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Solving total-variation image super-resolution problems via proximal symmetric alternating direction methods

Bin Gao^{1*}, Fenggang Sun², Ying Tong³ and Shiming Xu¹

*Correspondence: feimaxiao123@gmail.com
¹College of Communications Engineering, PLA University of Science and Technology, Nanjing, 210007, P.R. China
Full list of author information is available at the end of the article

Abstract

The single image super-resolution (SISR) problem represents a class of efficient models appealing in many computer vision applications. In this paper, we focus on designing a proximal symmetric alternating direction method of multipliers (SADMM) for the SISR problem. By taking full exploitation of the special structure, the method enjoys the advantage of being easily implementable by linearizing the quadratic term of subproblems in the SISR problem. With this linearization, the resulting subproblems easily achieve closed-form solutions. A global convergence result is established for the proposed method. Preliminary numerical results demonstrate that the proposed method is efficient and the computing time is saved by nearly 40% compared with several state-of-the-art methods.

Keywords: proximal symmetric alternating direction method of multipliers; linearized Peaceman-Rechford splitting method; convex minimization; strictly contractive; single image super-resolution

1 Introduction

SISR is a technique that aims at restoring a high-resolution (HR) image from a single degraded low-resolution (LR) image. Since the HR image contains more details than the LR one, the image of HR is preferred in many practical cases, such as video surveillance, the hyperspectral technique, and remote sensing. Super-resolution (SR) is a typical ill-posed inverse problem since a multiplicity of solutions exist for any input LR pixel [1]. To tackle such a problem, most of the SR methods reduce the size of the solution space by incorporating strong prior information, which can be obtained by training data, using various regularization methods, and capturing specific image features [2]. Motivated by those ideas, the SISR schemes can be broadly divided into three categories: interpolation-based methods, learning-based methods, and reconstruction-based methods.

Interpolation-based methods such as the bicubic approach are simple and easy to implement but tend to blur the high frequency details and produce aliasing artifacts at salient edges [3]. The learning-based algorithms recover the missing high frequency details by learning the relations between LR and HR image patches from a given database [4]. However, they highly rely on the similarity between the training set and the test images and generally have high computational complexity. Reconstruction-based methods that are

considered in this paper belong to the third category of SISR schemes [5–7]. As SISR essentially is a highly ill-posed problem, the performance of such approaches mainly relies on the prior knowledge. Among all the priors, smoothness priors such as the total variation (TV) prior have been widely used in many image processing applications [8]. To reduce its computational complexity, a fast SR alternating direction method of multipliers (FSR-ADMM) based on the TV model is proposed in [9]. Considering the efficiency of a symmetric alternating direction method of multipliers (SADMM) [10], our paper aims at constructing a fast SR symmetric alternating direction method of multipliers (FSR-SADMM).

In the SISR problem, the observed LR image is modeled as a noisy version of the blurred and downsampled HR image estimated as follows:

$$y = \Phi Hx + v, \tag{1.1}$$

where the vector $y \in \mathbb{R}^{N_l}$ ($N_l = m_l \times n_l$) corresponds to the LR observed image and $x \in \mathbb{R}^{N_h}$ ($N_h = m_h \times n_h$) denotes the HR image with $N_h > N_l$. $v \in \mathbb{R}^{N_l}$ represents a zero-mean additive white Gaussian noise (AWGN), $\Phi \in \mathbb{R}^{N_l \times N_h}$ and $H \in \mathbb{R}^{N_h \times N_h}$ stand for the downsampling and the blurring operations, respectively.

