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A pair of positive solutions for the Dirichlet $p(z)$-Laplacian with concave and convex nonlinearities

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Abstract We consider a nonlinear parametric Dirichlet problem driven by the anisotropic p-Laplacian with the combined effects of "concave" and "convex" terms. The "superlinear" nonlinearity need not satisfy the Ambrosetti-Rabinowitz condition. Using variational methods based on the critical point theory and the Ekeland variational principle, we show that for small values of the parameter, the problem has at least two nontrivial smooth positive solutions.

Keywords Variable exponent · Concave and convex terms · Positive solutions · Mountain pass theorem · Maximum principle · Ekeland variational principle

Mathematics Subject Classification (2000) 35J70

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the existence of multiple positive solutions for the following nonlinear, parametric and anisotropic Dirichlet elliptic problem:

$$(P)_\lambda \qquad \begin{cases} -\Delta_{p(z)}u(z) = \lambda u(z)^{q(z)-1} + f\big(z, u(z)\big) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ \lambda > 0. \end{cases}$$

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Here $\Delta_{p(z)}$ denotes the variable p-Laplacian, defined by

$$\Delta_{p(z)}u = \operatorname{div}\left(\|\nabla u\|^{p(z)-2}\right)\nabla u,$$

with $p \in C^1(\overline{\Omega})$ and $p_- = \min\limits_{\overline{\Omega}} p > 1$. Also $q \in C(\overline{\Omega})$ and

$$q_+ \;=\; \max_{\overline{\Omega}} q \;<\; p_-,$$

$\lambda > 0$ is a parameter and $f(z, \zeta)$ is a Carathéodory function (i.e., for every $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z, \zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z, \zeta)$ is continuous). We assume that the potential function

$$F(z, \zeta) \;=\; \int\limits_0^{\zeta} f(z, s)\, ds$$

exhibits a p_+-superlinear growth near $\pm\infty$, where $p_+ = \max\limits_{\overline{\Omega}} p$. Since $q_- < p_- \leqslant p_+$ we see that in problem $(P)_\lambda$, we have the combined effects of a $(q_+ - 1)$-sublinear (concave) term and of a $(p_+ - 1)$-superlinear (convex) term. A particular case of our setting is the following nonlinearity

$$\lambda \zeta^{q(z)-1} + \zeta^{r(z)-1},$$

with $q, r \in C(\overline{\Omega})$, $q_+ < r_- = \min\limits_{\overline{\Omega}} r \leqslant r_+ = \max\limits_{\overline{\Omega}} r < \widehat{p}^*$, where

$$\widehat{p}^* \;=\; \frac{N p_-}{N - p_-}.$$

Problems like $(P)_\lambda$ were studied in the case of constant exponents q and p, by Ambrosetti et al. [1] (for $p = 2$) and by Garcia Azorero et al. [7] (for $p > 1$) and Guo and Zhang [14] (for $p \geqslant 2$). In all three works the nonlinearity has the particular form $\lambda\zeta^{q-1} + \zeta^{r-1}$. Here we allow the exponents to be variable, the differential operator is anisotropic and the "convex" term is more general than ζ^{r-1} and need not satisfy the well known Ambrosetti-Rabinowitz condition, allowing for "superlinear" terms which exhibit slower growth near $\pm\infty$. Our method of proof combines minimax methods based on the critical point theory, with the use of the Ekeland variational principle. We show the existence of a λ^* such that for all $\lambda \in (0, \lambda^*)$ problem $(P)_\lambda$ admits at least two nontrivial smooth solutions. Also, analogous results for problems with constant exponents but with nonsmooth potentials were proved by Gasiński and Papageorgiou [9–11], Gasiński [8], Kristaly et al. [17] and Teng [18] (the last three papers deal with the multiplicity of solution) and Filippakis et al. [6] (where the authors obtained positive solutions). It would be interesting to extend the results of this paper to problems with a nonsmooth potential (hemivariational inequalities).

Partial differential equations involving variable exponents and nonstandard growth conditions, arise in many physical phenomena and have been used in elasticity, fluid mechanics, image restoration and in the calculus of variations. A rich bibliography on the theory and the applications of the subject, can be found in the recent comprehensive survey article of Harjulehto et al. [15]. In the next section, for the convenience of the reader, we recall the main mathematical tools which we will use in the analysis of problem $(P)_\lambda$ and also introduce the variable exponent Lebesgue and Sobolev spaces.

