Yu and Lin *Fixed Point Theory and Applications* 2013, 2013:283
http://www.fixedpointtheoryandapplications.com/content/2013/1/283

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Hierarchical problems with applications to mathematical programming with multiple sets split feasibility constraints

Zenn-Tsun Yu[1] and Lai-Jiu Lin[2*]

*Correspondence: maljlin@cc.ncue.edu.tw
[2]Department of Mathematics, National Changhua University of Education, Changhua, 50058, Taiwan
Full list of author information is available at the end of the article

Abstract

In this paper, we establish a strong convergence theorem for hierarchical problems, an equivalent relation between a multiple sets split feasibility problem and a fixed point problem. As applications of our results, we study the solution of mathematical programming with fixed point and multiple sets split feasibility constraints, mathematical programming with fixed point and multiple sets split equilibrium constraints, mathematical programming with fixed point and split feasibility constraints, mathematical programming with fixed point and split equilibrium constraints, minimum solution of fixed point and multiple sets split feasibility problems, minimum norm solution of fixed point and multiple sets split equilibrium problems, quadratic function programming with fixed point and multiple set split feasibility constraints, mathematical programming with fixed point and multiple set split feasibility inclusions constraints, mathematical programming with fixed point and split minimax constraints.

Keywords: hierarchical problems; multiple sets split feasibility problem; fixed point problem; mathematical programming; quadratic function programming; minimum norm solution

1 Introduction

The split feasibility problem (SFP) in finite dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. Since then, the split feasibility problem (SFP) has received much attention due to its applications in signal processing, image reconstruction, with particular progress in intensity-modulated radiation therapy, approximation theory, control theory, biomedical engineering, communications, and geophysics. For examples, one can refer to [1–5] and related literature. Since then, many researchers have studied (SFP) in finite dimensional or infinite dimensional Hilbert spaces. For example, one can see [2, 6–19].

A special case of problem (SFP) is the convexly constrained linear inverse problem in the finite dimensional Hilbert space [20]:

$$\text{(CLIP)}\quad \text{Find } \bar{x} \in C \text{ such that } A\bar{x} = b, \text{ where } b \in H_2,$$

which has extensively been investigated by using the Landweber iterative method [21]

$$x_{n+1} := x_n + \gamma A^T(b - Ax_n), \quad n \in \mathbb{N}.$$

In 2002, Byrne [2] first introduced the so-called CQ algorithm which generates a sequence $\{x_n\}$ by the following recursive procedure:

$$x_{n+1} = P_C\bigl(x_n - \rho_n A^*(I - P_Q)Ax_n\bigr), \tag{1}$$

where the stepsize ρ_n is chosen in the interval $(0, 2/\|A\|^2)$, and P_C and P_Q are the metric projections onto $C \subseteq \mathbb{R}^n$ and $Q \subseteq \mathbb{R}^m$, respectively. Compared with Censor and Elfving's algorithm [1] where the matrix inverse A is involved, the CQ algorithm (1) seems more easily executed since it only deals with metric projections with no need to compute matrix inverses.

In 2010, Xu [12] modified Byrne's CQ algorithm and proved the weak convergence theorem in infinite Hilbert spaces for their modified algorithm.

Let C be a nonempty closed convex subset of a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. A mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is said to be a quasi-nonexpansive mapping if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in \mathrm{Fix}(T)$, we denote by $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$ the set of fixed points of T. $A : C \to H$ is called strongly positive if

$$\langle x, Ax\rangle \geq \alpha\|x\|^2, \quad \forall x \in C.$$

Let f be a contraction on H and $\{\alpha_n\}$ be a sequence in $[0,1]$. In 2004, Xu [22] proved that under some condition on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n f x_n + (1 - \alpha_n)Tx_n$$

strongly converges to x^* in $\mathrm{Fix}(T)$, which is the unique solution of the variational inequality

$$\bigl\langle (I - f)x^*, x - x^*\bigr\rangle \geq 0$$

for all $x \in \mathrm{Fix}(T)$.

Xu [23] also studied the following minimization problem over the set of fixed points of a nonexpansive operator T on a real Hilbert space H:

$$\min_{x \in \mathrm{Fix}(T)} \frac{1}{2}\langle Bx, x\rangle - \langle a, x\rangle,$$

where a is a given point in H and B is a strongly positive bounded linear operator on H. In [23], Xu proved that the sequence $\{x_n\}$ defined by the following iterative method

$$x_{n+1} = (I - \alpha_n B)Tx_n + \alpha_n a$$

converges strongly to the unique solution of the minimization problem of a quadratic function. In [24], Marino *et al.* considered the following iterative method:

$$x_{n+1} = \alpha_n \gamma f x_n + (I - \alpha_n A)Tx_n. \tag{2}$$

They proved that the sequence generated by (2) converges strongly to the fixed point x^* of T which solves the following:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0$$

for all $x \in \mathrm{Fix}(T)$. For some more related works, see [25–27] and the references therein.

In this paper, we establish a strong convergence theorem for hierarchical problems, an equivalent relation between a multiple sets split feasibility problem and a fixed point problem. As applications of our results, we study the solution of mathematical programming with fixed point and multiple sets split feasibility constraints, mathematical programming with fixed point and multiple sets split equilibrium constraints, mathematical programming with fixed point and split feasibility constraints, mathematical programming with fixed point and split equilibrium constraints, minimum solution of fixed point and multiple sets split feasibility problems, minimum norm solution of fixed point and multiple sets split equilibrium problems, quadratic function programming with fixed point and multiple set split feasibility constraints, mathematical programming with fixed point and multiple set split feasibility inclusions constraints, mathematical programming with fixed point and split minimax constraints.

2 Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers, H be a (real) Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively, and let C be a nonempty closed convex subset of H. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. For each $x, y \in H$ and $\lambda \in [0,1]$, we have

$$\| \lambda x + (1 - \lambda)y \|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Hence, we also have

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2 \tag{3}$$

for all $x, y, u, v \in H$.

For $\alpha > 0$, a mapping $A : H \to H$ is called α-inverse-strongly monotone (α-ism) if

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in H.$$

If $0 < \lambda \leq 2\alpha$, $A : H \to H$ is an α-inverse-strongly monotone mapping, then $I - \lambda A : H \to H$ is nonexpansive. A mapping $T : C \to H$ is said to be a firmly nonexpansive mapping if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \| (I - T)x - (I - T)y \|^2$$

for every $x, y \in C$. Let $T : C \to H$ be a mapping. Then $p \in C$ is called an asymptotic fixed point of T [28] if there exists $\{x_n\} \subseteq C$ such that $x_n \rightharpoonup p$, and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $F(\hat{T})$ the set of asymptotic fixed points of T. A mapping $T : C \to H$ is said to be demiclosed if it satisfies $F(T) = F(\hat{T})$. A nonlinear operator $V : H \to H$ is called strongly monotone if there exists $\bar{\gamma} > 0$ such that $\langle x-y, Vx-Vy\rangle \geq \bar{\gamma}\|x-y\|^2$ for all $x,y \in H$. Such V is also called $\bar{\gamma}$-strongly monotone. A nonlinear operator $V : H \to H$ is called Lipschitzian continuous if there exists $L > 0$ such that $\|Vx - Vy\| \leq L\|x-y\|$ for all $x,y \in H$. Such V is also called L-Lipschitzian continuous.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x-y, u-v\rangle \geq 0$ for all $x,y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} : H \to D(B)$, which is called the resolvent of B for r, and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

The following lemmas are needed in this paper.

Lemma 2.1 [29] *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, and A^* be the adjoint of A. Let C be a nonempty closed convex subset of H_2, and let $G : H_2 \to H_2$ be a firmly nonexpansive mapping. Then $A^*(I - G)A$ is a $\frac{1}{\|A\|^2}$-ism, that is,*

$$\frac{1}{\|A\|^2}\left\|A^*(I-G)Ax - A^*(I-G)Ay\right\|^2 \leq \left\langle x-y, A^*(I-G)Ax - A^*(I-G)Ay\right\rangle$$

for all $x,y \in H_1$.

Lemma 2.2 [30] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $G : H \to H$ be a firmly nonexpansive mapping. Suppose that* $\mathrm{Fix}(G) \neq \emptyset$. *Then $\langle x - Gx, Gx - w\rangle \geq 0$ for each $x \in H$ and each $w \in \mathrm{Fix}(G)$.*

A mapping $T : H \to H$ is said to be averaged if $T = (1-\alpha)I + \alpha S$, where $\alpha \in (0,1)$ and $S : H \to H$ is a nonexpansive mapping. In this case, we also say that T is α-averaged. A firmly nonexpansive mapping is $\frac{1}{2}$-averaged.

Lemma 2.3 [31] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be a mapping. Then the following are satisfied:*

(i) *T is nonexpansive if and only if the complement $(I - T)$ is $1/2$-ism.*
(ii) *If S is υ-ism, then for $\gamma > 0$, γS is υ/γ-ism.*
(iii) *S is averaged if and only if the complement $I - S$ is υ-ism for some $\upsilon > 1/2$.*
(iv) *If S and T are both averaged, then the product (composite) ST is averaged.*
(v) *If the mappings $\{T_i\}_{i=1}^n$ are averaged and have a common fixed point, then $\bigcap_{i=1}^n \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_n)$.*

Lin and Takahashi [39] gave the following results in a Hilbert spaces.

Lemma 2.4 [32] *Let P_C be the metric projection of H onto C, and let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $t \geq 0$ satisfy $2\bar{\gamma} > tL^2$ and $1 > 2t\bar{\gamma}$. Then we know that*

$$z = P_C(I - tV)z \quad \Leftrightarrow \quad \langle Vz, y - z\rangle \geq 0 \quad \Leftrightarrow \quad z = P_C(I - V)z.$$

Such $z \in C$ exists always and is unique.

By Lemma 2.4, we have the following lemma.

Lemma 2.5 *Let $V : H \to H$ be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Let $\theta \in H$ and $V_1 : H \to H$ such that $V_1 x = Vx - \theta$. Then V_1 is a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous mapping. Furthermore, there exists a unique fixed point z_0 in C satisfying $z_0 = P_C(z_0 - Vz_0 + \theta)$. This point $z_0 \in C$ is also a unique solution of the hierarchical variational inequality*

$$\langle Vz_0 - \theta, q - z_0 \rangle \geq 0, \quad \forall q \in C.$$

Lemma 2.6 [33] *Let B be a maximal monotone mapping on H. Let J_r be the resolvent of B defined by $J_r = (I + rB)^{-1}$ for each $r > 0$. Then the following hold*:

(i) *For each $r > 0$, J_r is single-valued and firmly nonexpansive*;
(ii) *For each $r > 0$, $\mathcal{D}(J_r) = H$ and* $\mathrm{Fix}(J_r) = \{x \in \mathcal{D}(B) : 0 \in Bx\}$;

Lemma 2.7 [33] *Let B be a maximal monotone mapping on H. Let J_r be the resolvent of B defined by $J_r = (I + rB)^{-1}$ for each $r > 0$. Then the following holds*:

$$\frac{s-t}{s} \langle J_s x - J_t x, J_s x - x \rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$. In particular,

$$\|J_s x - J_t x\| \leq \frac{|s-t|}{s} \|J_s x - x\|$$

for all $s, t > 0$ and $x \in H$.

Let $\alpha, \beta \in \mathbb{R}$, T be a generalized hybrid mapping [34] if $\alpha \| Tx - Ty\|^2 + (1-\alpha)\| Ty - x\|^2 \leq \beta \| Tx - y\|^2 + (1-\beta)\|x - y\|^2$ for all $x, y \in C$.

Lemma 2.8 [35] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to H$ be a generalized hybrid mapping, then $F(T) = F(\hat{T})$.*

Remark 2.1 If T is a generalized hybrid mapping with $\mathrm{Fix}(T) \neq \emptyset$. By the definition of T and Lemma 2.8, we have that T is a quasi-nonexpansive mapping with $F(T) = F(\hat{T})$.

Lemma 2.9 [36] *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subseteq \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied for all* (*sufficiently large*) *numbers $k \in \mathbb{N}$*:

$$a_{m_k} \leq a_{m_k+1} \quad \textit{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.10 [37] *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ be a sequence of nonnegative real* *numbers with* $\sum_{n=1}^{\infty} u_n < \infty$, $\{t_n\}$ *be a sequence of real numbers with* $\limsup t_n \leq 0$. *Suppose that* $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n t_n + u_n$ *for each* $n \in \mathbb{N}$. *Then* $\lim_{n\to\infty} a_n = 0$.

We know that the equilibrium problem is to find $z \in C$ such that

$$\text{(EP)} \quad g(z,y) \geq 0 \quad \text{for each } y \in C,$$

where $g : C \times C \to \mathbb{R}$ is a bifunction. This problem includes fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, minimax inequalities, and saddle point problems as special cases. (For examples, one can see [38] and related literatures.)

The solution set of equilibrium problem (EP) is denoted by EP(g). For solving the equilibrium problem, let us assume that the bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $g(x,x) = 0$ for each $x \in C$;
(A2) g is monotone, *i.e.*, $g(x,y) + g(y,x) \leq 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} g(tz + (1-t)x, y) \leq g(x,y)$;
(A4) for each $x \in C$, the scalar function $y \to g(x,y)$ is convex and lower semicontinuous.

We have the following result from Blum and Oettli [38].

Theorem 2.1 [38] *Let C be a nonempty closed convex subset of a real Hilbert space H. Let* $g : C \times C \to \mathbb{R}$ *be a bifunction which satisfies conditions* (A1)-(A4). *Then, for each* $r > 0$ *and each* $x \in H$, *there exists* $z \in C$ *such that*

$$g(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in C$.

In 2005, Combettes and Hirstoaga [39] established the following important properties of a resolvent operator.

Theorem 2.2 [39] *Let C be a nonempty closed convex subset of a real Hilbert space H, and let* $g : C \times C \to \mathbb{R}$ *be a function satisfying conditions* (A1)-(A4). *For* $r > 0$, *define* $T_r^g : H \to C$ *by*

$$T_r^g x = \left\{ z \in C : g(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. *Then the following hold*:

(i) T_r^g *is single-valued*;
(ii) T_r^g *is firmly nonexpansive, that is,* $\|T_r^g x - T_r^g y\|^2 \leq \langle x - y, T_r^g x - T_r^g y\rangle$ *for all* $x, y \in H$;
(iii) $\{x \in H : T_r^g x = x\} = \{x \in C : g(x,y) \geq 0, \forall y \in C\}$;
(iv) $\{x \in C : g(x,y) \geq 0, \forall y \in C\}$ *is a closed and convex subset of C*.