Concisely, the SISR TV model corresponds to solving the following optimization problem:

$$\min_x \frac{1}{2} \underbrace{\|y - \Phi Hx\|_2^2}_{\text{data fidelity}} + \alpha \underbrace{\|x\|_{\text{TV}}}_{\text{TV regularization}}, \tag{1.2}$$

where $\|y - \Phi Hx\|_2^2$ stands for the data fidelity term while $\|x\|_{\text{TV}} = \phi(Ax) = \sqrt{\|\nabla_h x\|_2^2 + \|\nabla_v x\|_2^2}$ represents the regularization^a prior with $A = [\nabla_h, \nabla_v]^T \in \mathbb{R}^{2N_h \times N_h}$, and α denotes the regularization parameter balancing the regularization term and the data fidelity term. The direct ADMM in [6] is given hereinafter which first rewrites the problem (1.2) as

$$\begin{aligned} \min_{x,z,u} \quad & \frac{1}{2} \|y - \Phi z\|_2^2 + \alpha \phi(u) \\ \text{s.t.} \quad & Hx = z, \\ & Ax = u, \end{aligned}$$

and adopts the following iterative scheme:

$$x^{k+1} = \operatorname{argmin}_x \frac{\mu}{2} \|Hx - z^k + d_1^k\|_2^2 + \frac{\mu}{2} \|Ax - u^k + d_2^k\|_2^2, \tag{1.3a}$$

$$z^{k+1} = \operatorname{argmin}_z \frac{1}{2} \|y - \Phi z\|_2^2 + \frac{\mu}{2} \|Hx^{k+1} - z + d_1^k\|_2^2, \tag{1.3b}$$

$$u^{k+1} = \operatorname{argmin}_u \alpha \phi(u) + \frac{\mu}{2} \|Ax^{k+1} - u + d_2^k\|_2^2, \tag{1.3c}$$

$$d_1^{k+1} = d_1^k + (\Phi x^{k+1} - z^{k+1}), \tag{1.3d}$$

$$d_2^{k+1} = d_2^k + (Ax^{k+1} - u^{k+1}). \tag{1.3e}$$

Recently, Zhao *et al.* [9] proposed a fast single image super-resolution by adopting a new efficient analytical solution for ℓ_2 -norm regularized problems, which can reduce the number of iterations in each loop from five steps to three steps by tackling the downsampling operator Φ and the blurring operator H simultaneously. By rewriting (1.2) as the following constrained optimization problem:

$$\begin{aligned} \min_{x,u} \quad & \frac{1}{2} \|y - \Phi Hx\|_2^2 + \alpha\phi(u) \\ \text{s.t.} \quad & Ax = u, \end{aligned} \tag{1.4}$$

they proposed the easier ADMM-type iterative scheme, *i.e.*, FSR-ADMM:

$$x^{k+1} = \operatorname{argmin}_x \|y - \Phi Hx\|_2^2 + \mu \|Ax - u^k + d^k\|_2^2, \tag{1.5a}$$

$$u^{k+1} = \operatorname{argmin}_u \alpha\phi(u) + \frac{\mu}{2} \|Ax^{k+1} - u + d^k\|_2^2, \tag{1.5b}$$

$$d^{k+1} = d^k + (Ax^{k+1} - u^{k+1}). \tag{1.5c}$$

Note that (1.5a) is the classical least square problem and has the solution given by

$$x^{k+1} = (H^T \Phi^T \Phi H + \mu A^T A)^{-1} (H^T \Phi^T y + \mu A^T (u^k - d^k)). \tag{1.6}$$

As to such an expensive inversion process, the methods to alleviate the computational burden can be roughly categorized into two main categories, one is to ideally assume $A^T A = I$. Then such formula can be solved efficiently by a Thomas algorithm in $32N_h^2$ flops [11]. The other option is to establish a block circulant matrix with circulant blocks (BCCB) A . Under such conditions, the Woodbury formula can be utilized to decrease the order of computation complexity from $\mathcal{O}(8N_h^3)$ to $\mathcal{O}(2N_h \log 2N_h)$ [9]. Nevertheless, in realistic settings the problem is that one does not necessarily know in advance if the BCCB assumption is good for SISR because it may lead to the appearance of ringing artifacts emanating from the boundaries, which is a well-known mismatch and degradation consequence of such deconvolved images [12].

