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Asymptotic dimension and small subsets in locally compact topological groups

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Abstract We prove that for a coarse space X the ideal $\mathcal{S}(X)$ of small subsets of X coincides with the ideal $\mathcal{D}_<(X) = \{A \subset X : \mathrm{asdim}(A) < \mathrm{asdim}(X)\}$ provided that X is coarsely equivalent to a Euclidean space $\mathbb{R}^n$. Also we prove that for a locally compact Abelian group X, the equality $\mathcal{S}(X) = \mathcal{D}_<(X)$ holds if and only if the group X is compactly generated.

Keywords Asymptotic dimension · Locally compact group · Coarse structure · Small set

Mathematics Subject Classification (1991) 20F65 · 20F69 · 54F45 · 55M10

1 Introduction

In this paper we study the interplay between the ideal $\mathcal{S}(X)$ of small subsets of a coarse space X and the ideal $\mathcal{D}_<(X)$ of subsets of asymptotic dimension less than $\mathrm{asdim}(X)$ in X. We show that these two ideals coincide in spaces that are coarsely equivalent to $\mathbb{R}^n$, in particular, they coincide in each compactly generated locally compact abelian group.

Let us recall that a *coarse space* is a pair $(X, \mathcal{E})$ consisting of a set X and a coarse structure $\mathcal{E}$ on X, which is a family of subsets of $X \times X$ (called *entourages*) satisfying the following axioms:

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(A) each $\varepsilon \in \mathcal{E}$ contains the diagonal $\Delta_X = \{(x, y) \in X^2 : x = y\}$ and is *symmetric* in the sense that $\varepsilon = \varepsilon^{-1}$ where $\varepsilon^{-1} = \{(y, x) : (x, y) \in \varepsilon\}$;
(B) for any entourages $\varepsilon, \delta \in \mathcal{E}$ there is an entourage $\eta \in \mathcal{E}$ that contains the composition $\delta \circ \varepsilon = \{(x, z) \in X^2 : \exists y \in X \text{ with } (x, y) \in \varepsilon \text{ and } (y, z) \in \delta\}$;
(C) a subset $\delta \subset X^2$ belongs to $\mathcal{E}$ if $\Delta_X \subset \delta = \delta^{-1} \subset \varepsilon$ for some $\varepsilon \in \mathcal{E}$.

A subfamily $\mathcal{B} \subset \mathcal{E}$ is called a *base* of the coarse structure $\mathcal{E}$ if

$$\mathcal{E} = \{\varepsilon \subset X^2 : \exists \delta \in \mathcal{B} \text{ with } \Delta_X \subset \varepsilon = \varepsilon^{-1} \subset \delta\}.$$

A family $\mathcal{B}$ of subsets of X^2 is a base of a (unique) coarse structure if and only if it satisfies the axioms (A), (B).

Each subset A of a coarse space $(X, \mathcal{E})$ carries the induced coarse structure $\mathcal{E}_A = \{\varepsilon \cap A^2 : \varepsilon \in \mathcal{E}\}$. Endowed with this structure, the space $(A, \mathcal{E}_A)$ is called a *subspace* of $(X, \mathcal{E})$.

For an entourage $\varepsilon \subset X^2$, a point $x \in X$, and a subset $A \subset X$ let $B(x, \varepsilon) = \{y \in X : (x, y) \in \varepsilon\}$ be the ε-*ball* centered at x, $B(A, \varepsilon) = \bigcup_{a \in A} B(a, \varepsilon)$ be the ε-neighborhood of A in X, and $\operatorname{diam}(A) = A \times A$ be the *diameter* of A. For a family $\mathcal{U}$ of subsets of X we put $\operatorname{mesh}(\mathcal{U}) = \bigcup_{U \in \mathcal{U}} \operatorname{diam}(U)$.

Now we consider two basic examples of coarse spaces. The first of them is any metric space (X, d) carrying the *metric coarse structure* whose base consists of the entourages $\{(x, y) \in X^2 : d(x, y) < \varepsilon\}$ where $0 \le \varepsilon < \infty$. A coarse space is *metrizable* if its coarse structure is generated by some metric.

The second basic example is a topological group G endowed with the *left coarse structure* whose base consists of the entourages $\{(x, y) \in G^2 : x \in yK\}$ where $K = K^{-1}$ runs over compact symmetric subsets of G that contain the identity element 1_G of G. Let us observe that the left coarse structure on G coincides with the metric coarse structure generated by any left-invariant continuous metric d on G which is *proper* in the sense that each closed ball $B(e, R) = \{x \in G : d(x, e) \le R\}$ is compact. In particular, the coarse structure on $\mathbb{R}^n$, generated by the Euclidean metric coincides with the left coarse structure of the Abelian topological group $\mathbb{R}^n$.

Now we recall the definitions of large and small sets in coarse spaces. Such sets were introduced in [4] and studied in [13, §11] and [2]. A subset A of a coarse space $(X, \mathcal{E})$ is called

- *large* if $B(A, \varepsilon) = X$ for some $\varepsilon \in \mathcal{E}$;
- *small* if for each large set $L \subset X$ the set $L \setminus A$ remains large in X.

It follows that the family $\mathcal{S}(X)$ of small subsets of a coarse space $(X, \mathcal{E})$ is an ideal. A subfamily $\mathcal{I} \subset \mathcal{P}(X)$ of the power-set of a set X is called an *ideal* if $\mathcal{I}$ is *additive* (in the sense that $A \cup B \in \mathcal{I}$ for all $A, B \in \mathcal{I}$) and *downwards closed* (which means that $A \cap B \in \mathcal{I}$ for all $A \in \mathcal{I}$ and $B \subset X$).

Small sets can be considered as coarse counterparts of nowhere dense subsets in topological spaces, see [2]. It is well-known [8, 7.4.18] that the ideal of nowhere dense subsets in a Euclidean space $\mathbb{R}^n$ coincides with the ideal generated by closed subsets of topological dimension $< n$ in $\mathbb{R}^n$. The aim of this paper is to prove a coarse counterpart of this fundamental fact.

For this we need to recall [14, 9.4] the definition of the asymptotic dimension $\operatorname{asdim}(X)$ of a coarse space X.

Definition 1.1 The *asymptotic dimension* $\operatorname{asdim}(X)$ of a coarse space $(X, \mathcal{E})$ is the smallest number $n \in \omega$ such that for each entourage $\varepsilon \in \mathcal{E}$ there is a cover $\mathcal{U}$ of X such that

$\mathrm{mesh}(\mathcal{U}) \subset \delta$ for some $\delta \in \mathcal{E}$ and each ε-ball $B(x, \varepsilon)$, $x \in X$, meets at most $n + 1$ sets $U \in \mathcal{U}$. If such a number $n \in \omega$ does not exist, then we put $\mathrm{asdim}(X) = \infty$.

In Theorem 2.7 we shall prove that

$$\mathrm{asdim}(A \cup B) \leq \max\{\mathrm{asdim}(A), \mathrm{asdim}(B)\}$$

for any subspaces A, B of a coarse space X. This implies that for every number $n \in \omega \cup \{\infty\}$ the family $\{A \subset X : \mathrm{asdim}(A) < n\}$ is an ideal in $\mathcal{P}(X)$. In particular, the family

$$\mathcal{D}_<(X) = \{A \subset X : \mathrm{asdim}(A) < \mathrm{asdim}(X)\}$$

is an ideal in $\mathcal{P}(X)$. According to [5, 9.8.4], $\mathrm{asdim}(\mathbb{R}^n) = n$ for every $n \in \omega$.

The main result of this paper is:

Theorem 1.2 *For every $n \in \mathbb{N}$ the ideal $\mathcal{S}(X)$ of small subsets in the space $X = \mathbb{R}^n$ coincides with the ideal $\mathcal{D}_<(X)$.*

Theorem 1.2 will be proved in Sect. 5 with help of some tools of Combinatorial Topology. In light of this theorem the following problem arises naturally:

Problem 1.3 *Detect coarse spaces X for which $\mathcal{S}(X) = \mathcal{D}_<(X)$.*

It should be mentioned that the class of coarse spaces X with $\mathcal{S}(X) = \mathcal{D}_<(X)$ is closed under coarse equivalences.