We call such T_r^g the resolvent of g for $r > 0$.

3 Convergence theorems of hierarchical problems

Let H be a real Hilbert space, and let I be an identity mapping on H, C be a nonempty closed convex subset of H. For each $i = 1, 2$, let $\kappa_i > 0$ and let B_i be a κ_i-inverse-strongly monotone mapping of C into H. Let G_i be a maximal monotone mapping on H such that the domain of G_i is included in C for each $i = 1, 2$. Let $J_\lambda = (I + \lambda G_1)^{-1}$ and $T_r = (I + rG_2)^{-1}$ for each $\lambda > 0$ and $r > 0$. Let $\{\theta_n\} \subset H$ be a sequence. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$. Throughout this paper, we use these notations and assumptions unless specified otherwise.

The following strong convergence theorem for hierarchical problems is one of our main results of this paper.

Theorem 3.1 *Let $T : C \to H$ be a quasi-nonexpansive mapping with* $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$ *such that $F(T) \cap (B_1 + G_1)^{-1}0 \cap (B_2 + G_2)^{-1}0 \neq \emptyset$. Take $\mu \in \mathbb{R}$ as follows*:

$$0 < \mu < \frac{2\bar{\gamma}}{L^2}.$$

Let $\{x_n\} \subset H$ be defined by

$$(3.1)\qquad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_{\lambda_n}(I - \lambda_n B_1)T_{r_n}(I - r_n B_2)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \leq \lambda_n \leq b < 2\kappa_1$, *and* $0 < a \leq r_n \leq b < 2\kappa_2$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap(B_1+G_1)^{-1}0\cap(B_2+G_2)^{-1}0}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap (B_1 + G_1)^{-1}0 \cap (B_2 + G_2)^{-1}0.$$

Proof Take any $\bar{x} \in \mathrm{Fix}(T) \cap (B_1 + G_1)^{-1}0 \cap (B_2 + G_2)^{-1}0$ and let $\bar{x}$ be fixed. Then $\bar{x} = J_{\lambda_n}(I - \lambda_n B_1)\bar{x}$ and $\bar{x} = T_{r_n}(I - r_n B_2)\bar{x}$. Let $u_n = T_{r_n}(I - r_n B_2)x_n$. For each $n \in \mathbb{N}$, we have

$$\begin{aligned} &\|u_n - \bar{x}\|^2 \\ &\quad = \big\| T_{r_n}(I - r_n B_2)x_n - T_{r_n}(I - r_n B_2)\bar{x}\big\|^2 \\ &\quad \leq \big\|(x_n - \bar{x}) - r_n(B_2 x_n - B_2\bar{x})\big\|^2 \\ &\quad \leq \|x_n - \bar{x}\|^2 - 2r_n\langle x_n - \bar{x}, B_2 x_n - B_2\bar{x}\rangle + r_n^2\|B_2 x_n - B_2\bar{x}\|^2 \\ &\quad \leq \|x_n - \bar{x}\|^2 - 2r_n\kappa_2\|B_2 x_n - B_2\bar{x}\|^2 + r_n^2\|B_2 x_n - B_2\bar{x}\|^2 \\ &\quad \leq \|x_n - \bar{x}\|^2 - r_n(2\kappa_2 - r_n)\|B_2 x_n - B_2\bar{x}\|^2 \\ &\quad \leq \|x_n - \bar{x}\|^2, \end{aligned} \qquad (4)$$

and

$$\begin{aligned}
&\|y_n - \bar{x}\|^2 \\
&\quad = \big\|J_{\lambda_n}(I - \lambda_n B_1)u_n - J_{\lambda_n}(I - \lambda_n B_1)\bar{x}\big\|^2 \\
&\quad \le \big\|(u_n - \bar{x}) - \lambda_n(B_1 u_n - B_1\bar{x})\big\|^2 \\
&\quad \le \|u_n - \bar{x}\|^2 - 2\lambda_n\langle u_n - \bar{x}, B_1 u_n - B_1\bar{x}\rangle + \lambda_n^2\|B_1 u_n - B_1\bar{x}\|^2 \\
&\quad \le \|u_n - \bar{x}\|^2 - 2\lambda_n\kappa_1\|B_1 u_n - B_1\bar{x}\|^2 + \lambda_n^2\|B_1 u_n - B_1\bar{x}\|^2 \\
&\quad \le \|u_n - \bar{x}\|^2 - \lambda_n(2\kappa_1 - \lambda_n)\|B_1 u_n - B_1\bar{x}\|^2 \\
&\quad \le \|u_n - \bar{x}\|^2 \\
&\quad \le \|x_n - \bar{x}\|^2.
\end{aligned} \tag{5}$$

Since T is a quasi-nonexpansive mapping, we obtain that

$$\|s_n - \bar{x}\| = \|Ty_n - \bar{x}\| \le \|y_n - \bar{x}\| \le \|u_n - \bar{x}\| \le \|x_n - \bar{x}\|. \tag{6}$$

Let $z_n = \beta_n\theta_n + (I - \beta_n V)s_n$, we have that

$$\begin{aligned}
\|z_n - \bar{x}\| &= \big\|\beta_n\theta_n + (I - \beta_n V)s_n - \bar{x}\big\| \\
&\le \beta_n\|\theta_n - V\bar{x}\| + \big\|(I - \beta_n V)(s_n - \bar{x})\big\| \\
&\le \beta_n\|\theta_n - V\bar{x}\| + \big\|(I - \beta_n V)s_n - (I - \beta_n V)\bar{x}\big\|.
\end{aligned} \tag{7}$$

Put $\tau = \bar{\gamma} - \frac{L^2\mu}{2}$, we have that

$$\begin{aligned}
&\big\|(I - \beta_n V)s_n - (I - \beta_n V)\bar{x}\big\|^2 \\
&\quad = \|s_n - \bar{x}\|^2 - 2\beta_n\langle s_n - \bar{x}, Vs_n - V\bar{x}\rangle + \beta_n^2\|Vs_n - V\bar{x}\|^2 \\
&\quad \le \|s_n - \bar{x}\|^2 - 2\beta_n\bar{\gamma}\|s_n - \bar{x}\|^2 + \beta_n^2 L^2\|s_n - \bar{x}\|^2 \\
&\quad \le \big(1 - 2\beta_n\bar{\gamma} + \beta_n^2 L^2\big)\|s_n - \bar{x}\|^2 \\
&\quad \le \big(1 - 2\beta_n\tau - \beta_n\big(L^2\mu - \beta_n L^2\big)\big)\|s_n - \bar{x}\|^2 \\
&\quad \le \big(1 - 2\beta_n\tau + \beta_n^2\tau^2\big)\|s_n - \bar{x}\|^2 \\
&\quad \le (1 - \beta_n\tau)^2\|x_n - \bar{x}\|^2.
\end{aligned}$$

Since $1 - \beta_n\tau > 0$, we obtain that

$$\big\|(I - \beta_n V)s_n - (I - \beta_n V)\bar{x}\big\| \le (1 - \beta_n\tau)\|x_n - \bar{x}\|. \tag{8}$$

We have from (7) and (8) that

$$\|z_n - \bar{x}\| \le \beta_n\|\theta_n - V\bar{x}\| + (1 - \beta_n\tau)\|x_n - \bar{x}\|. \tag{9}$$

Thus, we obtain from the definition of x_n and (9) that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\| &= \big\|\alpha_n x_n + (1-\alpha_n)\big(\beta_n\theta_n + (I-\beta_n V)s_n\big) - \bar{x}\big\| \\
&\le \alpha_n\|x_n - \bar{x}\| + (1-\alpha_n)\big\|\big(\beta_n\theta_n + (I-\beta_n V)s_n\big) - \bar{x}\big\| \\
&\le \alpha_n\|x_n - \bar{x}\| + (1-\alpha_n)\big[\beta_n\|\theta_n - V\bar{x}\| + (1-\beta_n\tau)\|x_n - \bar{x}\|\big] \\
&\le \big[1-(1-\alpha_n)\beta_n\tau\big]\|x_n - \bar{x}\| + \beta_n(1-\alpha_n)\tau\frac{\|\theta_n - V\bar{x}\|}{\tau} \\
&\le \max\left\{\|x_n - \bar{x}\|, \frac{\|\theta_n - V\bar{x}\|}{\tau}\right\} \\
&\le \max\big\{\|x_n - \bar{x}\|, M\big\},
\end{aligned}$$

where $M = \max\{\frac{\|\theta_n - V\bar{x}\|}{\tau}, n \in \mathbb{N}\}$. By induction, we deduce

$$\|x_n - \bar{x}\| \le \max\big\{\|x_1 - \bar{x}\|, M\big\}.$$

This implies that the sequence $\{x_n\}$ is bounded. Furthermore, $\{u_n\}$, $\{z_n\}$, $\{y_n\}$ and $\{s_n\}$ are bounded.

By the definition of $\{x_n\}$, we have that

$$\begin{aligned}
x_{n+1} - x_n &= \alpha_n x_n + (1-\alpha_n)\big(\beta_n\theta_n + (I-\beta_n V)s_n\big) - x_n \\
&= (1-\alpha_n)\big[\big(\beta_n\theta_n + (I-\beta_n V)s_n\big) - x_n\big] \\
&= (1-\alpha_n)[\beta_n\theta_n - \beta_n V s_n + s_n - x_n].
\end{aligned} \tag{10}$$

By (10), we have that

$$\begin{aligned}
&\langle x_{n+1} - x_n, x_n - \bar{x}\rangle \\
&\quad = \big\langle(1-\alpha_n)[\beta_n\theta_n - \beta_n V s_n + s_n - x_n], x_n - \bar{x}\big\rangle \\
&\quad = (1-\alpha_n)\beta_n\langle\theta_n, x_n - \bar{x}\rangle - (1-\alpha_n)\beta_n\langle V s_n, x_n - \bar{x}\rangle + (1-\alpha_n)\langle s_n - x_n, x_n - \bar{x}\rangle.
\end{aligned} \tag{11}$$

By (3) and (11), we have that

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^2 - \|x_n - \bar{x}\|^2 - \|x_{n+1} - x_n\|^2 \\
&\quad = 2(1-\alpha_n)\beta_n\langle\theta_n, x_n - \bar{x}\rangle - 2(1-\alpha_n)\beta_n\langle V s_n, x_n - \bar{x}\rangle \\
&\qquad + (1-\alpha_n)\big[\|s_n - \bar{x}\|^2 - \|x_n - \bar{x}\|^2 - \|s_n - x_n\|^2\big].
\end{aligned} \tag{12}$$

By (5) and (12), we have that

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^2 - \|x_n - \bar{x}\|^2 - \|x_{n+1} - x_n\|^2 \\
&\quad \le 2(1-\alpha_n)\beta_n\langle\theta_n, x_n - \bar{x}\rangle - 2(1-\alpha_n)\beta_n\langle V s_n, x_n - \bar{x}\rangle \\
&\qquad - (1-\alpha_n)\|s_n - x_n\|^2.
\end{aligned} \tag{13}$$

By (10), we obtain that

$$\begin{aligned}&\|x_{n+1}-x_n\|^2\\&\quad\le(1-\alpha_n)^2\bigl[\beta_n\|\theta_n-Vs_n\|+\|s_n-x_n\|\bigr]^2\\&\quad=(1-\alpha_n)^2\bigl[\beta_n^2\|\theta_n-Vs_n\|^2+\|s_n-x_n\|^2+2\beta_n\|\theta_n-Vs_n\|\|s_n-x_n\|\bigr].\end{aligned}\tag{14}$$

By (13) and (14), we have that

$$\begin{aligned}&\|x_{n+1}-\bar{x}\|^2-\|x_n-\bar{x}\|^2\\&\quad\le2(1-\alpha_n)\beta_n\langle\theta_n,x_n-\bar{x}\rangle-2(1-\alpha_n)\beta_n\langle Vs_n,x_n-\bar{x}\rangle-(1-\alpha_n)\|s_n-x_n\|^2\\&\qquad+(1-\alpha_n)^2\bigl[\beta_n^2\|\theta_n-Vs_n\|^2+\|s_n-x_n\|^2+2\beta_n\|\theta_n-Vs_n\|\|s_n-x_n\|\bigr]\\&\quad\le2(1-\alpha_n)\beta_n\langle\theta_n,x_n-\bar{x}\rangle-2(1-\alpha_n)\beta_n\langle Vs_n,x_n-\bar{x}\rangle-(1-\alpha_n)\alpha_n\|s_n-x_n\|^2\\&\qquad+(1-\alpha_n)^2\bigl[\beta_n^2\|\theta_n-Vs_n\|^2+2\beta_n\|\theta_n-Vs_n\|\|s_n-x_n\|\bigr].\end{aligned}$$

Hence, we obtain that

$$\begin{aligned}&\|x_{n+1}-\bar{x}\|^2-\|x_n-\bar{x}\|^2+(1-\alpha_n)\alpha_n\|s_n-x_n\|^2\\&\quad\le2(1-\alpha_n)\beta_n\langle\theta_n,x_n-\bar{x}\rangle-2(1-\alpha_n)\beta_n\langle Vs_n,x_n-\bar{x}\rangle\\&\qquad+(1-\alpha_n)^2\bigl[\beta_n^2\|\theta_n-Vs_n\|^2+2\beta_n\|\theta_n-Vs_n\|\|s_n-x_n\|\bigr].\end{aligned}\tag{15}$$

We will divide the proof into two cases as follows.

