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Reanalysis of the $Y(3940)$, $Y(4140)$, $Z_c(4020)$, $Z_c(4025)$, and $Z_b(10650)$ as molecular states with QCD sum rules

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Abstract In this article, we calculate the contributions of the vacuum condensates up to dimension 10 in the operator product expansion, and study the $J^{PC}=0^{++}$, 1^{+-}, 2^{++} $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ molecular states with the QCD sum rules. In the calculations, we use the formula $\mu=\sqrt{M^2_{X/Y/Z}-(2\mathbb{M}_Q)^2}$ to determine the energy scales of the QCD spectral densities. The numerical results favor assigning the $Z_c(4020)$ and $Z_c(4025)$ to the $J^{PC}=0^{++}$, 1^{+-} or 2^{++} $D^*\bar{D}^*$ molecular states, the $Y(4140)$ to the $J^{PC}=0^{++}$ $D_s^*D_s^*$ molecular state, the $Z_b(10650)$ to the $J^{PC}=1^{+-}$ $B^*\bar{B}^*$ molecular state, and they disfavor assigning the $Y(3940)$ to the ($J^{PC}=0^{++}$) molecular state. The present predictions can be confronted with the experimental data in the future.

1 Introduction

In 2004, the Belle collaboration observed the near-threshold enhancement $Y(3940)$ in the $\omega J/\psi$ mass spectrum in the exclusive $B\to K\omega J/\psi$ decays [1]. In 2007, the BaBar collaboration confirmed the $Y(3940)$ in the exclusive $B\to K\omega J/\psi$ decays [2]. In 2010, the Belle collaboration confirmed the $Y(3940)$ in the process $\gamma\gamma\to\omega J/\psi$ [3]. Now the $X(3915)$ ($Y(3940)$) is listed in the Review of Particle Physics as the $\chi_{c0}(2P)$ state with the quantum numbers $J^{PC}=0^{++}$ [4].

In 2009, the CDF collaboration observed the narrow structure $Y(4140)$ near the $J/\psi\phi$ threshold in the exclusive $B^+\to J/\psi\phi K^+$ decays [5]. Latter, the Belle collaboration searched for the $Y(4140)$ in the process $\gamma\gamma\to\phi J/\psi$ and observed no evidence [6]. In 2012, the LHCb collaboration searched for the $Y(4140)$ state in $B^+\to J/\psi\phi K^+$ decays, and observed no evidence [7]. In 2013, the CMS collaboration observed a peaking structure consistent with the $Y(4140)$ in the $J/\psi\phi$ mass spectrum in the $B^\pm\to J/\psi\phi K^\pm$ decays, and fitted the structure to a S-wave relativistic Breit–Wigner line-shape with the statistical significance exceeding 5σ [8]. Also in 2013, the D0 collaboration observed the $Y(4140)$ in the $B^+\to J/\psi\phi K^+$ decays with the statistical significance of 3.1σ [9]. However, there is no suitable position in the $\bar{c}c$ spectroscopy for the $Y(4140)$.

The $Y(3940)$ and $Y(4140)$ appear near the $D^*\bar{D}^*$ and $D_s^*\bar{D}_s^*$ thresholds, respectively, and have analogous decays,

$$Y(3940)\to J/\psi\,\varphi,$$
$$Y(4140)\to J/\psi\,\phi. \quad (1)$$

It is natural to relate the $Y(3940)$ and $Y(4140)$ with the $D^*\bar{D}^*$ and $D_s^*\bar{D}_s^*$ molecular states, respectively [10–18]. Other assignments, such as the hybrid charmonium states [15,16,19] and tetraquark states [20] also have been suggested.

In 2011, the Belle collaboration observed the $Z_b(10610)$ and $Z_b(10650)$ in the $\pi^\pm\Upsilon(1,2,3\mathrm{S})$ and $\pi^\pm h_b(1,2\mathrm{P})$ invariant mass distributions in the $\Upsilon(5\mathrm{S})\to\pi^+\pi^-\Upsilon(1,2,3\mathrm{S})$, $\pi^+\pi^-h_b(1,2\mathrm{P})$ decays [21]. The quantum numbers $I^G(J^P)=1^+(1^+)$ are favored [21]. Later, the Belle collaboration updated the measured parameters $M_{Z_b(10610)}=(10607.2\pm2.0)$ MeV, $M_{Z_b(10650)}=(10652.2\pm1.5)$ MeV, $\Gamma_{Z_b(10610)}=(18.4\pm2.4)$ MeV, and $\Gamma_{Z_b(10650)}=(11.5\pm2.2)$ MeV [22]. In 2013, the Belle collaboration observed the $Z_b^0(10610)$ in a Dalitz analysis of the decays to $\Upsilon(2,3\mathrm{S})\pi^0$ in the $\Upsilon(5\mathrm{S})\to\Upsilon(1,2,3\mathrm{S})\pi^0\pi^0$ decays [23]. The $Z_b(10610)$ and $Z_b(10650)$ appear near the $B\bar{B}^*$ and $B^*\bar{B}^*$ thresholds, respectively. It is natural to relate the $Z_b(10610)$ and $Z_b(10650)$ with the $B\bar{B}^*$ and $B^*\bar{B}^*$ molecular states, respectively [24–36]. Other assignments, such as the tetraquark states [37–39], threshold cusps [40], the re-scattering effects [41,42], etc. are also suggested.

In 2013, the BESIII collaboration observed the $Z_c^\pm(4025)$ near the $(D^*\bar{D}^*)^\pm$ threshold in the $\pi^\mp$ recoil mass spectrum in the process $e^+e^-\to(D^*\bar{D}^*)^\pm\pi^\mp$ [43]. Furthermore, the

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BESIII collaboration observed the $Z_c(4020)$ in the $\pi^{\pm}h_c$ mass spectrum in the process $e^+e^- \to \pi^+\pi^- h_c$ [44]. The $Z_c(4020)$ and $Z_c(4025)$ appear near the $D^*\bar{D}^*$ threshold. It is natural to relate them with the $D^*\bar{D}^*$ molecular states [45–50]. Other assignments, such as the re-scattering effects [51,52], tetraquark states [53–55], etc. are also suggested.

The $Z_c(4020)$, $Z_c(4025)$, $Z_b(10610)$, $Z_b(10650)$ appear near the $D^*\bar{D}^*$, $D^*\bar{D}^*$, $B\bar{B}^*$, $B^*\bar{B}^*$ thresholds, respectively, and have analogous decays

$$\begin{aligned}
&Z_c^{\pm}(4020) \to \pi^{\pm} h_c, \\
&Z_c^{\pm}(4025) \to (D^*\bar{D}^*)^{\pm}, \\
&Z_b^{\pm}(10610) \to \pi^{\pm}\Upsilon(1,2,3\mathrm{S}),\ \pi^{\pm}h_b(1,2\mathrm{P}), \\
&Z_b^{\pm}(10650) \to \pi^{\pm}\Upsilon(1,2,3\mathrm{S}),\ \pi^{\pm}h_b(1,2\mathrm{P}). \qquad (2)
\end{aligned}$$

The S-wave $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ systems have the quantum numbers $J^{PC} = 0^{++}$, 1^{+-}, 2^{++}, the S-wave $\pi^{\pm}h_Q$ systems have the quantum numbers $J^{PC} = 1^{--}$, the S-wave $\pi^{\pm}\Upsilon$ systems have the quantum numbers $J^{PC} = 1^{+-}$. It is also possible for the P-wave $\pi^{\pm}h_Q$ systems to have the quantum numbers $J^{PC} = 0^{++}, 1^{+-}, 2^{++}$.

In this article, we take the $Y(3940)$, $Z_c(4020)$, $Z_c(4025)$ as the $D^*\bar{D}^*$ molecular states, the $Y(4140)$ as the $D_s^*\bar{D}_s^*$ molecular state, the $Z_b(10610)$ as the $B\bar{B}^*$ molecular state, the $Z_b(10650)$ as the $B^*\bar{B}^*$ molecular state, we study the $J^{PC} = 0^{++}, 1^{+-}, 2^{++}$ molecular states consisting of $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ with the QCD sum rules, and we make tentative assignments of the $Y(3940)$, $Y(4140)$, $Z_c(4020)$, $Z_c(4025)$, and $Z_b(10650)$ in the scenario of molecular states.

In Refs. [15,16], we study the scalar $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ molecular states with the QCD sum rules by carrying out the operator product expansion to the vacuum condensates up to dimension 10 and setting the energy scale to be $\mu = 1$ GeV. The predicted masses disfavor assigning the $Y(4140)$ to the scalar $D_s^*\bar{D}_s^*$ molecular state. In Refs. [17,18], Albuquerque et al. (Zhang and Huang) study the scalar $D_s^*\bar{D}_s^*$ molecular state with the QCD sum rules by carrying out the operator product expansion to the vacuum condensates up to dimension 8 (6), and their predictions favor assigning the $Y(4140)$ to the $J^P = 0^+$ molecular state, but they do not show or do not specify the energy scales of the QCD spectral densities. In Refs. [38,48], Cui et al. study the axial-vector $B^*\bar{B}^*$ ($D^*\bar{D}^*$) molecular state with the QCD sum rules by carrying out the operator product expansion to the vacuum condensates up to dimension 6, and their predictions favor assigning the $Z_b(10650)$ ($Z_c(4025)$) to the axial-vector $B^*\bar{B}^*$ ($D^*\bar{D}^*$) molecular state, but they do not show or do not specify the energy scales of the QCD spectral densities. Furthermore, in Refs. [17,18,38,48], some higher dimension vacuum condensates involving the gluon condensate, mixed condensate and four-quark condensate are neglected, which impairs the predictive ability, as the higher dimension vacuum condensates play an important role in determining the Borel windows.

In this article, we study the $J^{PC} = 0^{++}, 1^{+-}, 2^{++}$ molecular states consist of $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ with the QCD sum rules according to the routine in our previous works [36,39,54–56].

In Refs. [39,54–56], we focus on the scenario of tetraquark states, calculate the vacuum condensates up to dimension 10 in the operator product expansion, study the diquark–antidiquark-type scalar, vector, axial-vector, tensor hidden-charmed tetraquark states, and axial-vector hidden-bottom tetraquark states systematically with the QCD sum rules, and make reasonable assignments of the $X(3872)$, $Z_c(3900)$, $Z_c(3885)$, $Z_c(4020)$, $Z_c(4025)$, $Z(4050)$, $Z(4250)$, $Y(4360)$, $Y(4630)$, $Y(4660)$, $Z_b(10610)$, and $Z_b(10650)$. In Ref. [36], we focus on the scenario of molecular states, calculate the vacuum condensates up to dimension 10 in the operator product expansion, study the axial-vector hadronic molecular states with the QCD sum rules, and make tentative assignments of the $X(3872)$, $Z_c(3900)$, $Z_b(10610)$. The interested reader can consult Ref. [57–61] for more articles on the exotic X, Y, and Z particles. A hadron cannot be identified unambiguously by the mass alone. It is interesting to explore possible assignments in the scenario of molecular states.

In Refs. [36,39,54–56], we explore the energy-scale dependence of the hidden-charmed (bottom) tetraquark states and molecular states in detail for the first time, and we suggest the formula

$$\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_Q)^2}, \qquad (3)$$

with the effective masses $\mathbb{M}_Q$ to determine the energy scales of the QCD spectral densities in the QCD sum rules, which works very well.

In this article, we calculate the contributions of the vacuum condensates up to dimension 10 in a consistent way, study the $J^{PC} = 0^{++}$, 1^{+-}, 2^{++} molecular states consist of $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ in a systematic way, and make tentative assignments of the $Y(3940)$, $Y(4140)$, $Z_c(4020)$, $Z_c(4025)$, and $Z_b(10650)$ based on the QCD sum rules.

The article is arranged as follows: we derive the QCD sum rules for the masses and pole residues of the $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ molecular states in Sect. 2; in Sect. 3, we present the numerical results and discussions; Sect. 4 is reserved for our conclusion.

2 QCD sum rules for the $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ molecular states

In the following, we write down the two-point correlation functions $\Pi_{\mu\nu\alpha\beta}(p)$ and $\Pi(p)$ in the QCD sum rules,

$$\Pi_{\mu\nu\alpha\beta}(p) = i\int d^4x e^{ip\cdot x}\left\langle 0|T\left\{\eta_{\mu\nu}(x)\eta^{\dagger}_{\alpha\beta}(0)\right\}|0\right\rangle, \quad (4)$$

$$\Pi(p) = i\int d^4x e^{ip\cdot x}\left\langle 0|T\left\{\eta(x)\eta^{\dagger}(0)\right\}|0\right\rangle, \quad (5)$$

$$\eta^{\pm}_{\bar{u}d;\mu\nu}(x) = \frac{\bar{u}(x)\gamma_\mu Q(x)\bar{Q}(x)\gamma_\nu d(x) \pm \bar{u}(x)\gamma_\nu Q(x)\bar{Q}(x)\gamma_\mu d(x)}{\sqrt{2}},$$

$$\eta^{\pm}_{\bar{s}s;\mu\nu}(x) = \frac{\bar{s}(x)\gamma_\mu Q(x)\bar{Q}(x)\gamma_\nu s(x) \pm \bar{s}(x)\gamma_\nu Q(x)\bar{Q}(x)\gamma_\mu s(x)}{\sqrt{2}},$$

$$\eta_{\bar{u}d}(x) = \bar{u}(x)\gamma_\mu Q(x)\bar{Q}(x)\gamma^\mu d(x),$$
$$\eta_{\bar{s}s}(x) = \bar{s}(x)\gamma_\mu Q(x)\bar{Q}(x)\gamma^\mu s(x), \quad (6)$$

where $\eta_{\mu\nu}(x) = \eta^{\pm}_{\bar{u}d;\mu\nu}(x)$, $\eta^{\pm}_{\bar{s}s;\mu\nu}(x)$, $\eta(x) = \eta^{\pm}_{\bar{u}d}(x)$, $\eta^{\pm}_{\bar{s}s}(x)$, $Q = c, b$. Under the charge conjugation transformation $\widehat{C}$, the currents $\eta^{\pm}_{\mu\nu}(x)$ and $\eta(x)$ have the following properties:

