Research Article
Stability of Cubic Functional Equation in
the Spaces of Generalized Functions

Young-Su Lee and Soon-Yeong Chung

Received 24 April 2007; Accepted 13 September 2007

Recommended by H. Bevan Thompson

In this paper, we reformulate and prove the Hyers-Ulam-Rassias stability theorem of the cubic functional equation
\[ f(ax + y) + f(ax - y) = af(x + y) + af(x - y) + 2a(a^2 - 1)f(x) \]
for fixed integer \( a \) with \( a \neq 0, \pm 1 \) in the spaces of Schwartz tempered distributions and Fourier hyperfunctions.

Copyright © 2007 Y.-S. Lee and S.-Y. Chung. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

1. Introduction

In 1940, Ulam [1] raised a question concerning the stability of group homomorphisms: “Let \( f \) be a mapping from a group \( G_1 \) to a metric group \( G_2 \) with metric \( d(\cdot, \cdot) \) such that
\[ d(f(xy), f(x)f(y)) \leq \varepsilon. \] (1.1)
Then does there exist a group homomorphism \( L : G_1 \to G_2 \) and \( \delta_\varepsilon > 0 \) such that
\[ d(f(x), L(x)) \leq \delta_\varepsilon \] (1.2)
for all \( x \in G_1 \)?"

The case of approximately additive mappings was solved by Hyers [2] under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces. In 1978, Rassias [3] firstly generalized Hyers’ result to the unbounded Cauchy difference. During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4–12]). The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds and this terminology is also applied to the case of other functional equations.
Let both $E_1$ and $E_2$ be real vector spaces. Jun and Kim [13] proved that a function $f : E_1 \to E_2$ satisfies the functional equation
\[ f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \]  
if and only if there exists a mapping $B : E_1 \times E_1 \times E_1 \to E_2$ such that $f(x) = B(x,x,x)$ for all $x \in E_1$, where $B$ is symmetric for each fixed one variable and additive for each fixed two variables. The mapping $B$ is given by
\[ B(x,y,z) = \frac{1}{24} [f(x+y+z) + f(x-y-z) - f(x+y-z) - f(x-y+z)] \]  
for all $x, y, z \in E_1$. It is natural that (1.3) is called a cubic functional equation because the mapping $f(x) = ax^3$ satisfies (1.3). Also Jun et al. generalized cubic functional equation, which is equivalent to (1.3),
\[ f(ax + y) + f(ax - y) = af(x + y) + af(x - y) + 2a(a^2 - 1)f(x) \]  
for fixed integer $a$ with $a \neq 0, \pm 1$ (see [14]).

In this paper, we consider the general solution of (1.5) and prove the stability theorem of this equation in the space $\mathcal{F}'(\mathbb{R}^n)$ of Schwartz tempered distributions and the space $\mathcal{F}'(\mathbb{R}^n)$ of Fourier hyperfunctions. Following the notations as in [15, 16] we reformulate (1.5) and related inequality as
\[ u \circ A_1 + u \circ A_2 = au \circ B_1 + au \circ B_2 + 2a(a^2 - 1)u \circ P, \]  
\[ ||u \circ A_1 + u \circ A_2 - au \circ B_1 - au \circ B_2 - 2a(a^2 - 1)u \circ P|| \leq \epsilon (|x|^p + |y|^q), \]  
respectively, where $A_1, A_2, B_1, B_2, \text{ and } P$ are the functions defined by
\[ A_1(x,y) = ax + y, \quad A_2(x,y) = ax - y, \]
\[ B_1(x,y) = x + y, \quad B_2(x,y) = x - y, \quad P(x,y) = x, \]
and $p, q$ are nonnegative real numbers with $p,q \neq 3$. We note that $p$ need not be equal to $q$. Here $u \circ A_1, u \circ A_2, u \circ B_1, u \circ B_2, \text{ and } u \circ P$ are the pullbacks of $u$ in $\mathcal{F}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ by $A_1, A_2, B_1, B_2, \text{ and } P$, respectively. Also $| \cdot |$ denotes the Euclidean norm, and the inequality $||v|| \leq \psi(x,y)$ in (1.7) means that $|\langle v, \varphi \rangle| \leq ||\varphi||_{L_1}$ for all test functions $\varphi(x,y)$ defined on $\mathbb{R}^{2n}$.