Case 1: There exists a natural number N such that $\|x_{n+1}-\bar{x}\|\le\|x_n-\bar{x}\|$ for each $n\ge N$. So, $\lim_{n\to\infty}\|x_n-\bar{x}\|$ exists. Hence, it follows from (15), (i), and (ii) that

$$\lim_{n\to\infty}\|s_n-x_n\|=0.\tag{16}$$

By (14), (16), (i), and (ii), we have that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{17}$$

We also have that

$$\|z_n-s_n\|\le\bigl\|\beta_n\theta_n+(1-\beta_nV)s_n-s_n\bigr\|\le\beta_n\|\theta_n-Vs_n\|.\tag{18}$$

By (18), (iv), and (ii), we have that

$$\lim_{n\to\infty}\|z_n-s_n\|=0.\tag{19}$$

By (16) and (19), we have that

$$\lim_{n\to\infty}\|z_n-x_n\|=0.\tag{20}$$

By (10) and (6), we have that

$$\|s_n - \bar{x}\|^2 \le \|u_n - \bar{x}\|^2 \le \|x_n - \bar{x}\|^2 - r_n(2\kappa_2 - r_n)\|B_2u_n - B_2\bar{x}\|^2.$$

Therefore,

$$\begin{aligned} r_n(2\kappa_2 - r_n)\|B_2u_n - B_2\bar{x}\|^2 &\le \|x_n - \bar{x}\|^2 - \|s_n - \bar{x}\|^2 \\ &\le \|x_n - s_n\|\big(\|s_n - \bar{x}\| + \|x_n - \bar{x}\|\big). \end{aligned} \tag{21}$$

Thus, by (16), (21), and (iii), we have that

$$\lim_{n\to\infty} \|B_2u_n - B_2\bar{x}\| = 0. \tag{22}$$

Since T_{r_n} is firmly nonexpansive, we have from (3) that

$$\begin{aligned} &2\|u_n - \bar{x}\|^2 \\ &\quad = 2\big\|T_{r_n}(I - r_nB_2)x_n - T_{r_n}(I - r_nB_2)\bar{x}\big\|^2 \\ &\quad \le 2\big\langle u_n - \bar{x}, (I - r_nB_2)x_n - (I - r_nB_2)\bar{x}\big\rangle \\ &\quad \le 2\langle u_n - \bar{x}, x_n - \bar{x}\rangle - 2r_n\langle u_n - \bar{x}, B_2x_n - B_2\bar{x}\rangle \\ &\quad \le \|u_n - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 - \|u_n - x_n\|^2 \\ &\qquad - 2r_n\langle x_n - \bar{x}, B_2x_n - B_2\bar{x}\rangle - 2r_n\langle u_n - x_n, B_2x_n - B_2\bar{x}\rangle \\ &\quad \le \|u_n - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 - \|u_n - x_n\|^2 \\ &\qquad - 2\lambda_n\kappa_2\|B_2x_n - B_2\bar{x}\|^2 + 2r_n\langle x_n - u_n, B_2x_n - B_2\bar{x}\rangle \\ &\quad \le \|u_n - \bar{x}\|^2 + \|x_n - \bar{x}\|^2 - \|u_n - x_n\|^2 + 2r_n\|x_n - u_n\|\|B_2x_n - B_2\bar{x}\|. \end{aligned} \tag{23}$$

By (6) and (23), we have that

$$\begin{aligned} \|s_n - \bar{x}\|^2 &\le \|u_n - \bar{x}\|^2 \\ &\le \|x_n - \bar{x}\|^2 - \|u_n - x_n\|^2 + 2r_n\|x_n - u_n\|\|B_2x_n - B_2\bar{x}\|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|u_n - x_n\|^2 \\ &\quad \le \|x_n - \bar{x}\|^2 - \|s_n - \bar{x}\|^2 + 2r_n\|x_n - u_n\|\|B_2x_n - B_2\bar{x}\| \\ &\quad \le \|x_n - s_n\|\big(\|x_n - \bar{x}\| + \|s_n - \bar{x}\|\big) + 2r_n\|x_n - u_n\|\|B_2x_n - B_2\bar{x}\|. \end{aligned} \tag{24}$$

Thus, by (16), (22), and (24), we have that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{25}$$

By (5) and (6), we have that

$$\|s_n - \bar{x}\|^2 \le \|y_n - \bar{x}\|^2 \le \|x_n - \bar{x}\|^2 - \lambda_n(2\kappa_1 - \lambda_n)\|B_1u_n - B_1\bar{x}\|^2.$$

Therefore,

$$\lambda_n(2\kappa_1 - \lambda_n)\|B_1 u_n - B_1\bar{x}\|^2 \le \|x_n - \bar{x}\|^2 - \|s_n - \bar{x}\|^2$$
$$\le \|x_n - s_n\|\bigl(\|s_n - \bar{x}\| + \|x_n - \bar{x}\|\bigr). \tag{26}$$

Thus, by (16), (26), and (iii), we have that

$$\lim_{n\to\infty}\|B_1 u_n - B_1\bar{x}\| = 0. \tag{27}$$

Since J_{λ_n} is firmly nonexpansive, we have from (3) that

$$\begin{aligned}
&2\|y_n - \bar{x}\|^2\\
&\quad = 2\bigl\|J_{\lambda_n}(I - \lambda_n B_1)u_n - J_{\lambda_n}(I - \lambda_n B_1)\bar{x}\bigr\|^2\\
&\quad \le 2\bigl\langle y_n - \bar{x}, (I - \lambda_n B_1)u_n - (I - \lambda_n B_1)\bar{x}\bigr\rangle\\
&\quad \le 2\langle y_n - \bar{x}, u_n - \bar{x}\rangle - 2\lambda_n\bigl\langle y_n - \bar{x}, B_1 u_n - B_1\bar{x}\bigr\rangle\\
&\quad \le \|y_n - \bar{x}\|^2 + \|u_n - \bar{x}\|^2 - \|y_n - u_n\|^2\\
&\qquad - 2\lambda_n\langle u_n - \bar{x}, B_1 u_n - B_1\bar{x}\rangle - 2\lambda_n\langle y_n - u_n, B_1 u_n - B_1\bar{x}\rangle\\
&\quad \le \|y_n - \bar{x}\|^2 + \|u_n - \bar{x}\|^2 - \|y_n - u_n\|^2\\
&\qquad - 2\lambda_n\kappa_1\|B_1 u_n - B_1\bar{x}\|^2 + 2\lambda_n\langle u_n - y_n, B_1 u_n - B_1\bar{x}\rangle\\
&\quad \le \|y_n - \bar{x}\|^2 + \|u_n - \bar{x}\|^2 - \|y_n - u_n\|^2 + 2\lambda_n\|u_n - y_n\|\|B_1 u_n - B_1\bar{x}\|.
\end{aligned} \tag{28}$$

By (6) and (28), we have that

$$\begin{aligned}
\|s_n - \bar{x}\|^2 &\le \|y_n - \bar{x}\|^2\\
&\le \|u_n - \bar{x}\|^2 - \|y_n - u_n\|^2 + 2\lambda_n\|u_n - y_n\|\|B_1 u_n - B_1\bar{x}\|\\
&\le \|x_n - \bar{x}\|^2 - \|y_n - u_n\|^2 + 2\lambda_n\|u_n - y_n\|\|B_1 u_n - B_1\bar{x}\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|y_n - u_n\|^2\\
&\quad \le \|x_n - \bar{x}\|^2 - \|s_n - \bar{x}\|^2 + 2\lambda_n\|u_n - y_n\|\|B_1 u_n - B_1\bar{x}\|\\
&\quad \le \|x_n - s_n\|\bigl(\|x_n - \bar{x}\| + \|s_n - \bar{x}\|\bigr) + 2\lambda_n\|u_n - y_n\|\|B_1 u_n - B_1\bar{x}\|.
\end{aligned} \tag{29}$$

Thus, by (16), (27), and (29), we have that

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{30}$$

Since $\mathrm{Fix}(T) \cap (B_1 + G_1)^{-1}0 \cap (B_2 + G_2)^{-1}0$ is a nonempty closed convex subset of H, by Lemma 2.5, we can take $\bar{x}_0 \in \mathrm{Fix}(T) \cap (B_1 + G_1)^{-1}0 \cap (B_2 + G_2)^{-1}0$ such that

$$\bar{x}_0 = P_{\mathrm{Fix}(T)\cap(B_1+G_1)^{-1}0\cap(B_2+G_2)^{-1}0}(\bar{x}_0 - V\bar{x}_0 + \theta).$$

This point $\bar{x}_0$ is also a unique solution of the hierarchical variational inequality

$$\langle V\bar{x}_0 - \theta, q - \bar{x}_0\rangle \geq 0, \quad \forall q \in \operatorname{Fix}(T) \cap (B_1 + G_1)^{-1}0 \cap (B_2 + G_2)^{-1}0. \tag{31}$$

We want to show that

$$\limsup_{n\to\infty}\langle V\bar{x}_0 - \theta, z_n - \bar{x}_0\rangle \geq 0.$$

Without loss of generality, there exists a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightharpoonup w$ for some $w \in H$ and

$$\limsup_{n\to\infty}\langle V\bar{x}_0 - \theta, z_n - \bar{x}_0\rangle = \lim_{k\to\infty}\big\langle (V\bar{x}_0 - \theta, z_{n_k} - \bar{x}_0\big\rangle. \tag{32}$$

By (20) and (25), we have that

$$\lim_{n\to\infty}\|u_n - z_n\| = 0$$

and $u_{n_k} \rightharpoonup w$. On the other hand, since $0 < a \leq \lambda_n \leq b < 2\kappa_1$, there exists a subsequence $\{\lambda_{n_{k_j}}\}$ of $\{\lambda_{n_k}\}$ such that $\{\lambda_{n_{k_j}}\}$ converges to a number $\bar{\lambda} \in [a,b]$. By (30) and Lemma 2.7, we have that

$$\begin{aligned}
&\big\|u_{n_{k_j}} - J_{\bar{\lambda}}(I - \bar{\lambda}B_1)u_{n_{k_j}}\big\| \\
&\quad\leq \big\|u_{n_{k_j}} - J_{\lambda_{n_{k_j}}}(I - \lambda_{n_{k_j}}B_1)u_{n_{k_j}}\big\| + \big\|J_{\lambda_{n_{k_j}}}(I - \bar{\lambda}B_1)u_{n_{k_j}} - J_{\bar{\lambda}}(I - \bar{\lambda}B_1)u_{n_{k_j}}\big\| \\
&\qquad + \big\|J_{\lambda_{n_{k_j}}}(I - \lambda_{n_{k_j}}B_1)u_{n_{k_j}} - J_{\lambda_{n_{k_j}}}(I - \bar{\lambda}B_1)u_{n_{k_j}}\big\| \\
&\quad\leq \|u_{n_{k_j}} - y_{n_{k_j}}\| + |\lambda_{n_{k_j}} - \bar{\lambda}|\|B_1u_{n_{k_j}}\| \\
&\qquad + \frac{|\lambda_{n_{k_j}} - \bar{\lambda}|}{\bar{\lambda}}\big\|J_{\bar{\lambda}}(I - \bar{\lambda}B_1)u_{n_{k_j}} - (I - \bar{\lambda}B_1)u_{n_{k_j}}\big\| \to 0.
\end{aligned} \tag{33}$$

By (33), $u_{n_{k_j}} \rightharpoonup w$, Lemma 2.6 and 2.7, $w \in \operatorname{Fix}(J_{\bar{\lambda}}(I - \bar{\lambda}B_1)) = (B_1 + G_1)^{-1}0$. Without loss of generality and $0 < a \leq r_n \leq b < 2\kappa_2$, there exists a subsequence $\{r_{n_{k_j}}\}$ of $\{r_{n_k}\}$ such that $\{r_{n_{k_j}}\}$ converges to a number $\bar{r} \in [a,b]$. By (25) and Lemma 2.7, we have that

$$\begin{aligned}
&\big\|x_{n_{k_j}} - T_{\bar{r}}(I - \bar{r}B_2)x_{n_{k_j}}\big\| \\
&\quad\leq \big\|x_{n_{k_j}} - T_{r_{n_{k_j}}}(I - r_{n_{k_j}}B_2)x_{n_{k_j}}\big\| + \big\|T_{r_{n_{k_j}}}(I - r_{n_{k_j}}B_2)x_{n_{k_j}} - T_{r_{n_{k_j}}}(I - \bar{r}B_2)x_{n_{k_j}}\big\| \\
&\qquad + \big\|T_{r_{n_{k_j}}}(I - \bar{r}B_2)u_{n_{k_j}} - T_{\bar{r}}(I - \bar{r}B_2)u_{n_{k_j}}\big\| \\
&\quad\leq \|x_{n_{k_j}} - u_{n_{k_j}}\| + |r_{n_{k_j}} - \bar{r}|\|B_2u_{n_{k_j}}\| \\
&\qquad + \frac{|r_{n_{k_j}} - \bar{r}|}{\bar{r}}\big\|T_{\bar{r}}(I - \bar{r}B_2)u_{n_{k_j}} - (I - \bar{r}B_2)u_{n_{k_j}}\big\| \to 0.
\end{aligned} \tag{34}$$

By (34), $x_{n_{k_j}} \rightharpoonup w$, Lemma 2.8, we have that $w \in \mathrm{Fix}(T_{\bar{r}}(I - \bar{r}B_2)) = (B_2 + G_2)^{-1}0$. From (16), (25), and (30), we have that

$$\|Ty_n - y_n\| = \|s_n - y_n\| \leq \|s_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\| \to 0.$$

Since $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$, we have from $\|Ty_{n_j} - y_{n_j}\| \to 0$ and $y_{n_{k_j}} \rightharpoonup w$ that $w \in F(T)$. Hence, $w \in \mathrm{Fix}(T) \cap (B_1 + G_1)^{-1}0 \cap (B_2 + G_2)^{-1}0$. So, we have from (31) and (32) that

$$\limsup_{n\to\infty}\langle V\bar{x}_0 - \theta, z_n - \bar{x}_0\rangle = \lim_{k\to\infty}\langle V\bar{x}_0 - \theta, z_{n_k} - \bar{x}_0\rangle = \langle V\bar{x}_0 - \theta, w - \bar{x}_0\rangle \geq 0. \tag{35}$$

Let $z_n = \beta_n\theta_n + (1 - \beta_n V)s_n$. Then it follows from (7) that

$$\begin{aligned}\|z_n - \bar{x}_0\|^2 &= \left\|\beta_n\theta_n + (1 - \beta_n V)s_n - \bar{x}_0\right\|^2\\ &= \left\|\beta_n(\theta_n - V\bar{x}_0) + (1 - \beta_n V)(s_n - \bar{x}_0)\right\|^2\\ &\leq \left\|(1 - \beta_n V)(s_n - \bar{x}_0)\right\|^2 + 2\beta_n\langle\theta_n - V\bar{x}_0, z_n - \bar{x}_0\rangle\\ &\leq (1 - \beta_n\tau)^2\|x_n - \bar{x}_0\|^2 + 2\beta_n\langle\theta_n - V\bar{x}_0, z_n - \bar{x}_0\rangle.\end{aligned} \tag{36}$$