$$\widehat{C}\,\eta^{\pm}_{\mu\nu}(x)\,\widehat{C}^{-1} = \pm\,\eta^{\pm}_{\mu\nu}(x)\,|_{u\leftrightarrow d},$$
$$\widehat{C}\,\eta(x)\,\widehat{C}^{-1} = \eta(x)\,|_{u\leftrightarrow d}, \quad (7)$$

thereafter we will smear the subscripts $\bar{u}d$, $\bar{s}s$, and superscripts $\pm$ for simplicity. On the other hand, the currents $\eta^{\pm}_{\mu\nu}(x)$ and $\eta(x)$ are of the type $V_\mu \otimes V_\nu$, where the V_μ denotes the two-quark vector currents interpolating the conventional vector heavy mesons, so they have positive parity. The currents $\eta^{+}_{\mu\nu}(x)$ and $\eta(x)$ have both positive charge conjugation and positive parity, therefore couple potentially to the $J^{PC} = 2^{++}$ or 0^{++} states, while the currents $\eta^{-}_{\mu\nu}(x)$ have negative charge conjugation but positive parity, therefore couple potentially to the $J^{PC} = 1^{+-}$ states. We construct the color singlet–singlet-type currents $\eta_{\mu\nu}(x)$ and $\eta(x)$ to study the $D^*\bar{D}^*$, $D^*_s\bar{D}^*_s$, $B^*\bar{B}^*$, $B^*_s\bar{B}^*_s$ molecular states, and we assume that the operators $\eta_{\mu\nu}(x)$ and $\eta(x)$ couple potentially to the bound states, not to the scattering states. We can also construct the color octet–octet-type currents $\eta^{8}_{\mu\nu}(x)$ and $\eta^{8}(x)$, which have the same quantum numbers J^{PC} as their color singlet–singlet partners, to study the $D^*\bar{D}^*$, $D^*_s\bar{D}^*_s$, $B^*\bar{B}^*$, $B^*_s\bar{B}^*_s$ molecular states,

$$\eta^{8\pm}_{\bar{u}d;\mu\nu}(x) = \frac{\bar{u}(x)\gamma_\mu\lambda^a Q(x)\bar{Q}(x)\gamma_\nu\lambda^a d(x) \pm \bar{u}(x)\gamma_\nu\lambda^a Q(x)\bar{Q}(x)\gamma_\mu\lambda^a d(x)}{\sqrt{2}},$$

$$\eta^{8\pm}_{\bar{s}s;\mu\nu}(x) = \frac{\bar{s}(x)\gamma_\mu\lambda^a Q(x)\bar{Q}(x)\gamma_\nu\lambda^a s(x) \pm \bar{s}(x)\gamma_\nu\lambda^a Q(x)\bar{Q}(x)\gamma_\mu\lambda^a s(x)}{\sqrt{2}},$$

$$\eta^{8}_{\bar{u}d}(x) = \bar{u}(x)\gamma_\mu\lambda^a Q(x)\bar{Q}(x)\gamma^\mu\lambda^a d(x),$$
$$\eta^{8}_{\bar{s}s}(x) = \bar{s}(x)\gamma_\mu\lambda^a Q(x)\bar{Q}(x)\gamma^\mu\lambda^a s(x), \quad (8)$$

where the λ^a are the Gell–Mann matrices. In Ref. [36], we observe that the color octet–octet-type molecular states have larger masses than that of the corresponding color singlet–singlet-type molecular states. So in this article, we prefer the color singlet–singlet-type currents, which couple potentially to the color singlet–singlet-type molecular states have smaller masses. In Refs. [38,48], Cui, Liu and Huang take the currents $j_\mu(x)$,

$$j_\mu(x) = \epsilon_{\mu\nu\alpha\beta}\,\bar{u}(x)\gamma^\nu Q(x)\,iD^\alpha\,\bar{Q}(x)\gamma^\beta d(x), \quad (9)$$

where $D^\alpha = \partial^\alpha - ig_s G^\alpha(x)$, to study the $Z_b(10650)$ and $Z_c(4025)$ as the $B^*\bar{B}^*$ and $D^*\bar{D}^*$ molecular states, respectively, with $J^P = 1^+$. In Ref. [49], Chen et al. take the current $J_\mu(x)$,

$$J_\mu(x) = \bar{q}(x)\gamma^\alpha c(x)\bar{c}(x)\sigma_{\alpha\mu}\gamma_5 q(x) - \bar{q}(x)\sigma_{\alpha\mu}\gamma_5 c(x)\bar{c}(x)\gamma^\alpha q(x), \quad (10)$$

to study the $Z_c(4025)$ as the $D^*\bar{D}^*$ molecular state with $J^{PC} = 1^{+-}$. In this article, we use the simple $V_\mu \otimes V_\nu$-type currents to study the $J^{PC} = 0^{++}$, 1^{+-}, 2^{++} molecular states in a systematic way.

At the hadronic side, we can insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators $\eta_{\mu\nu}(x)$ and $\eta(x)$ into the correlation functions $\Pi_{\mu\nu\alpha\beta}(p)$ and $\Pi(p)$ to obtain the hadronic representation [62,63]. After isolating the ground state contributions of the scalar, axial-vector and tensor molecular states, we get the following results:

$$\begin{aligned}\Pi^{J=2}_{\mu\nu\alpha\beta}(p) &= \Pi_{J=2}(p)\left(\frac{\widetilde{g}_{\mu\alpha}\widetilde{g}_{\nu\beta}+\widetilde{g}_{\mu\beta}\widetilde{g}_{\nu\alpha}}{2} - \frac{\widetilde{g}_{\mu\nu}\widetilde{g}_{\alpha\beta}}{3}\right) \\ &\quad +\Pi_s(p)\,g_{\mu\nu}g_{\alpha\beta}, \\ &= \frac{\lambda^2_{Y/Z}}{M^2_{Y/Z}-p^2}\left(\frac{\widetilde{g}_{\mu\alpha}\widetilde{g}_{\nu\beta}+\widetilde{g}_{\mu\beta}\widetilde{g}_{\nu\alpha}}{2} - \frac{\widetilde{g}_{\mu\nu}\widetilde{g}_{\alpha\beta}}{3}\right) \\ &\quad +\cdots, \end{aligned} \quad (11)$$

$$\begin{aligned}\Pi^{J=1}_{\mu\nu\alpha\beta}(p) &= \Pi_{J=1}(p)\left(-\widetilde{g}_{\mu\alpha}p_\nu p_\beta - \widetilde{g}_{\nu\beta}p_\mu p_\alpha + \widetilde{g}_{\mu\beta}p_\nu p_\alpha + \widetilde{g}_{\nu\alpha}p_\mu p_\beta\right) \\ &\quad +\Pi_s(p)\left(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}\right), \\ &= \frac{\lambda^2_{Y/Z}}{M^2_{Y/Z}-p^2}\left(-\widetilde{g}_{\mu\alpha}p_\nu p_\beta - \widetilde{g}_{\nu\beta}p_\mu p_\alpha \right. \\ &\quad \left. + \widetilde{g}_{\mu\beta}p_\nu p_\alpha + \widetilde{g}_{\nu\alpha}p_\mu p_\beta\right) + \cdots, \end{aligned} \quad (12)$$

$$\Pi^{J=0}(p) = \Pi_{J=0}(p) = \frac{\lambda^2_{Y/Z}}{M^2_{Y/Z}-p^2} + \cdots, \quad (13)$$

where the notation $\widetilde{g}_{\mu\nu} = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}$, the components $\Pi_s(p)$ are irrelevant in the present analysis [64], and the pole residues $\lambda_{Y/Z}$ are defined by

$$\langle 0|\eta^{+}_{\mu\nu}(0)|Y/Z_{J=2}(p)\rangle = \lambda_{Y/Z}\,\varepsilon_{\mu\nu},$$
$$\langle 0|\eta^{-}_{\mu\nu}(0)|Y/Z_{J=1}(p)\rangle = \lambda_{Y/Z}\,(\varepsilon_\mu p_\nu - \varepsilon_\nu p_\mu),$$
$$\langle 0|\eta(0)|Y/Z_{J=0}(p)\rangle = \lambda_{Y/Z}, \quad (14)$$

the $\varepsilon_{\mu\nu}$ and ε_μ are the polarization vectors of the tensor and axial-vector molecular states, respectively, with the following properties:

$$\sum_\lambda \varepsilon^*_{\alpha\beta}(\lambda,p)\varepsilon_{\mu\nu}(\lambda,p)=\frac{\widetilde{g}_{\alpha\mu}\widetilde{g}_{\beta\nu}+\widetilde{g}_{\alpha\nu}\widetilde{g}_{\beta\mu}}{2}-\frac{\widetilde{g}_{\alpha\beta}\widetilde{g}_{\mu\nu}}{3},$$
$$\sum_\lambda \varepsilon^*_{\mu}(\lambda,p)\varepsilon_{\nu}(\lambda,p)=-\widetilde{g}_{\mu\nu}. \tag{15}$$

Here we add the superscripts and subscripts $J=2,\ 1,\ 0$ to denote the total angular momentum. In Ref. [50], Khemchandani et al. take the current $j_{\mu\nu}(x)=\bar{c}(x)\gamma_\mu u(x)\bar{d}(x)\gamma_\nu c(x)$ to interpolate the molecular states, and use the projectors $\mathcal{P}^0=\frac{\widetilde{g}_{\mu\nu}\widetilde{g}_{\alpha\beta}}{3}$, $\mathcal{P}^1=\frac{\widetilde{g}_{\mu\alpha}\widetilde{g}_{\nu\beta}-\widetilde{g}_{\mu\beta}\widetilde{g}_{\nu\alpha}}{2}$, $\mathcal{P}^2=\frac{\widetilde{g}_{\mu\alpha}\widetilde{g}_{\nu\beta}+\widetilde{g}_{\mu\beta}\widetilde{g}_{\nu\alpha}}{2}-\frac{\widetilde{g}_{\mu\nu}\widetilde{g}_{\alpha\beta}}{3}$ to separate the contributions of the $J^P=0^+,\ 1^+,\ 2^+$ molecular states, respectively. The present treatment differs from that of Ref. [50], while the present currents $\eta_{\mu\nu}(x)$ differ from those of Refs. [38,49].

In the following, we briefly outline the operator product expansion for the correlation functions $\Pi_{\mu\nu\alpha\beta}(p)$ and $\Pi(p)$ in perturbative QCD. We contract the s and Q quark fields in the correlation functions $\Pi_{\mu\nu\alpha\beta}(p)$ and $\Pi(p)$ with the Wick theorem, and obtain the results

$$\begin{aligned}\Pi_{\mu\nu\alpha\beta}(p)=\frac{i}{2}\int d^4x e^{ip\cdot x}\Big\{&\mathrm{Tr}\left[\gamma_\mu S_Q^{ij}(x)\gamma_\alpha S^{ji}(-x)\right]\\&\times\mathrm{Tr}\left[\gamma_\nu S^{mn}(x)\gamma_\beta S_Q^{nm}(-x)\right]\\&+\mathrm{Tr}\left[\gamma_\nu S_Q^{ij}(x)\gamma_\beta S^{ji}(-x)\right]\\&\times\mathrm{Tr}\left[\gamma_\mu S^{mn}(x)\gamma_\alpha S_Q^{nm}(-x)\right]\\&\pm\mathrm{Tr}\left[\gamma_\nu S_Q^{ij}(x)\gamma_\alpha S^{ji}(-x)\right]\\&\times\mathrm{Tr}\left[\gamma_\mu S^{mn}(x)\gamma_\beta S_Q^{nm}(-x)\right]\\&\pm\mathrm{Tr}\left[\gamma_\mu S_Q^{ij}(x)\gamma_\beta S^{ji}(-x)\right]\\&\times\mathrm{Tr}\left[\gamma_\nu S^{mn}(x)\gamma_\alpha S_Q^{nm}(-x)\right]\Big\},\end{aligned}\tag{16}$$

$$\begin{aligned}\Pi(p)=i\int d^4x e^{ip\cdot x}&\mathrm{Tr}\left[\gamma_\mu S_Q^{ij}(x)\gamma_\alpha S^{ji}(-x)\right]\\&\times\mathrm{Tr}\left[\gamma^\mu S^{mn}(x)\gamma^\alpha S_Q^{nm}(-x)\right],\end{aligned}\tag{17}$$

where the $\pm$ correspond to $\pm$ charge conjugations, respectively, the $S^{ij}(x)$ and $S_Q^{ij}(x)$ are the full s and Q quark propagators, respectively,

$$\begin{aligned}S^{ij}(x)=&\frac{i\delta_{ij}\not{x}}{2\pi^2x^4}-\frac{\delta_{ij}m_s}{4\pi^2x^2}-\frac{\delta_{ij}\langle\bar{s}s\rangle}{12}+\frac{i\delta_{ij}\not{x}m_s\langle\bar{s}s\rangle}{48}\\&-\frac{\delta_{ij}x^2\langle\bar{s}g_s\sigma Gs\rangle}{192}+\frac{i\delta_{ij}x^2\not{x}m_s\langle\bar{s}g_s\sigma Gs\rangle}{1152}\\&-\frac{ig_sG^a_{\alpha\beta}t^a_{ij}(\not{x}\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not{x})}{32\pi^2x^2}-\frac{i\delta_{ij}x^2\not{x}g_s^2\langle\bar{s}s\rangle^2}{7776}\\&-\frac{\delta_{ij}x^4\langle\bar{s}s\rangle\langle g_s^2GG\rangle}{27648}-\frac{1}{8}\langle\bar{s}_j\sigma^{\mu\nu}s_i\rangle\sigma_{\mu\nu}\\&-\frac{1}{4}\langle\bar{s}_j\gamma^\mu s_i\rangle\gamma_\mu+\cdots,\end{aligned}\tag{18}$$

$$\begin{aligned}S_Q^{ij}(x)=&\frac{i}{(2\pi)^4}\int d^4k e^{-ik\cdot x}\Big\{\frac{\delta_{ij}}{\not{k}-m_Q}\\&-\frac{g_sG^n_{\alpha\beta}t^n_{ij}}{4}\frac{\sigma^{\alpha\beta}(\not{k}+m_Q)+(\not{k}+m_Q)\sigma^{\alpha\beta}}{(k^2-m_Q^2)^2}\\&+\frac{g_sD_\alpha G^n_{\beta\lambda}t^n_{ij}(f^{\lambda\beta\alpha}+f^{\lambda\alpha\beta})}{3(k^2-m_Q^2)^4}\\&-\frac{g_s^2(t^at^b)_{ij}G^a_{\alpha\beta}G^b_{\mu\nu}(f^{\alpha\beta\mu\nu}+f^{\alpha\mu\beta\nu}+f^{\alpha\mu\nu\beta})}{4(k^2-m_Q^2)^5}\\&+\cdots\Big\},\end{aligned}$$
$$\begin{aligned}f^{\lambda\alpha\beta}=&(\not{k}+m_Q)\gamma^\lambda(\not{k}+m_Q)\gamma^\alpha(\not{k}+m_Q)\gamma^\beta(\not{k}+m_Q),\\f^{\alpha\beta\mu\nu}=&(\not{k}+m_Q)\gamma^\alpha(\not{k}+m_Q)\gamma^\beta(\not{k}+m_Q)\gamma^\mu(\not{k}\\&+m_Q)\gamma^\nu(\not{k}+m_Q),\end{aligned}\tag{19}$$

and $t^n=\frac{\lambda^n}{2}$, the λ^n are the Gell–Mann matrices, $D_\alpha=\partial_\alpha-ig_sG^n_\alpha t^n$ [63], then compute the integrals both in the coordinate and momentum spaces, and obtain the correlation functions $\Pi_{\mu\nu\alpha\beta}(p)$ and $\Pi(p)$ therefore the QCD spectral densities[1]. In Eq. (18), we retain the terms $\langle\bar{s}_j\sigma_{\mu\nu}s_i\rangle$ and $\langle\bar{s}_j\gamma_\mu s_i\rangle$ originate from the Fierz re-ordering of the $\langle s_i\bar{s}_j\rangle$