To alleviate the above dilemma and further apply FSR-ADMM into wider scenarios with proper matrix A , we propose a new method with two-fold solution. (1) Compute $(H^T \Phi^T \Phi H + \frac{\mu}{\tau} I_{N_h})^{-1}$ instead of computing $(H^T \Phi^T \Phi H + \mu A^T A)^{-1}$ so as to bypass the need of special condition of A . (2) Accelerate FSR-ADMM by introducing a new dual multiplier $\lambda^{k+\frac{1}{2}}$.

In light of the above analysis, this paper proposes the following iterative scheme based on semiproximal symmetric ADMM (FSR-SADMM):^b

$$x^{k+1} = \left(H^T \Phi^T \Phi H + \frac{\mu}{\tau} I_{N_h} \right)^{-1} \left(H^T \Phi^T y + \frac{\mu}{\tau} x^k - \mu A^T \left(Ax^k - u^k - \frac{\lambda^k}{\mu} \right) \right), \tag{1.7a}$$

$$\lambda^{k+\frac{1}{2}} = \lambda^k - r\mu (Ax^{k+1} - \lambda^k), \tag{1.7b}$$

$$u^{k+1} = \operatorname{argmin}_u \alpha\phi(u) + \frac{\mu}{2} \|Ax^{k+1} - u - \lambda^{k+\frac{1}{2}}/\mu\|_2^2, \tag{1.7c}$$

$$\lambda^{k+1} = \lambda^k - s\mu (Ax^{k+1} - \lambda^{k+1}). \tag{1.7d}$$

In a nutshell, the contributions of this article can be summarized as follows:

1. We propose a customized semiproximal term especially suitable for SR imaging system which can avoid computing some boring matrix inversion such as $(H^T\Phi^T\Phi H + \mu A^T A)^{-1}$ existing in (1.5a). Consequently, FSR-SADMM can be applied in wider scenarios without a strict condition of A .
2. Based on the iterative scheme of strictly contractive semiproximal Peaceman-Rachford splitting method, our proposed FSR-SADMM can significantly reduce the total iteration count while only involving additional dual update (*i.e.*, $\lambda^{k+\frac{1}{2}}$) which added negligible computational burden for each iteration. As a result, our proposed FSR-SADMM prefers to maintain a better convergence rate which can experimentally reduce the computing time by 40%.
3. We prove that the FSR-SADMM is convergent under mild conditions.

The rest of this paper is organized as follows. In Section 2, we present some preliminaries which are useful for the subsequent analysis. In Section 3, we illustrate the FSR-SADMM to reconstruct the single image super-resolution. Section 4 provides convergence analysis of the proposed method. Section 5 presents extensive numerical results that evaluate the performance of the proposed reconstruction algorithm in comparison with some state-of-the-art algorithms. Finally, concluding remarks are provided in Section 6.

2 Preliminaries

2.1 Variational reformulation of (1.4)

In this section, following He and Yuan’s approach [13], we reformulate the convex minimization model (1.4) as a variational form, which is useful for succedent algorithmic illustration and convergence analysis. Let us denote $z_1 = x$, $z_2 = u$, $B_1 = A$, $B_2 = -I_{2N_h}$, then (1.4) becomes

$$\begin{aligned} \min_{z_1 \in \mathbb{R}^{N_h}, z_2 \in \mathbb{R}^{2N_h}} \quad & \theta_1(z_1) + \theta_2(z_2) \\ \text{s.t.} \quad & B_1 z_1 + B_2 z_2 = 0, \end{aligned} \tag{2.1}$$

where $\theta_1(z_1) = \frac{1}{2\mu} \|y - \Phi H z_1\|_2^2$ and $\theta_2(z_2) = \alpha \phi(z_2)$. The Lagrangian function and augmented Lagrangian function of (2.1) can be written as

$$\mathcal{L}(z_1, z_2, \lambda) = \theta_1(z_1) + \theta_2(z_2) - \lambda^T (B_1 z_1 + B_2 z_2) \tag{2.2}$$