Thus, we obtain from the definition of x_n and (36) that

$$\begin{aligned}&\|x_{n+1} - \bar{x}_0\|^2\\ &\quad= \left\|\alpha_n x_n + (1 - \alpha_n)\big(\beta_n\theta_n + (1 - \beta_n V)s_n\big) - \bar{x}_0\right\|^2\\ &\quad\leq \alpha_n\|x_n - \bar{x}_0\|^2 + (1 - \alpha_n)\left\|\big(\beta_n\theta_n + (1 - \beta_n V)s_n\big) - \bar{x}_0\right\|^2\\ &\quad\leq \alpha_n\|x_n - \bar{x}_0\|^2 + (1 - \alpha_n)\big((1 - \beta_n\tau)^2\|x_n - \bar{x}_0\|^2 + 2\beta_n\langle\theta_n - V\bar{x}_0, z_n - \bar{x}_0\rangle\big)\\ &\quad\leq \big[\alpha_n + (1 - \alpha_n)(1 - \beta_n\tau)^2\big]\|x_n - \bar{x}_0\|^2 + 2\beta_n(1 - \alpha_n)\langle\theta_n - V\bar{x}_0, z_n - \bar{x}_0\rangle\\ &\quad\leq \big[1 - (1 - \alpha_n)\big(2\beta_n\tau - (\beta_n\tau)^2\big)\big]\|x_n - \bar{x}_0\|^2 + 2\beta_n(1 - \alpha_n)\langle\theta_n - V\bar{x}_0, z_n - \bar{x}_0\rangle\\ &\quad\leq \big[1 - 2(1 - \alpha_n)\beta_n\tau\big]\|x_n - \bar{x}_0\|^2 + (1 - \alpha_n)(\beta_n\tau)^2\|x_n - \bar{x}_0\|^2\\ &\qquad + 2\beta_n(1 - \alpha_n)\langle\theta_n - \theta, z_n - \bar{x}_0\rangle + 2\beta_n(1 - \alpha_n)\langle\theta - V\bar{x}_0, z_n - \bar{x}_0\rangle\\ &\quad\leq \big[1 - 2(1 - \alpha_n)\beta_n\tau\big]\|x_n - \bar{x}_0\|^2 + 2(1 - \alpha_n)\beta_n\tau\left(\frac{\beta_n\tau\|x_n - \bar{x}_0\|^2}{2}\right.\\ &\qquad\left. + \frac{\langle\theta_n - \theta, z_n - \bar{x}_0\rangle}{\tau} + \frac{\langle\theta - V\bar{x}_0, z_n - \bar{x}_0\rangle}{\tau}\right).\end{aligned} \tag{37}$$

By (35), (37), assumptions, and Lemma 2.10, we know that $\lim_{n\to\infty} x_n = \bar{x}_0$, where

$$\bar{x}_0 = P_{\mathrm{Fix}(T)\cap(B_1+G_1)^{-1}0\cap(B_2+G_2)^{-1}0}(\bar{x}_0 - V\bar{x}_0 + \theta).$$

Case 2: Suppose that there exists $\{n_i\}$ of $\{n\}$ such that $\|x_{n_i} - \bar{x}\| \leq \|x_{n_i+1} - \bar{x}\|$ for all $i \in \mathbb{N}$. By Lemma 2.9, there exists a nondecreasing sequence $\{m_j\}$ in $\mathbb{N}$ such that $m_j \to \infty$ and

$$\|x_{m_j} - \bar{x}\| \leq \|x_{m_j+1} - \bar{x}\| \quad\text{and}\quad \|x_j - \bar{x}\| \leq \|x_{m_j+1} - \bar{x}\|. \tag{38}$$

Hence, it follows from (15) and (38) that

$$\begin{aligned}&(1-\alpha_{m_j})\alpha_{m_j}\|s_{m_j}-x_{m_j}\|^2\\&\quad\le 2(1-\alpha_{m_j})\beta_{m_j}\langle\theta_{m_j},x_{m_j}-\bar{x}\rangle-2(1-\alpha_{m_j})\beta_{m_j}\langle Vs_{m_j},x_{m_j}-\bar{x}\rangle\\&\qquad+(1-\alpha_{m_j})^2\bigl[\beta_{m_j}^2\|\theta_{m_j}-Vs_{m_j}\|^2+2\beta_{m_j}\|\theta_{m_j}-Vs_{m_j}\|\|s_{m_j}-x_{m_j}\|\bigr]\end{aligned}\tag{39}$$

for each $j\in\mathbb{N}$. Hence, it follows from (39), (i), and (ii) that

$$\lim_{j\to\infty}\|s_{m_j}-x_{m_j}\|=0.\tag{40}$$

We want to show that

$$\limsup_{j\to\infty}\langle V\bar{x}_0-\theta,z_{m_j}-\bar{x}_0\rangle\ge 0.$$

Without loss of generality, there exists a subsequence $\{z_{m_{j_k}}\}$ of $\{z_{m_j}\}$ such that $z_{m_{j_k}}\rightharpoonup w$ for some $w\in H$ and

$$\limsup_{j\to\infty}\langle V\bar{x}_0-\theta,z_{m_j}-\bar{x}_0\rangle=\lim_{k\to\infty}\langle V\bar{x}_0-\theta,z_{m_{j_k}}-\bar{x}_0\rangle.\tag{41}$$

With the similar argument as in the proof of Case 1, we have $w\in\mathrm{Fix}(T)\cap(B_1+G_1)^{-1}0\cap(B_2+G_2)^{-1}0$. So, we have from (41) and (32) that

$$\limsup_{j\to\infty}\langle V\bar{x}_0-\theta,z_{m_j}-\bar{x}_0\rangle=\lim_{k\to\infty}\langle V\bar{x}_0-\theta,z_{m_{j_k}}-\bar{x}_0\rangle=\langle V\bar{x}_0-\theta,w-\bar{x}_0\rangle\ge 0.\tag{42}$$

With the similar argument as in the proof of Case 1, we have

$$\begin{aligned}&\|x_{m_j+1}-\bar{x}_0\|^2\\&\quad\le\bigl[1-2(1-\alpha_{m_j})\beta_{m_j}\tau\bigr]\|x_{m_j}-\bar{x}_0\|^2+(1-\alpha_{m_j})(\beta_{m_j}\tau)^2\|x_{m_j}-\bar{x}_0\|^2\\&\qquad+2\beta_{m_j}(1-\alpha_{m_j})\langle\theta_n-\theta,z_{m_j}-\bar{x}_0\rangle+2\beta_{m_j}(1-\alpha_{m_j})\langle\theta-V\bar{x}_0,z_{m_j}-\bar{x}_0\rangle.\end{aligned}\tag{43}$$

From $\|x_{m_j}-\bar{x}\|\le\|x_{m_j+1}-\bar{x}\|$, we have that

$$\begin{aligned}&2(1-\alpha_{m_j})\beta_{m_j}\tau\|x_{m_j}-\bar{x}_0\|^2\\&\quad\le(1-\alpha_{m_j})(\beta_{m_j}\tau)^2\|x_{m_j}-\bar{x}_0\|^2+2\beta_{m_j}(1-\alpha_{m_j})\langle\theta_n-\theta,z_{m_j}-\bar{x}_0\rangle\\&\qquad+2\beta_{m_j}(1-\alpha_{m_j})\langle\theta-V\bar{x}_0,z_{m_j}-\bar{x}_0\rangle.\end{aligned}\tag{44}$$

Since $(1-\alpha_{m_j})\beta_{m_j}>0$, we have that

$$2\tau\|x_{m_j}-\bar{x}_0\|^2\le\beta_{m_j}\tau\|x_{m_j}-\bar{x}_0\|^2+2\langle\theta_n-\theta,z_{m_j}-\bar{x}_0\rangle+2\langle\theta-V\bar{x}_0,z_{m_j}-\bar{x}_0\rangle.\tag{45}$$

By (42), (45), and assumptions, we know that

$$\lim_{j\to\infty}\|x_{m_j}-\bar{x}_0\|=0.$$

By (14), (40), and assumptions, we know that

$$\lim_{j\to\infty} \|x_{m_j+1} - x_{m_j}\| = 0.$$

Thus, we have that

$$\lim_{j\to\infty} \|x_{m_j+1} - \bar{x}_0\| = 0. \tag{46}$$

By (38) and (46), we have that

$$\lim_{j\to\infty} \|x_j - \bar{x}_0\| \le \lim_{j\to\infty} \|x_{m_j+1} - \bar{x}_0\| = 0.$$

Therefore, the proof is completed. □

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let I denote the identy mapping on H_1 and on H_2. Let G_i be a maximal monotone mapping on H_1 such that the domain of G_i is included in C for each $i = 1, 2$. Let $J_\lambda = (I + \lambda G_1)^{-1}$ and $T_r = (I + rG_2)^{-1}$ for each $\lambda > 0$ and $r > 0$. Let $A_i : H_1 \to H_2$ be a bounded linear operator, and let A_i^* be the adjoint of A_i for each $i = 1, 2$. Now, we recall the following multiple sets split feasibility problem:

(MSFP$_{\mathrm{FF}}$) Find $\bar{x} \in H_1$ such that $\bar{x} \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n})$, $A_1\bar{x} \in \mathrm{Fix}(F_1)$, and $A_2\bar{x} \in \mathrm{Fix}(F_2)$ for each $n \in \mathbb{N}$.

In order to study the convergence theorems for the solution set of multiple sets split feasibility problem (MSFP$_{\mathrm{FF}}$), we must give an essential result in this paper.

Theorem 3.2 *Given any $\bar{x} \in H_1$.*

(i) *If $\bar{x}$ is a solution of* (MSFP$_{\mathrm{FF}}$), *then* $J_{\lambda_n}(I - \rho_n A_1^*(I - F_1)A_1)T_{r_n}(I - \sigma_n A_2^*(I - F_2)A_2)\bar{x} = \bar{x}$ *for each $n \in \mathbb{N}$.*

(ii) *Suppose that $J_{\lambda_n}(I - \rho_n A_1^*(I - F_1)A_1)T_{r_n}(I - \sigma_n A_2^*(I - F_2)A_2)\bar{x} = \bar{x}$ with $0 < \rho_n < \frac{2}{\|A_1\|^2+2}$, $0 < \sigma_n < \frac{2}{\|A_2\|^2+2}$ for each $n \in \mathbb{N}$ and the solution set of* (MSFP$_{\mathrm{FF}}$) *is nonempty. Then $\bar{x}$ is a solution of* (MSFP$_{\mathrm{FF}}$).

Proof (i) Suppose that $\bar{x} \in H_1$ is a solution of (MSFP$_{\mathrm{FF}}$). Then $\bar{x} \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n})$, $A_1\bar{x} \in \mathrm{Fix}(F_1)$, and $A_2\bar{x} \in \mathrm{Fix}(F_2)$ for each $n \in \mathbb{N}$. It is easy to see that

$$J_{\lambda_n}\bigl(I - \rho_n A_1^*(I - F_1)A_1\bigr)T_{r_n}\bigl(I - \sigma_n A_2^*(I - F_2)A_2\bigr)\bar{x} = \bar{x}$$

for each $n \in \mathbb{N}$.

(ii) Since the solution set of (MSFP$_{\mathrm{FF}}$) is nonempty, there exists $\bar{w} \in H_1$ such that $\bar{w} \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n})$, $A_1\bar{w} \in \mathrm{Fix}(F_1)$, and $A_2\bar{w} \in \mathrm{Fix}(F_2)$. So,

$$\bar{w} \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}\bigl(I - \rho_n A_1^*(I - F_1)A_1\bigr) \cap \mathrm{Fix}(T_{r_n}) \cap \mathrm{Fix}\bigl(I - \sigma_n A_2^*(I - F_2)A_2\bigr) \neq \emptyset. \tag{47}$$

By Lemma 2.1, we have that

$$A_1^*(I - F_1)A_1 \text{ is } \frac{1}{\|A_1\|^2}\text{-ism}. \tag{48}$$

For each $n \in \mathbb{N}$, by (48), $0 < \rho_n < \frac{2}{\|A_1\|^2+2}$, and Lemma 2.3(ii), (iii), we know that

$$I - \rho_n A_1^*(I - F_1)A_1 \text{ is averaged.} \tag{49}$$

By Lemma 2.1 again, we have that

$$A_2^*(I - F_2)A_2 \text{ is } \frac{1}{\|A_2\|^2}\text{-ism.} \tag{50}$$

For each $n \in \mathbb{N}$, by (50), $0 < \sigma_n < \frac{2}{\|A_2\|^2+2}$, and Lemma 2.3(ii), (iii), we know that

$$I - \sigma_n A_2^*(I - F_2)A_2 \text{ is averaged.} \tag{51}$$

On the other hand, for each $n \in \mathbb{N}$, since J_{λ_n}, and T_{r_n} are firmly nonexpansive mappings, it is easy to see that

$$J_{\lambda_n} \text{ and } T_{r_n} \text{ are } \frac{1}{2} \text{ averaged.} \tag{52}$$

Hence, by (49), (51), (52), and Lemma 2.3(v), we have that for each $n \in \mathbb{N}$,

$$\begin{aligned} \bar{x} &\in \operatorname{Fix}\bigl(J_{\lambda_n}\bigl(I - \rho A_1^*(I - F_1)A_1\bigr)T_{r_n}\bigl(I - \sigma A_2^*(I - F_2)A_2\bigr)\bigr) \\ &= \operatorname{Fix}(J_{\lambda_n}) \cap \operatorname{Fix}\bigl(I - \rho A_1^*(I - F_1)A_1\bigr) \cap \operatorname{Fix}(T_{r_n}) \cap \operatorname{Fix}\bigl(I - \sigma A_2^*(I - F_2)A_2\bigr). \end{aligned}$$

This implies that for each $n \in \mathbb{N}$,

$$\bar{x} = J_{\lambda_n}\bigl(I - \rho A_1^*(I - F_1)A_1\bigr)\bar{x} \quad \text{and} \quad \bar{x} = T_{r_n}\bigl(I - \sigma A_2^*(I - F_2)A_2\bigr)\bar{x}.$$

By Lemma 2.2, for each $n \in \mathbb{N}$,

$$\begin{aligned} &\bigl\langle \bigl(\bar{x} - \rho A_1^*(I - F_1)A_1\bar{x}\bigr) - \bar{x}, \bar{x} - w\bigr\rangle \geq 0 \quad \text{for each } w \in \operatorname{Fix}(J_{\lambda_n}), \\ &\bigl\langle \bigl(\bar{x} - \rho A_2^*(I - F_2)A_2\bar{x}\bigr) - \bar{x}, \bar{x} - w\bigr\rangle \geq 0 \quad \text{for each } w \in \operatorname{Fix}(T_{r_n}). \end{aligned}$$

That is, for each $n \in \mathbb{N}$,

$$\begin{aligned} &\bigl\langle A_1^*(I - F_1)A_1\bar{x}, \bar{x} - w\bigr\rangle \leq 0 \quad \text{for each } w \in \operatorname{Fix}(J_{\lambda_n}), \\ &\bigl\langle A_2^*(I - F_2)A_2\bar{x}, \bar{x} - w\bigr\rangle \leq 0 \quad \text{for each } w \in \operatorname{Fix}(T_{r_n}). \end{aligned} \tag{53}$$

For each $n \in \mathbb{N}$, by (53) and the fact that A_i^* is the adjoint of A_i for each $i = 1, 2$,

$$\begin{aligned} &\langle A_1\bar{x} - F_1A_1\bar{x}, A_1\bar{x} - A_1w\rangle \leq 0 \quad \text{for each } w \in \operatorname{Fix}(J_{\lambda_n}), \\ &\langle A_2\bar{x} - F_2A_2\bar{x}, A_2\bar{x} - A_2w\rangle \leq 0 \quad \text{for each } w \in \operatorname{Fix}(T_{r_n}). \end{aligned} \tag{54}$$

On the other hand, by Lemma 2.2 again,

$$\begin{aligned} &\langle A_1\bar{x} - F_1A_1\bar{x}, v_1 - F_1A\bar{x}\rangle \leq 0 \quad \text{for each } v_1 \in \operatorname{Fix}(F_1), \\ &\langle A_2\bar{x} - F_2A_2\bar{x}, v_2 - F_2A_2\bar{x}\rangle \leq 0 \quad \text{for each } v_2 \in \operatorname{Fix}(F_2). \end{aligned} \tag{55}$$

For each $n \in \mathbb{N}$, by (54) and (55),

$$\begin{aligned} &\langle A_1\bar{x} - F_1A_1\bar{x}, v_1 - F_1A_1\bar{x} + A_1\bar{x} - A_1w\rangle \leq 0, \\ &\langle A_2\bar{x} - F_2A_2\bar{x}, v_2 - F_2A_2\bar{x} + A_2\bar{x} - Aw\rangle \leq 0 \end{aligned} \tag{56}$$

for each $w \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n})$, $v_2 \in \mathrm{Fix}(F_2)$, and $v_1 \in \mathrm{Fix}(F_1)$.