A function $f : X \to Y$ between two coarse spaces $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$ is called

- *coarse* if for each $\delta_X \in \mathcal{E}_X$ there is $\varepsilon_Y \in \mathcal{E}_Y$ such that for any pair $(x, y) \in \delta_X$ we get $(f(x), f(y)) \in \varepsilon_Y$;
- a *coarse equivalence* if f is coarse and there is a coarse map $g : Y \to X$ such that $\{(x, g \circ f(x)) : x \in X\} \subset \varepsilon_X$ and $\{(y, f \circ g(y)) : y \in Y\} \subset \varepsilon_Y$ for some entourages $\varepsilon_X \in \mathcal{E}_X$ and $\varepsilon_Y \in \mathcal{E}_Y$.

Two coarse spaces X, Y are called *coarsely equivalent* if there is a coarse equivalence $f : X \to Y$.

Proposition 1.4 *Assume that coarse spaces X, Y are coarsely equivalent. Then*

(1) $\mathrm{asdim}(X) = \mathrm{asdim}(Y)$*;*
(2) $\mathcal{D}_<(X) = \mathcal{S}(X)$ *if and only if* $\mathcal{D}_<(Y) = \mathcal{S}(Y)$.

This proposition will be proved in Sect. 3. Combined with Theorem 1.2 it implies:

Corollary 1.5 *If a coarse space X is coarsely equivalent to a Euclidean space $\mathbb{R}^n$, then $\mathcal{D}_<(X) = \mathcal{S}(X)$.*

Problem 1.3 can be completely resolved for locally compact Abelian topological groups G, endowed with their left coarse structure. First we establish the following general fact, which will be proved in Sect. 4.

Theorem 1.6 *For each topological group X endowed with its left coarse structure we get $\mathcal{D}_<(X) \subset \mathcal{S}(X)$.*

We recall that a topological group G is *compactly generated* if G is algebraically generated by some compact subset $K \subset G$.

Theorem 1.7 *For an Abelian locally compact topological group X the following conditions are equivalent:*

(1) $\mathcal{S}(X) = \mathcal{D}_<(X)$;
(2) X is compactly generated;
(3) X is coarsely equivalent to the Euclidean space $\mathbb{R}^n$ for some $n \in \omega$.

This theorem will be proved in Sect. 6.

Remark 1.8 The equivalence (1)⇔(2) in Theorem 1.7 does not hold beyond the class of Abelian groups. The simplest counterexample is the free group F_2 with two generators, endowed with the discrete topology. Any infinite cyclic subgroup $Z \subset F_2$ has infinite index in F_2 and hence is small, yet $\text{asdim}(Z) = \text{asdim}(F_2) = 1$.

A less trivial example is the wreath product $A \wr \mathbb{Z}$ of a non-trivial finite abelian group A and $\mathbb{Z}$. The group $A \wr \mathbb{Z}$ has asymptotic dimension 1 (see [9]) and the subgroup $\mathbb{Z}$ is small in $A \wr \mathbb{Z}$ and has $\text{asdim}(\mathbb{Z}) = 1 = \text{asdim}(A \wr \mathbb{Z})$. Let us recall that the group $A \wr \mathbb{Z}$ consists of ordered pairs $((a_i)_{i\in\mathbb{Z}}, n) \in \big(\oplus^{\mathbb{Z}} A\big) \times \mathbb{Z}$ and the group operation on $A \wr \mathbb{Z}$ is defined by

$$((a_i), n) * ((b_i), m) = ((a_{i+m} + b_i), n + m).$$

The group $A \wr \mathbb{Z}$ is finitely-generated and meta-abelian but is not finitely presented, see [3]. Groups which are coarsely equivalent to abelian groups were studied in [1].

Problem 1.9 *Is $\mathcal{S}(X) = \mathcal{D}_<(X)$ for each connected Lie group X? For each discrete poly-cyclic group X?*

2 The asymptotic dimension of coarse spaces

In this section we present various characterizations of the asymptotic dimension of coarse spaces. First we fix some notation. Let $(X, \mathcal{E})$ be a coarse space, $\varepsilon \in \mathcal{E}$ and $A \subset X$. We shall say that A has diameter less than ε if $\text{diam}(A) \subset \varepsilon$ where $\text{diam}(A) = A \times A$. A sequence $x_0, \dots, x_m \in X$ is called an *ε-chain* if $(x_i, x_{i+1}) \in \varepsilon$ for all $i < m$. In this case the finite set $C = \{x_0, \dots, x_m\}$ also will be called an *ε-chain*. A set $C \subset X$ is called *ε-connected* if any two points $x, y \in C$ can be linked by an ε-chain $x = x_0, \dots, x_m = y$. The maximal ε-connected subset $C(x, \varepsilon) \subset X$ containing a given point $x \in X$ is called the *ε-connected component* of x. It consists of all points $y \in X$ that can be linked with x by an ε-chain $x = x_0, \dots, x_m = y$.

A family $\mathcal{U}$ of subsets of X is called *ε-disjoint* if $(U \times V) \cap \varepsilon = \emptyset$ for any distinct sets $U, V \in \mathcal{U}$. Each natural number n is identified with the set $\{0, \dots, n-1\}$.

We shall study the interplay between the asymptotic dimension introduced in Definition 1.1 and the following modification:

Definition 2.1 The *colored asymptotic dimension* $\text{asdim}_{col}(X)$ of a coarse space $(X, \mathcal{E})$ is the smallest number $n \in \omega$ such that for every entourage $\varepsilon \in \mathcal{E}$ there is a cover $\mathcal{U}$ of X such that $\text{mesh}(\mathcal{U}) \subset \delta$ for some $\delta \in \mathcal{E}$ and $\mathcal{U}$ can be written as the union $\mathcal{U} = \bigcup_{i\in n+1} \mathcal{U}_i$ of $n + 1$ many ε-disjoint subfamilies $\mathcal{U}_i$. If such a number $n \in \omega$ does not exist, then we put $\text{asdim}_{col}(X) = \infty$.

Without lost of generality we can assume that the cover $\mathcal{U} = \bigcup_{i\in n+1} \mathcal{U}_i$ in the above definition consists of pairwise disjoint sets. In this case we can consider the coloring $\chi : X \to n + 1 = \{0, \dots, n\}$ such that $\chi^{-1}(i) = \bigcup \mathcal{U}_i$ for every $i \in n + 1$. For this coloring every χ-monochrome ε-connected subset $C \subset X$ lies in some $U \in \mathcal{U}$ and hence has diameter $\mathrm{diam}(C) \subset \mathrm{diam}(U) \subset \mathrm{mesh}(\mathcal{U}) \subset \delta$. A subset $A \subset X$ is *χ-monochrome* if $\chi(A)$ is a singleton. Thus we arrive to the following useful characterization of the colored asymptotic dimension.

Proposition 2.2 *A coarse space* $(X, \mathcal{E})$ *has* $\mathrm{asdim}_{col}(X) \le n$ *for some number* $n \in \omega$ *if and only if for any* $\varepsilon \in \mathcal{E}$ *there are a coloring* $\chi : X \to n+1$ *and an entourage* $\delta \in \mathcal{E}$ *such that each* χ*-monochrome* ε*-chain* $C \subset X$ *has* $\mathrm{diam}(C) \subset \delta$.

Proof The "only if" part follows from the above discussion. To prove the "if" part, for every $\varepsilon \in \mathcal{E}$ we need to construct a cover $\mathcal{U} = \bigcup_{i \in n+1} \mathcal{U}_i$ such that $\mathrm{mesh}(\mathcal{U}) \in \mathcal{E}$ and each family $\mathcal{U}_i$ is ε-disjoint. By our assumption, there is a coloring $\chi : X \to n+1$ and an entourage $\delta \in \mathcal{E}$ such that each χ-monochrome ε-chain $C \subset X$ has $\mathrm{diam}(C) \subset \delta$.