2 Mathematical background

Let

$$L_1^\infty(\Omega) = \{p \in L^\infty(\Omega) : \operatorname*{ess\text{-}sup}_\Omega p \geqslant 1\}.$$

For $p \in L_1^\infty(\Omega)$, we set

$$p_- = \operatorname*{ess\text{-}inf}_\Omega p \quad \text{and} \quad p_+ = \operatorname*{ess\text{-}sup}_\Omega p.$$

Also let $M(\Omega)$ be the vector space of all measurable functions $u\colon \Omega \longrightarrow \mathbb{R}$. As usual, we identify two measurable functions which differ on a Lebesgue-null set. Then for $p \in L_1^\infty(\Omega)$, we define

$$L^{p(z)}(\Omega) = \Big\{u \in M(\Omega) : \int_\Omega |u|^{p(z)}\, dz < +\infty\Big\}.$$

We equip $L^{p(z)}(\Omega)$ with the following norm (known in the literature as the *Luxemburg norm*):

$$\|u\|_{p(z)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda}\right)^{p(z)} dz \leqslant 1\right\}.$$

The variable exponent Sobolev space is defined by

$$W^{1,p(z)}(\Omega) = \{u \in L^{p(z)}(\Omega) : \|\nabla u\|_{p(z)} \in L^{p(z)}(\Omega)\}$$

and it is furnished with the norm

$$\|u\|_{1,p(z)} = \|u\|_{p(z)} + \|\nabla u\|_{p(z)}.$$

An equivalent norm on $W^{1,p(z)}(\Omega)$ is given by

$$\|u\| = \inf\left\{\lambda > 0 : \int_\Omega \left(\left(\frac{\|\nabla u\|}{\lambda}\right)^{p(z)} + \left(\frac{|u|}{\lambda}\right)^{p(z)}\right) dz \leqslant 1\right\}.$$

Also $W_0^{1,p(z)}(\Omega)$ is the $\|\cdot\|$-closure of the set of $W^{1,p(z)}(\Omega)$-functions which have compact support i.e., $\{u \in W^{1,p(z)}(\Omega) : u = u\chi_K \text{ with } K \subseteq \Omega \text{ compact}\}$. Clearly

$$C_c^\infty(\Omega) \subseteq W_0^{1,p(z)}(\Omega).$$

If p is globally log-Hölder continuous, then

$$W_0^{1,p(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|}.$$

For details we refer to Kováčik and Rákosnik [16] and Fan and Zhao [5].

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition*, if the following holds:

"Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that $\{\varphi(x_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|x_n\|)\varphi'(x_n) \longrightarrow 0 \quad \text{in } X^* \text{ as } n \to +\infty,$$

admits a strongly convergent subsequence."

Using this compactness-type condition, we can state the following theorem, known in the literature as the “mountain pass theorem”.

Theorem 1 *If X is a Banach space, $\varphi \in C^1(X)$ and satisfies the Cerami condition, $x_0, x_1 \in X$, $r > 0$, $\|x_0 - x_1\| > r$,*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = r\} = \eta_r,$$

$$c = \inf_{\gamma\in\Gamma} \max_{0\leqslant t\leqslant 1} \varphi(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0, 1]; X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geqslant \eta_r$ and c is a critical value of φ.

In the sequel we will use the pair $(W_0^{1,p(z)}(\Omega), W_0^{1,p(z)}(\Omega)^*)$ and by $\langle\cdot,\cdot\rangle$ we will denote the duality brackets for this pair. Suppose that $p \in C(\overline{\Omega})$ with $1 < p_- = \min_{\overline{\Omega}} p$ and consider the map

$$A\colon W_0^{1,p(z)}(\Omega) \longrightarrow W_0^{1,p(z)}(\Omega)^* = W^{-1,p'(z)}(\Omega)$$

(where $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$ for all $z \in \overline{\Omega}$), defined by

$$\langle A(u), y\rangle = \int_\Omega \|\nabla u\|^{p(z)-2}(\nabla u, \nabla y)\,dz \quad \forall u, y \in W_0^{1,p(z)}(\Omega). \tag{1}$$

From Fan and Zhang [4] or Gasiński and Papageorgiou [13], we have

Proposition 1 *The nonlinear map $A\colon W_0^{1,p(z)}(\Omega) \longrightarrow W^{-1,p'(z)}(\Omega)$ defined by (1) is continuous, strictly monotone (hence maximal monotone) and of type $(S)_+$, i.e., if $u_n \longrightarrow u$ weakly in $W_0^{1,p(z)}(\Omega)$ and*

$$\limsup_{n\to+\infty} \langle A(u_n), u_n - u\rangle \leqslant 0,$$

then

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p(z)}(\Omega).$$

In what follows, for every $r \in \mathbb{R}$, we set

$$r^\pm = \max\{\pm r, 0\}$$

and for every $p \in C^1(\overline{\Omega})$, we set

$$p_- = \min_{\overline{\Omega}} p \quad \text{and} \quad p_+ = \max_{\overline{\Omega}} p.$$

By $\|\cdot\|$ we will denote both the Sobolev norm and the Euclidean norm on $\mathbb{R}^N$. It will always be clear from the context which norm we use. By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Finally let us recall the Ekeland variational principle (see e.g., Gasiński and Papageorgiou [12, p. 582, Corollary 4.6.3]).