If $p < 0$ or $q < 0$, the right-hand side of (1.7) does not define a distribution and so inequality (1.7) makes no sense. If $p,q = 3$, it is not guaranteed whether Hyers-Ulam-Rassias stability of (1.5) is hold even in classical case (see [13, 14]). Thus we consider only the case $0 \leq p, q < 3$, or $p,q > 3$.

We prove as results that every solution $u$ in $\mathcal{F}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ of inequality (1.7) can be written uniquely in the form
\[ u = \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk}x_ix_jx_k + h(x), \quad a_{ijk} \in \mathbb{C}, \]
where \( h(x) \) is a measurable function such that
\[
| h(x) | \leq \frac{\epsilon}{2 |a|^3 - |a|^p} |x|^p. \tag{1.10}
\]

### 2. Preliminaries

We first introduce briefly spaces of some generalized functions such as Schwartz tempered distributions and Fourier hyperfunctions. Here we use the multi-index notations, \(|\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha! = \alpha_1! \cdots \alpha_n!\), \(x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}\), and \(\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}\) for \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\), \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n\), where \(\mathbb{N}_0\) is the set of nonnegative integers and \(\partial_j = \partial/\partial x_j\).

**Definition 2.1** [17, 18]. Denote by \(\mathcal{S}(\mathbb{R}^n)\) the Schwartz space of all infinitely differentiable functions \(\varphi\) in \(\mathbb{R}^n\) satisfying
\[
\|\varphi\|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)| < \infty \tag{2.1}
\]
for all \(\alpha, \beta \in \mathbb{N}_0^n\), equipped with the topology defined by the seminorms \(\|\cdot\|_{\alpha, \beta}\). A linear form \(u\) on \(\mathcal{S}(\mathbb{R}^n)\) is said to be **Schwartz tempered distribution** if there is a constant \(C \geq 0\) and a nonnegative integer \(N\) such that
\[
|\langle u, \varphi \rangle| \leq C \sum_{|\alpha|, |\beta| \leq N} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \varphi| \tag{2.2}
\]
for all \(\varphi \in \mathcal{S}(\mathbb{R}^n)\). The set of all Schwartz tempered distributions is denoted by \(\mathcal{S}'(\mathbb{R}^n)\).

Imposing growth conditions on \(\|\cdot\|_{\alpha, \beta}\) in (2.1), Sato and Kawai introduced the space \(\mathcal{F}(\mathbb{R}^n)\) of test functions for the Fourier hyperfunctions.

**Definition 2.2** [19]. Denote by \(\mathcal{F}(\mathbb{R}^n)\) the Sato space of all infinitely differentiable functions \(\varphi\) in \(\mathbb{R}^n\) such that
\[
\|\varphi\|_{A, B} = \sup_{x, \alpha, \beta} \left| x^\alpha \partial^\beta \varphi(x) \right| \exp \frac{k |x|}{h |\alpha|!} < \infty \tag{2.3}
\]
for some positive constants \(A, B\) depending only on \(\varphi\). We say that \(\varphi_j \to 0\) as \(j \to \infty\) if \(\|\varphi_j\|_{A, B} \to 0\) as \(j \to \infty\) for some \(A, B > 0\), and denote by \(\mathcal{F}'(\mathbb{R}^n)\) the strong dual of \(\mathcal{F}(\mathbb{R}^n)\) and call its elements **Fourier hyperfunctions**.

It can be verified that the seminorms (2.3) are equivalent to
\[
\|\varphi\|_{h, k} = \sup_{x \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n} \left| \partial^\alpha \varphi(x) \right| \exp \frac{k |x|}{h |\alpha|!} < \infty \tag{2.4}
\]
for some constants \(h, k > 0\). It is easy to see the following topological inclusion:
\[
\mathcal{F}(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n), \quad \mathcal{F}'(\mathbb{R}^n) \hookrightarrow \mathcal{F}'(\mathbb{R}^n). \tag{2.5}
\]
In order to solve (1.6), we employ the \( n \)-dimensional heat kernel, that is, the fundamental solution \( E_t(x) \) of the heat operator \( \partial_t - \Delta_x \) in \( \mathbb{R}^n_+ \times \mathbb{R}^n_+ \) given by

\[
E_t(x) = \begin{cases} 
(4\pi t)^{-n/2} \exp\left(-\frac{|x|^2}{4t}\right), & t > 0, \\
0, & t \leq 0.
\end{cases}
\] (2.6)

