ because the nonempty $\tau_{d^{-1}}$-closed subsets of X are the intervals of the form $[0,x]$, $x \in X$.

Let $\varphi : [0,\infty) \to [0,\infty)$ be such that $\varphi(t) = \sin(t/2)$, for all $t \in [0,\pi]$, and $\varphi(t) = t/2$, for all $t > \pi$. We wish to show that φ is a Bianchini-Grandolfi gauge function.

It is clear that φ is nondecreasing.

Moreover, $\sum_{n=0}^{\infty} \varphi^n(t) < \infty$, for all $t \geq 0$. Indeed, if $t > \pi$ we have $\varphi^n(t) \leq t/2^n$ whenever $n \in \omega$, while for $t \in [0,\pi]$, we have $\varphi(t) \leq t/2$ so,

$$\varphi^2(t) = \varphi(\varphi(t)) = \sin\frac{\varphi(t)}{2} \leq \sin\frac{t}{4} \leq \frac{t}{4}, \tag{3.12}$$

and following this process we deduce the known fact that $\varphi^n(t) \leq t/2^n$, for all $n \in \mathbb{N}$. We have shown that φ is a Bianchini-Grandolfi gauge function.

Finally, for each $x,y \in X$ and $u \in T(x) \setminus \{0\}$, there exists $n \in \mathbb{N}$ such that $u = \sin(x/2n)$. Choose $v = \sin(y/2n)$. Then $v \in T(y)$ and

$$\begin{aligned} q(u,v) &= \max\left\{\sin\frac{x}{2n}, \sin\frac{y}{2n}\right\} \leq \max\left\{\sin\frac{x}{2}, \sin\frac{y}{2}\right\} \\ &= \sin\frac{\max\{x,y\}}{2} = \varphi(\max\{x,y\}) = \varphi(q(x,y)). \end{aligned} \tag{3.13}$$

If $u = 0$, then $u \in T(y)$, and thus $q(u,u) = 0 \leq \varphi(q(x,y))$.

We have checked that conditions of Theorem 3.3 are fulfilled, and hence, there is $z \in T(z)$ with $q(z,z) = 0$. In fact $z = 0$ is the only point of X satisfying $q(z,z) = 0$ and $z \in T(z)$ (actually $\{z\} = T(z)$). The following consequence of Theorem 3.3, which is also illustrated by Example 3.4, improves and generalizes in several directions the Banach Contraction Principle for partial metric spaces obtained in Theorem 5.3 of [14].

Corollary 3.5. *Let (X, p) be a partial metric space such that the induced weightable T_0 qpm d_p is complete and let $T : X \to Cl_{d^s}(X)$ be a multivalued map such that for each $x, y \in X$ and $u \in T(x)$, there is $v \in T(y)$ satisfying*

$$p(u, v) \leq \varphi(p(x, y)), \tag{3.14}$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a Bianchini-Grandolfi gauge function. Then, there exists $z \in X$ such that $z \in T(z)$ and $p(z, z) = 0$.

Proof. Since $p = p_{d_p}$ (see Theorem 2.8), we deduce from Proposition 2.10 that p is a Q-function for the complete (weightable) T_0 qpm space (X, d_p). The conclusion follows from Theorem 3.3. □

Observe that if $k : [0, \infty) \to [0, 1)$ is a nondecreasing function such that $\limsup_{r \to t^+} k(r) < 1$, for all $t \geq 0$, then the function $\varphi : [0, \infty) \to [0, \infty)$ given by $\varphi(t) = k(t)t$, is a Bianchini-Grandolfi gauge function (compare [31, Proposition 8]). Therefore, the following variant of Theorem 3.1, which improves Corollary 2.4 of [7], is now a consequence of Theorem 3.3.

Corollary 3.6. *Let (X, d) be a complete T_0 qpm space. Then, for each generalized q-contractive multivalued map $T : X \to Cl_{d^s}(X)$ with q nondecreasing, there exists $z \in X$ such that $z \in T(z)$ and $q(z, z) = 0$.*

Remark 3.7. The proof of Theorem 3.3 shows that the condition that (X, d) is complete can be replaced by the more general condition that every Cauchy sequence in the metric space (X, d^s) is $\tau_{d^{-1}}$-convergent.

Acknowledgments

The authors thank one of the reviewers for suggesting the inclusion of a concrete example to which Theorem 3.3 applies. They acknowledge the support of the Spanish Ministry of Science and Innovation, Grant no. MTM2009-12872-C02-01.

References

[1] O. Kada, T. Suzuki, and W. Takahashi, “Nonconvex minimization theorems and fixed point theorems in complete metric spaces,” *Mathematica Japonica*, vol. 44, no. 2, pp. 381–391, 1996.

[2] J. Caristi and W. A. Kirk, “Geometric fixed point theory and inwardness conditions,” in *The Geometry of Metric and Linear Spaces*, vol. 490 of *Lecture Notes in Mathematics*, pp. 74–83, Springer, Berlin, Germany, 1975.

[3] I. Ekeland, “Nonconvex minimization problems,” *Bulletin of the American Mathematical Society*, vol. 1, no. 3, pp. 443–474, 1979.

[4] W. Takahashi, “Existence theorems generalizing fixed point theorems for multivalued mappings,” in *Fixed Point Theory and Applications*, M. A. Théra and J. B. Baillon, Eds., vol. 252 of *Pitman Research Notes in Mathematics Series*, pp. 397–406, Longman, Harlow, UK, 1991.