That is, for each $n \in \mathbb{N}$,

$$\begin{aligned} &\|A_1\bar{x} - F_1A_1\bar{x}\|^2 \leq \langle A_1\bar{x} - F_1A_1\bar{x}, A_1w - v_1\rangle, \\ &\|A_2\bar{x} - F_2A_2\bar{x}\|^2 \leq \langle A_2\bar{x} - F_2A_2\bar{x}, A_2w - v_2\rangle \end{aligned} \tag{57}$$

for each $w \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n})$, $v_1 \in \mathrm{Fix}(F_1)$, and $v_2 \in \mathrm{Fix}(F_2)$.

Since $\bar{w}$ is a solution of multiple sets split feasibility problem ($\mathrm{MSFP_{FF}}$), we know that $\bar{w} \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n})$, $A_1\bar{w} \in \mathrm{Fix}(F_1)$, and $A_2\bar{w} \in \mathrm{Fix}(F_2)$ for each $n \in \mathbb{N}$. So, it follows from (57) that $A_1\bar{x} = \mathrm{Fix}(F_1)$ and $A_2\bar{x} = \mathrm{Fix}(F_2)$. Furthermore, $\bar{x} \in \mathrm{Fix}(J_{\lambda_n})$ and $\bar{x} \in \mathrm{Fix}(T_{r_n})$ for each $n \in \mathbb{N}$. Therefore, $\bar{x}$ is a solution of ($\mathrm{MSFP_{FF}}$). □

Applying Theorem 3.1 and Theorem 3.2, we can find the solution of the following hierarchical problem.

Theorem 3.3 *Let $T : C \to H$ be a quasi-nonexpansive mapping with $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$. Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. For each $i = 1,2$, let F_i be a firmly nonexpansive mapping of H_2 into H_2, let $A_i : H_1 \to H_2$ be a bounded linear operator, and let A_i^* be the adjoint of A_i. Suppose that the solution set of* ($\mathrm{MSFP_{FF}}$) *is Ω and $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. Let $\{x_n\} \subset H$ be defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_{\lambda_n}(I - \lambda_n A_1^*(I - F_1)A_1)T_{r_n}(I - r_n A_2^*(I - F_2)A_2)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n\theta_n + (1 - \beta_n V)s_n) \end{cases} \tag{3.3}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \leq \lambda_n \leq b < \frac{2}{\|A_1\|^2+2}$, *and* $0 < a \leq r_n \leq b < \frac{2}{\|A_2\|^2+2}$;

(iv) $\lim_{n\to\infty} \theta_n = \theta$ *for some* $\theta \in H$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}(\bar{x} - V\bar{x} + \theta)$. This point $\bar{x}$ is also a unique solution of the following hierarchical problem: Find $\bar{x} \in \mathrm{Fix}(T) \cap \Omega$ such that

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega.$$

Proof Since F_i is firmly nonexpansive, it follows from Lemma 2.1 that we have that $A_i^*(I - F_i)A_i : C_1 \to H_1$ is $\frac{1}{\|A_i\|^2}$-ism for each $i = 1,2$. For each $i = 1,2$, put $B_i = A_i^*(I - F_i)A_i$ in Theorem 3.3. Then algorithm (3.1) in Theorem 3.1 follows immediately from algorithm (3.3) in Theorem 3.3.

Since the solution set of ($\mathrm{MSFP}_{\mathrm{FF}}$) is nonempty, by (47), we have for each $n \in \mathbb{N}$

$$\bar{w} \in \mathrm{Fix}\big(J_{\lambda_n}\big(\big(I - \lambda_n A_1^*(I - F_1)A_1\big)\big)\big) \cap \mathrm{Fix}\big(T_{r_n}\big(I - r_n A_2^*(I - F_2)A_2\big)\big) \neq \emptyset. \tag{58}$$

This implies that for each $n \in \mathbb{N}$,

$$\bar{w} \in \mathrm{Fix}\big(J_{\lambda_n}(I - \lambda_n B_1)\big) \cap \mathrm{Fix}\big(T_{r_n}(I - r_n B_2)\big) \neq \emptyset. \tag{59}$$

So,

$$\bar{w} \in (B_1 + G_1)^{-1}0 \cap (B_2 + G_2)^{-1}0 \neq \emptyset. \tag{60}$$

It follows from Theorem 3.1 that $\lim_{n\to\infty} x_n = \bar{x}$, where

$$\bar{x} = P_{\mathrm{Fix}(T)\cap(B_1+G_1)^{-1}0\cap(B_2+G_2)^{-1}0}(\bar{x} - V\bar{x} + \theta).$$

This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap (B_1 + G_1)^{-1}0 \cap (B_2 + G_2)^{-1}0,$$

that is, for each $n \in \mathbb{N}$,

$$\bar{x} = J_{\lambda_n}(I - \lambda_n B_1)\bar{x} = J_{\lambda_n}\big(I - \lambda_n A_1^*(I - F_1)A_1\big)\bar{x} \tag{61}$$

and

$$\bar{x} = T_{r_n}(I - r_n B_2)\bar{x} = T_{r_n}\big(I - r_n A_2^*(I - F_2)A_2\big)\bar{x}. \tag{62}$$

This implies that for each $n \in \mathbb{N}$,

$$\bar{x} = J_{\lambda_n}\big(I - \lambda_n A_1^*(I - F_1)A_1\big)T_{r_n}\big(I - r_n A_2^*(I - F_2)A_2\big)\bar{x}. \tag{63}$$

By assumptions, (63), and Theorem 3.2(ii), we know that $\bar{x}$ is a solution of ($\mathrm{MSFP}_{\mathrm{FF}}$). Furthermore, $\bar{x} \in \mathrm{Fix}(T)$. Therefore, $\bar{x} \in \mathrm{Fix}(T) \cap \Omega$. By the same argument as (61), (62), and (63), we also have

$$\langle V\bar{x} - \theta, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega.$$

Therefore, the proof is completed. □

Remark 3.1 In Theorem 3.3, we establish a strong convergence theorem for hierarchical problem ($\mathrm{MSFP}_{\mathrm{FF}}$) without calculating the inverse of the operator we consider.

4 Applications to mathematical programming with multiple sets split feasibility constraints

By Theorem 3.3, we obtain mathematical programming with fixed point and multiple sets split feasibility constraints.

Theorem 4.1 *Let $T : C \to H$ be a quasi-nonexpansive mapping with* $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$. *In Theorem* 3.3, *let $h : C \to \mathbb{R}$ be a convex Gâteaux differential function with Gâteaux derivative V. Let*

$$\Omega = \{q \in H_1 : q \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n}), A_1 q \in \mathrm{Fix}(F_1), A_2 q \in \mathrm{Fix}(F_2), n \in \mathbb{N}\}.$$

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}(\bar{x} - V\bar{x})$. *This point $\bar{x}$ is also a unique solution of the mathematical programming with fixed point and multiple sets split feasibility constraints*: $\min_{q\in\mathrm{Fix}(T)\cap\Omega} h(q)$.

Proof Put $\theta = 0$ in Theorem 3.3. Then, by Theorem 3.3, there exists $\bar{x} \in \mathrm{Fix}(T) \cap \Omega$ such that

$$\langle V\bar{x}, q - \bar{x}\rangle \geq 0, \quad \forall q \in F(T) \cap \Omega. \tag{64}$$

Since $h : C \to \mathbb{R}$ is a convex Gâteaux differential function with Gâteaux derivative V, we obtain that

$$\begin{aligned} \langle V\bar{x}, y - \bar{x}\rangle &= \lim_{t\to 0} \frac{h(\bar{x} + t(y - \bar{x})) - h(\bar{x})}{t} \\ &= \lim_{t\to 0} \frac{h((1-t)\bar{x} + ty) - h(\bar{x})}{t} \\ &\leq \lim_{t\to 0} \frac{(1-t)h(\bar{x}) + th(y) - h(\bar{x})}{t} \\ &= h(y) - h(\bar{x}) \end{aligned} \tag{65}$$

for all $y \in C$. By (64) and (65), it is easy to see that $h(\bar{x}) \leq h(q)$ for all $q \in \mathrm{Fix}(T) \cap \Omega$. □

We can apply Theorem 4.1 to study the mathematical programming of a quadratic function with fixed point and multiple sets split feasibility constraints.

Theorem 4.2 *Let $T : C \to H$ be a quasi-nonexpansive mapping with* $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$. *In Theorem* 3.3, *let $B : C \to C$ be a strongly positive self-adjoint bounded linear operator and $a \in H$. Let*

$$\Omega = \{q \in H_1 : q \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n}), A_1 q \in \mathrm{Fix}(F_1), A_2 q \in \mathrm{Fix}(F_2), n \in \mathbb{N}\}.$$

Let $\{x_n\} \subset H$ be defined by

$$(4.2) \quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_{\lambda_n}(I - \lambda_n A_1^*(I - F_1)A_1)T_{r_n}(I - r_n A_2^*(I - F_2)A_2)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (s_n - \beta_n(B(s_n) - a))) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < a \leq \lambda_n \leq b < \frac{2}{\|A_1\|^2+2}$, *and* $0 < a \leq r_n \leq b < \frac{2}{\|A_2\|^2+2}$;
(iv) $\lim_{n\to\infty} \theta_n = 0$.

Then $\lim_{n\to\infty} x_n = \bar{x}$. *This point* $\bar{x}$ *is also a unique solution of the mathematical programming of a quadratic function with fixed point and multiple sets split feasibility constraints*: $\min_{q\in \mathrm{Fix}(T)\cap\Omega} \frac{1}{2}\langle Bq, q\rangle - \langle a, q\rangle$.

Proof Let $h : C \to \mathbb{R}$ be defined by

$$h(x) = \frac{1}{2}\langle Bx, x\rangle - \langle a, x\rangle.$$

It is easy to see that h is a convex function. Since B is a strongly positive self-adjoint operator, there exists $\eta > 0$ such that $\langle Bx, x\rangle \geq \eta\|x\|^2$. This implies that

$$\langle Bx - By, x - y\rangle = \bigl\langle B(x - y), x - y\bigr\rangle \geq \eta\|x - y\|^2. \tag{66}$$

Therefore,

$$\langle Bx, x\rangle + \langle By, y\rangle \geq \langle Bx, y\rangle + \langle By, x\rangle. \tag{67}$$

From this we can show that h is a convex function. Indeed, for any $x, y \in C$ and any $\lambda \in [0,1]$. It follows from (67) that

$$\begin{aligned}
&h\bigl(\lambda x + (1-\lambda)y\bigr)\\
&\quad= \frac{1}{2}\bigl\langle B\bigl(\lambda x + (1-\lambda)y\bigr), \lambda x + (1-\lambda)y\bigr\rangle - \bigl\langle a, \lambda x + (1-\lambda)y\bigr\rangle\\
&\quad= \frac{1}{2}\bigl\langle \lambda Bx + (1-\lambda)By, \lambda x + (1-\lambda)y\bigr\rangle - \bigl\langle a, \lambda x + (1-\lambda)y\bigr\rangle\\
&\quad= \frac{1}{2}\lambda^2\langle Bx, x\rangle + \frac{1}{2}\lambda(1-\lambda)\langle Bx, y\rangle + \frac{1}{2}\lambda(1-\lambda)\langle By, x\rangle + \frac{1}{2}(1-\lambda)^2\langle By, y\rangle\\
&\qquad - \lambda\langle a, x\rangle - (1-\lambda)\langle a, y\rangle\\
&\quad\leq \frac{1}{2}\lambda^2\langle Bx, x\rangle + \frac{1}{2}\lambda(1-\lambda)\bigl(\langle Bx, x\rangle + \langle By, y\rangle\bigr) + \frac{1}{2}(1-\lambda)^2\langle By, y\rangle\\
&\qquad - \lambda\langle a, x\rangle - (1-\lambda)\langle a, y\rangle\\
&\quad\leq \frac{1}{2}\lambda\langle Bx, x\rangle + \frac{1}{2}(1-\lambda)\langle By, y\rangle - \lambda\langle a, x\rangle - (1-\lambda)\langle a, y\rangle\\
&\quad\leq \lambda h(x) + (1-\lambda)h(y).
\end{aligned}$$

