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Explicit description of spherical rigid hypersurfaces in \mathbb{C}^2

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Abstract

In this paper, we present a complete list of rigid spherical CR hypersurfaces in \mathbb{C}^2 . The construction is based on a renormalization of Stanton's family of rigid spheres.

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1 Introduction

The class of real hypersurfaces in complex space that are invariant with respect to an infinitesimal translation transversal to the complex tangent space is known as rigid hypersurfaces (see [1]). In this article, we consider rigid hypersurfaces in \mathbb{C}^2 with coordinates $z, w = u + iv$. In this case, a rigid hypersurface can be locally described by an equation of the form $v = h(z)$.

In 1991, Stanton [8] developed a normal form for rigid hypersurfaces. Other normal forms that reflect the presence of symmetries have been constructed by Kolář [5] and Ezhov et al. [2] more recently.

It is a natural question how to recognise hypersurfaces that are holomorphically equivalent to the model hypersurface, i.e. the sphere $v = |z|^2$, via its normal form. As an application of the normal form, Stanton derived a list of examples of normal forms equivalent to the sphere:

$$\frac{1}{2r} \sinh 2rv = |z|^2 \quad (1)$$

$$\frac{1}{2r} \sin 2rv \left(1 - \frac{2|b|^2\theta}{|c|^2} \right) = \frac{|z|^2 e^{-2\theta v}}{1 + 4|b|^2|z|^2 + 2i(b\bar{z} - \bar{b}z)} + \frac{|b|^2}{|c|^2} (e^{-2\theta v} - \cos 2rv) + \frac{\bar{b}z}{\bar{c}(1 - 2i\bar{b}z)} (e^{-2\theta v} - e^{2irv}) + \frac{b\bar{z}}{c(1 + 2ib\bar{z})} (e^{-2\theta v} - e^{-2irv}), \quad (2)$$

where b and $c = r + i\theta$ are complex parameters. The remaining examples from Stanton's list can be obtained by letting r and θ converge to 0. In particular, for $b = 0$ and $\theta = 0$, one finds the pendant to (1)

$$\frac{1}{2r} \sin 2rv = |z|^2.$$

Stanton has raised the question whether this list was complete. Her method was based on the analysis of the holomorphic mappings that transform the infinitesimal translation $\partial/\partial w$ into a vector field of a seven-parameter family of infinitesimal sphere automorphisms. She acknowledged that the listed examples correspond to a four-parameter subfamily.

The main aim of this paper is to provide a complete classification of spherical rigid hypersurfaces. Combined with Stanton's results, this also answers Rothschild's question which rigid hypersurfaces are equivalent to a hypersurface that is given by an equation:

$$v = p(z, \bar{z})$$

where p is a homogeneous polynomial (see [8]). Stanton has solved this problem for polynomials of degree bigger than two. Our result completes the case of degree 2.

The following argument gives the upper bound 4 for the number of parameters involved: In the Levi non-degenerate case, which applies for spherical rigid hypersurfaces, Stanton's normal form is similar to Chern-Moser's normal form, except for the terms of the defining function ϕ of bidegrees (2,2), (2,3), (3,2) and (3,3) with respect to z, \bar{z} being allowed to be any constants. These four real constants completely control the rigid hypersurface. Thus, the normal form equation becomes:

$$v = |z|^2 + c_{22}|z|^4 + c_{23}z^2\bar{z}^3 + c_{32}z^3\bar{z}^2 + c_{33}|z|^6 + \sum_{\substack{\min(j,k) \geq 2 \\ \max(j,k) \geq 4}} F_{jk}z^j\bar{z}^k.$$

The coefficients c_{22} , $c_{23} = \bar{c}_{32}$ and c_{33} depend in an algebraic way on Stanton's parameters b and c . It can be shown that not all coefficients can be attained in this way, which indicates that Stanton's list is incomplete. In the case when $c_{23} = 0$, Stanton's example reduces to:

$$\frac{1}{2r} \sin 2rv = e^{-2\theta v} |z|^2,$$

which realises only the coefficients $c_{33} \geq \frac{3}{2}c_{22}^2$. The coefficients $c_{33} < \frac{3}{2}c_{22}^2$ are realised by the family:

$$\frac{1}{2r} \sinh 2rv = e^{-2\theta v} |z|^2, \tag{3}$$

which for $\theta = 0$ coincides with (1) but was absent in Stanton's list for $\theta \neq 0$.