For each $x \in X$ let $C_\chi(x, \varepsilon)$ be the set of all points $y \in X$ that can be linked with x by a χ-monochrome ε-chain $x = x_0, x_1, \dots, x_m = y$. It follows that $\mathrm{diam}(C_\chi(x, \varepsilon)) \subset \delta$. For every $i \in n+1$ consider the ε-disjoint family $\mathcal{U}_i = \{C_\chi(x, \varepsilon) : x \in \chi^{-1}(i)\}$. It is clear that $\mathcal{U} = \bigcup_{i \in n+1} \mathcal{U}_i$ is a cover with $\mathrm{mesh}(\mathcal{U}) \subset \delta \in \mathcal{E}$, witnessing that $\mathrm{asdim}_{col}(X) \le n$. □

Now we are ready to prove the equivalence of two definitions of asymptotic dimension. For metrizable coarse spaces this equivalence was proved in [5, 9.3.7].

Proposition 2.3 *Each coarse space* $(X, \mathcal{E})$ *has* $\mathrm{asdim}(X) = \mathrm{asdim}_{col}(X)$.

Proof To prove that $\mathrm{asdim}(X) \le \mathrm{asdim}_{col}(X)$, put $n = \mathrm{asdim}_{col}(X)$ and take any entourage $\varepsilon \in \mathcal{E}$. By Definition 2.1, for the entourage $\varepsilon \circ \varepsilon \in \mathcal{E}$ we can find a cover $\mathcal{U} = \bigcup_{i \in n+1} \mathcal{U}_i$ with $\mathrm{mesh}(\mathcal{U}) \in \mathcal{E}$ such that each family $\mathcal{U}_i$ is $\varepsilon \circ \varepsilon$-disjoint. We claim that each ε-ball $B(x, \varepsilon)$, $x \in X$, meets at most one set of each family $\mathcal{U}_i$. Assuming that $B(x, \varepsilon)$ meets two distinct sets $U, V \in \mathcal{U}_i$, we can find points $u \in U$ and $v \in V$ with $(x, u), (x, v) \in \varepsilon$ and conclude that $(u, v) \in \varepsilon \circ \varepsilon$, which is not possible as $\mathcal{U}_i$ is $\varepsilon \circ \varepsilon$-disjoint. Now we see that the ball $B(x, \varepsilon)$ meets at most $n+1$ element of the cover $\mathcal{U}$ and hence $\mathrm{asdim}(X) \le n$.

The proof of the inequality $\mathrm{asdim}_{col}(X) \le \mathrm{asdim}(X)$ is a bit longer. If the dimension $n = \mathrm{asdim}(X)$ is infinite, then there is nothing to prove. So, we assume that $n \in \omega$. To prove that $\mathrm{asdim}_{col}(X) \le n$, fix an entourage $\varepsilon \in \mathcal{E}$. Let $\varepsilon^0 = \Delta_X$ and $\varepsilon^{k+1} = \varepsilon^k \circ \varepsilon$ for $k \in \omega$. Since $\mathrm{asdim}(X) \le n$, for the entourage $\varepsilon^{n+1} \in \mathcal{E}$ we can find a cover $\mathcal{U}$ of X such that $\delta = \mathrm{mesh}(\mathcal{U}) \in \mathcal{E}$ and each ε^{n+1}-ball $B(x, \varepsilon^{n+1})$ meets at most $n+1$ many sets $U \in \mathcal{U}$. For every $i \le n+1$ and $x \in X$ consider the subfamily $\mathcal{U}(x, \varepsilon^i) = \{U \in \mathcal{U} : B(x, \varepsilon^i) \cap U \neq \emptyset\}$ of $\mathcal{U}$. It follows that $1 \le |\mathcal{U}(x, \varepsilon^i)| \le |\mathcal{U}(x, \varepsilon^{i+1})| \le n+1$ for every $0 \le i \le n$. Consequently, $|\mathcal{U}(x, \varepsilon^i)| = i$ for some $i \le n+1$. Let $\chi(x)$ be the maximal number $k \le n$ such that $|\mathcal{U}(x, \varepsilon^{k+1})| = k+1$. In such a way we have defined a coloring $\chi : X \to n+1 = \{0, \dots, n\}$.

To finish the proof it suffices to show that any χ-monochrome ε-chain $C = \{x_0, \dots, x_m\} \subset X$ has $\mathrm{diam}(C) \subset \delta \circ \varepsilon^{n+1}$. Let $k = \chi(x_0)$ be the color of the chain C. It follows that $|\mathcal{U}(x_i, \varepsilon^{k+1})| = k+1$ for all $x_i \in C$. We claim that $\mathcal{U}(x_i, \varepsilon^{k+1}) = \mathcal{U}(x_{i+1}, \varepsilon^{k+1})$ for all $i < m$. Assuming the converse, we would get that $|\mathcal{U}(x_i, \varepsilon^{k+1}) \cup \mathcal{U}(x_{i+1}, \varepsilon^{k+1})| \ge k+3$ and then the family $\mathcal{U}(x_i, \varepsilon^{k+2}) \supset \mathcal{U}(x_i, \varepsilon^{k+1}) \cup \mathcal{U}(x_{i+1}, \varepsilon^{k+1})$ has cardinality $|\mathcal{U}(x_i, \varepsilon^{k+2})| \ge k+3$, which implies that $|\mathcal{U}(x_i, \varepsilon^i)| = i$ for some $k+3 \le i \le n+1$. But this contradicts the definition of $k = \chi(x_i)$. Hence $\mathcal{U}(x_i, \varepsilon^{k+1}) = \mathcal{U}(x_0, \varepsilon^{k+1})$ for all $i \le m$ and then $C \subset B(U, \varepsilon^{k+1})$ for every $U \in \mathcal{U}(x_0, \varepsilon^{k+1})$. Now we see that $\mathrm{diam}(C) \subset \mathrm{diam}(U) \circ \varepsilon^{k+1} \subset \delta \circ \varepsilon^{n+1}$. □

Propositions 2.2 and 2.3 imply:

Corollary 2.4 *A coarse space $(X, \mathcal{E})$ has asymptotic dimension* $\mathrm{asdim}(X) \le n$ *for some $n \in \omega$ if and only if for any $\varepsilon \in \mathcal{E}$ there are $\delta \in \mathcal{E}$ and a coloring $\chi : X \to n+1$ such that any χ-monochrome ε-chain $C \subset X$ has* $\mathrm{diam}(C) \subset \delta$.

This corollary can be generalized as follows (cf. [6]).

Proposition 2.5 *A coarse space $(X, \mathcal{E})$ has* $\mathrm{asdim}(X) \le n$ *for some $n \in \omega$ if and only if for any entourage $\varepsilon \in \mathcal{E}$ there is an entourage $\delta \in \mathcal{E}$ such that for any finite set $F \subset X$ there is a coloring $\chi : F \to n+1$ such that each χ-monochrome ε-chain $C \subset F$ has* $\mathrm{diam}(C) \subset \delta$.

Proof This proposition will follow from Corollary 2.4 as soon as for any $\varepsilon \in \mathcal{E}$ we find $\delta \in \mathcal{E}$ and a coloring $\chi : X \to n+1$ such that each χ-monochrome ε-chain in X has diameter less that δ.

By our assumption, there is an entourage $\delta \in \mathcal{E}$ such that for every finite subset $F \subset X$ there is a coloring $\chi_F : F \to n+1$ such that each χ_F-monochrome ε-chain in F has diameter less that δ. Extend χ_F to a coloring $\tilde{\chi}_F : X \to n+1$.

Let $\mathcal{F}$ denote the family of all finite subsets of X, partially ordered by the inclusion relation $\subset$. The colorings $\tilde{\chi}_F$, $F \in \mathcal{F}$, can be considered as elements of the compact Hausdorff space $K = \{0, \ldots, n\}^X$ endowed with the Tychonoff product topology. The compactness of K implies that the net $\{\tilde{\chi}_F\}_{F \in \mathcal{F}}$ has a cluster point $\chi \in K$, see [8, 3.1.23]. The latter means that for each finite set $F_0 \in \mathcal{F}$ and a neighborhood $O(\chi) \subset K$ there is a finite set $F \in \mathcal{F}$ such that $F \supset F_0$ and $\tilde{\chi}_F \in O(\chi)$.

We claim that the coloring $\chi : X \to n+1$ has the required property: each χ-monochrome ε-chain $C \subset X$ has $\mathrm{diam}(X) \subset \delta$. Observe that the finite set C determines a neighborhood $O_C(\chi) = \{f \in K : f|C = \chi|C\}$, which contains a coloring $\tilde{\chi}_F$ for some finite set $F \supset C$. The choice of the coloring $\chi_F = \tilde{\chi}_F|F$ guarantees that the set $C \subset F$ has $\mathrm{diam}(C) \subset \delta$. □

Proposition 2.5 admits the following self-generalization.