[1] It is convenient to introduce the external fields $\bar{\chi}$, χ, A^a_α, and the additional Lagrangian $\Delta\mathcal{L}$

$$\begin{aligned}\Delta\mathcal{L}=&\bar{s}(x)\left(i\gamma^\mu\partial_\mu-m_s\right)\chi(x)\\&+\bar{\chi}(x)\left(i\gamma^\mu\partial_\mu-m_s\right)s(x)+g_s\bar{s}(x)\gamma^\mu t^a s(x)A^a_\mu(x)+\cdots,\end{aligned}$$

in carrying out the operator product expansion [63,65,66]. We expand the heavy and light quark propagators S^Q_{ij} and S_{ij} in terms of the external fields $\bar{\chi}$, χ, and A^a_α,

$$\begin{aligned}S^Q_{ij}\left(x,\bar{\chi},\chi,A^a_\alpha\right)=&\frac{i}{(2\pi)^4}\int d^4k e^{-ik\cdot x}\Big\{\frac{\delta_{ij}}{\not{k}-m_Q}\\&-\frac{g_sA^a_{\alpha\beta}t^a_{ij}}{4}\frac{\sigma^{\alpha\beta}(\not{k}+m_Q)+(\not{k}+m_Q)\sigma^{\alpha\beta}}{(k^2-m_Q^2)^2}\\&+\cdots\Big\},\end{aligned}$$
$$\begin{aligned}S_{ij}\left(x,\bar{\chi},\chi,A^a_\mu\right)=&\frac{i\delta_{ij}\not{x}}{2\pi^2x^4}+\chi^i(x)\bar{\chi}^j(0)\\&-\frac{ig_sA^a_{\alpha\beta}t^a_{ij}\left(\not{x}\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not{x}\right)}{32\pi^2x^2}+\cdots,\end{aligned}$$

where $A^a_{\alpha\beta}=\partial_\alpha A^a_\beta-\partial_\beta A^a_\alpha+g_sf^{abc}A^b_\alpha A^c_\beta$. Then the correlation functions $\Pi(p)$ can be written as

$$\Pi(p)=\sum_{n=0}^{\infty}\mathcal{C}_n(p,\mu)\,\mathcal{O}_n\left(\bar{\chi},\chi,A^a_\alpha,\mu\right)$$

in the external fields $\bar{\chi}$, χ, and A^a_α, where the $\mathcal{C}_n(p,\mu)$ are the Wilson coefficients, the $\mathcal{O}_n\left(\bar{\chi},\chi,A^a_\alpha,\mu\right)$ are operators characterized by their dimensions n. We choose the energy scale $\mu\gg\Lambda_{QCD}$, the Wilson coefficients $\mathcal{C}_n(p^2,\mu)$ depend only on short-distance dynamics, and the perturbative calculations make sense. If we neglect the perturbative (or radiative) corrections, the operators $\mathcal{O}_n\left(\bar{\chi},\chi,A^a_\alpha,\mu\right)$ can also be to absorb the gluons emitted from the heavy quark lines to form $\langle\bar{s}_j g_s G^a_{\alpha\beta} t^a_{mn}\sigma_{\mu\nu}s_i\rangle$ and $\langle\bar{s}_j\gamma_\mu s_i g_s D_\nu G^a_{\alpha\beta}t^a_{mn}\rangle$ so as to extract the mixed condensate and four-quark condensates $\langle\bar{s}g_s\sigma Gs\rangle$ and $g_s^2\langle\bar{s}s\rangle^2$, respectively. The s-quark fields $s(x)$, $\bar{s}(x)$, and the gluon field $G^a_\mu(x)$ can be expanded in terms of the Taylor series of covariant derivatives,

$$
\begin{aligned}
s(x) &= \sum_{n=0}^{\infty}\frac{1}{n!}x^{\mu_1}x^{\mu_2}\cdots x^{\mu_n}\,D_{\mu_1}D_{\mu_2}\cdots D_{\mu_n}\,s(0),\\
\bar{s}(x) &= \sum_{n=0}^{\infty}\frac{1}{n!}x^{\mu_1}x^{\mu_2}\cdots x^{\mu_n}\,\bar{s}(0)\,D^\dagger_{\mu_1}D^\dagger_{\mu_2}\cdots D^\dagger_{\mu_n},\\
G^a_\mu(x) &= \sum_{n=0}^{\infty}\frac{1}{n!(n+2)}x^{\rho}x^{\mu_1}x^{\mu_2}\cdots x^{\mu_n}\,D_{\mu_1}D_{\mu_2}\\
&\quad\cdots D_{\mu_n}\,G^a_{\rho\mu}(0).
\end{aligned}
\tag{20}
$$

The bilinear fields $s_\alpha(x)\bar{s}_\beta(0)$ can be re-arranged into the following form in the Dirac spinor space:

$$
\begin{aligned}
s_\alpha(x)\bar{s}_\beta(0) &= -\frac{1}{4}\delta_{\alpha\beta}\bar{s}(0)s(x) - \frac{1}{4}(\gamma^\mu)_{\alpha\beta}\bar{s}(0)\gamma_\mu s(x)\\
&-\frac{1}{8}(\sigma^{\mu\nu})_{\alpha\beta}\bar{s}(0)\sigma_{\mu\nu}s(x) + \frac{1}{4}(\gamma^\mu\gamma_5)_{\alpha\beta}\bar{s}(0)\gamma_\mu\gamma_5 s(x)\\
&+\frac{1}{4}(i\gamma_5)_{\alpha\beta}\bar{s}(0)i\gamma_5 s(x).
\end{aligned}
\tag{21}
$$

The vacuum condensates $\langle\bar{s}g_s\sigma Gs\rangle$, $g_s^2\langle\bar{s}s\rangle^2$, and $\langle\bar{s}s\rangle\langle g_s^2 GG\rangle$ in the full s-quark propagator originate from the vacuum expectations of the operators $\bar{s}(0)\sigma^{\mu\nu}D^\alpha D^\beta s(0)$, $\bar{s}(0)\gamma^\mu D^\alpha D^\beta D^\lambda s(0)$, and $\bar{s}(0)D^\alpha D^\beta D^\lambda D^\tau s(0)$, respectively. We take into account the formulas $\left[D_\alpha, D_\beta\right] = -ig_s G_{\alpha\beta}$ and $D^\alpha G^a_{\alpha\mu} = -g_s\left(\bar{u}\gamma_\mu t^a u + \bar{d}\gamma_\mu t^a d + \bar{s}\gamma_\mu t^a s\right)$, then the terms with $n > 4$ in the Taylor expansion of the $s(x)$ and $\bar{s}(x)$ are of the order $\mathcal{O}(\alpha_s^k)$ with $k > 1$, and have no contribution in the present truncation. The operators $g_s G^n_{\alpha\beta}$, $g_s D_\alpha G^n_{\beta\lambda}$ and $g_s^2 G^a_{\alpha\beta}G^b_{\mu\nu}$ in the full Q-quark propagator are of the order $\mathcal{O}(\alpha_s^k)$ with $k = \frac{1}{2}$, 1, and 1, respectively. The terms with $n > 1$ in the Taylor expansion of the $G^a_\mu(x)$ are of the order $\mathcal{O}(\alpha_s^k)$ with $k > 1$, and have no contribution in the present truncation. In this article, we take the truncation $\mathcal{O}(\alpha_s^k)$ with $k \le 1$, the operators therefore the vacuum condensates have the dimensions less than or equal to 10.

Once the analytical expressions are obtained, we can take the quark–hadron duality below the continuum thresholds s_0 and perform a Borel transformation with respect to the variable $P^2 = -p^2$ to obtain the following QCD sum rules:

$$
\lambda^2_{Y/Z}\,\exp\left(-\frac{M^2_{Y/Z}}{T^2}\right) = \int_{4m_Q^2}^{s_0} \mathrm{d}s\,\rho(s)\,\exp\left(-\frac{s}{T^2}\right), \tag{22}
$$

where

$$
\begin{aligned}
\rho(s) &= \rho_0(s) + \rho_3(s) + \rho_4(s) + \rho_5(s) + \rho_6(s) + \rho_7(s)\\
&\quad+\rho_8(s) + \rho_{10}(s),
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\rho_0^{J=2}(s) &= \frac{1}{20480\pi^6}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,yz\,(1-y-z)^3\,(s\\
&\quad-\overline{m}_Q^2)^2\left(293s^2 - 190s\overline{m}_Q^2 + 17\overline{m}_Q^4\right)\\
&\quad+\frac{3}{20480\pi^6}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,yz\,(1-y-z)^2\,(s\\
&\quad-\overline{m}_Q^2)^4 + \frac{3m_s m_Q}{512\pi^6}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,(y+z)\\
&\quad\times(1-y-z)^2\left(s-\overline{m}_Q^2\right)^2\left(4s-\overline{m}_Q^2\right),
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\rho_3^{J=2}(s) &= -\frac{3m_Q\langle\bar{s}s\rangle}{64\pi^4}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,(y+z)(1-y-z)\\
&\quad\times\left(s-\overline{m}_Q^2\right)\left(3s-\overline{m}_Q^2\right)\\
&\quad+\frac{3m_s\langle\bar{s}s\rangle}{640\pi^4}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,yz\,(1-y-z)\\
&\quad\times\left(115s^2 - 112s\overline{m}_Q^2 + 17\overline{m}_Q^4\right)\\
&\quad+\frac{3m_s\langle\bar{s}s\rangle}{640\pi^4}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,yz\left(s-\overline{m}_Q^2\right)^2\\
&\quad-\frac{3m_s m_Q^2\langle\bar{s}s\rangle}{16\pi^4}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\left(s-\overline{m}_Q^2\right),
\end{aligned}
\tag{25}
$$

Footnote 1 continued

counted by the orders of the fine constant $\alpha_s(\mu) = \frac{g_s^2(\mu)}{4\pi}$, $\mathcal{O}\left(\alpha_s^k\right)$, with $k = 0, \frac{1}{2}, 1, \frac{3}{2}$, etc. In this article, we take the truncations $n \le 10$ and $k \le 1$, and factorize the higher dimensional operators into non-factorizable low dimensional operators with the same quantum numbers of the vacuum. Taking the following replacements:

$$\mathcal{O}_n\left(\bar{\chi}, \chi, A^a_\alpha, \mu\right) \to \langle\mathcal{O}_n\left(\bar{s}, s, G^a_\alpha, \mu\right)\rangle,$$

we obtain the correlation functions at the level of quark–gluon degrees of freedom. For example,

$$
\begin{aligned}
\chi^i(x)\bar{\chi}^j(0) &= -\frac{\delta_{ij}\bar{\chi}(0)\chi(0)}{12} - \frac{\delta_{ij}x^2\bar{\chi}(0)g_s\sigma A(0)\chi(0)}{192}\\
&\quad+\cdots \to -\frac{\delta_{ij}\langle\bar{s}s\rangle}{12} - \frac{\delta_{ij}x^2\langle\bar{s}g_s\sigma Gs\rangle}{192} + \cdots.
\end{aligned}
$$

For simplicity, we often take the following replacements:

$$
\begin{aligned}
S^Q_{ij}\left(x, \bar{\chi}, \chi, A^a_\alpha\right) &\to S^Q_{ij}\left(x, \bar{s}, s, G^a_\alpha\right),\\
S_{ij}\left(x, \bar{\chi}, \chi, A^a_\alpha\right) &\to S_{ij}\left(x, \bar{s}, s, G^a_\alpha\right),\\
\mathcal{O}_n\left(\bar{\chi}, \chi, A^a_\alpha\right) &\to \langle\mathcal{O}_n\left(\bar{s}, s, G^a_\alpha\right)\rangle,
\end{aligned}
$$

directly in the calculations by neglecting some intermediate steps, and resort to the routine taken in this article.