and

$$\mathcal{L}_\mu(z_1, z_2, \lambda) = \theta_1(z_1) + \theta_2(z_2) - \lambda^T (B_1 z_1 + B_2 z_2) + \frac{\mu}{2} \|B_1 z_1 + B_2 z_2\|^2, \tag{2.3}$$

respectively, where $\lambda \in \mathbb{R}^{2N_h}$ is a Lagrangian multiplier. Then seeking a saddle point of $L(z_1, z_2, \lambda)$ is to find $(z_1^*, z_2^*, \lambda^*)$ such that

$$\mathcal{L}_{\lambda \in \mathbb{R}^{2N_h}}(z_1^*, z_2^*, \lambda) \leq \mathcal{L}(z_1^*, z_2^*, \lambda^*) \leq \mathcal{L}_{z_1 \in \mathbb{R}^{N_h}, z_2 \in \mathbb{R}^{2N_h}}(z_1, z_2, \lambda^*). \tag{2.4}$$

That is, for any (z_1, z_2, λ) , we have

$$\theta_1(z_1) + \theta_2(z_2) - (\theta_1(z_1^*) + \theta_2(z_2^*)) - (z_1 - z_1^*)^T B_1^T \lambda^* - (z_2 - z_2^*)^T B_2^T \lambda^* \geq 0, \tag{2.5a}$$

$$(\lambda - \lambda^*)^\top (B_1 z_1 + B_2 z_2) \geq 0. \tag{2.5b}$$

Therefore, solving (1.4) is equivalent to finding $w = (z_1^*, z_2^*, \lambda^*)$ such that

$$\text{VI}(\Omega, F, \theta): \theta(u) - \theta(u^*) + (w - w^*)^\top F(w^*) \geq 0, \quad \forall w \in \Omega, \tag{2.6}$$

where

$$u := \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad w := \begin{pmatrix} z_1 \\ z_2 \\ \lambda \end{pmatrix}, \quad \theta(u) := \theta_1(z_1) + \theta_2(z_2), \tag{2.7}$$

and

$$F(w) := \begin{pmatrix} -B_1^\top \lambda \\ -B_2^\top \lambda \\ B_1 z_1 + B_2 z_2 \end{pmatrix}, \quad \Omega := \mathbb{R}^{N_h} \times \mathbb{R}^{2N_h} \times \mathbb{R}^{2N_h}. \tag{2.8}$$

Note that the mapping $F(w)$ defined in (2.8) is affine with a skew-symmetric matrix, it is monotone. We denote by Ω^* the solution set of $\text{VI}(\Omega, F, \theta)$.

2.2 Some notations

We use $\|\cdot\|$ to denote the 2-norm of a vector and let $\|z\|_G^2 = z^\top G z$ for $z \in \mathbb{R}^N$ and $G \in \mathbb{R}^{N \times N}$. For a real symmetric matrix S , we denote $S \geq 0$ ($S > 0$) if S is positive semidefinite (positive definite). For convenient analysis, we define the following matrices:

$$H = \begin{pmatrix} R & 0 & 0 \\ 0 & \frac{r+s-rs}{r+s} \mu B_2^\top B_2 & -\frac{r}{r+s} B_2^\top \\ 0 & -\frac{r}{r+s} B_2 & \frac{1}{(r+s)\mu} I_{2N_h} \end{pmatrix}, \tag{2.9}$$

$$M = \begin{pmatrix} I_{N_h} & 0 & 0 \\ 0 & I_{2N_h} & 0 \\ 0 & -s\mu B_2 & (r+s)I_{2N_h} \end{pmatrix}, \tag{2.10}$$

and

$$Q = \begin{pmatrix} R & 0 & 0 \\ 0 & \mu B_2^\top B_2 & -r B_2^\top \\ 0 & -B_2 & \frac{1}{\mu} I_{2N_h} \end{pmatrix}. \tag{2.11}$$

Below we prove three assertions regarding the matrices just defined. These assertions make it possible to present our convergence analysis for the new algorithm compactly with alleviated notation.