Theorem 2.6 *A coarse space $(X, \mathcal{E})$ has* $\mathrm{asdim}(X) \le n$ *for some $n \in \omega$ if and only if for any entourage $\varepsilon \in \mathcal{E}$ there is an entourage $\delta \in \mathcal{E}$ such that for any finite ε-connected subset $F \subset X$ there is a coloring $\chi : F \to n+1$ such that each χ-monochrome ε-chain $C \subset F$ has* $\mathrm{diam}(C) \subset \delta$.

Finally, let us prove Addition Theorem for the asymptotic dimension. For metrizable spaces this theorem is well known; see [14, 9.13] or [5, 9.7.1].

Theorem 2.7 *For any subspaces A, B of a coarse space $(X, \mathcal{E})$ we get*

$$\mathrm{asdim}(A \cup B) \le \max\{\mathrm{asdim}(A), \mathrm{asdim}(B)\}.$$

Proof Only the case of finite $n = \max\{\mathrm{asdim}(A), \mathrm{asdim}(B)\}$ requires the proof. Without loss of generality the sets A and B are disjoint. To show that $\mathrm{asdim}(A \cup B) \le n$ we shall apply Corollary 2.4. Fix any entourage $\varepsilon \in \mathcal{E}$. Since $\mathrm{asdim}(A) \le n$ there are an entourage $\delta_A \in \mathcal{E}$ and a coloring $\chi_A : A \to n+1$ such that each χ-monochrome ε-chain in A has diameter less that δ_A. Since $\mathrm{asdim}(B) \le n$, for the entourage $\varepsilon_B = \varepsilon \circ \delta_A \circ \varepsilon$ there are an entourage $\delta_B \in \mathcal{E}$ and a coloring $\chi_B : A \to \{0, \ldots, n\}$ such that each χ-monochrome ε_B-chain in B has diameter less that δ_B.

The union of the colorings χ_A and χ_B yields the coloring $\chi : A \cup B \to \{0, \ldots, n\}$ such that $\chi|A = \chi_A$ and $\chi|B = \chi_B$. We claim that each χ-monochrome ε-chain $C = \{x_0, \ldots, x_m\} \subset A \cup B$ has $\mathrm{diam}(C) \subset \delta$ where $\delta = \delta_A \circ \varepsilon \circ \delta_B \circ \varepsilon \circ \delta_A$. Without loss of generality, the points $x_0, \ldots, x_m$ of the chain C are pairwise distinct.

If $C \subset A$, then C, being a χ_A-monochrome ε-chain in A has $\operatorname{diam}(C) \subset \delta_A \subset \delta$ and we are done. So, we assume that $C \not\subset A$. In this case $b = |C \cap B| \geq 1$ and we can choose a strictly increasing sequence $0 \leq k_1 < k_2 < \cdots < k_b \leq m$ such that $\{x_{k_1}, \dots, x_{k_b}\} = C \cap B$. Then $\{x_0, \dots, x_{k_1-1}\}$, being a χ_A-monochrome ε-chain in A, has diameter less that δ_A. Consequently, the ε-chain $\{x_0, \dots, x_{k_1}\}$ has diameter less that $\delta_A \circ \varepsilon \subset \varepsilon_B$. By the same reason the ε-chain $\{x_{k_b}, \dots, x_m\}$ has diameter less that $\varepsilon \circ \delta_A \subset \varepsilon_B$ and for every $1 \leq i < b$ the ε-chain $\{x_{k_i}, \dots, x_{k_{i+1}}\} \subset \{x_{k_i}\} \cup A \cup \{x_{k_{i+1}}\}$ has diameter less than $\varepsilon \circ \delta_A \circ \varepsilon = \varepsilon_B$. Then $\{x_{k_1}, \dots, x_{k_b}\}$, being a χ_B-monochrome ε_B-chain in B, has diameter less that δ_B. Now we see that the ε-chain $C = \{x_0, \dots, x_m\}$ has $\operatorname{diam}(C) \subset \delta_A \circ \varepsilon \circ \delta_B \circ \varepsilon \circ \delta_A = \delta$. □

The characterization Theorem 2.6 will be applied to prove the following theorem which was known [7, 2.1] in the context of countable groups.

Theorem 2.8 *If G is a topological group endowed with its left coarse structure, then*

$$\operatorname{asdim}(G) = \sup\{\operatorname{asdim}(H) : H \text{ is a compactly generated subgroup of } G\}.$$

Proof Let $n = \sup\{\operatorname{asdim}(H) : H$ is a compactly generated subgroup of $G\}$. It is clear that $n \leq \operatorname{asdim}(G)$. The reverse inequality $\operatorname{asdim}(G) \leq n$ is trivial if $n = \infty$. So, we assume that $n < \infty$. To prove that $\operatorname{asdim}(G) \leq n$, we shall apply Theorem 2.6. Let $\mathcal{E}$ be the left coarse structure of the topological group G. Given any entourage $\varepsilon \in \mathcal{E}$, we should find an entourage $\delta \in \mathcal{E}$ such that for each finite ε-connected subset $F \subset G$ there is a coloring $\chi : F \to n+1$ such that each χ-monochrome ε-chain $C \subset F$ has $\operatorname{diam}(C) \subset \delta$.

By the definition of the coarse structure $\mathcal{E}$, for the entourage $\varepsilon \in \mathcal{E}$ there is a compact subset $K_\varepsilon = K_\varepsilon^{-1} \subset G$ such that $\varepsilon \subset \{(x, y) \in G^2 : x \in yK_\varepsilon\}$. Let H be the subgroup of G generated by the compact set K_ε, $\mathcal{E}_H$ be the left coarse structure of H, and $\varepsilon_H = \{(x, y) \in H^2 : x \in yK_\varepsilon\} \in \mathcal{E}_H$. Since $\operatorname{asdim}_{col}(H) = \operatorname{asdim}(H) \leq n$, by Proposition 2.2, there is a coloring $\chi_H : H \to n+1$ and an entourage $\delta_H \in \mathcal{E}_H$ such that each χ_H-monochrome ε-chain $C \subset H$ has diameter $\operatorname{diam}(C) \subset \delta_H$. By the definition of the coarse structure $\mathcal{E}_H$, there is a compact subset $K_\delta = K_\delta^{-1} \ni 1_H$ of H such that $\{(x, y) \in H \times H : x \in yK_\delta\}$.

We claim that the entourage $\delta = \{(x, y) \in G \times G : x \in yK_\delta\}$ satisfies our requirements. Let F be a finite ε-connected subset of G. Then for each point $x_0 \in F$ we get $F \in x_0 H$ and hence $x_0^{-1}F \subset H$. So, we can define a coloring $\chi : F \to n+1$ letting $\chi(x) = \chi_H(x_0^{-1}x)$ for $x \in F$. If $C \subset F$ is a χ-monochrome ε-chain, then $x_0^{-1}C$ is a χ_H-monochrome ε_H-chain in H and hence $\operatorname{diam}(x_0^{-1}C) \subset \delta_H$. The latter means that for any points $c, c' \in C$ we get $(x_0^{-1}c, x_0^{-1}c') \in \delta_H \subset \{(x, y) \in H \times H : x \in yK_\delta\}$ and hence $x_0^{-1}c \in x_0^{-1}c'K_\delta$ and $c \in c'K_\delta$, which means that $(c, c') \in \delta$ and hence $\operatorname{diam}(C) \subset \delta$. □

3 Proof of Proposition 1.4

Let $f : X \to Y$ be a coarse equivalence between two coarse spaces $(X, \mathcal{E}_X)$ and $(Y, \mathcal{E}_Y)$. Then there is a coarse map $g : Y \to X$ such that $\{(x, g \circ f(x)) : x \in X\} \subset \eta_X$ and $\{(y, f \circ g(y)) : y \in Y\} \subset \eta_Y$ for some entourages $\eta_X \in \mathcal{E}_X$ and $\eta_Y \in \mathcal{E}_Y$. It follows that $B(f(X), \eta_X) = Y$ and $B(g(Y), \eta_X) = X$.