Theorem 2 *If (V, d) is a complete metric space and $\vartheta : V \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a function not identically $+\infty$, lower semicontinuous and bounded below, then for every $\varepsilon > 0$, we can find $v_\varepsilon \in V$, such that*

(a) $\vartheta(v_\varepsilon) \leqslant \inf\limits_V \vartheta + \varepsilon$;
(b) $\vartheta(v_\varepsilon) \leqslant \vartheta(v) + \varepsilon d(v, v_\varepsilon)$ *for all* $v \in V$.

3 Positive solutions

The hypotheses on the data of problem $(P)_\lambda$ are the following:
$\underline{H_0}$: $p \in C^1(\overline{\Omega})$, $q \in C(\overline{\Omega})$ and

$$1 < q_- \leqslant q_+ < p_- \leqslant p_+ < N,$$

$\underline{H_1}$: $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z, 0) = 0$, $f(z, \zeta) \geqslant 0$ for all $\zeta \geqslant 0$, almost all $z \in \Omega$ and

(i) $\left|f(x, \zeta)\right| \leqslant a(z) + c|\zeta|^{r(z)-1}$ for almost all $z \in \Omega$, all $\zeta \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $c > 0$ and $r \in C(\overline{\Omega})$, such that

$$p_+ < r_+ < \widehat{p}^* = \frac{Np_-}{N - p_-};$$

(ii) if

$$F(z, \zeta) = \int\limits_0^\zeta f(z, s)\, ds,$$

then

$$\lim_{|\zeta| \to +\infty} \frac{F(z, \zeta)}{|\zeta|^{p_+}} = +\infty$$

uniformly for almost all $z \in \Omega$ and there exist $\tau \in C(\overline{\Omega})$ with $\tau(z) \in \Big((r_+ - p_-)\frac{N}{p_-},\ \widehat{p}^*\Big)$, $\tau(z) > q(z)$ for all $z \in \overline{\Omega}$ and $\beta_0 > 0$, such that

$$\beta_0 \leqslant \liminf_{\zeta \to +\infty} \frac{f(z, \zeta)\zeta - p_+ F(z, \zeta)}{\zeta^{\tau(z)}}$$

uniformly for almost all $z \in \Omega$;

(iii) we have

$$\lim_{\zeta \to 0^+} \frac{f(z, \zeta)}{\zeta^{p_+ - 1}} = 0$$

uniformly for almost all $z \in \Omega$.

Remark 1 Since we are interested in positive solutions and the asymptotic hypotheses $H_1(ii)$ and (iii) concern only the positive semiaxis, we may (and will) assume that

$$f(z, \zeta) = 0 \quad \text{for almost all} \quad z \in \Omega \quad \text{and all} \quad \zeta < 0.$$

Hypothesis $H_1(ii)$ implies that $F(z,\cdot)$ is p_+-superlinear near $+\infty$. But $f(z,\cdot)$ need not satisfy the usual in such cases Ambrosetti-Rabinowitz condition. Recall that the (unilateral) Ambrosetti-Rabinowitz condition says that there exist $\vartheta > p$ and $M > 0$, such that

$$0 < \vartheta F(z,\zeta) \leqslant f(z,\zeta)\zeta \quad \text{for almost all} \quad z \in \Omega \quad \text{and all} \quad \zeta \geqslant M.$$

This condition implies that

$$c\zeta^{\vartheta} \leqslant F(z,\zeta) \quad \text{for almost all} \quad z \in \Omega \quad \text{and all} \quad \zeta \geqslant M.$$

Our hypothesis permits for slower growth of $F(z,\cdot)$ near $+\infty$.

Example 1 The function

$$f(z,\zeta) = f(\zeta) = \zeta^{p_+-1}\left(\zeta \ln \zeta + \frac{\zeta}{1+\zeta}\right) \quad \forall \zeta \geqslant 0$$

satisfies hypothesis $H_1(ii)$ but not the Ambrosetti-Rabinowitz condition.

In the analysis of problem $(P)_\lambda$ in addition to the Sobolev space $W_0^{1,p(z)}(\Omega)$, we shall also use the Banach space

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : \ u|_{\partial\Omega} = 0\right\}.$$

This is an ordered Banach space with order cone

$$C_+ = \left\{u \in C_0^1(\overline{\Omega}) : \ u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior in $C_0^1(\overline{\Omega})$, given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : \ u(z) > 0 \text{ for all } z \in \Omega, \ \frac{\partial u}{\partial n}(z) < 0 \text{ for all } z \in \partial\Omega\right\}.$$

Here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

We will seek the positive solutions of $(P)_\lambda$ as critical points of the energy functional $\varphi_\lambda : W_0^{1,p(z)}(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\varphi_\lambda(u) = \int_\Omega \frac{1}{p(z)}\|\nabla u\|^{p(z)}\,dz - \lambda \int_\Omega \frac{1}{q(z)} u^+(z)^{q(z)}\,dz - \int_\Omega F\big(z,u(z)\big)\,dz \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

Hypotheses H_1 imply that $\varphi_\lambda \in C^1\big(W_0^{1,p(z)}(\Omega)\big)$.