Lemma 2.1 *If $R > 0$, $\mu \in (0, 1)$, $r \in (0, 1)$, $s \in (0, 1]$, and B_2 is full column rank, then the matrices H , M , and Q defined, respectively, in (2.9), (2.10), and (2.11) satisfy*

$$H > 0, \quad HM = Q, \tag{2.12}$$

and

$$G := Q^T + Q - M^T H M \geq 0. \tag{2.13}$$

Proof We consider two cases.

(I) $r \in (0, 1), s \in (0, 1)$. Since $R \geq 0, B_2$ is assumed to be full column rank, we only need to check that

$$\bar{H} = \begin{pmatrix} \mu(r + s - rs) & -r \\ -r & \frac{1}{\mu} \end{pmatrix} > 0.$$

Note that

$$\begin{cases} r + s - rs = r + s(1 - r) > 0, \\ r + s - rs - r^2 = (r + s)(1 - r) > 0. \end{cases}$$

Then we have

$$\bar{H} > 0.$$

Therefore, the assertion $H > 0$ is verified.

Under the definition of the matrices $H, M,$ and $Q,$ by a simple manipulation, we get

$$\begin{aligned} HM &= \begin{pmatrix} R & 0 & 0 \\ 0 & \frac{r+s-rs}{r+s} \mu B_2^T B_2 & -\frac{r}{r+s} B_2^T \\ 0 & -\frac{r}{r+s} B_2 & \frac{1}{(r+s)\mu} I_{2N_h} \end{pmatrix} \begin{pmatrix} I_{N_h} & 0 & 0 \\ 0 & I_{2N_h} & 0 \\ 0 & -s\mu B_2 & (r+s)I_{2N_h} \end{pmatrix} \\ &= \begin{pmatrix} R & 0 & 0 \\ 0 & \mu B_2^T B_2 & -r B_2^T \\ 0 & -B_2 & \frac{1}{\mu} I_{2N_h} \end{pmatrix} = Q. \end{aligned} \tag{2.14}$$

The second assertion $HM = Q$ is proved. Consequently, we have

$$M^T H M = M^T Q = \begin{pmatrix} I_{N_h} & 0 & 0 \\ 0 & I_{2N_h} & -s\mu B_2^T \\ 0 & 0 & (r+s)I_{2N_h} \end{pmatrix} \begin{pmatrix} R & 0 & 0 \\ 0 & \mu I_{2N_h} & -r B_2^T \\ 0 & -B_2 & \frac{1}{\mu} I_{2N_h} \end{pmatrix} \tag{2.15}$$

$$= \begin{pmatrix} R & 0 & 0 \\ 0 & (1+s)\mu B_2^T B_2 & -(r+s)B_2^T \\ 0 & -(r+s)B_2 & \frac{r+s}{\mu} I_{2N_h} \end{pmatrix}. \tag{2.16}$$

Using (2.10), (2.11), and the above equation, we get

$$\begin{aligned} G &= Q^T + Q - M^T H M \\ &= \begin{pmatrix} 2R & 0 & 0 \\ 0 & 2\mu B_2^T B_2 & -(1+r)B_2^T \\ 0 & -(1+r)B_2 & \frac{2}{\mu} I_{2N_h} \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
 & - \begin{pmatrix} R & 0 & 0 \\ 0 & (1+s)\mu B_2^T B_2 & -(r+s)B_2^T \\ 0 & -(r+s)B_2 & \frac{r+s}{\mu} I_{2N_h} \end{pmatrix} \\
 & = \begin{pmatrix} R & 0 & 0 \\ 0 & (1-s)\mu B_2^T B_2 & (s-1)B_2^T \\ 0 & (s-1)B_2 & \frac{2-r-s}{\mu} I_{2N_h} \end{pmatrix}.
 \end{aligned}$$

Similarly, note that

$$\begin{cases} 1-s > 0, \\ (1-s)(2-r-s) - (1-s)^2 = (1-s)\{(1-s) + (1-r)\} - (1-s)^2 > 0. \end{cases}$$

Therefore, the matrix G is positive definite.