1. First we prove that $\operatorname{asdim}(X) = \operatorname{asdim}(Y)$. Actually, this fact is known [14, p.129] and we present a proof for the convenience of the reader. By the symmetry, it suffices to show that $\operatorname{asdim}(X) \leq \operatorname{asdim}(Y)$. This inequality is trivial if $n = \operatorname{asdim}(Y)$ is infinite. So, assume that $n < \infty$. By Propositions 2.2 and 2.3, the inequality $\operatorname{asdim}(X) \leq n$ will be

proved as soon as for each $\varepsilon_X \in \mathcal{E}_X$ we find $\delta_X \in \mathcal{E}_X$ and a coloring $\chi_X : X \to n+1$ such that each χ_X-monochrome ε_X-chain $C \subset X$ has diameter $\mathrm{diam}(C) \subset \delta_X$.

Since the map $f : X \to Y$ is coarse, for the entourage ε_X there is an entourage ε_Y such that $\{(f(x), f(x')) : (x, x') \in \varepsilon_X\} \subset \varepsilon_Y$. Since $\mathrm{asdim}(Y) = n$, for the entourage ε_Y there is an entourage $\delta_Y \in \mathcal{E}_Y$ and a coloring $\chi_Y : Y \to n+1$ such that each χ_Y-monochrome ε_Y-chain $C_Y \subset Y$ has diameter $\mathrm{diam}(C_Y) \subset \delta_Y$.

Since the function $g : Y \to X$ is coarse, for the entourage δ_Y there is an entourage δ'_X such that $\{(g(y), g(y')) : (y, y') \in \delta_Y\} \subset \delta'_Y$. Put $\delta_X = \eta_X \circ \delta'_X \circ \eta_Y$ and consider the coloring $\chi_X = \chi_Y \circ f : X \to n+1$ of X. We claim that each χ_X-monochrome ε_X-chain $C_X \subset X$ has diameter $\mathrm{diam}(C_X) \subset \delta_X$. Then choice of ε_Y guarantees that the set $C_Y = f(C_X)$ is an ε_Y-chain. Being χ_X-monochrome, it has diameter $\mathrm{diam}(C_Y) \subset \delta_Y$. Then the set $C'_X = g(C_Y)$ has diameter $\mathrm{diam}(C'_X) \subset \delta'_X$. Now take any two points $c, c' \in C_X$ and observe that the pairs $(c, g \circ f(c))$ and $(c', g \circ f(c'))$ belong to the entourage η_X. Consequently,

$$(c, c') \in \{(c, g \circ f(c))\} \circ \{(g \circ f(c), g \circ f(c')\} \circ \{(g \circ f(c'), c')\} \subset \eta_X \circ \delta'_X \circ \eta_X = \delta_X$$

which means that the ε_X-chain C_X has diameter $\mathrm{diam}(C_X) \subset \delta_X$. So, $\mathrm{asdim}(X) \leq n$.

2. The second statement of Proposition 1.4, follows Claims 3.1 and 3.4 proved below.

Claim 3.1 *A subset $A \subset X$ and its image $f(A) \subset Y$ have the same asymptotic dimension* $\mathrm{asdim}(A) = \mathrm{asdim}(f(A))$.

Proof This claim follows from Proposition 1.4(1) proved above, since A and $f(A)$ are coarsely equivalent. □

Claim 3.2 *A subset $A \subset X$ is large in X if and only if its image $f(A)$ is large in Y.*

Proof If A is large in X, then $B(A, \varepsilon_X) = X$ for some $\varepsilon_X \in \mathcal{E}_X$. Since f is coarse, there exists $\varepsilon_Y \in \mathcal{E}_Y$ such that for each $(x_0, x_1) \in \varepsilon_X$ we get $(f(x_0), f(x_1)) \in \varepsilon_Y$. It follows that $B(f(A), \varepsilon_Y) \supset f(Y)$ and $B(f(A), \varepsilon_Y \circ \eta_Y) = B(B(f(A), \varepsilon_Y), \eta_Y) \supset B(f(X), \eta_Y) = Y$, which means that $f(A)$ is large.

Now assume conversely that the set $f(A)$ is large in Y. Then $g \circ f(A)$ is large in X. Since $g \circ f(A) \subset B(A, \eta_X)$, we conclude that A in large in X. So, A is large in X if and only if $f(A)$ is large in Y. □

Claim 3.3 *A subset $A \subset X$ is small if and only if for each entourage $\varepsilon_X \in \mathcal{E}_X$ the set $B(A, \varepsilon_x)$ is small.*

Proof The "if" part is trivial. To prove the "only if" part, assume that the set A is small in X. To show that $B(A, \varepsilon_X)$ is small in X, it is necessary to check that for each large subset $L \subset X$ the complement $L \setminus B(A, \varepsilon_X)$ is large in X. Consider the set $L' = (L \setminus B(A, \varepsilon_X)) \cup A$ and observe that $L \subset B(L', \varepsilon_X)$ and hence L' is large in X. Since A is small, the set $L' \setminus A = L \setminus B(A, \varepsilon_X)$ is large in X. □

Claim 3.4 *A subset $A \subset X$ is small in X if and only if its image $f(A)$ is small in Y.*

Proof Assume that A is small in X. To prove that $f(A)$ is small in Y, we need to check that for any large subset $L \subset Y$ the complement $L \setminus f(A)$ is large in Y. Claim 3.2 implies that the set $g(L)$ is large in X. By Claim 3.3, the set $B(A, \eta_X)$ is small in X and hence the complement $g(L) \setminus B(A, \eta_X)$ remains large in X. By Claim 3.2 $f(g(L) \setminus B(A, \eta_X))$

is large in Y. We claim that $f(g(L) \setminus B(A, \eta_X)) \subset B(L \setminus f(A), \eta_Y)$. Indeed, given point $y \in f(g(L) \setminus B(A, \eta_X))$, find a point $x \in g(L) \setminus B(A, \eta_X)$ such that $y = f(x)$ and a point $z \in L$ such that $x = g(z)$. We claim that $z \notin f(A)$. Assuming conversely that $z \in f(A)$, we get $x = g(z) \in g \circ f(A) \subset B(A, \eta_X)$, which contradicts the choice of x. So, $z \in L \setminus f(A)$ and $y = f \circ g(z) \in B(z, \eta_Y) \subset B(L \setminus f(A), \eta_Y)$.

Taking into account that the set $f(g(L) \setminus B(A, \eta_X)) \subset B(L \setminus f(A), \eta_Y)$ is large in Y, we conclude that the set $L \setminus f(A)$ is large in Y and hence $f(A)$ is small in Y.

Now assume that the set $f(X)$ is small in Y. Then the set $g \circ f(A)$ is small in X and so are the sets $B(g \circ f(A), \eta_X) \supset A$. □

4 Proof of Theorem 1.6

Let G be a topological group and $\mathcal{E}$ be its left coarse structure. The inclusion $\mathcal{D}_<(G) \subset \mathcal{S}(G)$ will follow as soon as we prove that each non-small subset $A \subset G$ has asymptotic dimension $\mathrm{asdim}(A) = \mathrm{asdim}(G)$. We divide the proof of this fact into 3 steps.

Claim 4.1 *There is an entourage $\varepsilon_A \in \mathcal{E}$ such that the set $G \backslash B(A, \varepsilon_A)$ is not large in G.*

Proof Since A is not small, there is a large set $L \subset X$ such that the complement $L \backslash A$ is not large. Since L is large in X, there is an entourage $\varepsilon_A \in \mathcal{E}$ such that $B(L, \varepsilon_A) = G$. We claim that the set $G \backslash B(A, \varepsilon_A)$ is not large. Assuming the opposite, we can find an entourage $\delta \in \mathcal{E}$ such that $B(G \backslash B(A, \varepsilon_A), \delta) = G$. Then for each $x \in G$ the ball $B(x, \delta)$ meets $G \setminus B(A, \varepsilon_A)$ at some point y. By the choice of ε_A, the ball $B(y, \varepsilon_A)$ meets the large set L at some point z. It follows from $y \notin B(A, \varepsilon_A)$ that $z \in L \cap B(y, \varepsilon) \subset L \cap (X \backslash A) = L \backslash A$ and hence $x \in B(L \backslash A, \varepsilon_A \circ \delta)$, which means that $L \setminus A$ is large in X. This is a required contradiction. □

Claim 4.2 $\mathrm{asdim}(B(A, \varepsilon_A)) = \mathrm{asdim}(A)$.