Proposition 2 *If hypotheses H_0 and H_1 hold and $\lambda > 0$, then φ_λ satisfies the Cerami condition.*

Proof Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p(z)}(\Omega)$ be a sequence, such that

$$\left|\varphi_\lambda(u_n)\right| \leqslant M_1 \quad \forall n \geqslant 1, \tag{2}$$

for some $M_1 > 0$ and

$$\big(1 + \|u_n\|\big)\varphi_\lambda'(u_n) \longrightarrow 0. \tag{3}$$

From (3), for all $y \in W_0^{1,p(z)}(\Omega)$, we have

$$\left|\langle A(u_n), y\rangle - \lambda \int_\Omega (u_n^+)^{q(z)-1} y\,dz - \int_\Omega f(z,u_n)y\,dz\right| \leqslant \frac{\varepsilon_n}{1+\|u_n\|}\|y\|, \tag{4}$$

with $\varepsilon_n \searrow 0$. In (4), we choose $y = -u_n^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\int_\Omega \|\nabla u_n^-\|^{p(z)}\,dz \leqslant \varepsilon_n \quad \forall n \geqslant 1 \tag{5}$$

(recall that $f(z,\zeta) = 0$ for almost all $z \in \Omega$ and all $\zeta \leqslant 0$). From (5), it follows that

$$u_n^- \longrightarrow 0 \quad \text{in } W_0^{1,p(z)}(\Omega) \tag{6}$$

(see Fan and Zhang [4] and Gasiński and Papageorgiou [13]).

Next in (4), we choose $y = u_n^+ \in W_0^{1,p(z)}(\Omega)$ and obtain

$$-\int_\Omega \|\nabla u_n^+\|^{p(z)}\,dz + \lambda\int_\Omega (u_n^+)^{q(z)}\,dz + \int_\Omega f(z,u_n^+)u_n^+\,dz \leqslant \varepsilon_n \quad \forall n \geqslant 1. \tag{7}$$

On the other hand, from (2) and (6), for all $n \geqslant 1$, we have

$$\int_\Omega \frac{p_+}{p(z)}\|\nabla u_n^+\|^{p(z)} - \lambda\int_\Omega \frac{p_+}{q(s)}(u_n^+)^{q(z)} - \int_\Omega p_+F(z,u_n^+)\,dz \leqslant p_+M_1. \tag{8}$$

Adding (7) and (8) and recalling that $\frac{p_+}{p(z)} \geqslant 1$ for all $z \in \overline{\Omega}$, we obtain

$$\int_\Omega \left[f(z,u_n^+)u_n^+ - p_+F(z,u_n^+)\right]dz \leqslant M_2 + \lambda\int_\Omega \left(\frac{p_+}{q(t)} - 1\right)(u_n^+)^{q(z)}\,dz, \tag{9}$$

for all $n \geqslant 1$ and for some $M_2 > 0$. By virtue of hypotheses $H_1(i)$ and (ii), we can find $\beta_1 \in (0,\beta_0)$ and $c_1 > 0$, such that

$$\beta_1(\zeta^+)^{\tau(z)} - c_1 \leqslant f(z,\zeta^+)\zeta^+ - pF(z,\zeta^+) \quad \text{for almost all} \quad z \in \Omega, \quad \text{all} \quad \zeta \in \mathbb{R}. \tag{10}$$

We use (10) in (9) and obtain

$$\beta_1\int_\Omega (u_n^+)^{\tau(z)}\,dz \leqslant M_3 + \lambda\int_\Omega \left(\frac{p_+}{q(z)} - 1\right)(u_n^+)^{q(z)}\,dz \quad \forall n \geqslant 1, \tag{11}$$

for some $M_3 > 0$. Since $\tau(z) > q(z)$ for all $z \in \overline{\Omega}$ (see hypothesis $H_1(ii)$), from (11), it follows that

$$\text{the sequence } \{u_n^+\}_{n\geqslant 1} \subseteq L^{\tau(z)}(\Omega) \text{ is bounded.} \tag{12}$$