We show that Stanton's mappings of a rigid sphere to the Heisenberg sphere can be modified by combining them with suitable sphere automorphisms so that all parameters can be covered.

This yields not only the desired classification but also provides the complete solution to a non-linear partial differential equation (PDE) that expresses the zero-curvature equation of a rigid hypersurface M in normal form:

$$v = h(z).$$

It is well known that local sphericity is equivalent to vanishing of the Cartan curvature (see, e.g.[3]). For rigid hypersurfaces the zero-curvature equation simplifies to

$$f_{z\bar{z}\bar{z}} - 3f_{z\bar{z}}f + 2f_zf^2 - f_zf_{\bar{z}} = 0 \tag{4}$$

where $f(z) = \frac{\partial}{\partial \bar{z}} \log \Delta h(z)$. Loboda has derived and studied PDE (4) in [6]. In particular he found the solution (3). The PDE (4) appears in relation to the study of shear-free congruences of null geodesics in a Lorentzian geometry by Robinson and Wilson in [7]. In

particular, they establish a correspondence between the Taub-NUT congruence and a CR manifold that is equivalent to our CR manifold $\sinh v = e^{-v}|z|^2$.

2 Modified normalisation

We apply a modified Chern-Moser normalisation procedure to the Heisenberg sphere $\text{Im } w_2 = |z_2|^2$ to obtain a hypersurface of the form:

$$v = |z|^2 + c_{22}|z|^4 + c_{23}z^2\bar{z}^3 + c_{32}z^3\bar{z}^2 + c_{33}|z|^6 + \dots$$

For given c_{22}, c_{23}, c_{33} , such normalisation mapping is uniquely determined up to automorphisms of the Heisenberg sphere. All rigid spheres can be found by this procedure, though the resulting hypersurface does not have to be rigid in the higher-order terms *a priori*. We construct the inverse mapping as a composition of two:

$$z_2 = p(w_1) + z_1 + 2i \sum_{j=2}^{\infty} T_j(w_1)z_1^j = p(w_1) + \frac{z_1}{1 - 2i\bar{p}'(w_1)z_1}$$

$$w_2 = q(w_1) + 2i \sum_{j=1}^{\infty} g_j(w_1)z_1^j = q(w_1) + \frac{2i\bar{p}(w_1)z_1}{1 - 2i\bar{p}'(w_1)z_1}$$

Here, we assume:

$$q' = 1 + 2ip'\bar{p}. \tag{5}$$

Condition (5) is imposed for convenience in computation and can be fixed at the next step.

The second mapping has the form:

$$z = e^{i\alpha(w_1)} \sqrt{h'(w_1)} z_1$$

$$w = h(w_1).$$

Then, p, α, h satisfy the equations:

$$6|p'|^2 + 2\alpha' - c_{22}h' = 0 \tag{6}$$

$$-c_{23}e^{-i\alpha}(h')^{3/2} - 2p'' + 4i|p'|^2p' = 0 \tag{7}$$

$$\frac{1}{3}h''' - \frac{(h'')^2}{2h'} + \frac{3c_{22}^2 - 2c_{33}}{2}(h')^3 + 2|p'|^4h' + \frac{2}{3}(ip''\bar{p}' - ip'\bar{p}'')h' = 0 \tag{8}$$

with initial conditions $p(0) = 0, h(0) = 0, h'(0) > 0$. Using combinations with automorphisms of the Heisenberg sphere we may assume that $\alpha(0) = 0, h'(0) = 1, p'(0) = 0, h''(0) = 0$. It follows that, for given c_{22}, c_{23}, c_{33} , system (6)-(8) has unique solutions α, p, h .