Since for each \( t > 0 \), \( E_t(\cdot) \) belongs to \( \mathcal{S}(\mathbb{R}^n) \), the convolution

\[
\tilde{u}(x,t) = (u \ast E_t)(x) = \langle u_y, E_t(x-y) \rangle, \quad x \in \mathbb{R}^n, \ t > 0,
\] (2.7)

is well defined for each \( u \in \mathcal{S}'(\mathbb{R}^n) \) and \( u \in \mathcal{F}'(\mathbb{R}^n) \), which is called the Gauss transform of \( u \). Also we use the following result which is called the heat kernel method (see [20]).

Let \( u \in \mathcal{S}'(\mathbb{R}^n) \). Then its Gauss transform \( \tilde{u}(x,t) \) is a \( C^\infty \)-solution of the heat equation

\[
\left( \frac{\partial}{\partial t} - \Delta \right) \tilde{u}(x,t) = 0
\] (2.8)

satisfying the following.

(i) There exist positive constants \( C, M, \) and \( N \) such that

\[
|\tilde{u}(x,t)| \leq Ct^{-M}(1 + |x|)^N \quad \text{in} \quad \mathbb{R}^n \times (0,\delta). \] (2.9)

(ii) \( \tilde{u}(x,t) \to u \) as \( t \to 0^+ \) in the sense that for every \( \varphi \in \mathcal{S}(\mathbb{R}^n) \),

\[
\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x,t)\varphi(x)dx.
\] (2.10)

Conversely, every \( C^\infty \)-solution \( U(x,t) \) of the heat equation satisfying the growth condition (2.9) can be uniquely expressed as \( U(x,t) = \tilde{u}(x,t) \) for some \( u \in \mathcal{S}'(\mathbb{R}^n) \).

Similarly, we can represent Fourier hyperfunctions as initial values of solutions of the heat equation as a special case of the results (see [21]). In this case, the estimate (2.9) is replaced by the following.

For every \( \epsilon > 0 \) there exists a positive constant \( C_\epsilon \) such that

\[
|\tilde{u}(x,t)| \leq C_\epsilon \exp\left(\epsilon \left(|x| + \frac{1}{t}\right)\right) \quad \text{in} \quad \mathbb{R}^n \times (0,\delta). \] (2.11)

We refer to [17, Chapter VI] for pullbacks and to [16, 18, 20] for more details of \( \mathcal{S}'(\mathbb{R}^n) \) and \( \mathcal{F}'(\mathbb{R}^n) \).

3. General solution in \( \mathcal{S}'(\mathbb{R}^n) \) and \( \mathcal{F}'(\mathbb{R}^n) \)

Jun and Kim (see [22]) showed that every continuous solution of (1.5) in \( \mathbb{R} \) is a cubic function \( f(x) = f(1)x^3 \) for all \( x \in \mathbb{R} \). Using induction argument on the dimension \( n \), it is easy to see that every continuous solution of (1.5) in \( \mathbb{R}^n \) is a cubic form

\[
f(x) = \sum_{1 \leq i,j,k \leq n} a_{ijk}x_ix_jx_k, \quad a_{ijk} \in \mathbb{C}.
\] (3.1)
In this section, we consider the general solution of the cubic functional equation in the spaces of \( \mathcal{F}(\mathbb{R}^n) \) and \( \mathcal{F}^*(\mathbb{R}^n) \). It is well known that the semigroup property of the heat kernel

\[
(E_t \ast E_s)(x) = E_{t+s}(x)
\]  

(3.2)

holds for convolution. Semigroup property will be useful to convert (1.6) into the classical functional equation defined on upper-half plane.