Let $\vartheta_0 \in (r_+, \widehat{p}^*)$ (see hypothesis $H_1(i)$). Also it is clear from hypothesis $H_1(ii)$ that we can always assume that $\tau_- < r_+ < \vartheta_0$. So, we can find $t \in (0,1)$, such that

$$\frac{1}{r_+} = \frac{1-t}{\tau_-} + \frac{t}{\vartheta_0}.$$

Applying the interpolation inequality (see e.g., Gasiński and Papageorgiou [12, p. 905]), we have

$$\|u_n^+\|_{r_+} \leqslant \|u_n^+\|_{\tau_-}^{1-t}\|u_n^+\|_{\vartheta_0}^{t} \quad \forall n \geqslant 1,$$

so

$$\|u_n^+\|_{r_+}^{r_+} \leqslant \|u_n^+\|_{\tau_-}^{(1-t)r_+}\|u_n^+\|_{\vartheta_0}^{tr_+} \quad \forall n \geqslant 1$$

and thus

$$\|u_n^+\|_{r_+}^{r_+} \leqslant M_4\|u_n^+\|_{\vartheta_0}^{tr_+} \quad \forall n \geqslant 1, \tag{13}$$

for some $M_4 > 0$ [see (12)]. By virtue of hypothesis $H_1(i)$, we have

$$f(z,\zeta)\zeta \leqslant c_2(\zeta + \zeta^{r_+}) \quad \forall \zeta \geqslant 0, \tag{14}$$

for some c_2. In (4), we choose $y = u_n^+ \in W_0^{1,p(z)}(\Omega)$. Then, recalling that $q_+ < p_+$ and using hypothesis H_0 and (13), we have

$$\begin{aligned}\int_\Omega \|\nabla u_n^+\|^{p(z)}\,dz &\leqslant \lambda\int_\Omega (u_n^+)^{q(z)}\,dz + \int_\Omega f(z,u_n^+)u_n^+\,dz + c_3\\ &\leqslant c_4\big(1 + \|u_n^+\| + \|u_n^+\|_{r_+}^{r_+}\big)\\ &\leqslant c_5\big(1 + \|u_n^+\| + \|u_n^+\|^{tr_+}\big) \quad \forall n \geqslant 1,\end{aligned}$$

for some $c_3, c_4, c_5 > 0$. So

$$\|u_n^+\|^{p_-} \leqslant c_6\big(1 + \|u_n^+\| + \|u_n^+\|^{tr_+}\big) \quad \forall n \geqslant 1, \tag{15}$$

for some $c_6 > 0$ (see Fan and Zhang [4] or Gasiński and Papageorgiou [13]). Note that

$$tr_+ = \frac{\vartheta_0(r_+ - \tau_-)}{\vartheta_0 - \tau_-} < p_-.$$

So, from (15), we infer that

$$\text{the sequence } \{u_n^+\}_{n\geqslant 1} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded}$$

and so

$$\text{the sequence } \{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded}$$

[see (6)]. Passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \quad \text{in } W_0^{1,p(z)}(\Omega),$$

hence

$$u_n \longrightarrow u \quad \text{in } L^{r(z)}(\Omega)$$

(recall that $r_+ < \widehat{p}^*$ and use the Sobolev embedding theorem for variable exponent spaces; see Fan and Zhao [5]). In (4), we choose $y = u_n - u \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned}\Big|\big\langle A(u_n), u_n - u\big\rangle - \lambda\int_\Omega (u_n^+)^{q(z)-1}(u_n - u)\,dz&\\ - \int_\Omega f(z,u_n)(u_n - u)\,dz\Big| &\leqslant \varepsilon_n' \quad \forall n \geqslant 1,\end{aligned}$$

with $\varepsilon_n' \searrow 0$, so

$$\lim_{n\to+\infty} \langle A(u_n),\ u_n - u\rangle = 0$$

and thus

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p(z)}(\Omega)$$

(see Proposition 1). Therefore φ_λ satisfies the Cerami condition.

Proposition 3 *If hypotheses H_0 and H_1 hold, then we can find $\lambda^* > 0$, such that for $\lambda \in (0,\lambda^*)$, there exists $\varrho = \varrho(\lambda) > 0$, such that*

$$\inf\{\varphi_\lambda(u) :\ \|u\| = \varrho\} = \eta_\lambda^+ > 0.$$

Proof By virtue of hypotheses $H_1(i)$ and (iii), for a given $\varepsilon > 0$, we can find $c_7 = c_7(\varepsilon) > 0$, such that

$$f(z,\zeta) \leqslant \varepsilon(\zeta^+)^{p_+-1} + c_7(\zeta^+)^{r_+-1} \quad \text{for almost all } z \in \Omega \text{ and all } \zeta \in \mathbb{R},$$

so

$$F(z,\zeta) \leqslant \frac{\varepsilon}{p_+}(\zeta^+)^{p_+} + \frac{c_7}{r_+}(\zeta^+)^{r_+} \quad \text{for almost all } z \in \Omega \text{ and all } \zeta \in \mathbb{R}. \tag{16}$$