Though it is easy to successively compute the coefficients of the power series of α, p, h from recursive formulae, we have not succeeded in solving this system of ODE, except for $c_{23} = 0$. In this case, $p \equiv 0$ and:

$$h(u) = \frac{2}{\sqrt{9c_{22}^2 - 6c_{33}}} \arctan \frac{\sqrt{9c_{22}^2 - 6c_{33}}u}{2}$$

$$\alpha(u) = \frac{c_{22}}{2}h(u).$$

3 Stanton's mapping

Rigid surfaces can be characterised by the presence of a translation symmetry transversal to the complex tangent spaces. In suitable local coordinates, such symmetry is generated

as the flow of the vector field $\frac{\partial}{\partial u}$. Stanton constructed holomorphic mappings in the ambient space that pull a suitable infinitesimal automorphism X of the Heisenberg sphere back to $\frac{\partial}{\partial u}$. The relevant infinitesimal automorphisms of the Heisenberg sphere are well-known and form a seven-parametric family consisting of:

$$X = 2 \operatorname{Re} \left((b + cz + aw + 2i\bar{a}z^2 + \rho zw) \frac{\partial}{\partial z} + (1 + 2i\bar{b}z + 2rw + 2i\bar{a}zw + \rho w^2) \frac{\partial}{\partial w} \right),$$

where $b, a \in \mathbb{C}$ and $c = r + i\theta \in \mathbb{C}^*$ and $\rho \in \mathbb{R}$.

The resulting system is:

$$\begin{aligned} \frac{\partial Z}{\partial w} &= b + cZ + aW + 2i\bar{a}Z^2 + \rho ZW \\ \frac{\partial W}{\partial w} &= 1 + 2i\bar{b}Z + 2rW + 2i\bar{a}ZW + \rho W^2 \end{aligned} \quad (9)$$

with initial conditions $Z(z, 0) \equiv \frac{z}{1-2i\bar{b}z}$, $W(z, 0) \equiv 0$. Stanton solved this system in the particular case $a = \rho = 0$, that is when (9) is linear. The solutions are:

$$\begin{aligned} Z &= \frac{b}{c}(e^{cw} - 1) + \frac{e^{cw}z}{1 - 2i\bar{b}z} \\ W &= \left(1 - \frac{2i|b|^2}{c}\right) \frac{e^{2rw} - 1}{2r} + \frac{2i\bar{b}}{\bar{c}} \left(\frac{z}{1 - 2i\bar{b}z} + \frac{b}{c}\right) (e^{2rw} - e^{cw}). \end{aligned} \quad (10)$$

Substituting this mapping into the sphere equation, $v = |z|^2$ yields (2). We show that not all values of the parameters c_{22}, c_{23}, c_{33} can be realised by these mappings. Direct computation shows that the explicit sixth-order expansion of (2) is:

$$\begin{aligned} v &= |z|^2 + (6|b|^2 - 2\theta) |z|^4 + (2\bar{c} + 4i|b|^2) bz^2\bar{z}^3 + (2c - 4i|b|^2) \bar{b}z^3\bar{z}^2 + \\ &+ \left(\frac{2}{3}r^2 + 6\theta^2 + 56|b|^4 - \frac{112}{3}\theta|b|^2\right) |z|^6 + \dots \end{aligned} \quad (11)$$

Hence, the parameters take the form:

$$c_{22} = 6|b|^2 - 2\theta \quad (12)$$

$$c_{23} = 2(r - i\theta)b + 4i|b|^2 \quad (13)$$

$$c_{33} = \frac{2}{3}r^2 + 6\theta^2 + 56|b|^4 - \frac{112}{3}\theta|b|^2. \quad (14)$$

Stanton's family realises all those c_{22}, c_{23}, c_{33} for which the system of algebraic equations (12)-(14) has a solution $b \in \mathbb{C}, r, \theta \in \mathbb{R}$. A method to solve this system is to first express θ using Equation (12) then express r^2 through c_{33} and $|b|^2$ using Equation (14) and plug it into the absolute square of Equation (13). Solve the resulting cubic equation on $|b|^2$ and finally find r, θ . However, the resulting values for $|b|^2$ or r^2 could turn out to be negative.