Convolving the tensor product \( E_t(\xi)E_s(\eta) \) of \( n \)-dimensional heat kernels in both sides of (1.6), we have

\[
\left[ (u \circ A_1) \ast (E_t(\xi)E_s(\eta)) \right](x,y) = \left\langle u_{\xi}, a^{-n} \int E_t(x - \frac{\xi - \eta}{a}) E_s(y - \eta) d\eta \right\rangle
\]

\[
= \left\langle u_{\xi}, a^{-n} \int E_t(ax + y - \xi - \eta) E_s(\eta) d\eta \right\rangle = \left\langle u_{\xi}, \int E_{ax+t}(ax + y - \xi) E_s(\eta) d\eta \right\rangle
\]

\[
= \left\langle u_{\xi}, (E_{ax+t} \ast E_s)(ax + y - \xi) \right\rangle = \left\langle u_{\xi}, E_{ax+t+s}(ax + y - \xi) \right\rangle = \tilde{u}(ax + y, a^2 t + s),
\]

(3.3)

and similarly we get

\[
\left[ (u \circ A_2) \ast (E_t(\xi)E_s(\eta)) \right](x,y) = \tilde{u}(ax - y, a^2 t + s),
\]

\[
\left[ (u \circ B_1) \ast (E_t(\xi)E_s(\eta)) \right](x,y) = \tilde{u}(x + y, t + s),
\]

\[
\left[ (u \circ B_2) \ast (E_t(\xi)E_s(\eta)) \right](x,y) = \tilde{u}(x - y, t + s),
\]

\[
\left[ (u \circ P) \ast (E_t(\xi)E_s(\eta)) \right](x,y) = \tilde{u}(x, t).
\]

Thus (1.6) is converted into the classical functional equation

\[
\tilde{u}(ax + y, a^2 t + s) + \tilde{u}(ax - y, a^2 t + s) = a\tilde{u}(x + y, t + s) + a\tilde{u}(x - y, t + s) + 2a(a^2 - 1)\tilde{u}(x, t)
\]

(3.5)

for all \( x, y \in \mathbb{R}^n \), \( t, s > 0 \).

**Lemma 3.1.** Let \( f : \mathbb{R}^n \times (0, \infty) \rightarrow \mathbb{C} \) be a continuous function satisfying

\[
f(ax + y, a^2 t + s) + f(ax - y, a^2 t + s) = af(x + y, t + s) + af(x - y, t + s) + 2a(a^2 - 1)f(x, t)
\]

(3.6)

for fixed integer \( a \) with \( a \neq 0, \pm 1 \). Then the solution is of the form

\[
f(x, t) = \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk}x_i x_j x_k + t \sum_{1 \leq i \leq n} b_i x_i, \quad a_{ijk}, b_i \in \mathbb{C}.
\]

(3.7)
Proof. In view of (3.6) and given the continuity, \( f(x,0^+) := \lim_{t \to 0^+} f(x,t) \) exists. Define 
\( h(x,t) := f(x,t) - f(x,0^+) \), then \( h(x,0^+) = 0 \) and

\[
h(ax + y, a^2 t + s) + h(ax - y, a^2 t + s) \\
= ah(x + y, t + s) + ah(x - y, t + s) + 2a(a^2 - 1) h(x,t)
\] (3.8)

for all \( x, y \in \mathbb{R}^n, t, s > 0 \). Setting \( y = 0, s \to 0^+ \) in (3.8), we have

\[
h(ax, a^2 t) = a^3 h(x,t).
\] (3.9)

Putting \( y = 0, s = a^2 s \) in (3.8), and using (3.9), we get

\[
a^2 h(x,t) + s) = h(x, t + a^2 s) + (a^2 - 1) h(x,t).
\] (3.10)

Letting \( t \to 0^+ \) in (3.10), we obtain

\[
a^2 h(x,s) = h(x, a^2 s).
\] (3.11)

Replacing \( t \) by \( a^2 t \) in (3.10) and using (3.11), we have

\[
h(x,a^2 t + s) = h(x,t + s) + (a^2 - 1) h(x,t).
\] (3.12)

Switching \( t \) with \( s \) in (3.12), we get

\[
h(x,t + a^2 s) = h(x,t + s) + (a^2 - 1) h(x,s).
\] (3.13)

Adding (3.10) to (3.13), we obtain

\[
h(x,t + s) = h(x,t) + h(x,s),
\] (3.14)

which shows that

\[
h(x,t) = h(x,1)t.
\] (3.15)

Letting \( t \to 0^+, s = 1 \) in (3.8), we have

\[
h(ax + y,1) + h(ax - y,1) = ah(x + y,1) + ah(x - y,1).
\] (3.16)