Let $V(x) = B(x) - a$ for all $x \in C$. It is easy to see that V is the Gâteaux derivative of h. Indeed, for any $u \in H$, $x \in C$ and any $t \in [0,1]$. Since B is a self-adjoint bounded linear operator, we see that for each $u \in H$,

$$\begin{aligned}
&h'(x)(u)\\
&\quad= \lim_{t\to 0}\frac{h(x + tu) - h(x)}{t}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{t\to 0} \frac{\frac{1}{2}\langle B(x+tu), x+tu\rangle - \langle a, x+tu\rangle - \frac{1}{2}\langle Bx, x\rangle + \langle a, x\rangle}{t} \\
&= \lim_{t\to 0} \frac{\frac{1}{2}\langle tBu + Bx, tu + x\rangle - \langle a, tu + x\rangle - \frac{1}{2}\langle Bx, x\rangle + \langle a, x\rangle}{t} \\
&= \lim_{t\to 0} \frac{\frac{1}{2}[t^2\langle Bu, u\rangle + t\langle Bu, x\rangle + t\langle Bx, u\rangle + \langle Bx, x\rangle] - t\langle a, u\rangle - \langle a, x\rangle - \frac{1}{2}\langle Bx, x\rangle + \langle a, x\rangle}{t} \\
&= \lim_{t\to 0} \frac{1}{2}\bigl[t\langle Bu, u\rangle + \langle Bu, x\rangle + \langle Bx, u\rangle\bigr] - \langle a, u\rangle \\
&= \langle Bx, u\rangle - \langle a, u\rangle = \langle Bx - a, u\rangle = \langle Vx, u\rangle.
\end{aligned}$$

Therefore, V is the Gâteaux derivative of h. Since B is a strongly positive bounded linear operator in H, we have that

$$\|Vx - Vy\| = \|Bx - a - By + a\| = \|Bx - By\| \le \|B\|\|x - y\|.$$

This implies that V is Lipschitz, and we have that

$$\langle Vx - Vy, x - y\rangle = \langle Bx - a - By + a, x - y\rangle = \bigl\langle B(x - y), x - y\bigr\rangle \ge \eta\|x - y\|^2.$$

This implies that V is strongly monotone. Therefore, Theorem 4.2 follows from Theorem 4.1. □

Theorem 4.3 *Let $T: C \to H$ be a quasi-nonexpansive mapping with* $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$. *In Theorem* 3.3, *let $h: C \to \mathbb{R}$ be a convex Gâteaux differential function with Gâteaux derivative V. Let Φ_i be a maximal monotone mapping on H_2 such that the domain of Φ_i is included in Q for each $i = 1, 2$, where Q is a closed convex subset of H_2. Let*

$$\Omega = \bigl\{q \in H_1 : q \in G_1^{-1}0 \cap G_2^{-1}0, A_1 q \in \Phi_1^{-1}0, A_2 q \in \Phi_2^{-1}0\bigr\}.$$

Then $\lim_{n\to\infty} x_n = \bar{x}$. This point $\bar{x}$ is also a unique solution of the mathematical programming with fixed point and multiple sets split feasibility problem constraints: $\min_{q\in \mathrm{Fix}(T)\cap\Omega} h(q)$.

Proof Let $J_\lambda = (I + \lambda G_1)^{-1}$, $T_r = (I + rG_2)^{-1}$, $F_1 = (I + \lambda\Phi_1)^{-1}$, and $F_2 = (I + r\Phi_2)^{-1}$ for each $\lambda > 0$ and $r > 0$ in Theorem 4.2. Then $\mathrm{Fix}(J_\lambda) = G_1^{-1}0$, $\mathrm{Fix}(T_r) = G_2^{-1}0$, $\mathrm{Fix}(F_1) = \Phi_1^{-1}0$, and $\mathrm{Fix}(F_2) = \Phi_2^{-1}0$. Therefore, Theorem 4.3 follows from Theorem 4.2. □

Takahashi *et al.* [40] showed the following result.

Lemma 4.1 [40] *Let C be a nonempty closed convex subset of a Hilbert space H, and let $g: C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)-(A4). *Define A_g as follows*:

$$\text{(L4.1)}\quad A_g x = \begin{cases} \{z \in H : g(x, y) \ge \langle y - x, z\rangle, \forall y \in C\}, & \forall x \in C, \\ \emptyset, & \forall x \notin C. \end{cases}$$

Then $\mathrm{EP}(g) = A_g^{-1}0$ *and A_g is a maximal monotone operator with the domain of $A_g \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r^g of g coincides with the resolvent of A_g, i.e., $T_r^g x = (I + rA_g)^{-1}x$.*

Now, we consider the following multiple sets split equilibrium problem:

(MSEP) Find $\bar{x} \in H_1$ such that $\bar{x} \in \mathrm{EP}(g_1) \cap \mathrm{EP}(g_2)$, $A_1\bar{x} \in \mathrm{EP}(f_1)$ and $A_2\bar{x} \in \mathrm{EP}(f_2)$.

Applying Theorems 2.2 and 4.1, Lemma 4.1, we can find the minimum norm solution of (MSEP) and mathematical programming with fixed point and multiple sets split equilibrium constraints.

Theorem 4.4 *Let $T : C \to H$ be a quasi-nonexpansive mapping with* $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$. *Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $g_1 : C \times C \to \mathbb{R}$, $g_2 : C \times C \to \mathbb{R}$, $f_1 : Q \times Q \to \mathbb{R}$, and $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions* (A1)-(A4), *and let $T^{g_1}_{\lambda_n}$, $T^{g_2}_{r_n}$, $T^{f_1}_{\lambda_n}$, $T^{f_2}_{r_n}$ be the resolvent of g_1, g_2, f_1, f_2, respectively, for $\lambda_n > 0$, $r_n > 0$. For $i = 1, 2$, let $A_i : H_1 \to H_2$ be a bounded linear operator, and let A_i^* be the adjoint of A_i. Let $h : C \to \mathbb{R}$ be a convex Gâteaux differential function with Gâteaux derivative V. Suppose that Ω is the solution set of* (MSEP) *and* $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. *Let $\{x_n\} \subset H$ be defined by*

$$\begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = T^{g_1}_{\lambda_n}(I - \lambda_n A_1^*(I - T^{f_1}_{\lambda_n})A_1) T^{g_2}_{r_n}(I - r_n A_2^*(I - T^{f_2}_{r_n})A_2)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n V)s_n) \end{cases} \tag{4.2}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \leq \lambda_n \leq b < \frac{2}{\|A_1\|^2+2}$, *and* $0 < a \leq r_n \leq b < \frac{2}{\|A_2\|^2+2}$;

(iv) $\lim_{n\to\infty} \theta_n = 0$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}(\bar{x} - V\bar{x})$. This point $\bar{x}$ is also a unique solution of the mathematical programming with fixed point and multiple sets split equilibrium constraints: $\min_{q\in\mathrm{Fix}(T)\cap\Omega} h(q)$.

Proof Define A_g as (L4.1). By Lemma 4.1, we know that $\mathrm{EP}(g) = A_g^{-1}0$ and A_g is a maximal monotone operator with the domain of $A_g \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T^g_r of g coincides with the resolvent of A_g, *i.e.*, $T^g_r x = (I + rA_g)^{-1}x$. By Theorem 2.2, $T^{f_1}_{\lambda_n}$, $T^{f_2}_{r_n}$ are firmly nonexpansive mappings.

Put $G_1 = A_{g_1}$, $G_2 = A_{g_2}$, $F_1 = T^{f_1}_{\lambda_n}$ and $F_2 = T^{f_2}_{r_n}$ in Theorem 3.3. Then $J_{\lambda_n}x = (I + \lambda_n A_{g_1})^{-1}x = T^{g_1}_{\lambda_n}x$, $T_{r_n}x = (I + r_n A_{g_2})^{-1}x = T^{g_2}_{r_n}x$. By Theorem 2.2, we have that $\mathrm{Fix}(J_{\lambda_n}) = \mathrm{Fix}(T^{g_1}_{\lambda_n}) = \mathrm{EP}(g_1)$, $\mathrm{Fix}(T_{r_n}) = \mathrm{Fix}(T^{g_2}_{r_n}) = \mathrm{EP}(g_2)$, $\mathrm{Fix}(F_1) = \mathrm{Fix}(T^{f_1}_{\lambda_n}) = \mathrm{EP}(f_1)$ and $\mathrm{Fix}(F_2) = \mathrm{Fix}(T^{f_2}_{r_n}) = \mathrm{EP}(f_2)$. Therefore, the solution set of (MSEP) coincides with the solution set of ($\mathrm{MSFP}_{\mathrm{FF}}$). Therefore, by Theorem 4.1, we get the result. □

The following unique minimum norm common solution of a fixed point problem and multiple sets split equilibrium problem is a special case of Theorem 4.4.

Corollary 4.1 *Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $g_1 : C \times C \to \mathbb{R}$, $g_2 : C \times C \to \mathbb{R}$, $f_1 : Q \times Q \to \mathbb{R}$, and $f_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions* (A1)-(A4), *and let $T^{g_1}_{\lambda_n}$, $T^{g_2}_{r_n}$, $T^{f_1}_{\lambda_n}$, $T^{f_2}_{r_n}$ be the resolvent of g_1, g_2, f_1, f_2, respectively, for $\lambda_n > 0$, $r_n > 0$. For $i = 1, 2$, let $A_i : H_1 \to H_2$ be a bounded linear* *operator, and let* A_i^* *be the adjoint of* A_i. *Suppose that* Ω *is the solution set of* (MSEP) *and* $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. *Let* $\{x_n\} \subset H$ *be defined by*

$$(4.2)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = T_{\lambda_n}^{g_1}(I - \lambda_n A_1^*(I - T_{\lambda_n}^{f_1})A_1)T_{r_n}^{g_2}(I - r_n A_2^*(I - T_{r_n}^{f_2})A_2)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, *and* $\{r_n\} \subset (0,\infty)$. *Assume that*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < a \leq \lambda_n \leq b < \frac{2}{\|A_1\|^2+2}$, *and* $0 < a \leq r_n \leq b < \frac{2}{\|A_2\|^2+2}$;
(iv) $\lim_{n\to\infty} \theta_n = 0$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}0$. *This point* $\bar{x}$ *is also a unique minimum norm solution of the fixed point and multiple sets split equilibrium constraints*: $\min_{q\in\mathrm{Fix}(T)\cap\Omega} \|q\|$.

Proof Let $h(x) = \frac{1}{2}\|x\|^2$, and let V be the Gâteaux derivative of h. It is easy to see $V(x) = x$ for each $x \in H$. Then Corollary 4.1 follows immediately from Theorem 4.4. □

Now, we consider the following split equilibrium problem:

(SEP) Find $\bar{x} \in C$ such that $\bar{x} \in \mathrm{EP}(g_1)$ and $A_1\bar{x} \in \mathrm{EP}(f_1)$.

Applying Theorems 2.2 and 4.1, Lemma 4.1, we can find the unique minimum norm common solution of fixed point and split equilibrium constraints and the solution of mathematical programming with fixed point and split equilibrium constraints.

Theorem 4.5 *Let* C *and* Q *be nonempty closed convex subsets of Hilbert spaces* H_1 *and* H_2, *respectively. Let* $g_1 : C \times C \to \mathbb{R}$ *and* $f_1 : Q \times Q \to \mathbb{R}$ *be bifunctions satisfying conditions* (A1)-(A4), *and let* $T_{\lambda_n}^{g_1}$, $T_{\lambda_n}^{f_1}$ *be the resolvent of* g_1, f_1, *respectively, for* $\lambda_n > 0$, $r_n > 0$. *Let* $A_1 : H_1 \to H_2$ *be a bounded linear operator, and let* A_1^* *be the adjoint of* A_1. *Let* $T : C \to C$ *be a quasi-nonexpansive mapping with* $F(T) = F(\hat{T})$. *Let* $V : C \to C$ *be a* $\bar{\gamma}$*-strongly monotone and L-Lipschitzian continuous operator with* $\bar{\gamma} > 0$ *and* $L > 0$ *and* $\{\theta_n\} \subseteq C$. *Let* $h : C \to \mathbb{R}$ *be a convex Gâteaux differential function with Gâteaux derivative* V. *Suppose that* Ω *is the solution set of* (SEP) *and* $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. *Let* $\{x_n\} \subset H$ *be defined by*

$$(4.3)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = T_{\lambda_n}^{g_1}(I - \lambda_n A_1^*(I - T_{\lambda_n}^{f_1})A_1)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, *and* $\{r_n\} \subset (0,\infty)$. *Assume that*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < a \leq \lambda_n \leq b < \frac{2}{\|A_1\|^2+2}$;
(iv) $\lim_{n\to\infty} \theta_n = 0$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}(\bar{x} - V\bar{x})$. *This point* $\bar{x}$ *is also a unique solution of the mathematical programming with fixed point and split equilibrium constraints*: $\min_{q\in\mathrm{Fix}(T)\cap\Omega} h(q)$.

Proof Define A_g as (L4.1). By Lemma 4.1, we know that $\mathrm{EP}(g) = A_g^{-1}0$ and A_g is a maximal monotone operator with the domain of A_g included in C. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r^g of g coincides with the resolvent of A_g, *i.e.*, $T_r^g x = (I + rA_g)^{-1}x$. By Theorem 2.2, $T_{\lambda_n}^{f_1}$ is a firmly nonexpansive mapping; we also know that the identity mapping I is a firmly nonexpansive mapping.

Put $G_1 = A_{g_1}$, $G_2 = A_{g_2}$, $F_1 = T_{\lambda_n}^{f_1}$, and $F_2 = I$ in Theorem 3.3. Then $J_{\lambda_n}x = (I + \lambda_n A_{g_1})^{-1}x = T_{\lambda_n}^{g_1}x$, $T_{r_n}x = (I + r_n A_{g_2})^{-1}x = T_{r_n}^{g_2}x$. Let $g_2(x,y) = 0$, $\forall x, y \in C$. Then $T_{r_n}x = (I + r_n A_{g_2})^{-1}x = T_{r_n}^{g_2}x = P_C x$. By Theorem 2.2, we have that $\mathrm{Fix}(J_{\lambda_n}) = \mathrm{Fix}(T_{\lambda_n}^{g_1}) = \mathrm{EP}(g_1)$, $\mathrm{Fix}(T_{r_n}) = \mathrm{Fix}(P_C) = C$, $\mathrm{Fix}(F_1) = \mathrm{Fix}(T_{\lambda_n}^{f_1}) = \mathrm{EP}(f_1)$, and $\mathrm{Fix}(F_2) = \mathrm{Fix}(I) = H_2$. So, we have that the solution set of (SEP) coincides with the solution set of ($\mathrm{MSFP_{FF}}$). On the other hand, we have that $T_{r_n}^{g_2}(I - r_n A_1^*(I - F_2)A_1)x_n = P_C x_n = x_n$. Then algorithm (3.3) in Theorem 3.3 follows immediately from algorithm (4.3) in Theorem 4.5. Therefore, by Theorems 3.3 and 4.1, we get the result. □

Put $h(x) = \frac{1}{2}\|x\|^2$ for each $x \in H$ in Theorem 4.5, we obtain a unique minimum norm common solution of a fixed point problem and a split equilibrium constraints.