$$\rho_4^{J=2}(s) = -\frac{m_Q^2}{15360\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \times\left(\frac{z}{y^2}+\frac{y}{z^2}\right)(1-y-z)^3 \times\left\{56s-17\overline{m}_Q^2+10s^2\delta\left(s-\overline{m}_Q^2\right)\right\} -\frac{m_Q^2}{5120\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \times\left(\frac{z}{y^2}+\frac{y}{z^2}\right)(1-y-z)^2\left(s-\overline{m}_Q^2\right) -\frac{1}{10240\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz\,(y+z) \times(1-y-z)^2\left(185s^2-208s\overline{m}_Q^2+43\overline{m}_Q^4\right) +\frac{1}{5120\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz\,(y+z) \times(1-y-z)\left(s-\overline{m}_Q^2\right)^2, \quad (26)$$

$$\rho_5^{J=2}(s) = \frac{3m_Q\langle\bar{s}g_s\sigma Gs\rangle}{128\pi^4}\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz\,(y+z) \times\left(2s-\overline{m}_Q^2\right) -\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{640\pi^4}\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz\,yz\left\{56s -17\overline{m}_Q^2+10s^2\delta\left(s-\overline{m}_Q^2\right)\right\} -\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{640\pi^4}\int_{y_i}^{y_f} dy\,y(1-y)\left(s-\widetilde{m}_Q^2\right) +\frac{3m_s m_Q^2\langle\bar{s}g_s\sigma Gs\rangle}{64\pi^4}\int_{y_i}^{y_f} dy, \quad (27)$$

$$\rho_6^{J=2}(s) = \frac{m_Q^2\langle\bar{s}s\rangle^2}{8\pi^2}\int_{y_i}^{y_f} dy - \frac{m_s m_Q\langle\bar{s}s\rangle^2}{16\pi^2}\int_{y_i}^{y_f} dy\left\{1 +s\delta\left(s-\widetilde{m}_Q^2\right)\right\} +\frac{g_s^2\langle\bar{s}s\rangle^2}{4320\pi^4}\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz\,yz\left\{56s -17\overline{m}_Q^2+10\overline{m}_Q^4\delta\left(s-\overline{m}_Q^2\right)\right\} +\frac{g_s^2\langle\bar{s}s\rangle^2}{4320\pi^4}\int_{y_i}^{y_f} dy\,y(1-y)\left(s-\widetilde{m}_Q^2\right) -\frac{g_s^2\langle\bar{s}s\rangle^2}{6480\pi^4}\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz\,(1-y-z) \times\left\{45\left(\frac{z}{y}+\frac{y}{z}\right)\left(2s-\overline{m}_Q^2\right)+\left(\frac{z}{y^2}+\frac{y}{z^2}\right) \times m_Q^2\left[19+20\overline{m}_Q^2\delta\left(s-\overline{m}_Q^2\right)\right]+(y+z) \times\left[18\left(3s-\overline{m}_Q^2\right)+10\overline{m}_Q^4\delta\left(s-\overline{m}_Q^2\right)\right]\right\}, \quad (28)$$

$$\rho_7^{J=2}(s) = \frac{m_Q^3\langle\bar{s}s\rangle}{192\pi^2 T^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \times\left(\frac{y}{z^3}+\frac{z}{y^3}+\frac{1}{y^2}+\frac{1}{z^2}\right)(1-y-z)\,\overline{m}_Q^2 \times\delta\left(s-\overline{m}_Q^2\right) - \frac{m_Q\langle\bar{s}s\rangle}{64\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \times\left(\frac{y}{z^2}+\frac{z}{y^2}\right)(1-y-z)\left\{1+\overline{m}_Q^2\delta\left(s-\overline{m}_Q^2\right)\right\} +\frac{m_Q\langle\bar{s}s\rangle}{32\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \times\left\{1+\frac{\overline{m}_Q^2}{3}\delta\left(s-\overline{m}_Q^2\right)\right\} -\frac{m_Q\langle\bar{s}s\rangle}{384\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f} dy \times\left\{1+\widetilde{m}_Q^2\,\delta\left(s-\widetilde{m}_Q^2\right)\right\}, \quad (29)$$

$$\rho_8^{J=2}(s) = -\frac{m_Q^2\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{16\pi^2}\int_0^1 dy\left(1+\frac{\widetilde{m}_Q^2}{T^2}\right) \times\delta\left(s-\widetilde{m}_Q^2\right), \quad (30)$$

$$\rho_{10}^{J=2}(s) = \frac{m_Q^2\langle\bar{s}g_s\sigma Gs\rangle^2}{128\pi^2 T^6}\int_0^1 dy\,\widetilde{m}_Q^4\,\delta\left(s-\widetilde{m}_Q^2\right) -\frac{m_Q^4\langle\bar{s}s\rangle^2}{144T^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\left\{\frac{1}{y^3}+\frac{1}{(1-y)^3}\right\} \times\delta\left(s-\widetilde{m}_Q^2\right) +\frac{m_Q^2\langle\bar{s}s\rangle^2}{48T^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1 dy\left\{\frac{1}{y^2}+\frac{1}{(1-y)^2}\right\}$$

$$
\begin{aligned}
&\times\delta\left(s-\widetilde{m}_Q^2\right)\\
&+\frac{\langle\bar{s}g_s\sigma Gs\rangle^2}{5184\pi^2T^2}\int_0^1 \mathrm{d}y\,\widetilde{m}_Q^2\delta\left(s-\widetilde{m}_Q^2\right)\\
&+\frac{m_Q^2\langle\bar{s}s\rangle^2}{144T^6}\left\langle\frac{\alpha_sGG}{\pi}\right\rangle\int_0^1 \mathrm{d}y\,\widetilde{m}_Q^4\,\delta\left(s-\widetilde{m}_Q^2\right),
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\rho_0^{J=1}(s) = &\frac{1}{4096\pi^6 s}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,yz\,(1-y-z)^3\\
&\times\left(s-\overline{m}_Q^2\right)^2\left(49s^2-30s\overline{m}_Q^2+\overline{m}_Q^4\right)\\
&+\frac{1}{4096\pi^6 s}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,yz\,(1-y-z)^2\\
&\times\left(s-\overline{m}_Q^2\right)^3\left(3s+\overline{m}_Q^2\right)\\
&+\frac{9m_sm_Q}{1024\pi^6}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,(y+z)\\
&\times(1-y-z)^2\left(s-\overline{m}_Q^2\right)^2,
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
\rho_3^{J=1}(s) = &-\frac{3m_Q\langle\bar{s}s\rangle}{64\pi^4}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\\
&\times(y+z)(1-y-z)\left(s-\overline{m}_Q^2\right)\\
&+\frac{m_s\langle\bar{s}s\rangle}{128\pi^4 s}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}zyz\\
&\times(1-y-z)\left(55s^2-48s\overline{m}_Q^2+3\overline{m}_Q^4\right)\\
&+\frac{m_s\langle\bar{s}s\rangle}{128\pi^4 s}\int_{y_i}^{y_f}\mathrm{d}y\\
&\times\int_{z_i}^{1-y}\mathrm{d}z\,yz\left(s-\overline{m}_Q^2\right)\left(s+\overline{m}_Q^2\right),
\end{aligned} \tag{33}
$$

$$
\begin{aligned}
\rho_4^{J=1}(s) = &-\frac{m_Q^2}{3072\pi^4 s}\left\langle\frac{\alpha_sGG}{\pi}\right\rangle\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\\
&\times\left(\frac{z}{y^2}+\frac{y}{z^2}\right)(1-y-z)^3\\
&\times\left\{8s-\overline{m}_Q^2+\frac{5\overline{m}_Q^4}{3}\delta\left(s-\overline{m}_Q^2\right)\right\}\\
&-\frac{m_Q^2}{3072\pi^4 s}\left\langle\frac{\alpha_sGG}{\pi}\right\rangle\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\\
&\times\left(\frac{z}{y^2}+\frac{y}{z^2}\right)(1-y-z)^2\,\overline{m}_Q^2\\
&-\frac{1}{6144\pi^4 s}\left\langle\frac{\alpha_sGG}{\pi}\right\rangle\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\\
&\times(y+z)\,(1-y-z)^2\left(5s^2-3\overline{m}_Q^4\right)\\
&+\frac{1}{3072\pi^4 s}\left\langle\frac{\alpha_sGG}{\pi}\right\rangle\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}\\
&\times z\,(y+z)\,(1-y-z)\left(s^2-\overline{m}_Q^4\right),
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\rho_5^{J=1}(s) = &\frac{3m_Q\langle\bar{s}g_s\sigma Gs\rangle}{256\pi^4}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\,(y+z)\\
&-\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{128\pi^4 s}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\\
&\times yz\left\{8s-\overline{m}_Q^2+\frac{5s^2}{3}\delta\left(s-\overline{m}_Q^2\right)\right\}\\
&-\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{384\pi^4 s}\int_{y_i}^{y_f}\mathrm{d}y\,y(1-y)\,\widetilde{m}_Q^2,
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
\rho_6^{J=1}(s) = &-\frac{m_sm_Q\langle\bar{s}s\rangle^2}{32\pi^2}\int_0^1\mathrm{d}y\,\delta\left(s-\widetilde{m}_Q^2\right),\\
&+\frac{g_s^2\langle\bar{s}s\rangle^2}{864\pi^4 s}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\\
&\times yz\left\{8s-\overline{m}_Q^2+\frac{5\overline{m}_Q^4}{3}\delta\left(s-\overline{m}_Q^2\right)\right\}\\
&+\frac{g_s^2\langle\bar{s}s\rangle^2}{2592\pi^4 s}\int_{y_i}^{y_f}\mathrm{d}y\,y(1-y)\,\widetilde{m}_Q^2\\
&-\frac{g_s^2\langle\bar{s}s\rangle^2}{864\pi^4}\int_{y_i}^{y_f}\mathrm{d}y\int_{z_i}^{1-y}\mathrm{d}z\\
&\times(1-y-z)\left\{3\left(\frac{z}{y}+\frac{y}{z}\right)\right.
\end{aligned}
$$

$$+\left(\frac{z}{y^2}+\frac{y}{z^2}\right) m_Q^2 \delta\left(s-\overline{m}_Q^2\right)+(y+z) \times\left[8+2\overline{m}_Q^2 \delta\left(s-\overline{m}_Q^2\right)\right]\Big\}, \quad (36)$$

$$\rho_7^{J=1}(s)=\frac{m_Q^3\langle\bar{s}s\rangle}{384\pi^2 s}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z \times\left(\frac{y}{z^3}+\frac{z}{y^3}+\frac{1}{y^2}+\frac{1}{z^2}\right)(1-y-z)\left(1+\frac{s}{T^2}\right) \times\delta\left(s-\overline{m}_Q^2\right)-\frac{m_Q\langle\bar{s}s\rangle}{128\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z \times\left(\frac{y}{z^2}+\frac{z}{y^2}\right)(1-y-z)\delta\left(s-\overline{m}_Q^2\right) -\frac{m_Q\langle\bar{s}s\rangle}{192\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z\, \delta\left(s-\overline{m}_Q^2\right) -\frac{m_Q\langle\bar{s}s\rangle}{768\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z \times\delta\left(s-\overline{m}_Q^2\right), \quad (37)$$

$$\rho_0^{J=0}(s)=\frac{3}{1024\pi^6}\int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z\, yz\,(1-y-z)^3 \times\left(s-\overline{m}_Q^2\right)^2\left(7s^2-6s\overline{m}_Q^2+\overline{m}_Q^4\right) +\frac{3}{1024\pi^6}\int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z\, yz\,(1-y-z)^2 \times\left(s-\overline{m}_Q^2\right)^3\left(3s-\overline{m}_Q^2\right) +\frac{3m_s m_Q}{512\pi^6}\int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z\,(y+z) \times(1-y-z)^2\left(s-\overline{m}_Q^2\right)^2\left(5s-2\overline{m}_Q^2\right), \quad (38)$$

$$\rho_3^{J=0}(s)=-\frac{3m_Q\langle\bar{s}s\rangle}{32\pi^4}\int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z\,(y+z) \times(1-y-z)\left(s-\overline{m}_Q^2\right)\left(2s-\overline{m}_Q^2\right) +\frac{3m_s\langle\bar{s}s\rangle}{32\pi^4}\int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z\, yz \times(1-y-z)\left(10s^2-12s\overline{m}_Q^2+3\overline{m}_Q^4\right) +\frac{3m_s\langle\bar{s}s\rangle}{32\pi^4}\int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z\, yz\left(s-\overline{m}_Q^2\right) \times\left(2s-\overline{m}_Q^2\right)-\frac{3m_s m_Q^2\langle\bar{s}s\rangle}{8\pi^4}\int_{y_i}^{y_f} \mathrm{d}y\left(s-\widetilde{m}_Q^2\right), \quad (39)$$

$$\rho_4^{J=0}(s)=-\frac{m_Q^2}{256\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z \times\left(\frac{z}{y^2}+\frac{y}{z^2}\right)(1-y-z)^3 \times\left\{2s-\overline{m}_Q^2+\frac{\overline{m}_Q^4}{6}\delta\left(s-\overline{m}_Q^2\right)\right\} -\frac{m_Q^2}{512\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z \times\left(\frac{z}{y^2}+\frac{y}{z^2}\right)(1-y-z)^2\left(3s-2\overline{m}_Q^2\right) -\frac{1}{512\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z \times(y+z)(1-y-z)^2 \times\left(10s^2-12s\overline{m}_Q^2+3\overline{m}_Q^4\right) +\frac{1}{256\pi^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle \int_{y_i}^{y_f} \mathrm{d}y \times\int_{z_i}^{1-y} \mathrm{d}z\,(y+z)(1-y-z)\left(s-\overline{m}_Q^2\right) \times\left(2s-\overline{m}_Q^2\right), \quad (40)$$

$$\rho_5^{J=0}(s)=\frac{3m_Q\langle\bar{s}g_s\sigma Gs\rangle}{128\pi^4}\int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z \times(y+z)\left(3s-2\overline{m}_Q^2\right) -\frac{3m_s\langle\bar{s}g_s\sigma Gs\rangle}{32\pi^4}\int_{y_i}^{y_f} \mathrm{d}y \int_{z_i}^{1-y} \mathrm{d}z \times yz\left\{2s-\overline{m}_Q^2+\frac{s}{6}\delta\left(s-\overline{m}_Q^2\right)\right\} -\frac{m_s\langle\bar{s}g_s\sigma Gs\rangle}{64\pi^4}\int_{y_i}^{y_f} \mathrm{d}y\, y(1-y)$$