Also letting \( t = 1, s \to 0^+ \) in (3.8), and using (3.11), we get

\[
a^2 h(ax + y,1) + a^2 h(ax - y,1) = ah(x + y,1) + ah(x - y,1) + 2a(a^2 - 1) h(x,1).
\] (3.17)

Now taking (3.16) into (3.17), we obtain

\[
h(x + y,1) + h(x - y,1) = 2h(x,1).
\] (3.18)
Replacing \(x, y\) by \((x + y)/2\), \(y = (x - y)/2\) in (3.18), respectively, we see that \(h(x, 1)\) satisfies Jensen functional equation

\[
2h\left(\frac{x+y}{2}, 1\right) = h(x, 1) + h(y, 1). \tag{3.19}
\]

Putting \(x = y = 0\) in (3.16), we get \(h(0, 1) = 0\). This shows that \(h(x, 1)\) is additive.

On the other hand, letting \(t = s \to 0^+\) in (3.6), we can see that \(f(x, 0^+)\) satisfies (1.5). Given the continuity, the solution \(f(x, t)\) is of the form

\[
f(x, t) = \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k + t \sum_{1 \leq i \leq n} b_i x_i, \quad a_{ijk}, b_i \in \mathbb{C}, \tag{3.20}
\]

which completes the proof.

As a direct consequence of the above lemma, we present the general solution of the cubic functional equation in the spaces of \(\mathcal{F}'(\mathbb{R}^n)\) and \(\mathcal{B}'(\mathbb{R}^n)\).

**Theorem 3.2.** Suppose that \(u\) in \(\mathcal{F}'(\mathbb{R}^n)\) or \(\mathcal{B}'(\mathbb{R}^n)\) satisfies the equation

\[
u \circ A_1 + u \circ A_2 = au \circ B_1 + au \circ B_2 + 2a(a^2 - 1)u \circ P \tag{3.21}
\]

for fixed integer \(a\) with \(a \neq 0, \pm 1\). Then the solution is the cubic form

\[
u = \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k, \quad a_{ijk} \in \mathbb{C}. \tag{3.22}
\]

**Proof.** Convolving the tensor product \(E_t(\xi)E_t(\eta)\) of \(n\)-dimensional heat kernels in both sides of (3.21), we have the classical functional equation

\[
\hat{u}(ax + y, a^2 t + s) + \hat{u}(ax - y, a^2 t + s) = a\hat{u}(x + y, t + s) + a\hat{u}(x - y, t + s) + 2a(a^2 - 1)\hat{u}(x, t) \tag{3.23}
\]

for all \(x, y \in \mathbb{R}^n, t, s > 0\), where \(\hat{u}\) is the Gauss transform of \(u\). By Lemma 3.1, the solution \(\hat{u}\) is of the form

\[
\hat{u}(x, t) = \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k + t \sum_{1 \leq i \leq n} b_i x_i, \quad a_{ijk}, b_i \in \mathbb{C}. \tag{3.24}
\]

Thus we get

\[
\langle \hat{u}, \varphi \rangle = \left\langle \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k + t \sum_{1 \leq i \leq n} b_i x_i, \varphi \right\rangle \tag{3.25}
\]

for all test functions \(\varphi\). Now letting \(t \to 0^+\), it follows from the heat kernel method that

\[
\langle u, \varphi \rangle = \left\langle \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k, \varphi \right\rangle \tag{3.26}
\]

for all test functions \(\varphi\). This completes the proof. \(\square\)
4. Stability in $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}'(\mathbb{R}^n)$

We are going to prove the stability theorem of the cubic functional equation in the spaces of $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}'(\mathbb{R}^n)$.

We note that the Gauss transform
\[
\psi_p(x, t) := \int |\xi|^p E_t(x - \xi) \, d\xi
\] (4.1)
is well defined and $\psi_p(x, t) \to |x|^p$ locally uniformly as $t \to 0^+$. Also $\psi_p(x, t)$ satisfies semi-homogeneous property
\[
\psi_p(rx, r^2t) = r^p \psi_p(x, t)
\] (4.2)
for all $r \geq 0$.

We are now in a position to state and prove the main result of this paper.