Corollary 4.2 *Let* C *and* Q *be nonempty closed convex subsets of Hilbert spaces* H_1 *and* H_2, *respectively. Let* $g_1 : C \times C \to \mathbb{R}$ *and* $f_1 : Q \times Q \to \mathbb{R}$ *be bifunctions satisfying conditions* (A1)-(A4), *and let* $T_{\lambda_n}^{g_1}$, $T_{\lambda_n}^{f_1}$ *be the resolvent of* g_1, f_1, *respectively, for* $\lambda_n > 0$, $r_n > 0$. *Let* $A_1 : H_1 \to H_2$ *be a bounded linear operator, and let* A_1^* *be the adjoint of* A_1. *Let* $T : C \to C$ *be a quasi-nonexpansive mapping with* $F(T) = F(\hat{T})$. *Let* $V : C \to C$ *be a* $\bar{\gamma}$*-strongly monotone and* L*-Lipschitzian continuous operator with* $\bar{\gamma} > 0$ *and* $L > 0$ *and* $\{\theta_n\} \subseteq C$. *Suppose that the solution set of* (SEP) *is* Ω *and* $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. *Let* $\{x_n\} \subset H$ *be defined by*

$$(4.3)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = T_{\lambda_n}^{g_1}(I - \lambda_n A_1^*(I - T_{\lambda_n}^{f_1})A_1)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, *and* $\{r_n\} \subset (0,\infty)$. *Assume that*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < a \le \lambda_n \le b < \frac{2}{\|A_1\|^2+2}$;
(iv) $\lim_{n\to\infty} \theta_n = 0$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}(0)$. *This point* $\bar{x}$ *is also a unique minimum norm common solution of fixed point and split equilibrium constraints*: *Find* $\min_{q\in\mathrm{Fix}(T)\cap\Omega} \|q\|$.

Now, we recall the following split feasibility problem:

(SFP) Find $\bar{x} \in H_1$ such that $\bar{x} \in C$ and $A\bar{x} \in Q$.

Applying Theorem 4.5, we can find a unique minimum norm common solution with fixed point and split feasibility constraints, and the solution of mathematical programming with fixed point and split feasibility constraints.

Theorem 4.6 *Let C and Q be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A_1 : H_1 \to H_2$ be a bounded linear operator, and let A_1^* be the adjoint of A_i. Let $T : C \to C$ be a quasi-nonexpansive mapping with $F(T) = F(\hat{T})$. Let $V : C \to C$ be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$ and $\{\theta_n\} \subseteq C$. Let $h : C \to \mathbb{R}$ be a convex Gâteaux differential function with Gâteaux derivative V. Suppose that the solution set of* (SFP) *is Ω and* $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. *Let $\{x_n\} \subset H$ be defined by*

$$
\begin{cases}
x_1 \in C \text{ chosen arbitrarily}, \\
y_n = P_C(I - \lambda_n A_1^*(I - P_Q)A_1)x_n, \\
s_n = Ty_n, \\
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n V)s_n)
\end{cases}
\tag{4.4}
$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset (0,1)$, and $\{r_n\} \subset (0,\infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < a \leq \lambda_n \leq b < \frac{2}{\|A_1\|^2+2}$;
(iv) $\lim_{n\to\infty} \theta_n = 0$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}(\bar{x} - V\bar{x})$. This point $\bar{x}$ is also a unique solution of the mathematical programming with fixed point and split feasibility constraints: $\min_{q\in \mathrm{Fix}(T)\cap\Omega} h(q)$.

Proof Put $g_1(x,y) = 0$, $\forall x,y \in C$ and $f_1(x,y) = 0$, $\forall x,y \in Q$ in Theorem 4.5. Then $T_{\lambda_n}^{g_1} = P_C$, $T_{r_n}^{f_1} = P_Q$. Therefore, algorithm (4.3) in Theorem 4.5 follows immediately from algorithm (4.4) in Theorem 4.6.

By Theorem 2.2, we have that $\mathrm{EP}(g_1) = \mathrm{Fix}(T_{\lambda_n}^{g_1}) = \mathrm{Fix}(P_C) = C$ and $\mathrm{EP}(f_1) = \mathrm{Fix}(T_{r_n}^{f_1}) = \mathrm{Fix}(P_Q) = Q$. So, the solution set of (SEP) coincides with the solution set of (SFP). Therefore, by Theorem 4.5, we get the result. □

For each $i = 1,2$, let X_i, Y_i be two Hilbert spaces, a function $F : X_i \times Y_i \to \mathbb{R} \cup \{-\infty,\infty\}$ is said to be convex-concave iff it is convex in the variable x and concave in the variable y. To such a function, Rockafellar associated the operator T_F, defined by $T_F = \partial_1 F \times \partial_2(-F)$, where ∂_1 (resp. ∂_2) stands for the subdifferential of F with respect to the first (resp. the second) variable. T_F is a maximal monotone operator if and only if F is closed and proper in the Rockafellar sense (see [41]). Moreover, it is well known that $(\bar{x}_1,\bar{y}_1) \in H_1 = X_1 \times Y_1$ is a saddlepoint of F, namely

$$
F(\bar{x}_1,y) \leq F(\bar{x}_1,\bar{y}_1) \leq F(x,\bar{y}_1) \quad \text{for all } x \in X_i, y \in Y_i,
$$

if and only if the following monotone variational inclusion holds true, that is, $(0,0) \in T_F(\bar{x}_1,\bar{y}_1)$. If $(\bar{x}_1,\bar{y}_1) \in H_1 = X_1 \times Y_1$ is a saddlepoint of F, then

$$
\inf_{x\in X_1} \sup_{y\in Y_1} F(x,y) = F(\bar{x}_1,\bar{y}_1) = \sup_{y\in Y_1} \inf_{x\in X_1} F(x,y).
$$

For each $i = 1,2$, let $\varphi_i : X_1 \times Y_1 \to \mathbb{R} \cup \{-\infty,\infty\}$ and $\psi_i : X_2 \times Y_2 \to \mathbb{R} \cup \{-\infty,\infty\}$ be convex-concave functions. Now, we consider the following multiple sets split minimax problem:

(MSMMP) Find $(\bar{x}_1, \bar{y}_1) \in H_1 = X_1 \times Y_1$ such that for each $i = 1, 2$,

$$\inf_{x \in X_1} \sup_{y \in Y_1} \varphi_i(x, y) = \varphi_i(\bar{x}_1, \bar{y}_1) = \sup_{y \in Y_1} \inf_{x \in X_1} \varphi_i(x, y) \quad \text{and}$$

$$\inf_{u \in X_2} \sup_{v \in Y_2} \psi_i(u, v) = \psi_i\big(A_i(\bar{x}_1, \bar{y}_1)\big) = \sup_{v \in Y_2} \inf_{u \in X_2} \psi_i(u, v).$$

By Theorem 3.3, we can find the unique minimum norm common solution of fixed point and multiple sets split minimax problems (MSMMP) and the solution of mathematical programming with fixed point and multiple sets split minimax (MSMMP) constraints.

Theorem 4.7 *Let $T : C_1 \times C_2 \to X_1 \times X_2$ be a quasi-nonexpansive mapping with* $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$. *For each $i = 1, 2$, let $A_1 : X_1 \times Y_1 \to X_2 \times Y_2$, $A_2 : X_1 \times Y_1 \to X_2 \times Y_2$ be a bounded linear operator, A_i^* be the adjoint of A_i. For each $i = 1, 2$, let $\varphi_i : X_1 \times Y_1 \to \mathbb{R} \cup \{-\infty, \infty\}$ and $\psi_i : X_2 \times Y_2 \to \mathbb{R} \cup \{-\infty, \infty\}$ be convex-concave functions which are proper and closed in the Rockafellar sense. Let C_1, C_2 be closed convex subsets of Hilbert spaces X_1, Y_1, and let $C = C_1 \times C_2$ be a closed convex subset of $H_1 = X_1 \times Y_1$. Let $J_{\lambda_n} = (I + \lambda_n T_{\varphi_1})^{-1}$, $T_{r_n} = (I + r_n T_{\varphi_2})^{-1}$, $F_1 = (I + \lambda_n T_{\psi_1})^{-1}$, $F_2 = (I + \lambda_n T_{\psi_2})^{-1}$. Let $h : C_1 \times C_2 \to \mathbb{R}$ be a convex Gâteaux differential function with Gâteaux derivative V. Let V be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$ and suppose that the solution of multiple sets split minimax problem* (MSMMP) *is* Ω *and* $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. *Let $\{x_n\}$ be defined by*

$$(4.7) \qquad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_{\lambda_n}(I - \lambda_n A_1^*(I - F_1)A_1)T_{r_n}(I - r_n A_2^*(I - F_2)A_2)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, $\{\beta_n\} \subset (0, 1)$, and $\{r_n\} \subset (0, \infty)$. Assume that:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(iii) $0 < a \leq \lambda_n \leq b < \frac{2}{\|A_1\|^2 + 2}$, *and* $0 < a \leq r_n \leq b < \frac{2}{\|A_2\|^2 + 2}$;
(iv) $\lim_{n\to\infty} \theta_n = 0$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}(\bar{x} - V\bar{x})$. This point $\bar{x}$ is also a unique solution of the mathematical programming with fixed point and multiple split minimax constraints: $\min_{(q_1, q_2) \in \mathrm{Fix}(T)\cap\Omega} h(q_1, q_2)$.

Proof Since $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. There exists $(\bar{a}_1, \bar{b}_1) \in H_1 = X_1 \times Y_1$ such that for each $i = 1, 2$,

$$\begin{aligned} &\inf_{x \in X_1} \sup_{y \in Y_1} \varphi_i(x, y) = \varphi_i(\bar{a}_1, \bar{b}_1) = \sup_{y \in Y_1} \inf_{x \in X_1} \varphi_i(x, y) \quad \text{and} \\ &\inf_{u \in X_2} \sup_{v \in Y_2} \psi_i(u, v) = \psi_i(A_i(\bar{a}_1, \bar{b}_1) = \sup_{v \in Y_2} \inf_{u \in X_2} \psi_i(u, v). \end{aligned} \tag{68}$$

That is,

$$\begin{aligned} &(0, 0) \in T_{\varphi_1}(\bar{a}_1, \bar{b}_1) \cap T_{\varphi_2}(\bar{a}_1, \bar{b}_1), \qquad (0, 0) \in T_{\psi_1}(\bar{u}_1, \bar{v}_1) \quad \text{and} \quad (0, 0) \in T_{\psi_2}(\bar{u}_2, \bar{v}_2), \\ &\quad \text{where } (\bar{u}_1, \bar{v}_1) = A_1(\bar{a}_1, \bar{b}_1) \text{ and } (\bar{u}_2, \bar{v}_2) = A_2(\bar{a}_1, \bar{b}_1). \end{aligned} \tag{69}$$

That is,

$$(\bar{a}_1, \bar{b}_1) \in T_{\varphi_1}^{-1}(0,0) \cap T_{\varphi_2}^{-1}(0,0), \qquad (\bar{u}_1, \bar{v}_1) \in T_{\psi_1}^{-1}(0,0), \quad \text{and} \quad (\bar{u}_2, \bar{v}_2) \in T_{\psi_2}^{-1}(0,0),$$
$$\text{where } (\bar{u}_1, \bar{v}_1) = A_1(\bar{a}_1, \bar{b}_1) \text{ and } (\bar{u}_2, \bar{v}_2) = A_2(\bar{a}_1, \bar{b}_1). \tag{70}$$

That is,

$$(\bar{a}_1, \bar{b}_1) \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n}), \qquad (\bar{u}_1, \bar{v}_1) \in \mathrm{Fix}(F_1), \quad \text{and} \quad (\bar{u}_2, \bar{v}_2) \in \mathrm{Fix}(F_2),$$
$$\text{where } (\bar{u}_1, \bar{v}_1) = A_1(\bar{a}_1, \bar{b}_1) \text{ and } (\bar{u}_2, \bar{v}_2) = A_2(\bar{a}_1, \bar{b}_1). \tag{71}$$

That is,

$$(\bar{a}_1, \bar{b}_1) \in \mathrm{Fix}(J_{\lambda_n}) \cap \mathrm{Fix}(T_{r_n}), \qquad A_1(\bar{a}_1, \bar{b}_1) \in \mathrm{Fix}(F_1), \quad \text{and} \quad A_2(\bar{a}_1, \bar{b}_1) \in \mathrm{Fix}(F_2),$$
$$\text{where } (\bar{u}_1, \bar{v}_1) = A_1(\bar{a}_1, \bar{b}_1) \text{ and } (\bar{u}_2, \bar{v}_2) = A_2(\bar{a}_1, \bar{b}_1). \tag{72}$$

This implies that $\mathrm{Fix}(T) \cap \Omega_1 \neq \emptyset$, where Ω_1 is the solution set of ($\mathrm{MSFP_{FF}}$).

By Theorem 3.3, we have that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega_1}(\bar{x} - V\bar{x})$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x}, q - \bar{x} \rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega_1.$$

By (68), (69), (70), (71), and (72), and Theorem 4.1, we get the result. □

Now, we recall the following split minimax problem:

(SMMP) Find $(\bar{x}, \bar{y}) \in H_1 = X_1 \times Y_1$ such that

$$\inf_{x\in X_1} \sup_{y\in Y_1} \varphi_1(x,y) = \varphi_1(\bar{x}, \bar{y}) = \sup_{y\in Y_1} \inf_{x\in X_1} \varphi_1(x,y) \quad \text{and}$$
$$\inf_{u\in X_2} \sup_{v\in Y_2} \psi_1(u,v) = \psi_1\big(A_1(\bar{x}, \bar{y})\big) = \sup_{v\in Y_2} \inf_{u\in X_2} \psi_1(u,v).$$

By Theorem 3.3, we can find the solution of split minimax problem (SMMP) and mathematical programming with fixed point and split minimax problem (SMMP) constraints.