$$\times\left(3s-2\overline{m}_Q^2\right)+\frac{3m_s m_Q^2\langle\bar{s}g_s\sigma Gs\rangle}{32\pi^4}\int_{y_i}^{y_f}dy, \quad (41)$$

$$\rho_6^{J=0}(s)=\frac{m_Q^2\langle\bar{s}s\rangle^2}{4\pi^2}\int_{y_i}^{y_f}dy-\frac{m_s m_Q\langle\bar{s}s\rangle^2}{8\pi^2}\times\int_{y_i}^{y_f}dy\left\{1+\frac{s}{2}\delta\left(s-\widetilde{m}_Q^2\right)\right\}$$
$$+\frac{g_s^2\langle\bar{s}s\rangle^2}{72\pi^4}\int_{y_i}^{y_f}dy\int_{z_i}^{1-y}dz\times yz\left\{2s-\overline{m}_Q^2+\frac{\overline{m}_Q^4}{6}\delta\left(s-\overline{m}_Q^2\right)\right\}$$
$$+\frac{g_s^2\langle\bar{s}s\rangle^2}{432\pi^4}\int_{y_i}^{y_f}dy\,y(1-y)\left(3s-2\widetilde{m}_Q^2\right)$$
$$-\frac{g_s^2\langle\bar{s}s\rangle^2}{432\pi^4}\int_{y_i}^{y_f}dy\int_{z_i}^{1-y}dz\,(1-y-z)\times\left\{3\left(\frac{z}{y}+\frac{y}{z}\right)\left(3s-2\overline{m}_Q^2\right)+\left(\frac{z}{y^2}+\frac{y}{z^2}\right)\times m_Q^2\left[2+\overline{m}_Q^2\delta\left(s-\overline{m}_Q^2\right)\right]+(y+z)\times\left[12\left(2s-\overline{m}_Q^2\right)+2\overline{m}_Q^4\delta\left(s-\overline{m}_Q^2\right)\right]\right\}, \quad (42)$$

$$\rho_7^{J=0}(s)=\frac{m_Q^3\langle\bar{s}s\rangle}{192\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f}dy\int_{z_i}^{1-y}dz\times\left(\frac{y}{z^3}+\frac{z}{y^3}+\frac{1}{y^2}+\frac{1}{z^2}\right)(1-y-z)\times\left(1+\frac{s}{T^2}\right)\delta\left(s-\overline{m}_Q^2\right)$$
$$-\frac{m_Q\langle\bar{s}s\rangle}{64\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f}dy\int_{z_i}^{1-y}dz\times\left(\frac{y}{z^2}+\frac{z}{y^2}\right)(1-y-z)\times\left\{2+\overline{m}_Q^2\delta\left(s-\overline{m}_Q^2\right)\right\}$$
$$+\frac{m_Q\langle\bar{s}s\rangle}{32\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f}dy\int_{z_i}^{1-y}dz\times\left\{2+\overline{m}_Q^2\delta\left(s-\overline{m}_Q^2\right)\right\}$$
$$-\frac{m_Q\langle\bar{s}s\rangle}{384\pi^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_{y_i}^{y_f}dy\times\left\{2+\widetilde{m}_Q^2\,\delta\left(s-\widetilde{m}_Q^2\right)\right\}, \quad (43)$$

$$\rho_8^{J=0}(s)=-\frac{m_Q^2\langle\bar{s}s\rangle\langle\bar{s}g_s\sigma Gs\rangle}{8\pi^2}\int_0^1dy\times\left(1+\frac{\widetilde{m}_Q^2}{T^2}\right)\delta\left(s-\widetilde{m}_Q^2\right), \quad (44)$$

$$\rho_{10}^{J=0}(s)=\frac{m_Q^2\langle\bar{s}g_s\sigma Gs\rangle^2}{64\pi^2T^6}\int_0^1dy\times\widetilde{m}_Q^4\,\delta\left(s-\widetilde{m}_Q^2\right)-\frac{m_Q^4\langle\bar{s}s\rangle^2}{72T^4}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1dy\times\left\{\frac{1}{y^3}+\frac{1}{(1-y)^3}\right\}\delta\left(s-\widetilde{m}_Q^2\right)$$
$$+\frac{m_Q^2\langle\bar{s}s\rangle^2}{24T^2}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1dy\times\left\{\frac{1}{y^2}+\frac{1}{(1-y)^2}\right\}\delta\left(s-\widetilde{m}_Q^2\right)$$
$$+\frac{m_Q^2\langle\bar{s}g_s\sigma Gs\rangle^2}{288\pi^2T^2}\int_0^1dy\frac{1}{y(1-y)}\delta\left(s-\widetilde{m}_Q^2\right)$$
$$+\frac{m_Q^2\langle\bar{s}s\rangle^2}{72T^6}\left\langle\frac{\alpha_s GG}{\pi}\right\rangle\int_0^1dy\widetilde{m}_Q^4\,\delta\left(s-\widetilde{m}_Q^2\right), \quad (45)$$

the subscripts 0, 3, 4, 5, 6, 7, 8, 10 denote the dimensions of the vacuum condensates, $y_f=\frac{1+\sqrt{1-4m_Q^2/s}}{2}$, $y_i=\frac{1-\sqrt{1-4m_Q^2/s}}{2}$, $z_i=\frac{ym_Q^2}{ys-m_Q^2}$, $\overline{m}_Q^2=\frac{(y+z)m_Q^2}{yz}$, $\widetilde{m}_Q^2=\frac{m_Q^2}{y(1-y)}$, $\int_{y_i}^{y_f}dy\rightarrow\in_0^1 dy$, $\int_{z_i}^{1-y}dz\rightarrow\int_0^{1-y}dz$ when the δ functions $\delta\left(s-\overline{m}_Q^2\right)$ and $\delta\left(s-\widetilde{m}_Q^2\right)$ appear. In this article, we carry out the operator product expansion to the vacuum condensates up to dimension 10, and assume vacuum saturation for the higher dimensional vacuum condensates. The condensates $\langle\frac{\alpha_s}{\pi}GG\rangle$, $\langle\bar{s}s\rangle\langle\frac{\alpha_s}{\pi}GG\rangle$, $\langle\bar{s}s\rangle^2\langle\frac{\alpha_s}{\pi}GG\rangle$, $\langle\bar{s}g_s\sigma Gs\rangle^2$, and $g_s^2\langle\bar{s}s\rangle^2$ are the vacuum expectations of the operators of the order $\mathcal{O}(\alpha_s)$. The four-quark condensate $g_s^2\langle\bar{q}q\rangle^2$ comes from the terms $\langle\bar{s}\gamma_\mu t^a sg_s D_\eta G^a_{\lambda\tau}\rangle$, $\langle\bar{s}_j D^\dagger_\mu D^\dagger_\nu D^\dagger_\alpha s_i\rangle$ and $\langle\bar{s}_j D_\mu D_\nu D_\alpha s_i\rangle$, rather than comes from the perturbative corrections of $\langle\bar{s}s\rangle^2$. The condensates $\langle g_s^3 GGG\rangle$, $\langle\frac{\alpha_s GG}{\pi}\rangle^2$,

$\langle\frac{\alpha_s GG}{\pi}\rangle\langle\bar{s}g_s\sigma Gs\rangle$ have the dimensions 6, 8, 9, respectively, but they are the vacuum expectations of the operators of the order $\mathcal{O}(\alpha_s^{3/2})$, $\mathcal{O}(\alpha_s^2)$, $\mathcal{O}(\alpha_s^{3/2})$, respectively, and discarded. We take the truncations $n \leq 10$ and $k \leq 1$ in a consistent way, the operators of the orders $\mathcal{O}(\alpha_s^k)$ with $k > 1$ are discarded. Furthermore, the values of the condensates $\langle g_s^3 GGG\rangle$, $\langle\frac{\alpha_s GG}{\pi}\rangle^2$, $\langle\frac{\alpha_s GG}{\pi}\rangle\langle\bar{s}g_s\sigma Gs\rangle$ are very small, and they can be neglected safely. In Refs. [36,39,54–56], the same truncations are taken to study the hidden-charmed and hidden-bottom tetraquark states and molecular states with the QCD sum rules, and to obtain the energy-scale formula, such truncations work well.

Differentiating Eq. (22) with respect to $\frac{1}{T^2}$, then eliminating the pole residues $\lambda_{Y/Z}$, we obtain the QCD sum rules for the masses of the scalar, axial-vector and tensor $D_s^*\bar{D}_s^*$ and $B_s^*\bar{B}_s^*$ molecular states,

$$M_{Y/Z}^2 = \frac{\int_{4m_Q^2}^{s_0} \mathrm{d}s \frac{d}{d(-1/T^2)}\rho(s)\exp\left(-\frac{s}{T^2}\right)}{\int_{4m_Q^2}^{s_0} \mathrm{d}s\rho(s)\exp\left(-\frac{s}{T^2}\right)}. \quad (46)$$

We can obtain the QCD sum rules for the $D^*\bar{D}^*$ and $B^*\bar{B}^*$ molecular states with the simple replacements,

$$m_s \to 0,$$
$$\langle\bar{s}s\rangle \to \langle\bar{q}q\rangle,$$
$$\langle\bar{s}g_s\sigma Gs\rangle \to \langle\bar{q}g_s\sigma Gq\rangle. \quad (47)$$

For the tetraquark and molecular states, it is more reasonable to refer to the $\lambda_{X/Y/Z}$ as the pole residues (not the decay constants). We cannot obtain the true values of the pole residues $\lambda_{X/Y/Z}$ by measuring the leptonic decays as in the cases of the $D_s(D)$ and $J/\psi(\Upsilon)$, $D_s(D) \to \ell\nu$ and $J/\psi(\Upsilon) \to \mathrm{e}^+\mathrm{e}^-$, and have to calculate the $\lambda_{X/Y/Z}$ using some theoretical methods. It is hard to obtain the true values. In this article, we focus on the masses to study the molecular states, and the unknown contributions of the perturbative corrections to the pole residues in the numerator and denominator are expected to be canceled out with each other efficiently, as we obtain the hadronic masses $M_{Y/Z}$ through a ratio; see Eq. (46). Neglecting perturbative $\mathcal{O}(\alpha_s)$ corrections cannot impair the predictive ability qualitatively.

3 Numerical results and discussions

The vacuum condensates are taken to be the standard values $\langle\bar{q}q\rangle = -(0.24\pm 0.01\,\mathrm{GeV})^3$, $\langle\bar{s}s\rangle = (0.8\pm 0.1)\langle\bar{q}q\rangle$, $\langle\bar{q}g_s\sigma Gq\rangle = m_0^2\langle\bar{q}q\rangle$, $\langle\bar{s}g_s\sigma Gs\rangle = m_0^2\langle\bar{s}s\rangle$, $m_0^2 = (0.8\pm 0.1)\,\mathrm{GeV}^2$, $\langle\frac{\alpha_s GG}{\pi}\rangle = (0.33\,\mathrm{GeV})^4$ at the energy scale $\mu = 1\,\mathrm{GeV}$ [62,63,67,68]. The quark condensates and mixed quark condensates evolve with the renormalization group equation, $\langle\bar{q}q\rangle(\mu) = \langle\bar{q}q\rangle(Q)\left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{4}{9}}$, $\langle\bar{s}s\rangle(\mu) = \langle\bar{s}s\rangle(Q)\left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{4}{9}}$, $\langle\bar{q}g_s\sigma Gq\rangle(\mu) = \langle\bar{q}g_s\sigma Gq\rangle(Q)\left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{2}{27}}$, $\langle\bar{s}g_s\sigma Gs\rangle(\mu) = \langle\bar{s}g_s\sigma Gs\rangle(Q)\left[\frac{\alpha_s(Q)}{\alpha_s(\mu)}\right]^{\frac{2}{27}}$.

In the article, we take the $\overline{MS}$ masses $m_c(m_c) = (1.275\pm 0.025)\,\mathrm{GeV}$, $m_b(m_b) = (4.18\pm 0.03)\,\mathrm{GeV}$, and $m_s(\mu = 2\,\mathrm{GeV}) = (0.095\pm 0.005)\,\mathrm{GeV}$ from the Particle Data Group [4], and take into account the energy-scale dependence of the $\overline{MS}$ masses from the renormalization group equation,

$$m_s(\mu) = m_s(2\mathrm{GeV})\left[\frac{\alpha_s(\mu)}{\alpha_s(2\mathrm{GeV})}\right]^{\frac{4}{9}},$$
$$m_c(\mu) = m_c(m_c)\left[\frac{\alpha_s(\mu)}{\alpha_s(m_c)}\right]^{\frac{12}{25}},$$
$$m_b(\mu) = m_b(m_b)\left[\frac{\alpha_s(\mu)}{\alpha_s(m_b)}\right]^{\frac{12}{23}},$$
$$\alpha_s(\mu) = \frac{1}{b_0 t}\left[1 - \frac{b_1}{b_0^2}\frac{\log t}{t} + \frac{b_1^2(\log^2 t - \log t - 1) + b_0 b_2}{b_0^4 t^2}\right], \quad (48)$$

where $t = \log\frac{\mu^2}{\Lambda^2}$, $b_0 = \frac{33-2n_f}{12\pi}$, $b_1 = \frac{153-19n_f}{24\pi^2}$, $b_2 = \frac{2857-\frac{5033}{9}n_f+\frac{325}{27}n_f^2}{128\pi^3}$, $\Lambda = 213\,\mathrm{MeV}$, $296\,\mathrm{MeV}$ and $339\,\mathrm{MeV}$ for the flavors $n_f = 5$, 4, and 3, respectively [4].

In the conventional QCD sum rules [62,63], there are two criteria (pole dominance and convergence of the operator product expansion) for choosing the Borel parameter T^2 and threshold parameter s_0. We impose the two criteria on the hidden-charmed (or bottom) molecular states, and search for the optimal values.