**Theorem 4.1.** Let $a$ be a fixed integer with $a \neq 0, \pm 1$ and let $\epsilon$, $p$, $q$ be real numbers such that $\epsilon \geq 0$ and $0 \leq p, q < 3$, or $p, q > 3$. Suppose that $u$ in $\mathcal{S}'(\mathbb{R}^n)$ or $\mathcal{F}'(\mathbb{R}^n)$ satisfy the inequality
\[
||u \circ A_1 - u \circ A_2 - au \circ B_1 - au \circ B_2 - 2a(a^2 - 1)u \circ P|| \leq \epsilon (|x|^p + |y|^q).
\] (4.3)

Then there exists a unique cubic form
\[
c(x) = \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k
\] (4.4)
such that
\[
||u - c(x)|| \leq \epsilon \frac{2|a|^3 - |a|^p}{|a|^3 - |a|^p} |x|^p.
\] (4.5)

**Proof.** Let $v := u \circ A_1 - u \circ A_2 - au \circ B_1 - au \circ B_2 - 2a(a^2 - 1)u \circ P$. Convolving the tensor product $E_t(\xi)E_s(\eta)$ of $n$-dimensional heat kernels in $v$, we have
\[
\left| \left[ v \ast (E_t(\xi)E_s(\eta)) \right](x, y) \right| = \left| \left\langle v, E_t(x - \xi)E_s(y - \eta) \right\rangle \right|
\leq \epsilon \| (|\xi|^p + |\eta|^q) E_t(x - \xi)E_s(y - \eta) \|_{L^1}
= \epsilon (\psi_p(x, t) + \psi_q(y, s)).
\] (4.6)

Also we see that, as in Theorem 3.2,
\[
\left[ v \ast (E_t(\xi)E_s(\eta)) \right](x, y) = \tilde{u}(ax + y, a^2t + s) + \tilde{u}(ax - y, a^2t + s)
- a\tilde{u}(x + y, t + s) - a\tilde{u}(x - y, t + s) - 2a(a^2 - 1)\tilde{u}(x, t),
\] (4.7)
where $\tilde{u}$ is the Gauss transform of $u$. Thus inequality (4.3) is converted into the classical functional inequality

$$\left| \tilde{u}(ax+y, a^2t+s) + \tilde{u}(ax-y, a^2t+s) - a\tilde{u}(x+y, t+s) - a\tilde{u}(x-y, t+s) - 2a(a^2 - 1)\tilde{u}(x, t) \right| \leq \epsilon (\psi_p(x, t) + \psi_q(y, s))$$  

(4.8)

for all $x, y \in \mathbb{R}^n, t, s > 0$.

We first prove for $0 \leq p, q < 3$. Letting $y = 0, s \to 0^+$ in (4.8) and dividing the result by $2|a|^3$, we get

$$\left| \frac{\tilde{u}(ax, a^2t)}{a^3} - \tilde{u}(x, t) \right| \leq \frac{\epsilon}{2|a|^3} \psi_p(x, t).$$  

(4.9)

By virtue of the semihomogeneous property of $\psi_p$, substituting $x, t$ by $ax, a^2t$, respectively, in (4.9) and dividing the result by $|a|^3$, we obtain

$$\left| \frac{\tilde{u}(a^2x, a^4t)}{a^6} - \frac{\tilde{u}(ax, a^2t)}{a^3} \right| \leq \frac{\epsilon}{2|a|^3} |a|^{p-3} \psi_p(x, t).$$  

(4.10)

Using induction argument and triangle inequality, we have

$$\left| \frac{\tilde{u}(a^n x, a^{2n}t)}{a^{3n}} - \tilde{u}(x, t) \right| \leq \frac{\epsilon}{2|a|^3} \psi_p(x, t) \sum_{j=0}^{n-1} |a|^{(p-3)j}$$  

(4.11)

for all $n \in \mathbb{N}, x \in \mathbb{R}^n, t > 0$. Let us prove the sequence $\{a^{-3n}\tilde{u}(a^n x, a^{2n}t)\}$ is convergent for all $x \in \mathbb{R}^n, t > 0$. Replacing $x, t$ by $amx, a^{2m}t$, respectively, in (4.11) and dividing the result by $|a|^{3m}$, we see that

$$\left| \frac{\tilde{u}(a^{n+m} x, a^{2(m+n)}t)}{a^{3(m+n)}} - \frac{\tilde{u}(a^m x, a^{2m}t)}{a^{3m}} \right| \leq \frac{\epsilon}{2|a|^3} \psi_p(x, t) \sum_{j=m}^{n-1} |a|^{(p-3)j}.$$  