(II) $r \in (0, 1)$ and $s = 1$. Note that

$$H = \begin{pmatrix} R & 0 & 0 \\ 0 & \frac{\mu}{r+1} B_2^T B_2 & -\frac{r}{r+1} B_2^T \\ 0 & -\frac{r}{r+1} B_2 & \frac{1}{(r+1)\mu} I_{2N_h} \end{pmatrix}$$

and

$$G = \begin{pmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1-r}{\mu} I_{2N_h} \end{pmatrix}.$$

Thus H and G are positive definite and positive semidefinite, respectively. Here, we would emphasize that we do not require the positive definiteness of G . Instead, positive semidefiniteness of G is enough for our algorithmic analysis. □

3 Algorithm

3.1 FSR-SADMM

In this section, we will present our new algorithm to solve (1.4). But we first present the iterative scheme by using the standard strictly contractive Peaceman-Rachford splitting method with two different relaxation factors:

$$z_1^{k+1} = \operatorname{argmin}_{z_1 \in \mathbb{R}^{N_h}} \mathcal{L}_\mu(z_1, z_2^k, \lambda^k), \tag{3.1a}$$

$$\lambda^{k+\frac{1}{2}} = \lambda^k - r\mu(B_1 z_1^{k+1} + B_2 z_2^k), \tag{3.1b}$$

$$z_2^{k+1} = \operatorname{argmin}_{z_2 \in \mathbb{R}^{2N_h}} \mathcal{L}_\mu(z_1^{k+1}, z_2, \lambda^{k+\frac{1}{2}}), \tag{3.1c}$$

$$\lambda^{k+1} = \lambda^{k+\frac{1}{2}} - s\mu(B_1 z_1^{k+1} + B_2 z_2^{k+1}). \tag{3.1d}$$

By introducing a customized semiproximal term especially for TV super-resolution imaging, our FSR-SADMM has the iterative scheme

$$z_1^{k+1} = \operatorname{argmin}_{z_1 \in \mathbb{R}^{N_h}} \mathcal{L}_\mu(z_1, z_2^k, \lambda^k) + \frac{1}{2} \|z_1 - z_1^k\|_R^2, \tag{3.2a}$$

$$\lambda^{k+\frac{1}{2}} = \lambda^k - r\mu(B_1 z_1^{k+1} + B_2 z_2^k), \tag{3.2b}$$

$$z_2^{k+1} = \operatorname{argmin}_{z_2 \in \mathbb{R}^{2N_h}} \mathcal{L}_\mu(z_1^{k+1}, z_2, \lambda^{k+\frac{1}{2}}), \tag{3.2c}$$

$$\lambda^{k+1} = \lambda^{k+\frac{1}{2}} - s\mu(B_1 z_1^{k+1} + B_2 z_2^{k+1}), \tag{3.2d}$$

where $R = \frac{\mu}{\tau} I_{N_h} - \mu B_1^\top B_1$ is a customized semidefinite matrix.

3.2 Related methods

The methods proposed in [14] and [15] belong to the category of SADMM-type approaches with logarithmic-quadratic proximal regularization, which have larger step sizes compared with SADMM. The former is with the step sizes of $r \in (0, 1), s \in (0, 1)$ while the latter $s \in (0, 2), r \in (0, 2 - s)$, which are different from $r \in (0, 1), s \in (0, 1]$ considered in this paper. The parameters r and s in SADMM have also been studied intensively by He *et al.* [16] and Gu *et al.* [15]. However, both of them cannot fully utilize the special structure of SISR scenarios involved in this paper. And computing $(H^\top \Phi^\top \Phi H + \mu A^\top A)^{-1}$ cannot be avoided.