In Refs. [36,39,54–56], we study the acceptable energy scales of the QCD spectral densities in the QCD sum rules for the hidden charmed (bottom) tetraquark states and molecular states in detail for the first time, and suggest a formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_Q)^2}$ to determine the energy scales of the QCD spectral densities. The heavy tetraquark system $Q\bar{Q}q'\bar{q}$ could be described by a double-well potential with two light quarks $q'\bar{q}$ lying in the two wells, respectively. In the heavy quark limit, the Q-quark can be taken as a static well potential, which binds the light quark q' to form a diquark in the color antitriplet channel or binds the light antiquark $\bar{q}$ to form a meson in the color singlet channel (or a meson-like state in the color octet channel). Then the heavy tetraquark states are characterized by the effective heavy quark masses $\mathbb{M}_Q$ (or constituent quark masses) and the virtuality $V = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_Q)^2}$ (or bound energy, being not as robust). The effective masses $\mathbb{M}_Q$, just like the mixed condensates, appear as parameters and their values are fitted by the QCD sum rules. The effective masses $\mathbb{M}_Q$ have uncertainties, the optimal val-

Table 1 The Borel parameters, continuum threshold parameters, pole contributions, energy scales, masses, and pole residues of the scalar, axial-vector, and tensor molecular states. The symbolic quark constituents are shown in the bracket

J^{PC}	μ (GeV)	T^2 (GeV^2)	s_0 (GeV^2)	Pole (%)	$M_{Y/Z}$ (GeV)	$\lambda_{Y/Z}$ (GeV$^{5(4)}$)
0^{++} ($c\bar{c}u\bar{d}$)	1.6	2.5−2.9	20 ± 1	(43−68)	$4.01^{+0.09}_{-0.09}$	$3.97^{+0.67}_{-0.60} \times 10^{-2}$
1^{+-} ($c\bar{c}u\bar{d}$)	1.7	2.8−3.2	20 ± 1	(45−68)	$4.04^{+0.07}_{-0.08}$	$6.37^{+0.96}_{-0.89} \times 10^{-3}$
2^{++} ($c\bar{c}u\bar{d}$)	1.6	2.6−3.0	20 ± 1	(45−69)	$4.01^{+0.10}_{-0.08}$	$3.05^{+0.47}_{-0.44} \times 10^{-2}$
0^{++} ($c\bar{c}s\bar{s}$)	1.8	2.8−3.2	22 ± 1	(46−69)	$4.14^{+0.08}_{-0.08}$	$5.75^{+0.96}_{-0.85} \times 10^{-2}$
1^{+-} ($c\bar{c}s\bar{s}$)	1.9	3.2−3.6	22 ± 1	(48−68)	$4.16^{+0.05}_{-0.04}$	$8.80^{+0.60}_{-0.57} \times 10^{-3}$
2^{++} ($c\bar{c}s\bar{s}$)	1.8	3.0−3.4	22 ± 1	(47−68)	$4.13^{+0.08}_{-0.08}$	$4.34^{+0.67}_{-0.60} \times 10^{-2}$
0^{++} ($b\bar{b}u\bar{d}$)	2.8	6.8−7.8	124 ± 2	(44−65)	$10.65^{+0.15}_{-0.09}$	$2.07^{+0.45}_{-0.32} \times 10^{-1}$
1^{+-} ($b\bar{b}u\bar{d}$)	2.9	7.0−8.0	124 ± 2	(45−65)	$10.67^{+0.09}_{-0.08}$	$1.34^{+0.20}_{-0.18} \times 10^{-2}$
2^{++} ($b\bar{b}u\bar{d}$)	2.8	7.2−8.2	124 ± 2	(44−64)	$10.66^{+0.14}_{-0.09}$	$1.67^{+0.31}_{-0.23} \times 10^{-1}$
0^{++} ($b\bar{b}s\bar{s}$)	2.9	7.0−8.0	126 ± 2	(45−66)	$10.70^{+0.11}_{-0.08}$	$2.49^{+0.41}_{-0.35} \times 10^{-1}$
1^{+-} ($b\bar{b}s\bar{s}$)	3.0	7.2−8.2	126 ± 2	(47−66)	$10.73^{+0.09}_{-0.07}$	$1.63^{+0.23}_{-0.21} \times 10^{-2}$
2^{++} ($b\bar{b}s\bar{s}$)	3.0	7.8−8.8	128 ± 2	(48−66)	$10.71^{+0.08}_{-0.08}$	$2.31^{+0.31}_{-0.27} \times 10^{-1}$

ues in the diquark–antidiquark systems are not necessarily the ideal values in the meson–meson systems. The QCD sum rules have three typical energy scales μ^2, T^2, V^2. It is natural to take the energy scale, $\mu^2 = V^2 = \mathcal{O}(T^2)$. The effective masses $\mathbb{M}_c = 1.84$ GeV and $\mathbb{M}_b = 5.14$ GeV are the optimal values for the hadronic molecular states, and can reproduce the experimental data $M_{X(3872)} = 3.87$ GeV, $M_{Z_c(3900)} = 3.90$ GeV, $M_{Z_b(10610)} = 10.61$ GeV approximately [36]. In this article, we take the effective masses $\mathbb{M}_c = 1.84$ GeV and $\mathbb{M}_b = 5.14$ GeV, and the predictions indicate that they are also the optimal values to reproduce the experimental values of the masses of the $Z_c(4020)$, $Z_c(4025)$, $Y(4140)$, and $Z_b(10650)$.

The energy-scale formula serves as additional constraints on choosing the Borel parameters and threshold parameters, as the predicted masses should satisfy the formula. The optimal Borel parameters and continuum threshold parameters therefore for the pole contributions and energy scales of the QCD spectral densities are shown explicitly in Table 1.

In Fig. 1, the masses of the scalar $D^*\bar{D}^*$ and $D_s^*\bar{D}_s^*$ molecular states are plotted with variations of the Borel parameters T^2 and energy scales μ for the continuum threshold parameters $s^0_{D^*\bar{D}^*} = 20\,\text{GeV}^2$ and $s^0_{D_s^*\bar{D}_s^*} = 22\,\text{GeV}^2$, respectively. From the figure, we can see that the masses decrease monotonously with increase of the energy scales, the energy scales $\mu = (1.5–1.6)$ GeV and $\mu = (1.7–1.9)$ GeV can reproduce the experimental values of the masses $M_{Z_c(4025)}$ (or $M_{Z_c(4020)}$) and $M_{Y(4140)}$, respectively. The formula $\mu = \sqrt{M^2_{X/Y/Z} - (2\mathbb{M}_Q)^2}$ leads to the values $\mu = 1.6$ GeV and $\mu = 1.8$ GeV for the scalar $D^*\bar{D}^*$ and $D_s^*\bar{D}_s^*$ molecular states, respectively. The agreement is excellent. The masses $M_{Y(3940)} < M_{Z_c(4025)}$, the energy scale of the QCD spectral density of the $Y(3940)$ should be smaller than that of the $Z_c(4025)$ according to the energy formula. From Fig. 1, we can see that the predicted mass is larger than 3.95 GeV even for the energy scale $\mu = 1.8$ GeV, and we cannot satisfy the relation $\sqrt{s_0} \approx M_{Y(3940)} + 0.5$ GeV with reasonable $M_{Y(3940)}$ compared to the experimental data. Now the $X(3915)$ is listed in the Review of Particle Physics as the $\chi_{c0}(2\text{P})$ state with $J^{PC} = 0^{++}$ [4]. The present result supports the assignment of the Particle Data Group. In Ref. [15,16], we study the scalar $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ molecular states with the QCD sum rules by carrying out the operator product expansion to the vacuum condensates up to dimension 10 and setting the energy scale to be $\mu = 1$ GeV. The predicted masses are about (250–500) MeV above the corresponding $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, and $B_s^*\bar{B}_s^*$ thresholds. If larger energy scales are taken, the conclusion should be modified.

In Figs. 2, 3, the contributions of different terms in the operator product expansion are plotted with variations of the Borel parameters T^2 for the energy scales and central values of the threshold parameters shown in Table 1. The contributions of the condensates do not decrease monotonously with increase of dimensions. However, in the Borel windows shown in Table 1, the D_4, D_7, D_{10} play a less important role, $D_3 \gg |D_5| \gg D_6 \gg |D_8|$ for the $J = 2$ molecular states and $J = 0$ $D_s^*\bar{D}_s^*$ molecular state, $D_3 \gg |D_5| \gg D_6$ for the $J = 1$ molecular states, $D_3 \gg |D_5| \sim D_6 \gg |D_8|$ for the $J = 0$ $D^*\bar{D}^*$ and $B_s^*\bar{B}_s^*$ molecular states, $D_3 > D_6 > |D_5| \sim |D_8|$ for the $J = 0$ $B^*\bar{B}^*$ molecular state, the D_6, D_8, D_{10} decrease monotonously and quickly with increase of the Borel parameters for the $J = 0, 2$ molecular states, where the D_i with $i = 0, 3, 4, 5, 6, 7, 8, 10$ denote the contributions of the vacuum condensates of dimensions $D = i$, and the total contributions are normalized to be 1. The convergence of the operator product expansion does not mean that the perturbative terms make dominant con-

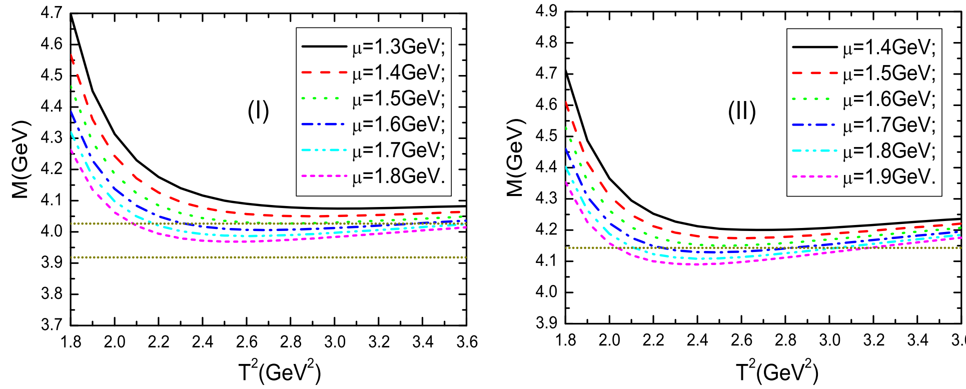

Fig. 1 The masses with variations of the Borel parameters T^2 and energy scales μ, where the horizontal lines denote the experimental values of the masses of the $Z_c(4025)$, $Y(3940)$, and $Y(4140)$, respectively, (I) and (II) denote the scalar $D^*\bar{D}^*$ and $D_s^*\bar{D}_s^*$ molecular states, respectively

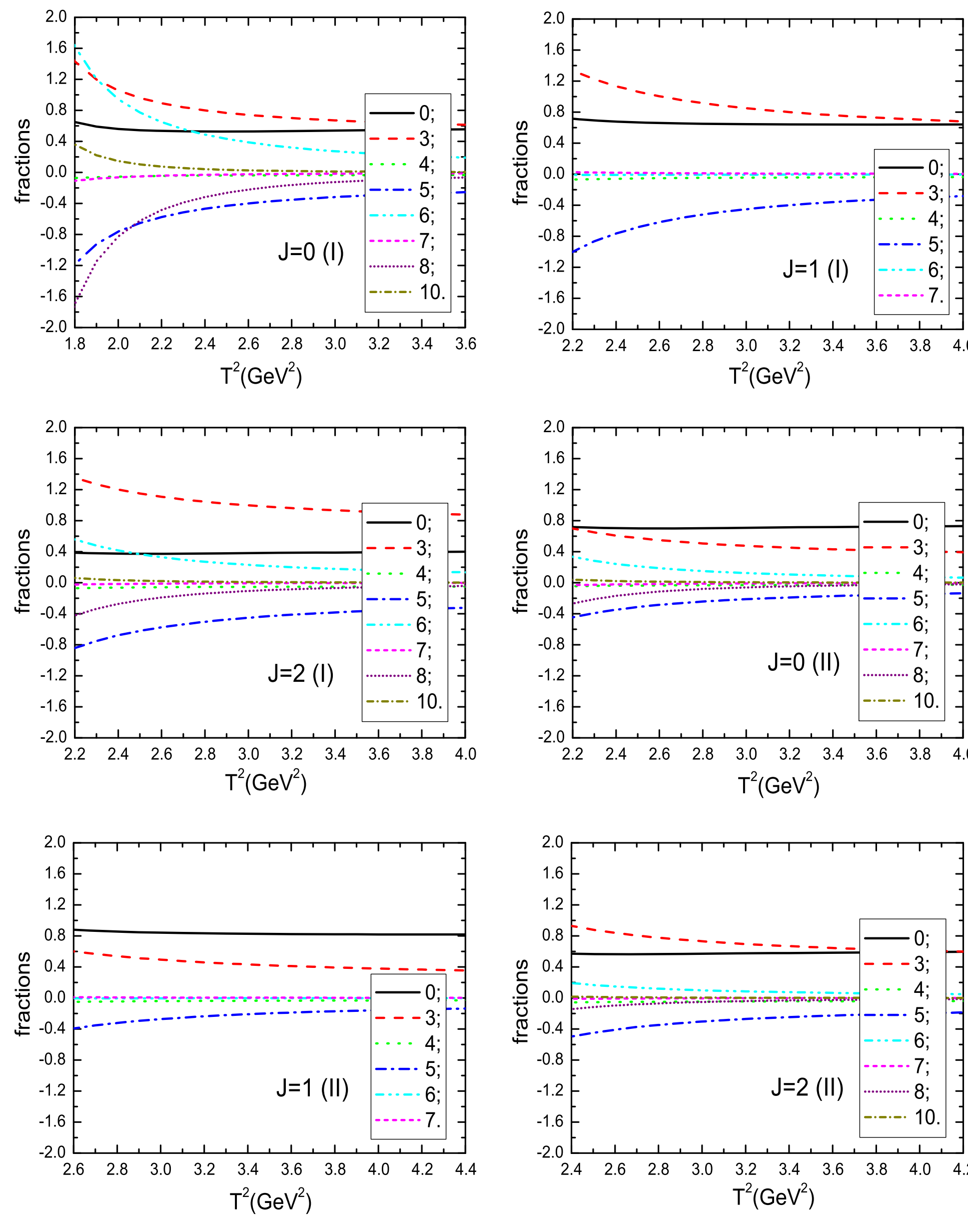

Fig. 2 The contributions of different terms in the operator product expansion with variations of the Borel parameters T^2, where the 0, 3, 4, 5, 6, 7, 8, 10 denotes the dimensions of the vacuum condensates, the $J = 0, 1, 2$ denote the angular momentum of the molecular states, the (I) and (II) denote the $D^*\bar{D}^*$ and $D_s^*\bar{D}_s^*$ molecular states, respectively