(4.12)

Letting $m \to \infty$, we have $\{a^{-3n}\tilde{u}(a^n x, a^{2n}t)\}$ is a Cauchy sequence. Therefore, we may define

$$G(x, t) = \lim_{n \to \infty} a^{-3n}\tilde{u}(a^n x, a^{2n}t)$$  

(4.13)

for all $x \in \mathbb{R}^n, t > 0$.

Now we verify that the given mapping $G$ satisfies (3.6). Replacing $x, y, t, s$ by $a^n x, a^n y, a^{2n} t, a^{2n} s$ in (4.8), respectively, and then dividing the result by $|a|^{3n}$, we get

$$|a|^{-3n} \left| \tilde{u}(a^n(ax + y), a^{2n}(a^2t + s)) + \tilde{u}(a^n(ax - y), a^{2n}(a^2t + s)) 
- a\tilde{u}(a^n(x + y), a^{2n}(t + s)) - a\tilde{u}(a^n(x + y), a^{2n}(t + s)) - 2a(a^2 - 1)\tilde{u}(a^n x, a^{2n} t) \right|$$

$$\leq |a|^{-3n}(\psi_p(a^n x, a^{2n} t) + \psi_q(a^n y, a^{2n} s))$$

$$= (|a|^{(p-3)n}\psi_p(x, t) + |a|^{(q-3)n}\psi_q(y, s))$$  

(4.14)
Now letting \( n \to \infty \), we see by definition of \( G \) that \( G \) satisfies
\[
G(ax + y, a^2 t + s) + G(ax - y, a^2 t + s) = aG(x + y, t + s) + aG(x - y, t + s) + 2a(a^2 - 1)G(x, t)
\]  
(4.15)
for all \( x, y \in \mathbb{R}^n, t, s > 0 \). By Lemma 3.1, \( G(x, t) \) is of the form
\[
G(x, t) = \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k + t \sum_{1 \leq i \leq n} b_i x_i, \quad a_{ijk}, b_i \in \mathbb{C}.
\]  
(4.16)
Letting \( n \to \infty \) in (4.11) yields
\[
|G(x, t) - \tilde{u}(x, t)| \leq \frac{\epsilon}{2(|a|^3 - |a|^p)} \psi_p(x, t).
\]  
(4.17)
To prove the uniqueness of \( G(x, t) \), we assume that \( H(x, t) \) is another function satisfying
(4.15) and (4.17). Setting \( y = 0 \) and \( s \to 0^+ \) in (4.15), we have
\[
G(ax, a^2 t) = a^3 G(x, t).
\]  
(4.18)
Then it follows from (4.15), (4.17), and (4.18) that
\[
|G(x, t) - H(x, t)| = |a|^{-3n} |G(a^n x, a^{2n} t) - H(a^n x, a^{2n} t)| \leq |a|^{-3n} |G(a^n x, a^{2n} t) - \tilde{u}(a^n x, a^{2n} t)|
\]
\[
+ |a|^{-3n} |\tilde{u}(a^n x, a^{2n} t) - H(a^n x, a^{2n} t)| \leq \frac{\epsilon}{|a|^{3n} (|a|^3 - |a|^p)} \psi_p(x, t)
\]  
(4.19)
for all \( n \in \mathbb{N}, x \in \mathbb{R}^n, t > 0 \). Letting \( n \to \infty \), we have \( G(x, t) = H(x, t) \) for all \( x \in \mathbb{R}^n, t > 0 \). This proves the uniqueness.