Consider two examples: 1. Let $c_{23} = 0$. Then, $b = 0$, and (14) becomes:

$$r^2 = \frac{3}{2}c_{33} - \frac{9}{4}c_{22}^2.$$

By allowing r^2 to become negative, the solution:

$$\frac{1}{2r} \sin 2rv = e^{-2\theta v} |z|^2$$

turns into (1).

2. For $c_{22} = 0$ and $c_{23} = 2$, only $c_{33} \geq -2$ is feasible. Indeed, from Equation (12), we get $\theta = 3|b|^2$. Then, (13) is equivalent to:

$$\frac{1}{b} = r - i|b|^2.$$

This implies:

$$r^2 + |b|^4 = \frac{1}{|b|^2}$$

and hence:

$$r^2 = \frac{1}{|b|^2} - |b|^4.$$

This is only possible if $|b| \leq 1$. Now (14) becomes:

$$c_{33} = \frac{2}{3} \frac{1}{|b|^2} - \frac{8}{3} |b|^4 \geq -2.$$

Finally, we point out that Stanton's mapping can be interpreted as modified normalisation with:

$$\begin{aligned} \alpha(u) &= \frac{-\theta}{2r} \log(1 + 2ru) \\ h(u) &= \frac{1}{2r} \log(1 + 2ru) \\ p(u) &= \frac{b}{c} \left(e^{\frac{c}{2r} \log(1+2ru)} - 1 \right). \end{aligned}$$

Notice that the initial conditions are $p'(0) = b$ and $h''(0) = -2r$.

On the other hand, the proposition below shows that the solutions of (9) with different vector fields X yield the missing rigid spheres.

Proposition 1. *For any set of parameters c_{22}, c_{23}, c_{33} , there exists a rigid sphere. It can be realised by the solutions $Z(z, w), W(z, w)$ of (9) with parameters $c = i\theta, a, \rho$ and $\text{Re } c = b = 0$, where:*

$$\begin{aligned} \theta &= -\frac{c_{22}}{2} \\ a &= -\frac{c_{23}}{2} \\ \rho &= -\frac{3}{2}c_{33} + \frac{9}{4}c_{22}^2 \end{aligned}$$

and initial conditions $Z(z, 0) = z, W(z, 0) = 0$.

Proof. For any choice of parameters c_{22}, c_{23}, c_{33} , we find the corresponding parameters θ, a, ρ and hence the infinitesimal sphere automorphism X . According to the Picard-Lindelöf theorem, the system (9) has a unique solution with initial conditions $Z(z, 0) = z, W(z, 0) = 0$.

Direct computations with jets show that:

$$\begin{aligned} Z(z, w) &= z + i\theta zw + \frac{aw^2}{2} + 2i\bar{a}z^2w + \frac{\rho - \theta^2}{2}zw^2 + \frac{i\theta a}{6}w^3 + \dots \\ W(z, w) &= w + i\bar{a}zw^2 + \frac{\rho w^3}{3} + \dots \end{aligned}$$

where the dots indicate terms of order higher than 3. Substituting these truncated mappings into the sphere equation:

$$\frac{W - \bar{W}}{2i} - Z\bar{Z} = 0$$

yields:

$$v = |z|^2 - 2\theta|z|^4 - 2az^2\bar{z}^3 - 2\bar{a}z^3\bar{z}^2 + \left(6\theta^2 - \frac{2\rho}{3}\right)|z|^3 + \dots$$

where the dots indicate terms of bidegree (2,4) and (4,2) and higher order. □

It is an immediate consequence that the modified Chern-Moser normalisation of the Heisenberg sphere from Section 2 yields indeed rigid hypersurfaces.