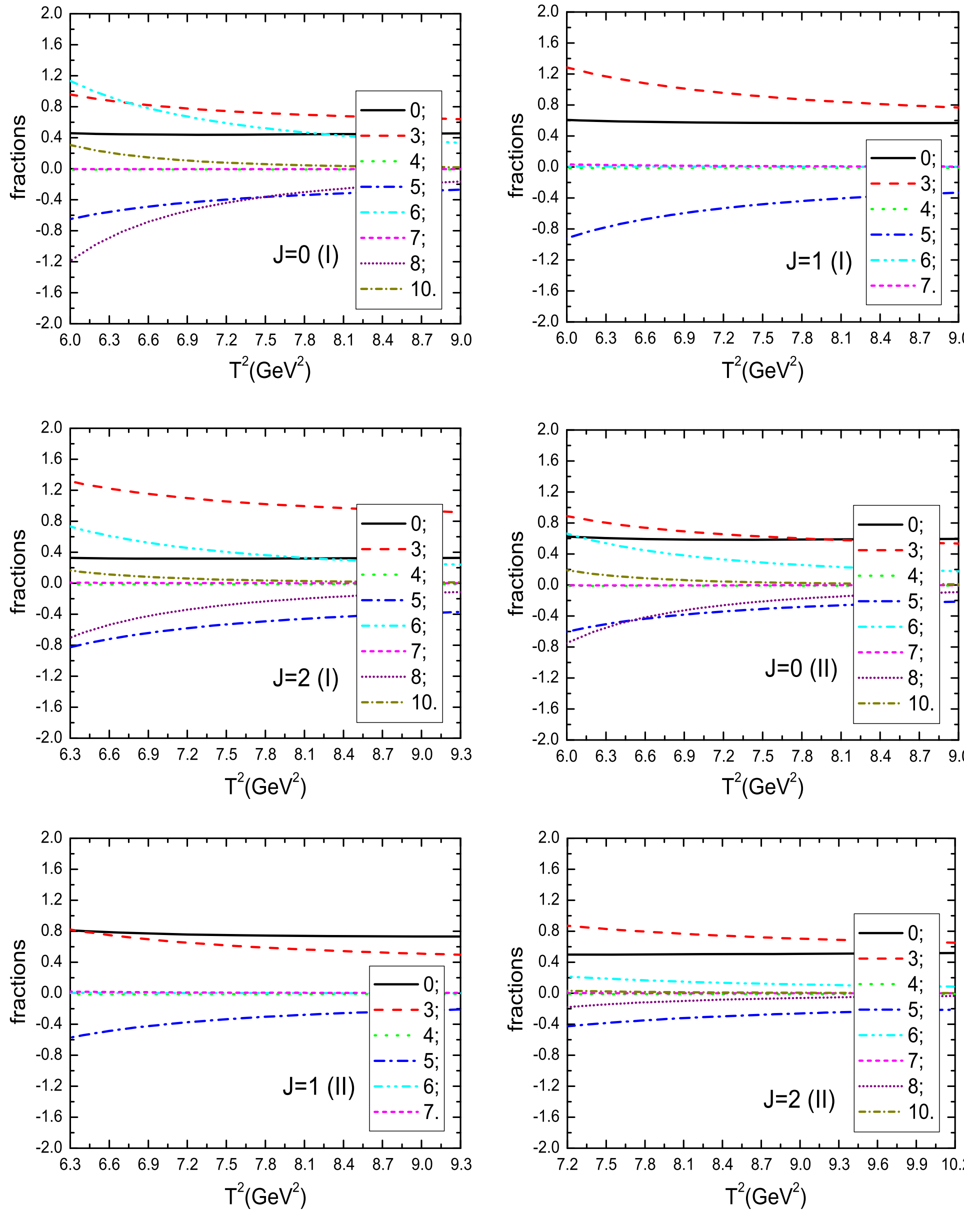

Fig. 3 The contributions of different terms in the operator product expansion with variations of the Borel parameters T^2, where the 0, 3, 4, 5, 6, 7, 8, 10 denotes the dimensions of the vacuum condensates, the $J = 0, 1, 2$ denote the angular momentum of the molecular states, the (I) and (II) denote the $B^*\bar{B}^*$ and $B_s^*\bar{B}_s^*$ molecular states, respectively

tributions, as the continuum hadronic spectral densities are approximated by $\rho_{QCD}(s)\Theta(s - s_0)$ in the QCD sum rules for the heavy molecular states, where the $\rho_{QCD}(s)$ denotes the full QCD spectral densities; the contributions of the quark condensates $\langle\bar{q}q\rangle$ and $\langle\bar{s}s\rangle$ (of dimension 3) can be very large. In summary, the two criteria (pole dominance and convergence of the operator product expansion) of the QCD sum rules are fully satisfied, so we expect to make reasonable predictions.

We take into account all uncertainties of the input parameters, and obtain the values of the masses and pole residues of the scalar, axial-vector and tensor molecular states, which are shown explicitly in Figs. 4, 5 and Table 1.

The uncertainties of the effective masses $\mathbb{M}_Q$ and energy scales μ have the correlation,

$$4\mathbb{M}_Q\delta\mathbb{M}_Q = -\mu\delta\mu. \tag{49}$$

If we take the uncertainty $\delta\mu = 0.3$ GeV, the induced uncertainties are $\delta\mathbb{M}_c \approx 0.07$ GeV, $\delta\mathbb{M}_b \approx 0.04$ GeV, $\delta M_{Y/Z_b} \approx 100$ MeV, $\delta M_{Y/Z_c} \approx 50$ MeV, $\delta M_{Y/Z}/M_{Y/Z} \approx 1\,\%$, $\delta\lambda_{Y/Z_c}/\lambda_{Y/Z_c} \approx 10\,\%$ and $\delta\lambda_{Y/Z_b}/\lambda_{Y/Z_b} \approx 20\,\%$; see Table 2. The uncertainties $\delta M_{Y/Z}/M_{Y/Z} \ll \delta\lambda_{Y/Z}/\lambda_{Y/Z}$, we obtain the hadronic masses $M_{Y/Z}$ through a ratio; see Eq. (46), the energy-scale dependence of the hadronic masses $M_{Y/Z}$ originate from the numerator and denominator are canceled out with each other efficiently, the predicted masses are robust. On the other hand, if we take the uncertainties of the experimental values of the masses of the $Z_c(4020)$, $Z_c(4025)$, $Y(4140)$, $Z_b(10650)$ as the input parameters [5,22,43,44], the allowed uncertainties are $|\delta\mu| \ll 0.1$ GeV,

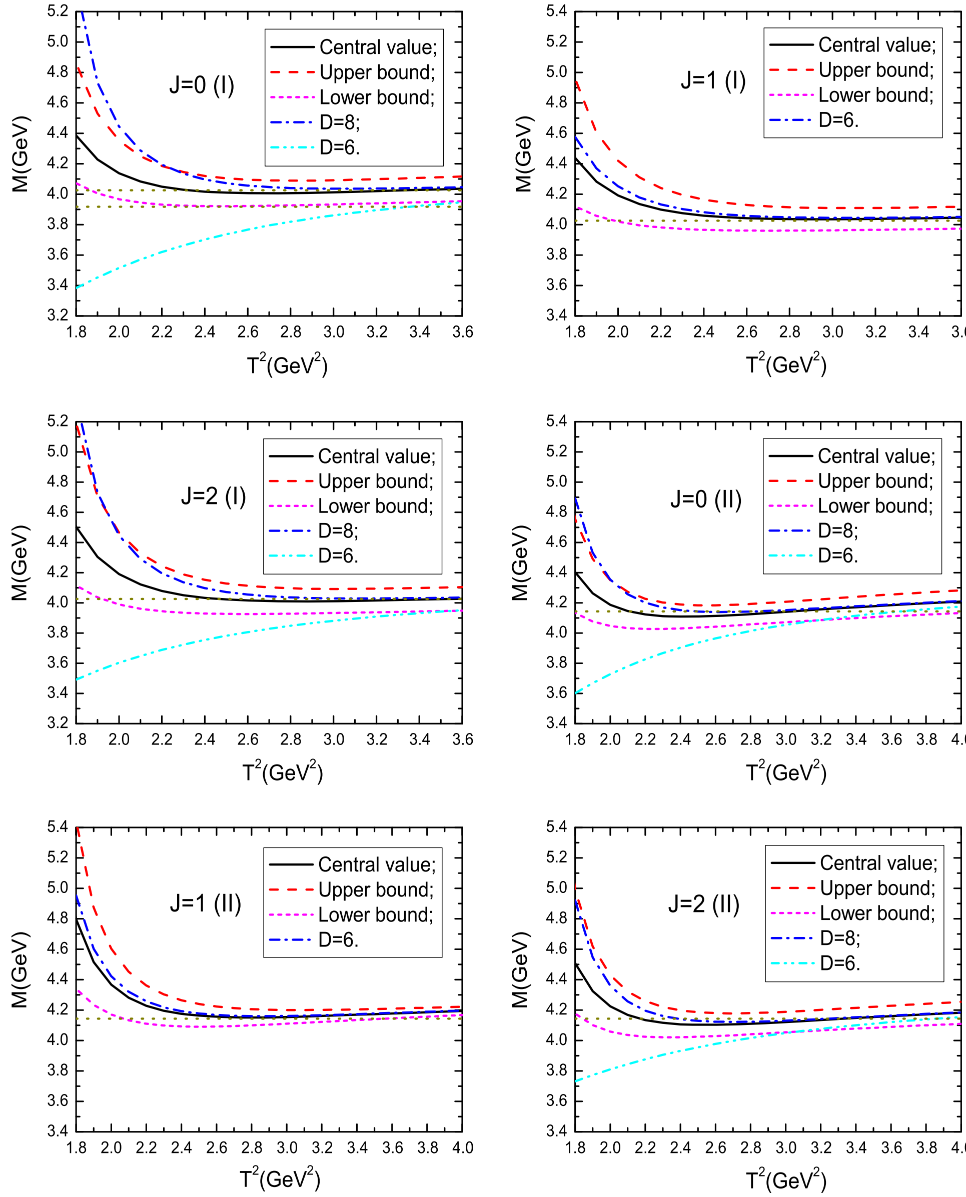

Fig. 4 The masses with variations of the Borel parameters T^2, where the horizontal lines denote the experimental values of the masses of the $Z_c(4025)$, $Y(3940)$, and $Y(4140)$, the $J = 0, 1, 2$ denote the angular momentum of the molecular states, the (I) and (II) denote the $D^*\bar{D}^*$ and $D_s^*\bar{D}_s^*$ molecular states, respectively. The $D = 8$ and $D = 6$ denote the vacuum condensates are taken into account up to dimensions 8 and 6, respectively

$\delta\mathbb{M}_c \ll 0.03$ GeV, $\delta\mathbb{M}_b \ll 0.02$ GeV. In Refs. [17,18,38,50], the authors study the $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$ molecular states by choosing the $\overline{MS}$ masses $m_Q(m_Q)$ and the vacuum condensates $\langle\bar{q}q\rangle_{\mu=1\,\mathrm{GeV}}$, $\langle\bar{q}g_s\sigma Gq\rangle_{\mu=1\,\mathrm{GeV}}$, etc. In this article, we calculate the QCD spectral densities at a special energy scale μ consistently, the energy scales μ are determined by the parameters $\mathbb{M}_Q$, which have very small allowed uncertainties. The correlation functions $\Pi(p)$ can be written as

$$\Pi(p) = \int_{4m_Q^2(\mu)}^{s_0} \mathrm{d}s \frac{\rho_{QCD}(s,\mu)}{s-p^2} + \int_{s_0}^{\infty} \mathrm{d}s \frac{\rho_{QCD}(s,\mu)}{s-p^2}, \quad (50)$$

through dispersion relation at the QCD side, and they are scale independent,

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\Pi(p) = 0, \quad (51)$$

which does not mean

$$\frac{\mathrm{d}}{\mathrm{d}\mu} \int_{4m_Q^2(\mu)}^{s_0} \mathrm{d}s \frac{\rho_{QCD}(s,\mu)}{s-p^2} \to 0, \quad (52)$$

due to the following two reasons inherited from the QCD sum rules:

- Perturbative corrections are neglected, the higher dimensional vacuum condensates are factorized into lower dimensional ones; therefore the energy-scale dependence of the higher dimensional vacuum condensates is modified;