Proof Observe that the identity embedding $i : A \to B(A, \varepsilon_A)$ is a coarse equivalence. The coarse inverse $j : B(A, \varepsilon_A) \to A$ to i can be defined by choosing a point $j(x) \in B(x, \varepsilon_A) \cap A$ for each $x \in B(A, \varepsilon_A)$. Now we equality $\mathrm{asdim}(B(A, \varepsilon_A)) = \mathrm{asdim}(A)$ follows from the invariance of the asymptotic dimension under coarse equivalences, see Proposition 1.4. □

Claim 4.3 $\mathrm{asdim}(G) = \mathrm{asdim}(A)$.

Proof The inequality $\mathrm{asdim}(A) \leq \mathrm{asdim}(G)$ is trivial. So, it suffices to check that $\mathrm{asdim}(G) \leq n$ where $n = \mathrm{asdim}(A) = \mathrm{asdim}(B(A, \varepsilon_A))$. If n is infinite, then there is nothing to prove. So, we assume that $n \in \omega$.

For the proof of the inequality $\mathrm{asdim}(G) \leq n$, we shall apply Theorem 2.6. Given any $\varepsilon \in \mathcal{E}$ we should find $\delta \in \mathcal{E}$ such that for each finite ε-connected subset $F \subset G$ there is a coloring $\chi : F \to n+1$ such that each χ-monochrome ε-chain $C \subset F$ has $\mathrm{diam}(C) \subset \delta$. By the definition of the left coarse structure $\mathcal{E}$ we lose no generality assuming that $\varepsilon = \{(x, y) \in G \times G : x \in yK_\varepsilon\}$ for some compact subset $K_\varepsilon = K_\varepsilon^{-1} \subset G$ containing the neutral element 1_G of G. In this case the entourage ε is left invariant in the sense that for each pair $(x, y) \in \varepsilon$ and each $z \in G$ the pair (zx, zy) belongs to ε.

Since $\mathrm{asdim}_{col}(B(A, \varepsilon_A)) = \mathrm{asdim}(B(A, \varepsilon_A)) \leq n$, for the entourage $\varepsilon \in \mathcal{E}$, there are an entourage $\delta \in \mathcal{E}$ and a coloring $\chi_A : B(A, \varepsilon_A) \to n + 1$ such that each χ-monochrome ε-chain $C \subset B(A, \varepsilon_A)$ has $\mathrm{diam}(C) \subset \delta$, see Proposition 2.2. By the definition of the left coarse structure $\mathcal{E}$, we lose no generality assuming that $\delta = \{(x, y) \in G \times G : x \in yK_\delta\}$ for some compact set $K_\delta = K_\delta^{-1} \ni 1_G$ of G, which implies that the entourage δ is left invariant.

Now take any finite ε-connected subset $F \subset G$. Replacing F by $F \cup F^{-1} \cup \{1_G\}$ we can assume that $F = F^{-1} \ni 1_G$. Since the set $G \setminus B(A, \varepsilon_A)$ is not large, there is a point $z \notin (G \setminus B(A, \varepsilon_A))F$. Then zF^{-1} is disjoint with $G \setminus B(A, \varepsilon_A)$ and hence $zF = zF^{-1} \subset B(A, \varepsilon_A)$. So, it is legal to define a coloring $\chi : F \to n+1$ by the formula $\chi(x) = \chi_A(zx)$ for $x \in F$. Taking into account the left invariance of the entourages ε and δ, it is easy to see that each χ-monochrome ε-chain $C \subset F$ has diameter $\mathrm{diam}(C) \subset \delta$. By Propositions 2.2 and 2.3, $\mathrm{asdim}(G) = \mathrm{asdim}_{col}(G) \le n = \mathrm{asdim}(A)$. □

5 Proof of Theorem 1.2

We need to prove that a subset $A \subset \mathbb{R}^n$ is small if and only if it has asymptotic dimension $\mathrm{asdim}(A) < \mathrm{asdim}(\mathbb{R}^n) = n$. The "if" part of this characterization follows from the inclusion $\mathcal{D}_<(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ proved in Theorem 1.6. To prove the "only if" part, we need to recall some (standard) notions of Combinatorial Topology [10,12].

On the Euclidean space $\mathbb{R}^n$ we shall consider the metric generated by the sup-norm $\|x\| = \max_{i \in n} |x(i)|$.

By the *standard n-dimensional simplex* we understand the compact convex subset

$$\Delta = \left\{ (x_0, \dots, x_n) \in [0,1]^{n+1} : \sum_{i=0}^{n} x_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

of the Euclidean space $\mathbb{R}^{n+1}$ endowed with the sup-norm. For each $i \le n$ by $v_i : n+1 \to \{0,1\} \subset \mathbb{R}$ we denote the vertex of Δ defined by $v_i^{-1}(1) = \{i\}$. For each vertex v_i of Δ consider its *star*

$$\mathcal{S}t_\Delta(v_i) = \{x \in \Delta : x(i) > 0\}$$

and its *barycentric star*

$$\mathcal{S}t'_\Delta(v_i) = \left\{ x \in \Delta : x(i) = \max_{j \le n} x(j) \right\} \subset \mathcal{S}t_\Delta(v_i).$$

It is clear that $\bigcup_{i=0}^n \mathcal{S}t'_\Delta(v_i) = \Delta$ while $\bigcap_{i=0}^n \mathcal{S}t'_\Delta(v_i) = \{b_\Delta\}$ is the singleton containing the barycenter

$$b_\Delta = \tfrac{1}{n+1} \sum_{i=0}^{n} v_i$$

of the simplex Δ.

Claim 5.1 $\bigcap_{i=0}^n B(\mathcal{S}t'_\Delta(v_i), \varepsilon) \subset B(b_\Delta, n\varepsilon)$ *for each positive real number* ε.

Proof Given any vector $x \in \bigcap_{i=0}^n B(\mathcal{S}t'_\Delta(v_i), \varepsilon)$, for every $i \le n$ we can find a vector $y \in \mathcal{S}t'_\Delta(v_i)$ with $\|x - y\| < \varepsilon$. Then $|x_i - y_i| \le \|x - y\| < \varepsilon$ and hence $x_i > y_i - \varepsilon = \max_{j \le n} y_j - \varepsilon \ge \frac{1}{n+1} - \varepsilon$. On the other hand,

$$x_i = 1 - \sum_{j \ne i} x_j < 1 - \sum_{j \ne i} \left(\frac{1}{n+1} - \varepsilon \right) = 1 - \frac{n}{n+1} + n\varepsilon = \frac{1}{n+1} + n\varepsilon.$$

So, $\|x - b_\Delta\| < n\varepsilon$. □

Now we are going to generalize Claim 5.1 to arbitrary simplexes. By an *n-dimensional simplex* in $\mathbb{R}^n$ we understand the convex hull $\sigma = \mathrm{conv}(\sigma^{(0)})$ of an affinely independent subset $\sigma^{(0)} \subset \mathbb{R}^n$ of cardinality $|\sigma^{(0)}| = n+1$. Each point $v \in \sigma^{(0)}$ is called a *vertex* of the simplex σ. The arithmetic mean

$$b_\sigma = \frac{1}{n+1}\sum_{v\in\sigma^{(0)}} v$$

of the vertices is called the *barycenter* of the simplex σ. By $\partial\sigma$ we denote the boundary of the simplex σ in $\mathbb{R}^n$. Observe that the homothetic copy $\frac{1}{2}b_\sigma + \frac{1}{2}\sigma = \{\frac{1}{2}b_\sigma + \frac{1}{2}x : x \in \sigma\}$ of σ is contained in the interior $\sigma \setminus \partial\sigma$ of σ. For each vertex $v \in \sigma^{(0)}$ let

$$\mathcal{S}t_\sigma(v) = \sigma \setminus \mathrm{conv}(\sigma^{(0)}\setminus\{v\})$$

be the *star* of v in σ.