Theorem 4.8 *Let $T : C_1 \times C_2 \to X_1 \times X_2$ be a quasi-nonexpansive mapping with* $\mathrm{Fix}(T) = \mathrm{Fix}(\hat{T})$. *Let $A_1 : X_1 \times Y_1 \to X_2 \times Y_2$ be a bounded linear operator. Let A_1^* be the adjoint of A_1. Let A_1, A_1^*, T, V be defined as in Theorem* 4.7. *Let φ_1, ψ_1 be defined as in Theorem* 4.7. *Let C_1, C_2 closed convex subsets of Hilbert spaces X_1, X_2, let $C = C_1 \times C_2$ be a closed convex subset of $H_1 = X_1 \times Y_1$, and let $J_{\lambda_n} = (I + \lambda_n T_{\varphi_1})^{-1}$, $F_1 = (I + \lambda_n T_{\psi_1})^{-1}$. Let $V : C \to C$ be a $\bar{\gamma}$-strongly monotone and L-Lipschitzian continuous operator with $\bar{\gamma} > 0$ and $L > 0$ and $\{\theta_n\} \subseteq C$. Let $h : C \to \mathbb{R}$ be a convex Gâteaux differential function with Gâteaux derivative V. Suppose that the solution of split minimax problem* (SMMP) *is* Ω

and $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$. *Let* $\{x_n\}$ *be defined by*

$$(4.8)\quad \begin{cases} x_1 \in C \text{ chosen arbitrarily}, \\ y_n = J_{\lambda_n}(I - \lambda_n A_1^*(I - F_1)A_1)x_n, \\ s_n = Ty_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)(\beta_n \theta_n + (1 - \beta_n V)s_n) \end{cases}$$

for each $n \in \mathbb{N}$, $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\} \subset (0, 1)$, *and* $\{\beta_n\} \subset (0, 1)$. *Assume that*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;

(iii) $0 < a \leq \lambda_n \leq b < \frac{2}{\|A_1\|^2+2}$;

(iv) $\lim_{n\to\infty} \theta_n = 0$.

Then $\lim_{n\to\infty} x_n = \bar{x}$, *where* $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}(\bar{x} - V\bar{x})$. *This point* $\bar{x}$ *is also a unique solution of the mathematical programming with fixed point and split minimax problem constraints*: $\min_{(q_1,q_2)\in \mathrm{Fix}(T)\cap\Omega} h(q_1, q_2)$.

Proof Define A_g as (L4.1). By Lemma 4.1, we know that $\mathrm{EP}(g) = A_g^{-1}0$ and A_g is a maximal monotone operator with the domain of $A_g \subset C$. Furthermore, for any $x \in H$ and $r > 0$, the resolvent T_r^g of g coincides with the resolvent of A_g, *i.e.*, $T_r^g x = (I + rA_g)^{-1}x$. By Theorem 2.2, $T_{\lambda_n}^{f_1}$ is a firmly nonexpansive mapping, we also know that the identity mapping I is a firmly nonexpansive mapping.

Put $G_2 = A_{g_2}$ and $F_2 = I$ in Theorem 3.3. Then $T_{r_n}x = (I + r_n A_{g_2})^{-1}x = T_{r_n}^{g_2}x$. Let $g_2(x, y) = 0$, $\forall x, y \in C$. Then $T_{r_n}x = (I + r_n A_{g_2})^{-1}x = T_{r_n}^{g_2}x = P_C x$. So, we have that $\mathrm{Fix}(T_{r_n}) = \mathrm{Fix}(P_C) = C$ and $\mathrm{Fix}(F_2) = \mathrm{Fix}(I) = H_2$. So, we have that $T_{r_n}^{g_2}(I - r_n A_1^*(I - F_2)A_1)x_n = P_C x_n = x_n$. Then algorithm (3.3) in Theorem 3.3 follows immediately from algorithm (4.8) in Theorem 4.8.

Since $\mathrm{Fix}(T) \cap \Omega \neq \emptyset$, there exists $(\bar{a}_1, \bar{b}_1) \in H_1 = X_1 \times Y_1$ such that

$$\begin{aligned} &\inf_{x\in X_1} \sup_{y\in Y_1} \varphi_1(x, y) = \varphi_1(\bar{a}_1, \bar{b}_1) = \sup_{y\in Y_1} \inf_{x\in X_1} \varphi_1(x, y) \quad \text{and} \\ &\inf_{u\in X_2} \sup_{v\in Y_2} \psi_1(u, v) = \psi_1\bigl(A_1(\bar{a}_1, \bar{b}_1)\bigr) = \sup_{v\in Y_2} \inf_{u\in X_2} \psi_1(u, v). \end{aligned} \tag{73}$$

That is,

$$(0, 0) \in T_{\varphi_1}(\bar{a}_1, \bar{b}_1) \quad \text{and} \quad (0, 0) \in T_{\psi_1}(\bar{u}_1, \bar{v}_1), \quad \text{where } (\bar{u}_1, \bar{v}_1) = A_1(\bar{a}_1, \bar{b}_1). \tag{74}$$

That is,

$$(\bar{a}_1, \bar{b}_1) \in T_{\varphi_1}^{-1}(0, 0) \quad \text{and} \quad (\bar{u}_1, \bar{v}_1) \in T_{\psi_1}^{-1}(0, 0), \quad \text{where } (\bar{u}_1, \bar{v}_1) = A_1(\bar{a}_1, \bar{b}_1). \tag{75}$$

That is,

$$(\bar{a}_1, \bar{b}_1) \in \mathrm{Fix}(J_{\lambda_n}) \quad \text{and} \quad (\bar{u}_1, \bar{v}_1) \in \mathrm{Fix}(F_1), \quad \text{where } (\bar{u}_1, \bar{v}_1) = A_1(\bar{a}_1, \bar{b}_1). \tag{76}$$

That is,

$$(\bar{a}_1, \bar{b}_1) \in \mathrm{Fix}(J_{\lambda_n}) \quad \text{and} \quad A_1(\bar{a}_1, \bar{b}_1) \in \mathrm{Fix}(F_1). \tag{77}$$

This implies that $\mathrm{Fix}(T) \cap \Omega_1 \neq \emptyset$, where Ω_1 is the solution set of ($\mathrm{SFP}_{\mathrm{FF}}$).

By Theorem 3.3, we have that $\lim_{n\to\infty} x_n = \bar{x}$, where $\bar{x} = P_{\mathrm{Fix}(T)\cap\Omega}(\bar{x} - V\bar{x})$. This point $\bar{x}$ is also a unique solution of the following hierarchical variational inequality:

$$\langle V\bar{x}, q - \bar{x}\rangle \geq 0, \quad \forall q \in \mathrm{Fix}(T) \cap \Omega.$$

By (73), (74), (75), (76), and (77), and Theorem 4.1, we get the result. □

Now, we recall the following multiple split minimax-equilibrium problem:

(MSMMP) Find $(\bar{x}_1, \bar{y}_1) \in H_1 = X_1 \times Y_1$ such that for each $i = 1, 2$, $(\bar{a}_1, \bar{b}_1) \in H_1 = X_1 \times Y_1$ such that

$$\inf_{x\in X_1} \sup_{y\in Y_1} \varphi_i(x,y) = \varphi_i(\bar{x}_1, \bar{y}_1) = \sup_{y\in Y_1} \inf_{x\in X_1} \varphi_i(x,y) \quad \text{and}$$
$$(\bar{u}_1, \bar{v}_1) = A_1(\bar{x}_1, \bar{y}_1) \in \mathrm{EP}(f_1) \quad \text{and}$$
$$(\bar{u}_2, \bar{v}_2) = A_2(\bar{x}_1, \bar{y}_1) \in \mathrm{EP}(f_2).$$

By the same argument as in Theorems 4.4 and 4.7, we can find the solution of multiple split minimax problem (MSMMEP) and mathematical programming with fixed point and multiple split minimax problem (MSMMEP) constraints.

By the same argument as in Theorems 4.5 and 4.8, we can find the solution of the split minimax-equilibrium problem (SMMEP) and mathematical programming with fixed point and multiple split minimax-equilibrium problem (SMMEP) constraints.

5 Concluding remark

Applying Theorems 4.3-4.8 and following the same arguments as in Theorem 4.2, we can study the mathematical programming of a quadratic function with various types of fixed point and multiple split feasibility constraints.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors discussed the problem, carried out and checked the results, read and approved the manuscript.

Author details

[1]Department of Electronic Engineering, Nan Kai University of Technology, Nantou, 542, Taiwan. [2]Department of Mathematics, National Changhua University of Education, Changhua, 50058, Taiwan.

Received: 5 July 2013 Accepted: 18 September 2013 Published: 08 Nov 2013

References

1. Censor, Y, Elfving, T: A multiprojection algorithm using Bregman projection in a product space. Numer. Algorithms **8**, 221-239 (1994)
2. Byrne, C: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. **18**, 441-453 (2002)
3. Censor, Y, Bortfeld, T, Martin, B, Trofimov, A: A unified approach for inversion problems in intensity modulated radiation therapy. Phys. Med. Biol. **51**, 2353-2365 (2003)
4. López, G, Martín-Márquez, V, Xu, HK: Iterative algorithms for the multiple-sets split feasibility problem. In: Censor, Y, Jiang, M, Wang, G (eds.) Biomedical Mathematics: Promising Directions in Imaging, Therapy Planning and Inverse Problems, pp. 243-279. Medical Physics Publishing, Madison (2010)
5. Stark, H: Image Recovery: Theory and Applications. Academic Press, Orlando (1987)
6. Byrne, C: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl. **20**, 103-120 (2004)

7. Dang, Y, Gao, Y: The strong convergence of a KM-CQ-like algorithm for a split feasibility problem. Inverse Probl. **27**, 015007 (2011)
8. Masad, E, Reich, S: A note on the multiple-set split convex feasibility problem in Hilbert space. J. Nonlinear Convex Anal. **8**, 367-371 (2008)
9. Qu, B, Xiu, N: A note on the CQ algorithm for the split feasibility problem. Inverse Probl. **21**, 1655-1665 (2005)
10. Wang, F, Xu, HK: Approximating curve and strong convergence of the CQ algorithm for the split feasibility problem. J. Inequal. Appl. **2010**, 102085 (2010)
11. Xu, HK: A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem. Inverse Probl. **22**, 2021-2034 (2006)
12. Xu, HK: Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces. Inverse Probl. **26**, 105018 (2010)
13. Yang, Q: The relaxed CQ algorithm for solving the split feasibility problem. Inverse Probl. **20**, 1261-1266 (2004)
14. Yang, Q: On variable-step relaxed projection algorithm for variational inequalities. J. Math. Anal. Appl. **302**, 166-179 (2005)
15. Zhao, J, Yang, Q: Self-adaptive projection methods for the multiple-sets split feasibility problem. Inverse Probl. **27**, 035009 (2011)
16. Ceng, LC, Ansari, QH, Yao, JC: An extragradient method for split feasibility and fixed point problems. Comput. Math. Appl. **64**, 633-642 (2012)
17. Ceng, LC, Ansari, QH, Yao, JC: Mann type iterative methods for finding a common solution of split feasibility and fixed point problems. Positivity **16**(3), 471-495 (2012)
18. Ceng, LC, Ansari, QH, Yao, JC: Relaxed extragradient methods for finding minimum-norm solutions of the split feasibility problem. Nonlinear Anal. **75**(4), 2116-2125 (2012)
19. Chang, SS, Joseph Lee, HW, Chan, CK, Wang, L, Qin, LJ: Split feasibility problem for quasi-nonexpansive multi-valued mappings and total asymptotically strict pseudo-contractive mapping. Appl. Math. Comput. **219**, 10416-10424 (2013)
20. Eicke, B: Iteration methods for convexly constrained ill-posed problems in Hilbert space. Numer. Funct. Anal. Optim. **13**, 413-429 (1992)
21. Landweber, L: An iteration formula for Fredholm integral equations of the first kind. Am. J. Math. **73**, 615-624 (1951)
22. Xu, HK: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**, 279-291 (2004)
23. Xu, HK: An iterative approach to quadratic optimization. J. Optim. Theory Appl. **116**, 659-678 (2003)
24. Marino, G, Xu, HK: A general iterative method for nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **318**, 43-52 (2006)
25. Zegeye, H, Shazad, N: Strong convergence theorem for a common point of solution of variational inequality and fixed point problem. Adv. Fixed Point Theory **2**, 374-397 (2012)
26. Wang, Y, Xu, W: Strong convergence of a modified iterative algorithm for hierarchical fixed point problems and variational inequalities. Adv. Fixed Point Theory Appl. **2013**, 121 (2013)
27. Wang, ZM, Lou, W: A new iterative algorithm of common solutions to quasi-variational inclusion and fixed point problems. J. Math. Comput. Sci. **3**, 57-72 (2013)
28. Reich, S: A weak convergence theorem for the alternative method with Bregman distance. In: Kartsatos, AG (ed.) Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 313-318. Marcel Dekker, New York (1996)
29. Yu, ZT, Lin, LJ, Chuang, CS: A unified study of the split feasible problems with applications. J. Nonlinear Convex Anal. **15** (2014)
30. Bauschke, HH, Combettes, PL: A weak-to-strong convergence principle for Fejermonotone methods in Hilbert spaces. Math. Oper. Res. **26**, 248-264 (2001)
31. Combettes, PL: Solving monotone inclusions via compositions of nonexpansive averaged operators. Optimization **53**(5-6), 475-504 (2004)
32. Lin, LJ, Takahashi, W: A general iterative method for hierarchical variational inequality problems in Hilbert spaces and applications. Positivity **16**, 429-453 (2012)
33. Takahashi, W: Nonlinear Functional Analysis-Fixed Point Theory and Its Applications. Yokohama Publishers, Yokohama (2000)
34. Kocourek, P, Takahashi, W, Yao, JC: Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces. Taiwan. J. Math. **14**, 2497-2511 (2010)
35. Takahashi, W, Yao, JC, Kocourek, P: Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces. J. Nonlinear Convex Anal. **11**, 567-586 (2010)
36. Maingé, PE: Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. **16**, 899-912 (2008)
37. Aoyama, K, Kimura, Y, Takahashi, W, Toyoda, M: Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 2350-2360 (2007)
38. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123-146 (1994)
39. Combettes, PL, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117-136 (2005)
40. Takahashi, S, Takahashi, W, Toyoda, M: Strong convergence theorems for maximal monotone operators with nonlinear mappings in Hilbert spaces. J. Optim. Theory Appl. **147**, 27-41 (2010)
41. Rockafellar, RT: Monotone operators associated with saddle functions and minimax problems. In: Browder, FE (ed.) Nonlinear Analysis, Part I. Symposia in Pure Mathematics, vol. 18, pp. 397-407. Am. Math. Soc., Providence (1970)

10.1186/1687-1812-2013-283
Cite this article as: Yu and Lin: **Hierarchical problems with applications to mathematical programming with multiple sets split feasibility constraints.** *Fixed Point Theory and Applications* 2013, 2013:283