For every $u \in W_0^{1,p(z)}(\Omega)$ with $\|u\| \leqslant 1$, we have

$$\begin{aligned}
\varphi_\lambda(u) &= \int_\Omega \frac{1}{p(z)}\|\nabla u\|^{p(z)}\,dz - \lambda\int_\Omega \frac{1}{q(z)}(u^+)^{q(z)}\,dz - \int_\Omega F\big(z,u(z)\big)\,dz \\
&\geqslant \frac{1}{p_+}\|u\|^{p_+} - \lambda c_8\|u\|^{q_-} - \frac{\varepsilon c_9}{p_+}\|u\|^{p_+} - c_{10}\|u\|^{r_+} \\
&= \big(c_{11} - \lambda c_8\|u\|^{q_- - p_+} - c_{10}\|u\|^{r_+ - p_+}\big)\|u\|^{p_+},
\end{aligned} \tag{17}$$

for some $c_8, c_9, c_{10} > 0$ and with $c_{11} = \frac{1}{p_+}(1 - \varepsilon c_9) > 0$ (for $\varepsilon > 0$ small). We consider the function

$$\xi(t) = \lambda c_8 t^{q_- - p_+} + c_{10} t^{r_+ - p_+} \quad \forall t > 0.$$

Since $q_+ < p_+ < r_+$, it is clear that

$$\lim_{t\to 0^+}\xi(t) = \lim_{t\to+\infty}\xi(t) = +\infty.$$

Because ξ is continuous on $(0,+\infty)$, we can find $t_0 > 0$, such that $0 < \xi(t_0) = \inf_{\mathbb{R}_+}\xi$. Hence we have

$$0 = \xi'(t_0) = \lambda c_8(q_- - p_+)t_0^{q_- - p_+ - 1} + c_{10}(r_+ - p_+)t_0^{r_+ - p_+ - 1},$$

so

$$\lambda c_8(p_+ - q_-)t_0^{q_- - p_+ - 1} = c_{10}(r_+ - p_+)t_0^{r_+ - p_+ - 1}$$

and thus

$$t_0 = \left(\frac{\lambda c_8(p_+ - q_-)}{c_{10}(r_+ - p_+)}\right)^{\frac{1}{r_+ - q_-}}.$$

We consider $\xi(t_0)$. Then clearly we can find $\lambda^* > 0$, such that for all $\lambda \in (0, \lambda^*)$, we have $\xi(t_0) < c_{11}$. Then from (17), we infer that

$$\varphi_\lambda(u) \geqslant \eta_\lambda^+ > 0 \quad \forall u \in W_0^{1,p(z)}(\Omega),\ \|u\| = \varrho(\lambda) \leqslant 1.$$

Proposition 4 *If hypotheses H_0 and H_1 hold and $u \in C_+ \setminus \{0\}$ with $\|u\|_{p_+}, \|u\|_{q_+}, \|u\| \geqslant 1$, then $\varphi_\lambda(tu) \longrightarrow -\infty$ as $t \to +\infty$.*

Proof By virtue of hypotheses $H_1(i)$ and (ii), for a given $\varepsilon > 0$, we can find $c_\varepsilon > 0$, such that

$$F(z,\zeta) \geqslant \frac{1}{p_+\varepsilon}(\zeta^+)^{p_+} - c_\varepsilon \quad \text{for almost all} \quad z \in \Omega \quad \text{and all} \quad \zeta \in \mathbb{R}. \tag{18}$$

Then, for $t \geqslant 1$, we have

$$F\big(z, tu(z)\big) \geqslant \frac{t^{p_+}u(z)^{p_+}}{p_+\varepsilon} - c_\varepsilon \quad \text{for almost all} \quad z \in \Omega$$

[see (18)]. Thus

$$\frac{F\big(z, tu(z)\big)}{t^{p_+}} \geqslant \frac{u(z)^{p_+}}{p_+\varepsilon} - \frac{c_\varepsilon}{t^{p_+}} \quad \text{for almost all} \quad z \in \Omega.$$

Recalling that $\|u\|_{p_+} \leqslant 1$, we have

$$\int_\Omega \frac{F\big(z, tu(z)\big)}{t^{p_+}}\,dz \geqslant \frac{1}{p_+\varepsilon} - \frac{c_\varepsilon}{t^{p_+}}|\Omega|_N,$$

so

$$\liminf_{t\to+\infty} \int_\Omega \frac{F\big(z, tu(z)\big)}{t^{p_+}}\,dz \geqslant \frac{1}{p_+\varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \searrow 0$ and conclude that

$$\int_\Omega \frac{F\big(z, tu(z)\big)}{t^{p_+}}\,dz \longrightarrow +\infty \quad \text{as } t \to +\infty. \tag{19}$$

Then

$$\begin{aligned}\varphi_\lambda(tu) &= \int_\Omega \frac{1}{p(z)}\big\|\nabla(tu)\big\|^{p(z)}\,dz - \lambda\int_\Omega \frac{1}{q(z)}(tu^+)^{q(z)}\,dz - \int_\Omega F(z,tu)\,dz \\ &\geqslant \frac{t^{p_-}}{p_+}c_{12}\|u\|^{p_-} - \frac{\lambda t^{q_-}}{q_-}c_{13}\|u\|^{q_+} - \int_\Omega F(z,tu)\,dz,\end{aligned}$$

for some $c_{12}, c_{13} > 0$. Thus

$$\frac{\varphi_\lambda(tu)}{t^{p_+}} \geqslant \frac{c_{12}}{p_+t^{p_+-p_-}}\|u\|^{p_-} - \frac{\lambda c_{13}}{q_-t^{p_+-q_-}}\|u\|^{q_+} - \int_\Omega \frac{F(z,tu)}{t^{p_+}}\,dz.$$