In fact, n-dimensional simplexes can be alternatively defined as images of the standard n-dimensional simplex Δ under injective affine maps $f : \Delta \to \mathbb{R}^n$.

A map $f : \Delta \to \mathbb{R}^n$ is called *affine* if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ for any points $x, y \in \Delta$ and a real number $t \in [0, 1]$. It is well-known that each affine function $f : \Delta \to \mathbb{R}^n$ is uniquely defined by its restriction $f|\Delta^{(0)}$ to the set $\Delta^{(0)} = \{v_i\}_{i\le n}$ of vertices of Δ.

A map $f : \Delta \to \mathbb{R}^n$ will be called *b_Δ-affine* if for every $i \le n$ the restriction $f|\mathrm{conv}(\{b_\Delta\}\cup\Delta^{(0)}\setminus\{v_i\})$ is affine. A b_Δ-affine function $f : \Delta \to \mathbb{R}^n$ is uniquely determined by its restriction $f|\Delta^{(0)} \cup \{b_\Delta\}$.

A function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called *Lipschitz* if it its *Lipschitz constant*

$$\mathrm{Lip}(f) = \sup\left\{\frac{d_Y(f(x), f(x'))}{d_X(x, x')} : x, x' \in X,\quad x \neq x'\right\}$$

is finite. A bijective function $f : X \to Y$ is *bi-Lipschitz* if f and f^{-1} are Lipschitz.

Claim 5.2 *For any n-dimensional simplex σ in $\mathbb{R}^n$ there is a real constant L such that each b_Δ-affine function $f : \Delta \to \sigma$ with $f(\Delta^{(0)}) = \sigma^{(0)}$ and $f(b) \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$ is bijective, bi-Lipschitz and has* $\mathrm{Lip}(f)\cdot\mathrm{Lip}(f^{-1}) \le L$.

This claim can be easily derived from the fact that each b_Δ-affine function $f : \Delta \to \sigma$ with $f(\Delta^{(0)}) = \sigma^{(0)}$ is Lipschitz and its Lipschitz constant $\mathrm{Lip}(f)$ depends continuously on $f(b_\Delta)$.

Given an n-dimensional simplex $\sigma \subset \mathbb{R}^n$ and a point $b' \in \sigma \setminus \partial\sigma$ in its interior, fix a b_Δ-affine function $f : \Delta \to \sigma$ such that $f(\Delta^{(0)}) = \sigma^{(0)}$ and $f(b_\Delta) = b'$. For each vertex $v \in \sigma^{(0)}$ consider its *b'-barycentric star*

$$\mathcal{S}t'_{\sigma,b'}(v) = f\big(\mathcal{S}t'_\Delta(f^{-1}(v))\big) \subset \mathcal{S}t_\sigma(v).$$

It is easy to see that the set $\mathcal{S}t_{\sigma,b'}(v)$ does not depend on the choice of the b_Δ-affine function f.

Claim 5.3 *For any n-dimensional simplex σ in $\mathbb{R}^n$ there is a real constant L such that for each point $b' \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$ and each $\varepsilon > 0$ we get $\sigma \cap \bigcap_{v\in\sigma^{(0)}} B(\mathcal{S}t'_{\sigma,b'}(v), \varepsilon) \subset B(b', L\varepsilon)$.*

Proof By Claim 5.2, there is a real constant C such that each bijective b_Δ-affine function $f : \Delta \to \sigma$ with $f(\Delta^{(0)}) = \sigma^{(0)}$ and $f(b_\Delta) \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$ has $\mathrm{Lip}(f)\cdot\mathrm{Lip}(f^{-1}) \le C$. Put $L = nC$. Given any point $b' \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$, choose a bijective b_Δ-affine function

$f : \Delta \to \sigma$ such that $f(\Delta^{(0)}) = \sigma^{(0)}$ and $f(b_\sigma) = b'$. The choice of C guarantees that $\mathrm{Lip}(f) \cdot \mathrm{Lip}(f^{-1}) \le C$. Now observe that

$$\begin{aligned}
\sigma \cap \bigcap_{v\in\sigma^{(0)}} B(\mathcal{S}t'_{\sigma,b'}(v), \varepsilon) &= \bigcap_{v\in\sigma^{(0)}} f \circ f^{-1}\Big(B(\mathcal{S}t'_{\sigma,b'}(v), \varepsilon)\Big) \\
&\subset \bigcap_{v\in\sigma^{(0)}} f\Big(B(f^{-1}(\mathcal{S}t'_{\sigma,b'}(v)), \mathrm{Lip}(f^{-1})\varepsilon)\Big) \\
&= \bigcap_{v\in\sigma^{(0)}} f\left(B(\mathcal{S}t'_\Delta(f^{-1}(v)), \mathrm{Lip}(f^{-1})\varepsilon)\right) \\
&= f\left(\bigcap_{v\in\Delta^{(0)}} B\left(\mathcal{S}t'_\Delta(v), \mathrm{Lip}(f^{-1})\varepsilon\right)\right) \\
&\subset f\left(B(b_\Delta, n\mathrm{Lip}(f^{-1})\varepsilon)\right) \subset B\left(f(b_\Delta), \mathrm{Lip}(f)\mathrm{Lip}(f^{-1})n\varepsilon\right) \\
&= B(b', Cn\varepsilon) = B(b', L\varepsilon).
\end{aligned}$$

□

Now consider the binary unit cube $K = \{0, 1\}^n \subset \mathbb{R}^n$ endowed with the partial ordering $\le$ defined by $x \le y$ iff $x(i) \le y(i)$ for all $i < n$. Given two vectors $x, y \in \{0, 1\}^n$, we write $x < y$ if $x \le y$ and $x \ne y$.

For every increasing chain $v_0 < v_1 < \ldots < v_n$ of points of the binary cube $K = \{0, 1\}^n$, consider the simplex $\mathrm{conv}\{v_0, \ldots, v_n\}$ and let $\mathcal{T}_K$ be the (finite) set of these simplexes. Next, consider the family $\mathcal{T} = \{\sigma + z : \sigma \in \mathcal{T}_K,\ z \in \mathbb{Z}^n\}$ of translations of the simplexes from the family $\mathcal{T}_K$, and observe that $\bigcup \mathcal{T} = \mathbb{R}^n$. For each point $v \in \mathbb{Z}^n$ let

$$\mathcal{S}t_{\mathcal{T}}(v) = \bigcup\{\mathcal{S}t_\sigma(v) : v \in \sigma \in \mathcal{T}\}$$

be the $\mathcal{T}$-*star* of v in the triangulation $\mathcal{T}$ of the space $\mathbb{R}^n$.

Now we are able to prove the "only if" part of Theorem 1.2. Assume that a subset $A \subset \mathbb{R}^n$ is small. Then there is a function $\varphi : (0, \infty) \to (0, \infty)$ such that for each $\delta \in (0, \infty)$ and a point $x \in \mathbb{R}^n$ there is a point $y \in \mathbb{R}^n$ with $B(y, \delta) \subset B(x, \varphi(\delta)) \setminus A$. The inequality $\mathrm{asdim}(A) < n$ will follow as soon as given any $\delta < \infty$ we construct a cover $\mathcal{U}$ of A with finite $\mathrm{mesh}(\mathcal{U}) = \sup_{U\in\mathcal{U}} \mathrm{diam}(U)$ such that each δ-ball $B(a, \delta)$, $a \in A$, meets at most n elements of the cover $\mathcal{U}$.

By Claim 5.3, there is a constant L such that for each simplex $\sigma \in \mathcal{T}$, each point $b' \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$ and each $\varepsilon > 0$ we get $\sigma \cap \bigcap_{v\in\sigma^{(0)}} B(\mathcal{S}t'_{\sigma,b'}(v), \varepsilon) \subset B(b', L\varepsilon)$.

Given any $\delta < \infty$, choose $\varepsilon > 0$ so small that for any simplex $\sigma \in \mathcal{T}$ the following conditions hold:

(1) $B\big(b_\sigma, \varepsilon\varphi(L\delta)\big) \subset \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$;
(2) for any $b' \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$ and any vertex $v \in \sigma^{(0)}$ the $2\varepsilon\delta$-neighborhood $B(\mathcal{S}t'_{\sigma,b'}(v), 2\varepsilon\delta)$ lies in the $\mathcal{T}$-star $\mathcal{S}t_{\mathcal{T}}(v)$ of v.