4 Twisting Stanton's mapping

One way to produce solutions of (9) with non-trivial a, ρ is to apply the adjoint action of $SU(2, 1)$ on the coefficient matrix of the linear system in a suitable way. Geometrically, this amounts to compose Stanton's mapping with a sphere automorphism. The composition of Stanton's mapping $z_1(z, w), w_1(z, w)$ with parameters $c = r + i\theta, b$ and the sphere automorphism:

$$z_2 = \frac{z_1 - bw_1}{1 + 2i\bar{b}z_1 + (r - i|b|^2)w_1}$$

$$w_2 = \frac{w_1}{1 + 2i\bar{b}z_1 + (r - i|b|^2)w_1}$$

is:

$$z_2 = \frac{P_1}{Q}, \quad w_2 = \frac{P_2}{Q}$$

with:

$$P_1 = (a + 4(\theta - \phi)\phi z) \frac{\cosh rw - e^{i\theta w} + i\theta \frac{\sinh rw}{r}}{r^2 + \theta^2} + \left(-2i\phi \frac{\sinh rw}{r} + e^{i\theta w}\right) z$$

$$P_2 = 2i(\phi - \bar{a}z) \frac{\cosh rw - e^{i\theta w} + i\theta \frac{\sinh rw}{r}}{r^2 + \theta^2} + \frac{\sinh rw}{r}$$

$$Q = 2(\phi - \theta)(\phi - \bar{a}z) \frac{\cosh rw - e^{i\theta w} + i\theta \frac{\sinh rw}{r}}{(r^2 + \theta^2)} + i(\phi - 2\bar{a}z) \frac{\sinh rw}{r} + \cosh rw \quad (15)$$

where $a = -b(r - i\theta + 2i|b|^2)$ and $\phi = |b|^2$.

Notice that the Taylor series of:

$$\cosh rw - e^{i\theta w} + i\theta \frac{\sinh rw}{r}$$

has coefficients $a_0 = a_1 = 0$:

$$a_{2n+1} = \frac{i\theta(r^{2n} - (-1)^n\theta^{2n})}{(2n+1)!}, \quad a_{2n} = \frac{r^{2n} - (-1)^n\theta^{2n}}{(2n)!}$$

for $n \geq 1$ and therefore is divisible by $r^2 + \theta^2$. Clearly, $\sinh rw$ is divisible by r . Therefore, P_1, P_2, Q are entire functions with respect to z, w, r, θ, a, ϕ . Moreover, they are even functions with respect to r .

The vector field $\frac{\partial}{\partial w}$ is pulled back to:

$$(i\tau z_2 + aw_2 + 2i\bar{a}z_2^2 + \rho z_2 w_2) \frac{\partial}{\partial z_2} + (1 + 2i\bar{a}z_2 w_2 + \rho w_2^2) \frac{\partial}{\partial w_2},$$

where:

$$\begin{aligned} \tau &= \theta - 3\phi = -\frac{c_{22}}{2} \\ a &= -b(r - i\theta + 2i\phi) = -\frac{c_{23}}{2} \\ \rho &= -3\phi^2 - r^2 + 2\phi\theta = -\frac{3}{2}c_{33} + \frac{9}{4}c_{22}^2. \end{aligned} \tag{16}$$

We make formula (15) universal by allowing imaginary r and negative ϕ . Solving (16) for θ, b, r , we find:

$$\begin{aligned} \theta &= \tau + 3\phi \\ r^2 &= -\rho + (2\tau + 3\phi)\phi \end{aligned} \tag{17}$$

Now ϕ can be determined from the equation:

$$|a|^2 = 4\phi^3 + 4\tau\phi^2 + (\tau^2 - \rho)\phi. \tag{18}$$

Notice that the cubic equation has real coefficients and therefore has at least one real solution. Let ϕ be any real solution.

For $\tau = 0$, the solution ϕ can be given by the formula:

$$\phi = \frac{1}{2} \left(\left(|a|^2 + \sqrt{|a|^4 - \frac{\rho^3}{27}} \right)^{\frac{1}{3}} + \left(|a|^2 - \sqrt{|a|^4 - \frac{\rho^3}{27}} \right)^{\frac{1}{3}} \right).$$

Here, we take the principal branch of the cubic root on the right half-plane and the real cubic root on the real axis. Notice that ϕ is a continuous real-valued function, though not necessarily non-negative, and that $\phi = 0$ for $a = 0$.