Using also (19), we have

$$\lim_{t\to+\infty} \frac{\varphi_\lambda(tu)}{t^{p_+}} = -\infty,$$

so

$$\varphi_\lambda(tu) \longrightarrow -\infty \quad \text{as } t \to +\infty.$$

Now, we have all the necessary tools to state and prove the multiplicity theorem for the positive solutions of problem $(P)_\lambda$.

Theorem 3 *If hypotheses H_0 and H_1 hold, then there exists $\lambda^* > 0$, such that for all $\lambda \in (0, \lambda^*)$, problem $(P)_\lambda$ has at least two positive smooth solutions $u_0, \widehat{u} \in int\, C_+$.*

Proof Propositions 2, 3 and 4 permit the use of the mountain pass theorem (see Theorem 1). So, for $\lambda \in (0, \lambda^*)$ (see Proposition 3), we can find $u_0 \in W_0^{1,p(z)}(\Omega)$, such that

$$\varphi_\lambda(0) = 0 < \eta_\lambda^+ \leqslant \varphi_\lambda(u_0) \tag{20}$$

and

$$\varphi_\lambda'(u_0) = 0. \tag{21}$$

From (20), we see that $u_0 \neq 0$. From (21), we have

$$A(u_0) = \lambda (u_0^+)^{q(\cdot)-1} + N_f(u_0), \tag{22}$$

where

$$N_f(u)(\cdot) = f\big(\cdot, u(\cdot)\big) \quad \forall u \in W_0^{1,p(z)}(\Omega).$$

On (22) we act with $-u_0^- \in W_0^{1,p(z)}(\Omega)$ and obtain

$$\int_\Omega \|\nabla u_0^-\|^{p(z)}\, dz = 0$$

(recall that $f(z, \zeta) = 0$ for almost all $z \in \Omega$ and all $\zeta \leqslant 0$), so $u_0^- = 0$ and thus

$$u_0 \geqslant 0, \quad u_0 \neq 0.$$

Then (22) becomes

$$A(u_0) = \lambda u_0^{q(\cdot)-1} + N_f(u_0),$$

so

$$\begin{cases} -\Delta_{p(z)} u_0(z) = \lambda u_0(z)^{q(z)-1} + f\big(z, u_0(z)\big) \text{ in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

From the nonlinear regularity theory (see e.g., Fan [3] and Gasiński and Papageorgiou [12]), we have $u_0 \in C_+ \setminus \{0\}$. Since $\Delta_{p(z)} u_0(z) \leqslant 0$ for almost all $z \in \Omega$, invoking the nonlinear maximum principle of Zhang [19], we infer that $u_0 \in \operatorname{int} C_+$.

Proposition 3 implies that for every $\lambda \in (0, \lambda^*)$, we can find $\varrho = \varrho(\lambda) \leqslant 1$, such that

$$0 < \inf_{\|u\|=\varrho} \varphi_\lambda(u). \tag{23}$$

We will show that

$$\inf_{\|u\|\leqslant\varrho} \varphi_\lambda(u) < 0. \tag{24}$$

To this end, let $u \in C_+ \setminus \{0\}$ be such that $\|u\| \leqslant 1$ and let $t \in (0,1)$. As $F \geqslant 0$ (see hypotheses H_1), we have

$$\begin{aligned}\varphi_\lambda(tu) &= \int_\Omega \frac{1}{p(z)} \big\|\nabla(tu)\big\|^{p(z)}\,dz - \lambda \int_\Omega \frac{1}{q(z)}(tu)^{q(z)}\,dz - \int_\Omega F\big(z, tu(z)\big)\,dz \\ &\leqslant \frac{t^{p_-}}{p_-} c_{14}\|u\|^{p_-} - \frac{\lambda t^{q_+}}{q_+}\int_\Omega u^{q(z)}\,dz \\ &\leqslant t^{q_+}\left(\frac{t^{p_- - q_+}}{p_-}c_{14} - \frac{\lambda}{q_+}\int_\Omega u^{q(z)}dz\right). \end{aligned} \tag{25}$$

Since $q_+ < p_-$, for $t \in (0,1)$ small enough, from (25), we have

$$\varphi_\lambda(tu) \leqslant 0 \quad \text{and} \quad t\|u\| \leqslant \varrho,$$

so (24) holds.