Now consider the closed cover

$$\widetilde{\mathcal{T}} = \left\{\tfrac{1}{\varepsilon}\sigma : \sigma \in \mathcal{T}\right\}$$

of the space $\mathbb{R}^n$ and observe that for each simplex $\sigma \in \widetilde{\mathcal{T}}$ we get

(1_ε) $B\big(b_\sigma, \varphi(L\delta)\big) \subset \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$;

(2_ε) for any $b' \in \frac{1}{2}b_\sigma + \frac{1}{2}\sigma$ and any vertex $v \in \sigma^{(0)}$ the 2δ-neighborhood $B(\mathcal{S}t'_{\sigma,b'}(v), 2\delta)$ lies in the $\widetilde{\mathcal{T}}$-star $\mathcal{S}t_{\widetilde{\mathcal{T}}}(v)$ of v.

By the choice of the function φ, for each simplex $\sigma \in \widetilde{\mathcal{T}}$, there is a point $b'_\sigma \in \mathbb{R}^n$ such that $B(b'_\sigma, L\delta) \subset B(b_\sigma, \varphi(L\delta)) \setminus A$. The condition (1_ε) guarantees that

$$b'_\sigma \in B(b_\sigma, \varphi(L\delta)) \subset \tfrac{1}{2}b_\sigma + \tfrac{1}{2}\sigma.$$

For every point $v \in \frac{1}{\varepsilon}\mathbb{Z}^n$ consider the set

$$\mathcal{S}t'(v) = \bigcup \left\{\mathcal{S}t_{\sigma,b'_\sigma}(v) : \sigma \in \widetilde{\mathcal{T}},\ v \in \sigma^{(0)}\right\} \subset \mathcal{S}t_{\widetilde{\mathcal{T}}}(v)$$

and observe that $\mathcal{U} = \{\mathcal{S}t'(v) : v \in \frac{1}{\varepsilon}\mathbb{Z}^n\}$ is a cover of the Euclidean space $\mathbb{R}^n$. It follows that

$$\mathrm{mesh}(\mathcal{U}) = \sup_{v \in \varepsilon^{-1}\mathbb{Z}^n} \mathrm{diam}(\mathcal{S}t'(v)) \le 2 \sup_{v \in \varepsilon^{-1}\mathbb{Z}^n} \mathrm{diam}(\sigma) \le 2\varepsilon^{-1}\mathrm{diam}([0,1]^n) < \infty.$$

It remains to check that each ball $B(a, \delta)$, $a \in A$, meets at most n sets $U \in \mathcal{U}$.

Assume conversely that there are a point $a \in A$ and a set $V \subset \varepsilon^{-1}\mathbb{Z}^n$ of cardinality $|V| = n+1$ such that $B(a,\delta) \cap \mathcal{S}t'(v) \ne \emptyset$ for each $v \in V$. Then $a \in \bigcap_{v\in V} B(\mathcal{S}t'(v), \delta)$. It follows from $a \in \bigcap_{v\in V} B(\mathcal{S}t'(v), \delta) \subset \bigcap_{v \in V} \mathcal{S}t_{\widetilde{\mathcal{T}}}(v)$ that V coincides with the set $\sigma^{(0)}$ of vertices of some simplex $\sigma \in \widetilde{\mathcal{T}}$ and a lies in the interior of the simplex σ.

Next, we show that $a \in B(\mathcal{S}t'_{\sigma,b'_\sigma}(v), \delta)$ for each $v \in V$. In the opposite case, $a \in B(\mathcal{S}t'_{\tau,b'_\tau}(v), \delta) \subset B(\tau, \delta)$ for some simplex $\tau \in \widetilde{\mathcal{T}} \setminus \{\sigma\}$ such that $v \in \tau^{(0)} \setminus \sigma^{(0)}$. Choose a vertex $u \in \sigma^{(0)} \setminus \tau^{(0)}$ and observe that the condition (2_ε) implies that $a \in B(\mathcal{S}t'(u), \delta) \cap B(\tau, \delta) = \emptyset$, which is a contradiction.

Finally, the choice of L and b'_σ yields the desired contradiction

$$a \in \sigma \cap \bigcap_{v \in \sigma^{(0)}} B(\mathcal{S}t'_{\sigma,b'_\sigma}(v), \delta) \subset B(b'_\sigma, L\delta) \subset \mathbb{R}^n \setminus A,$$

completing the proof of the theorem.

6 Proof of Theorem 1.7

Given an Abelian locally compact topological group G endowed with its left coarse structure, we need to prove the equivalence of the following statements:

(1) $\mathcal{S}(G) = \mathcal{D}_<(G)$;
(2) G is compactly generated;
(3) G is coarsely equivalent to a Euclidean space $\mathbb{R}^n$ for some $n \in \omega$.

We shall prove the implications (1)⇒(2)⇒(3)⇒(1). The implication (3)⇒(1) follows from Corollary 1.5.

To prove that (2)⇒(3), assume that the group G is compactly generated. By Theorem 24 [11, p.85], G is topologically isomorphic to the direct sum $\mathbb{R}^n \times \mathbb{Z}^m \times K$ for some $n, m \in \omega$ and a compact subgroup $K \subset G$. Since the projection $\mathbb{R}^n \times \mathbb{Z}^m \times K \to \mathbb{R}^n \times \mathbb{Z}^m$ and the embedding $\mathbb{Z}^n \times \mathbb{Z}^m \to \mathbb{R}^n \times \mathbb{Z}^m$ are coarse equivalences, we conclude that G is coarsely equivalent to $\mathbb{Z}^{n+m}$ and to $\mathbb{R}^{n+m}$.

To prove that (1)⇒(2), assume that $\mathcal{S}(G) = \mathcal{D}_<(G)$. First we prove that G has finite asymptotic dimension. By the Principal Structure Theorem 25 [11, p.26], G contains an open subgroup G_0 that is topologically isomorphic to $\mathbb{R}^n \times K$ for some $n \in \omega$ and some compact subgroup K of G_0. The subgroup G_0 has asymptotic dimension $\mathrm{asdim}(G_0) = \mathrm{asdim}(\mathbb{R}^n) = n < \infty$. If $\mathrm{asdim}(G) = \infty$, then the quotient group G/G_0 has infinite asymptotic dimension and hence has infinite free rank. Then the group G/G_0 contains a subgroup isomorphic to the free abelian group $\oplus^\omega\mathbb{Z}$ with countably many generators. It follows that G also contains a discrete subgroup H isomorphic to $\oplus^\omega\mathbb{Z}$. Replacing H by a smaller subgroup, if necessary, we can assume that H has infinite index in G and hence is small in G. Since $\mathrm{asdim}(H) = \infty = \mathrm{asdim}(G)$, we conclude that $\mathcal{S}(G) \neq \mathcal{D}_<(G)$, which is a desired contradiction showing that $\mathrm{asdim}(G) < \infty$.

By Theorem 2.8, there is a compactly generated subgroup $H \subset G$ with $\mathrm{asdim}(H) = \mathrm{asdim}(G)$. Since $H \notin \mathcal{D}_<(G) = \mathcal{S}(G)$, the subset H is not small in G. Repeating the proof of Claim 4.1, we can show that the set $G \setminus B(H, \varepsilon)$ is not large for some entourage $\varepsilon \in \mathcal{E}$. By the definition of the left coarse structure $\mathcal{E}$, there is a compact subset $K \subset G$ such that $B(H, \varepsilon) \subset HK$. We claim that $K^{-1}HK = G$. Assuming the opposite, we can find a point $x \in G \setminus K^{-1}HK$ and consider the finite set $F = \{x, x^{-1}, xx^{-1}\} = F^{-1}$. Since the set $G \setminus HK$ is not large, there is a point $z \in (G \setminus HK)F$. For this point z we get $zF \cap (G \setminus HK) = \emptyset$ and hence $z \in zF \subset HK$. Then $x \in z^{-1}zF \subset z^{-1}HK \subset K^{-1}HHK = K^{-1}HK$, which is a contradiction. Now the compact generacy of the subgroup H implies the compact generacy of the group $G = K^{-1}HK$.

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