Theorem 1. *Let:*

$$\begin{aligned} Z(z, w, a, \rho, \tau, \phi) &= \frac{P_1}{Q} \\ W(z, w, a, \rho, \tau, \phi) &= \frac{P_2}{Q} \end{aligned}$$

where P_1, P_2, Q are as in (15) and θ and r^2 are expressed as functions of τ, ρ, ϕ by (17).

Then, Z, W satisfy the system (9) on the real algebraic set given by (18).

It follows that all rigid spheres can be found as inverse images of $\text{Im } W = |Z|^2$ under the mappings Z, W with suitable parameters τ, a, ρ, θ and therefore have the form:

$$(1 + 2\phi|z|^2) \frac{\sin 2rv}{2r} - e^{-2\theta v}|z|^2 - (\phi + \bar{a}z + a\bar{z} + 4\phi(\phi - \theta)|z|^2) \frac{e^{-2\theta v} - \cos 2rv + \frac{\theta \sin 2rv}{r}}{r^2 + \theta^2} = 0.$$

Proof. Direct computation shows that the expressions:

$$\begin{aligned} \frac{\partial Z}{\partial w} - i\tau Z - aW - 2i\bar{a}Z^2 - \rho ZW \\ \frac{\partial W}{\partial w} - 1 - 2i\bar{a}ZW - \rho W^2 \end{aligned}$$

factorise with a factor $4\phi^3 + 4\tau\phi^2 + (\tau^2 - \rho)\phi - |a|^2 = 4\phi^3 - 4\theta\phi^2 + (\theta^2 + r^2)\phi - |a|^2$.

In fact:

$$Q \frac{\partial P_1}{\partial w} - P_1 \frac{\partial Q}{\partial w} - a P_2 Q - 2i\bar{a} P_1^2 + (r^2 + 3\phi^2 - 2\theta\phi) P_1 P_2 - i(\theta - 3\phi) P_1 Q =$$

$$\frac{2(4\phi^3 - 4\theta\phi^2 + (\theta^2 + r^2)\phi - |a|^2) \left(e^{i\theta w} - \cosh(rw) - i\theta \frac{\sinh(rw)}{r} \right)}{r^2 + \theta^2} \times$$

$$\left[-i(2\bar{a}(\theta - \phi)z^2 + (\theta^2 - 6\theta\phi + r^2 + 6\phi^2)z - a)(e^{i\theta w} + \cosh(rw)) \right.$$

$$\left. + (2\bar{a}(\theta\phi + r^2)z^2 + 3\phi(\theta^2 - 2\phi\theta - r^2)z\phi + \theta a) \frac{\sinh(rw)}{r} \right]$$

and:

$$Q \frac{\partial P_2}{\partial w} - P_2 \frac{\partial Q}{\partial w} - Q^2 - 2i\bar{a} P_1 P_2 + (r^2 + 3\phi^2 - 2\theta\phi) P_2^2 =$$

$$\frac{2(4\phi^3 - 4\theta\phi^2 + (\theta^2 + r^2)\phi - |a|^2) \left(e^{i\theta w} - \cosh(rw) - i\theta \frac{\sinh(rw)}{r} \right)}{r^2 + \theta^2} \times$$

$$\left[2(\bar{a}z - \phi)(e^{i\theta w} - \cosh(rw)) - i(2\bar{a}\theta z + \theta^2 - 2\phi\theta + r^2) \frac{\sinh(rw)}{r} \right].$$

Therefore, Z, W satisfy the system (9) on the real algebraic set (18).

Now the rigid sphere formula can be obtained either from Stanton's formula (2) by replacing r, θ, b by their expressions in τ, a, ρ, θ or by inserting Z, W into the standard Heisenberg sphere equation. \square

Notice that the Taylor series of $e^{-2\theta v} - \cos 2rv + \frac{\theta \sin 2rv}{r}$ has coefficients $a_0 = a_1 = 0$:

$$a_{2n+1} = -\frac{2^{2n+1}\theta(r^{2n} - (-1)^n\theta^{2n})}{(2n+1)!}, \quad a_{2n} = \frac{2^{2n}(r^{2n} - (-1)^n\theta^{2n})}{(2n)!}$$

for $n \geq 1$ and therefore is divisible by $r^2 + \theta^2$. Clearly, $\sin 2rv$ is divisible by r . It follows that the rigid sphere formula is an entire function with respect to all variables and parameters.