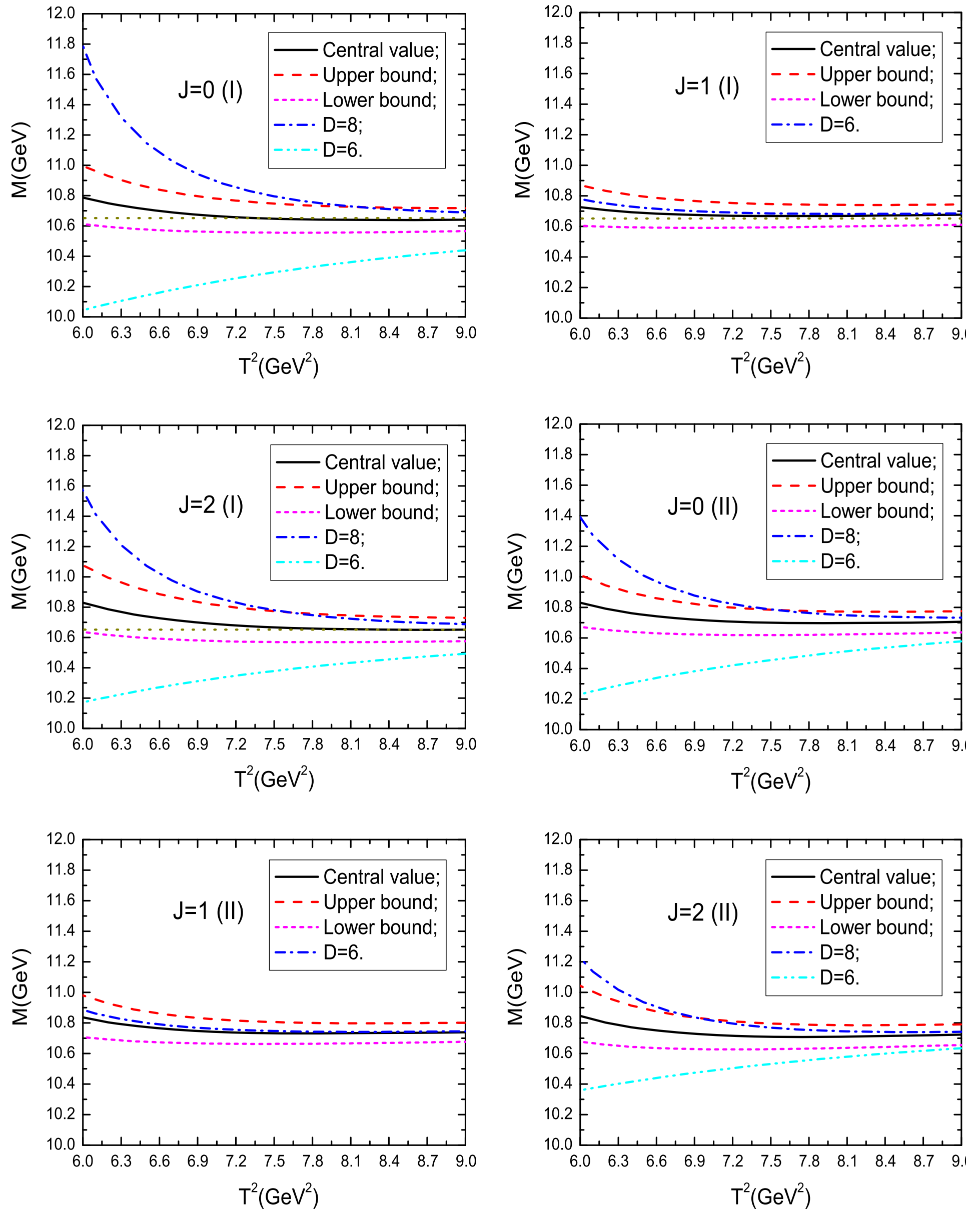

Fig. 5 The masses with variations of the Borel parameters T^2, where the horizontal lines denote the experimental value of the mass of the $Z_b(10650)$, the $J = 0, 1, 2$ denote the angular momentum of the molecular states, the (I) and (II) denote the $B^*\bar{B}^*$ and $B_s^*\bar{B}_s^*$ molecular states, respectively. The $D = 8$ and $D = 6$ denote the vacuum condensates are taken into account up to dimensions 8 and 6, respectively

- truncations s_0 set in, the correlation between the threshold $4m_Q^2(\mu)$ and continuum threshold s_0 is unknown, the quark–hadron duality is an assumption.

We cannot obtain energy-scale independent QCD sum rules, but we have an energy-scale formula to determine the energy scales consistently.

The present predictions $M^{J=2}_{D^*\bar{D}^*} = \left(4.01^{+0.10}_{-0.08}\right)$ GeV, $M^{J=1}_{D^*\bar{D}^*} = \left(4.04^{+0.07}_{-0.08}\right)$ GeV, $M^{J=0}_{D^*\bar{D}^*} = \left(4.01^{+0.09}_{-0.09}\right)$ GeV are consistent with the experimental values $M_{Z_c(4025)} = (4026.3 \pm 2.6 \pm 3.7)$ MeV, $M_{Z_c(4020)} = (4022.9 \pm 0.8 \pm 2.7)$ MeV from the BESIII collaboration [43,44]. More experimental data on the spin and parity are still needed to identify the $Z_c(4020)$ and $Z_c(4025)$ unambiguously. In Ref. [50], K. P. Khemchandani et al. carry out the operator product expansion up to dimension 6 and obtain the values $M^{J=2}_{D^*\bar{D}^*} = (3946 \pm 104)$ MeV, $M^{J=1}_{D^*\bar{D}^*} = (3950 \pm 105)$ MeV, $M^{J=0}_{D^*\bar{D}^*} = (3943 \pm 104)$ MeV. The central values are smaller than ours about 50 MeV. In the calculations, we observe that the vacuum condensates of dimensions 7, 8, 10 play an important role in determining the Borel windows, and warrant platforms for the masses and pole residues. The conclusion survives in the QCD sum rules for the tetraquark states and molecular states consist of two heavy quarks and two light quarks. There appear terms of the orders $\mathcal{O}\left(\frac{1}{T^2}\right)$, $\mathcal{O}\left(\frac{1}{T^4}\right)$, $\mathcal{O}\left(\frac{1}{T^6}\right)$ in the QCD spectral densities, if we take into account the vacuum condensates whose dimensions

Table 2 The uncertainties originate from the uncertainty of the energy scale $\delta\mu = 0.3\,\text{GeV}$

J^{PC}	μ (GeV)	$\delta\mathbb{M}_Q$ (GeV)	$\delta M_{Y/Z}$ (GeV)	$\delta\lambda_{Y/Z}/\lambda_{Y/Z}$ (%)
0^{++} $(c\bar{c}u\bar{d})$	1.6 ± 0.3	± 0.07	${}^{+0.07}_{-0.05}$	${}^{+9}_{-12}$
1^{+-} $(c\bar{c}u\bar{d})$	1.7 ± 0.3	± 0.07	${}^{+0.06}_{-0.05}$	${}^{+9}_{-12}$
2^{++} $(c\bar{c}u\bar{d})$	1.6 ± 0.3	± 0.07	${}^{+0.08}_{-0.05}$	${}^{+8}_{-11}$
0^{++} $(c\bar{c}s\bar{s})$	1.8 ± 0.3	± 0.07	${}^{+0.05}_{-0.03}$	${}^{+7}_{-9}$
1^{+-} $(c\bar{c}s\bar{s})$	1.9 ± 0.3	± 0.08	${}^{+0.05}_{-0.03}$	${}^{+6}_{-9}$
2^{++} $(c\bar{c}s\bar{s})$	1.8 ± 0.3	± 0.07	${}^{+0.05}_{-0.03}$	${}^{+6}_{-8}$
0^{++} $(b\bar{b}u\bar{d})$	2.8 ± 0.3	± 0.04	${}^{+0.14}_{-0.10}$	${}^{+18}_{-19}$
1^{+-} $(b\bar{b}u\bar{d})$	2.9 ± 0.3	± 0.04	${}^{+0.12}_{-0.10}$	${}^{+19}_{-20}$
2^{++} $(b\bar{b}u\bar{d})$	2.8 ± 0.3	± 0.04	${}^{+0.13}_{-0.11}$	${}^{+17}_{-19}$
0^{++} $(b\bar{b}s\bar{s})$	2.9 ± 0.3	± 0.04	${}^{+0.12}_{-0.10}$	${}^{+16}_{-18}$
1^{+-} $(b\bar{b}s\bar{s})$	3.0 ± 0.3	± 0.04	${}^{+0.11}_{-0.09}$	${}^{+18}_{-19}$
2^{++} $(b\bar{b}s\bar{s})$	3.0 ± 0.3	± 0.04	${}^{+0.11}_{-0.08}$	${}^{+15}_{-16}$

are larger than 6 [36,39,54–56]. The terms associate with $\frac{1}{T^2}$, $\frac{1}{T^4}$, $\frac{1}{T^6}$ in the QCD spectral densities manifest themselves at small values of the Borel parameter T^2, we have to choose large values of the T^2 to warrant convergence of the operator product expansion and appearance of the Borel platforms. In the Borel windows, the higher dimension vacuum condensates play a less important role. In summary, the higher dimension vacuum condensates play an important role in determining the Borel windows therefore the ground state masses and pole residues, so we should take them into account consistently. In Figs. 4, 5, we also plot the masses by taking into account the vacuum condensates up to dimension 6 and 8, respectively. From the figures, we can see that neglecting the vacuum condensates of the dimensions 7, 8, 10 cannot lead to platforms flat enough so as to extract robust values.

The present predictions $M^{J=2}_{D_s^*\bar{D}_s^*} = \left(4.13^{+0.08}_{-0.08}\right)$ GeV, $M^{J=1}_{D_s^*\bar{D}_s^*} = \left(4.16^{+0.05}_{-0.04}\right)$ GeV, $M^{J=0}_{D_s^*\bar{D}_s^*} = \left(4.14^{+0.08}_{-0.08}\right)$ GeV are consistent with the experimental value $M_{Y(4140)} = (4143.0 \pm 2.9 \pm 1.2)$ MeV from the CDF collaboration [5]. The CMS collaboration fitted the peaking structure in the $J/\psi\phi$ mass spectrum to a S-wave relativistic Breit–Wigner line-shape with a statistical significance exceeding 5σ [8]. We can tentatively assign the $Y(4140)$ to the scalar $D_s^*\bar{D}_s^*$ molecular state, while there lack experimental candidates for the axial-vector and tensor $D_s^*\bar{D}_s^*$ molecular states. We can search for the axial-vector and tensor $D_s^*\bar{D}_s^*$ molecular states in the $J/\psi\phi$ mass spectrum and measure the angular correlation to determine the spin and parity.

The present predictions $M^{J=2}_{B^*\bar{B}^*} = \left(10.66^{+0.14}_{-0.09}\right)$ GeV, $M^{J=1}_{B^*\bar{B}^*} = \left(10.67^{+0.09}_{-0.08}\right)$ GeV, $M^{J=0}_{B^*\bar{B}^*} = \left(10.65^{+0.15}_{-0.09}\right)$ GeV are consistent with the experimental value $M_{Z_b(10650)} = (10652.2 \pm 1.5)$ MeV from the Belle collaboration [22], while the Belle data favors the $J^{PC} = 1^{+-}$ assignment. We can tentatively assign the $Z_b(10650)$ to the axial-vector $B^*\bar{B}^*$ molecular state, while there lack experimental candidates for the scalar and tensor $B^*\bar{B}^*$ molecular states. We can search for the scalar and tensor $B^*\bar{B}^*$ molecular states in the $\Upsilon\varphi$ mass spectrum and measure the angular correlations to determine the spin and parity.

There also lack experimental candidates for the $B_s^*\bar{B}_s^*$ molecular states, we can search for them in the $\Upsilon\phi$ mass spectrum and measure the angular correlations to determine the spin and parity.

In Refs. [39,54,55], we resort to the same routine to study the heavy tetraquark states, the predicted masses favor assigning the $Z_c(4020)$ and $Z_c(4025)$ to the 1^{+-} or 2^{++} tetraquark states, and the $Z_b(10650)$ to the 1^{+-} tetraquark state. A hadron cannot be identified unambiguously by the mass alone [38], so it is interesting to explore possible assignments in the scenario of molecular states. The predicted masses of the heavy molecular states also favor assigning the $Z_c(4020)$ and $Z_c(4025)$ to the 1^{+-} or 2^{++} molecular states, the $Z_b(10650)$ to the 1^{+-} molecular state. The $Z_c(4020)$, $Z_c(4025)$, $Z_b(10650)$ maybe have both tetraquark and molecule components, which should be interpolated by the tetraquark-type currents and molecule-type currents, respectively. In the present work and Refs. [36,39,54–56], we obtain the pole residues (or the current-hadron coupling constants), which can be taken as basic input parameters to study the strong decays of the heavy tetraquark states or molecular states with the three-point QCD sum rules. Then we obtain more knowledge to identify the $Z_c(4020)$, $Z_c(4025)$, $Z_b(10650)$. In the scenario of meta-stable Feshbach resonances, the $Z_c(4025)$ and $Z_b(10650)$ are taken as the $h_c(2\text{P})\pi - D^*\bar{D}^*$ and $\chi_{b1}\rho - B^*\bar{B}^*$ hadrocharmonium-molecule mixed states, respectively, where the $\chi_{b1}\rho$ is a P-wave system [69]. The hadrocharmonium system admits bound states giving rise to a discrete spectrum of levels, a resonance occurs if one of such levels falls close to some opencharm (open-bottom) threshold, as the coupling between channels leads to an attractive interaction and favors the formation of a meta-stable Feshbach resonance. We can borrow some ideas from the meta-stable Feshbach resonances, the couplings between the tetraquark states and molecular states leads to an attractive interaction and favors the formation of the $Z_c(4020)$, $Z_c(4025)$, $Z_b(10650)$, as they couple potentially both to the tetraquark-type and molecule-type currents.

4 Conclusion

In this article, we calculate the contributions of the vacuum condensates up to dimension 10 and discard the perturbative corrections in the operator product expansion, and we study the $J^{PC} = 0^{++}$, 1^{+-} and 2^{++} $D^*\bar{D}^*$, $D_s^*\bar{D}_s^*$, $B^*\bar{B}^*$, $B_s^*\bar{B}_s^*$ molecular states in detail with the QCD sum rules. In the calculations, we use the formula $\mu = \sqrt{M_{X/Y/Z}^2 - (2\mathbb{M}_Q)^2}$ suggested in our previous work to determine the energy scales of the QCD spectral densities. The present predictions favor assigning the $Z_c(4020)$ and $Z_c(4025)$ to the $J^{PC} = 0^{++}$, 1^{+-} or 2^{++} $D^*\bar{D}^*$ molecular states, the $Y(4140)$ to the $J^{PC} = 0^{++}$ $D_s^* D_s^*$ molecular state, the $Z_b(10650)$ to the $J^{PC} = 1^{+-}$ $B^*\bar{B}^*$ molecular state, and they disfavor assigning the $Y(3940)$ to the $(J^{PC} = 0^{++})$ molecular state. The present predictions can be confronted with the experimental data in the future at BESIII, LHCb, and Belle-II. The pole residues can be taken as basic input parameters to study the relevant processes of the $J^{PC} = 0^{++}$, 1^{+-} and 2^{++} molecular states with the three-point QCD sum rules.

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