[5] S. Al-Homidan, Q. H. Ansari, and J.-C. Yao, “Some generalizations of Ekeland-type variational principle with applications to equilibrium problems and fixed point theory,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 1, pp. 126–139, 2008.

[6] N. Hussain, M. H. Shah, and M. A. Kutbi, “Coupled coincidence point theorems for nonlinear contractions in partially ordered quasi-metric spaces with a Q-function,” *Fixed Point Theory and Applications*, vol. 2011, Article ID 703938, 21 pages, 2011.

[7] A. Latif and S. A. Al-Mezel, "Fixed point results in quasimetric spaces," *Fixed Point Theory and Applications*, vol. 2011, Article ID 178306, 8 pages, 2011.

[8] C. Alegre, "Continuous operators on asymmetric normed spaces," *Acta Mathematica Hungarica*, vol. 122, no. 4, pp. 357–372, 2009.

[9] M. Ali-Akbari, B. Honari, M. Pourmahdian, and M. M. Rezaii, "The space of formal balls and models of quasi-metric spaces," *Mathematical Structures in Computer Science*, vol. 19, no. 2, pp. 337–355, 2009.

[10] S. Cobzaş, "Compact and precompact sets in asymmetric locally convex spaces," *Topology and its Applications*, vol. 156, no. 9, pp. 1620–1629, 2009.

[11] L. M. García-Raffi, S. Romaguera, and E. A. Sánchez-Pérez, "The Goldstine theorem for asymmetric normed linear spaces," *Topology and its Applications*, vol. 156, no. 13, pp. 2284–2291, 2009.

[12] L. M. García-Raffi, S. Romaguera, and M. P. Schellekens, "Applications of the complexity space to the general probabilistic divide and conquer algorithms," *Journal of Mathematical Analysis and Applications*, vol. 348, no. 1, pp. 346–355, 2008.

[13] R. Heckmann, "Approximation of metric spaces by partial metric spaces," *Applied Categorical Structures*, vol. 7, no. 1-2, pp. 71–83, 1999.

[14] S. G. Matthews, "Partial metric topology," in *Proceedings of the 14th Summer Conference on General Topology and Its Applications*, vol. 728 of *Annals of the New York Academy of Sciences*, pp. 183–197, The New York Academy of Sciences, 1994.

[15] S. Romaguera, "On computational models for the hyperspace," in *Advances in Mathematics Research. Vol. 8*, Advances in Mathematics, pp. 277–294, Nova Science, New York, NY, USA, 2009.

[16] S. Romaguera and M. Schellekens, "Partial metric monoids and semivaluation spaces," *Topology and its Applications*, vol. 153, no. 5-6, pp. 948–962, 2005.

[17] S. Romaguera and P. Tirado, "The complexity probabilistic quasi-metric space," *Journal of Mathematical Analysis and Applications*, vol. 376, no. 2, pp. 732–740, 2011.

[18] S. Romaguera and O. Valero, "A quantitative computational model for complete partial metric spaces via formal balls," *Mathematical Structures in Computer Science*, vol. 19, no. 3, pp. 541–563, 2009.

[19] S. Romaguera and O. Valero, "Domain theoretic characterisations of quasi-metric completeness in terms of formal balls," *Mathematical Structures in Computer Science*, vol. 20, no. 3, pp. 453–472, 2010.

[20] M. Schellekens, "The Smyth completion: a common foundation for denotational semantics and complexity analysis," in *Mathematical Foundations of Programming Semantics (New Orleans, LA, 1995)*, vol. 1 of *Electronic Notes in Theoretical Computer Science*, pp. 535–556, Elsevier, Amsterdam, The Netherlands, 1995.

[21] M. P. Schellekens, "A characterization of partial metrizability: domains are quantifiable," *Theoretical Computer Science*, vol. 305, no. 1–3, pp. 409–432, 2003.

[22] P. Waszkiewicz, "Quantitative continuous domains," *Applied Categorical Structures*, vol. 11, no. 1, pp. 41–67, 2003.

[23] P. Waszkiewicz, "Partial metrisability of continuous posets," *Mathematical Structures in Computer Science*, vol. 16, no. 2, pp. 359–372, 2006.

[24] P. D. Proinov, "A generalization of the Banach contraction principle with high order of convergence of successive approximations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2361–2369, 2007.

[25] P. Fletcher and W. F. Lindgren, *Quasi-Uniform Spaces*, vol. 77 of *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1982.

[26] H.-P. A. Künzi, "Nonsymmetric distances and their associated topologies: about the origins of basic ideas in the area of asymmetric topology," in *Handbook of the History of General Topology, Vol. 3*, vol. 3 of *History of Topology*, pp. 853–968, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001.

[27] I. L. Reilly, P. V. Subrahmanyam, and M. K. Vamanamurthy, "Cauchy sequences in quasipseudometric spaces," *Monatshefte für Mathematik*, vol. 93, no. 2, pp. 127–140, 1982.

[28] E. Rakotch, "A note on contractive mappings," *Proceedings of the American Mathematical Society*, vol. 13, pp. 459–465, 1962.

[29] R. M. Bianchini and M. Grandolfi, "Trasformazioni di tipo contrattivo generalizzato in uno spazio metrico. ," *Atti della Accademia Nazionale dei Lincei*, vol. 45, pp. 212–216, 1968.

[30] P. D. Proinov, "New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems," *Journal of Complexity*, vol. 26, no. 1, pp. 3–42, 2010.

[31] T. H. Chang, "Common fixed point theorems for multivalued mappings," *Mathematica Japonica*, vol. 41, no. 2, pp. 311–320, 1995.