We find an example of a rigid sphere that is not in Stanton's family by setting:

$$\tau = 0, \quad a = \sqrt{2}, \quad \rho = 6.$$

Then:

$$\theta = -3, \quad \phi = -1, \quad r^2 = -3$$

and:

$$(1 - 2|z|^2) \frac{\sinh 2\sqrt{3}v}{2\sqrt{3}} - e^{6v}|z|^2 + \left(1 - \sqrt{2}(z + \bar{z}) + 8|z|^2\right) \frac{e^{6v} - \cosh 2\sqrt{3}v - \sqrt{3} \sinh 2\sqrt{3}v}{6} = 0.$$

5 The zero-curvature equation

Local equivalence of a real hypersurface M in \mathbb{C}^2 to a sphere can be characterised by vanishing of its Cartan curvature. In [3], an explicit expression of the Cartan curvature has been computed. In the case of rigid hypersurfaces, this expression considerably simplifies. If M is given by the equation:

$$v = h(z)$$

vanishing of the Cartan curvature is equivalent to the non-linear PDE:

$$f_{z\bar{z}\bar{z}} - 3f_{z\bar{z}}f + 2f_z f^2 - f_z f_{\bar{z}} = 0 \tag{19}$$

with $f = \frac{\partial}{\partial \bar{z}} \log \Delta h$. The condition that h is a real function translates into $\frac{\partial f}{\partial z} = \frac{\partial \bar{f}}{\partial \bar{z}}$.

Notice that the normal form conditions on h are encoded in the choice of the solution of the auxiliary equation:

$$\frac{\partial}{\partial \bar{z}} \log \Delta h = f$$

together with the condition that f does not contain antiholomorphic terms.

Though we did not succeed in solving the zero-curvature equation directly, we can demonstrate the special cases when M is circular, i.e. $v = f(|z|^2)$, or M is a tube, i.e. $v = h(x)$.

In the circular case:

$$v = h(|z|^2) = |z|^2 + a_2|z|^4 + a_3|z|^6 + \dots$$

and the equation on $g = \log(th'' + h')$ with t replacing $|z|^2$ is:

$$tg^{IV} + 3g''' - g'(3tg''' + 7g'') - t(g'')^2 + 2t(g')^2g'' + 2(g')^3 = 0.$$

Formal power series solutions:

$$g(t) = c_1t + c_2t^2 + \dots$$

are determined by c_1, c_2 . They correspond to the surfaces:

$$\frac{\sin \alpha v}{\alpha} = e^{\beta v |z|^2} \quad \text{and} \quad \frac{\sinh \alpha v}{\alpha} = e^{\beta v |z|^2}$$

with $\beta = \frac{c_1}{4}$, $\alpha = \sqrt{-\frac{2c_2}{3} - \frac{c_1^2}{12}}$ and $\alpha = \sqrt{\frac{2c_2}{3} + \frac{c_1^2}{12}}$ respectively.

The spherical tubes are well-known (see, for instance, [4]). They are never in rigid normal form, and they are affinely equivalent to one of the following:

$$v = x^2, \quad v = e^x, \quad \sin v = e^x, \quad e^v + e^x = 1 \tag{20}$$

This corresponds to the solutions for (19) listed below, respectively:

$$f = 0, \quad f = \frac{1}{2}, \quad f = \tan x, \quad f = -\tanh x$$

If follows:

$$h = \frac{1}{2}x^2, \quad h = e^x, \quad h = -\log \cos x, \quad h = \log \cosh x$$

which yields the tubes:

$$v = x^2, \quad v = e^x, \quad e^v = \cos x, \quad e^v = \cosh x$$

which are affinely equivalent to the tubes (20).

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