Let

$$\vartheta_\lambda = \inf_{\partial B_\varrho} \varphi_\lambda - \inf_{\overline{B}_\varrho} \varphi_\lambda > 0$$

[see (23) and (24)] and let $\varepsilon \in (0, \vartheta_\lambda)$. By virtue of the Ekeland variational principle (see Theorem 2), we can find $u_\varepsilon \in \overline{B}_\varrho$, such that

$$\varphi_\lambda(u_\varepsilon) \leqslant \inf_{\overline{B}_\varrho} \varphi_\lambda + \varepsilon \tag{26}$$

and

$$\varphi_\lambda(u_\varepsilon) \leqslant \varphi_\lambda(y) + \varepsilon\|y - u_\varepsilon\| \quad \forall y \in \overline{B}_\varrho. \tag{27}$$

Since $\varepsilon \in (0, \vartheta_\lambda)$, from (26), we see that

$$\varphi_\lambda(u_\varepsilon) < \inf_{\overline{B}_\varrho} \varphi_\lambda. \tag{28}$$

From (23) and (28), it follows that

$$u_\varepsilon \in B_\varrho = \left\{u \in W_0^{1,p(z)}(\Omega) : \|u\| < \varrho\right\}.$$

Let

$$h_\lambda^\varepsilon(u) = \varphi_\lambda(y) + \varepsilon\|y - u_\varepsilon\|.$$

From (27), we see that u_ε is a minimizer on $\overline{B}_\varrho$ of the locally Lipschitz function h_λ^ε. Since $u_\varepsilon \in B_\varrho$, it follows that $0 \in \partial h_\lambda^\varepsilon(u_\varepsilon)$, where $\partial h_\lambda^\varepsilon(u_\varepsilon)$ denotes the generalized subdifferential in the sense of Clarke of the locally Lipschitz functional h_λ^ε (see Clarke [2]; Gasiński and Papageorgiou [12]). Hence

$$-\varepsilon\|h\| \leqslant \left\langle \varphi'_\lambda(u_\varepsilon),\ h\right\rangle \quad \forall h \in W_0^{1,p(z)}(\Omega),$$

so

$$\left\|\varphi'_\lambda(u_\varepsilon)\right\| = \varepsilon. \tag{29}$$

Let $\varepsilon_n = \frac{1}{n}$ and let $u_n = u_{\varepsilon_n} \in B_\varrho$, for $n \geqslant 1$. Then

$$\varphi_\lambda(u_n) \longrightarrow \inf_{\overline{B}_\varrho} \varphi_\lambda \tag{30}$$

[see (26)] and

$$\varphi_\lambda'(u_n) \longrightarrow 0 \quad \text{in } W^{-1,p'(z)}(\Omega) \tag{31}$$

[see (29)]. By virtue of Proposition 2, it follows that

$$u_n \longrightarrow \widehat{u} \quad \text{in } W_0^{1,p(z)}(\Omega)$$

(at least for a subsequence). Hence

$$\varphi_\lambda(u_n) \longrightarrow \varphi_\lambda(\widehat{u}) = \inf_{\overline{B}_\varrho} \varphi_\lambda < 0$$

[see (30) and (24)], so $\widehat{u} \neq 0$.

Moreover, since $\varphi_\lambda(\widehat{u}) < 0 < \eta_\lambda^+ \leqslant \varphi_\lambda(u_0)$ [see (24) and (20)], we see that $\widehat{u} \neq u_0$. Because $\varphi_\lambda \in C^1\big(W_0^{1,p(z)}(\Omega)\big)$, from (31), we have

$$\varphi_\lambda'(\widehat{u}) = 0,$$

so

$$A(\widehat{u}) = \lambda(\widehat{u}^+)^{q(\cdot)-1} + N_f(\widehat{n}). \tag{32}$$

Acting on (32) with $-\widehat{u}^- \in W_0^{1,p(z)}(\Omega)$, we obtain $\widehat{u}_0 \geqslant 0$, $\widehat{u}_0 \neq 0$. From (32), we have

$$\begin{cases} -\Delta_{p(z)}\widehat{u}(z) = \lambda\widehat{u}(z)^{q(z)-1} + f\big(z, \widehat{u}(z)\big) \text{ in } \Omega, \\ \widehat{u}|_{\partial\Omega} = 0. \end{cases}$$

Nonlinear regularity theory (see e.g., Fan [3] and Gasiński and Papageorgiou [13]) implies that $\widehat{u} \in C^1(\overline{\Omega})$ and since $\Delta_{p(z)}\widehat{u}(z) \leqslant 0$ for almost all $z \in \Omega$, the nonlinear maximum principle of Zhang [19] implies that $\widehat{u} \in \operatorname{int} C_+$.

Remark 2 If $f(z,\zeta) = \zeta^{r(z)-1}$ for all $z \in \overline{\Omega}$, $\zeta \geqslant 0$ with $r \in C(\overline{\Omega})$ as in hypothesis $H_1(i)$, then Theorem 3 extends to the anisotropic p-Laplacian the work of Garcia Azorero et al. [7] and Guo and Zhang [14].

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