It follows from the inequality (4.17) that
\[
|\langle G(x, t) - \tilde{u}(x, t), \varphi \rangle| \leq \frac{\epsilon}{2(|a|^3 - |a|^p)} \langle \psi_p(x, t), \varphi \rangle
\]  
(4.20)
for all test functions \( \varphi \). Letting \( t \to 0^+ \), we have the inequality
\[
\left\| u - \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k \right\| \leq \frac{\epsilon}{2 |a|^3 - |a|^p}.
\]  
(4.21)
Now we consider the case \( p, q > 3 \). For this case, replacing \( x, y, t \) by \( x/a, 0, t/a^2 \) in
(4.8), respectively, and letting \( s \to 0^+ \) and then multiplying the result by \( |a|^3 \), we have
\[
\left| \tilde{u}(x, t) - a^3 \tilde{u} \left( \frac{x}{a}, \frac{t}{a^2} \right) \right| \leq \frac{\epsilon}{2 |a|^3} |a|^{-p} \psi_p(x, t).
\]  
(4.22)
Substituting \( x, t \) by \( x/a, t/a^2 \), respectively, in (4.22) and multiplying the result by \( |a|^3 \) we get
\[
\left| a^3 \tilde{u} \left( \frac{x}{a}, \frac{t}{a^2} \right) - a^6 \tilde{u} \left( \frac{x}{a^2}, \frac{t}{a^4} \right) \right| \leq \frac{\epsilon}{2 |a|^3} |a|^{2(3-p)} \psi_p(x, t).
\]  
(4.23)
Using induction argument and triangle inequality, we obtain
\[
\left| \tilde{u}(x,t) - a^{3n} \tilde{u}\left( \frac{x}{a^n}, \frac{t}{a^{2n}} \right) \right| \leq \frac{\epsilon}{2|a|^3} \psi_p(x,t) \sum_{j=1}^{n} |a|^{(3-p)j} \tag{4.24}
\]
for all \( n \in \mathbb{N}, x \in \mathbb{R}^n, t > 0 \). Following the same method as in the case \( 0 \leq p, q < 3 \), we see that
\[
G(x,t) := \lim_{n \to \infty} a^{3n} \tilde{u}\left( \frac{x}{a^n}, \frac{t}{a^{2n}} \right) \tag{4.25}
\]
is the unique function satisfying (4.15). Letting \( n \to \infty \) in (4.24), we get
\[
\left| \tilde{u}(x,t) - C(x,t) \right| \leq \frac{\epsilon}{2 |a|^p - |a|^3} \psi_p(x,t). \tag{4.26}
\]
Now letting \( t \to 0^+ \) in (4.26), we have the inequality
\[
\left\| u - \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k \right\| \leq \frac{\epsilon}{2 |a|^p - |a|^3} |x|^p. \tag{4.27}
\]
This completes the proof. \( \square \)

Remark 4.2. The above norm inequality
\[
\left\| u - c(x) \right\| \leq \frac{\epsilon}{2 |a|^p - |a|^3} |x|^p \tag{4.28}
\]
implies that \( u - c(x) \) is a measurable function. Thus all the solution \( u \) in \( \mathcal{F}'(\mathbb{R}^n) \) or \( \mathcal{F}'(\mathbb{R}^n) \) can be written uniquely in the form
\[
u = c(x) + h(x), \tag{4.29}
\]
where \( |h(x)| \leq (\epsilon/(2 |a|^p - |a|^3)) |x|^p \).

Corollary 4.3. Let \( a \) be a fixed integer with \( a \neq 0, \pm 1 \) and \( \epsilon \geq 0 \). Suppose that \( u \) in \( \mathcal{F}'(\mathbb{R}^n) \) or \( \mathcal{F}'(\mathbb{R}^n) \) satisfy the inequality
\[
\left\| u \circ A_1 - u \circ A_2 - au \circ B_1 - au \circ B_2 - 2a(a^2 - 1) u \circ P \right\| \leq \epsilon. \tag{4.30}
\]
Then there exists a unique cubic form
\[
c(x) = \sum_{1 \leq i \leq j \leq k \leq n} a_{ijk} x_i x_j x_k \tag{4.31}
\]
such that
\[
\left\| u - c(x) \right\| \leq \frac{\epsilon}{2(a^3 - 1)}. \tag{4.32}
\]
Acknowledgment

This work was supported by Korea Research Foundation Grant (KRF-2003-041-C00023).

References


Young-Su Lee: Department of Mathematics, Sogang University, Seoul 121-741, South Korea
*Email address: masuri@sogang.ac.kr*

Soon-Yeong Chung: Department of Mathematics and Program of Integrated Biotechnology, Sogang University, Seoul 121-741, South Korea
*Email address: sychung@sogang.ac.kr*