4 Global convergence

To make the analysis more elegant, we reformulate FSR-SADMM (3.2a)-(3.2d) into the form^c

$$z_1^{k+1} = \operatorname{argmin}_{z_1 \in \mathbb{R}^{N_h}} \left\{ \theta_1(z_1) - (\lambda^k)^\top B_1 z_1 + \frac{\mu}{2} \|B_1 z_1 + B_2 z_2^k\|^2 + \frac{1}{2} \|z_1 - z_1^k\|_R^2 \right\}, \tag{4.1a}$$

$$\lambda^{k+\frac{1}{2}} = \lambda^k - r\mu(B_1 z_1^{k+1} + B_2 z_2^k), \tag{4.1b}$$

$$z_2^{k+1} = \operatorname{argmin}_{z_2 \in \mathbb{R}^{N_h}} \left\{ \theta_2(z_2) - (\lambda^{k+\frac{1}{2}})^\top B_2 z_2 + \frac{\mu}{2} \|B_1 z_1^{k+1} + B_2 z_2\|^2 \right\}, \tag{4.1c}$$

$$\lambda^{k+1} = \lambda^{k+\frac{1}{2}} - s\mu(B_1 z_1^{k+1} + B_2 z_2^{k+1}). \tag{4.1d}$$

Now we analyze the convergence for our proposed FSR-SADMM (4.1a)-(4.1d). We prove its global convergence from the contraction perspective. In order to further alleviate the notation in our analysis, we define an auxiliary sequence \tilde{w}^k as

$$\tilde{w}^k = \begin{pmatrix} \tilde{z}_1^k \\ \tilde{z}_2^k \\ \tilde{\lambda}^k \end{pmatrix} = \begin{pmatrix} z_1^{k+1} \\ z_2^{k+1} \\ \lambda^k - \mu(B_1 z_1^{k+1} + B_2 z_2^k) \end{pmatrix}, \tag{4.2}$$

where (z_1^{k+1}, z_2^{k+1}) is produced by (4.1a) and (4.1c), and we immediately get

$$z_1^{k+1} = \tilde{z}_1^k, \quad z_2^{k+1} = \tilde{z}_2^k, \quad \lambda^{k+\frac{1}{2}} = \lambda^k - r(\lambda^k - \tilde{\lambda}^k),$$

and

$$\begin{aligned} \lambda^{k+1} &= \lambda^{k+\frac{1}{2}} - s\mu(B_1 \tilde{z}_1^k + B_2 \tilde{z}_2^k) \\ &= \lambda^k - r(\lambda^k - \tilde{\lambda}^k) - s[\mu(B_1 \tilde{z}_1^k + B_2 \tilde{z}_2^k) - \mu B_2(z_2^k - \tilde{z}_2^k)] \end{aligned}$$

$$\begin{aligned}
 &= \lambda^k - r(\lambda^k - \tilde{\lambda}^k) - s[\lambda^k - \tilde{\lambda}^k - \mu B_2(z_2^k - \tilde{z}_2^k)] \\
 &= \lambda^k - [(r + s)(\lambda^k - \tilde{\lambda}^k) - s\mu B_2(z_2^k - \tilde{z}_2^k)].
 \end{aligned}$$

Moreover, we have the following relationship:

$$\begin{pmatrix} z_1^{k+1} \\ z_2^{k+1} \\ \lambda^{k+1} \end{pmatrix} = \begin{pmatrix} z_1^k \\ z_2^k \\ \lambda^k \end{pmatrix} - \begin{pmatrix} I_N & 0 & 0 \\ 0 & I_N & 0 \\ 0 & -s\mu B_2 & (r + s)I_{2N_h} \end{pmatrix} \begin{pmatrix} z_1^k - \tilde{z}_1^k \\ z_2^k - \tilde{z}_2^k \\ \lambda^k - \tilde{\lambda}^k \end{pmatrix},$$

which can be reformulated as a compact form under the notation of w^k and \tilde{w}^k :

$$w^{k+1} = w^k - M(w^k - \tilde{w}^k), \tag{4.3}$$

where M is defined in (2.10).

Now we start to prove some properties for the sequence $\{\tilde{w}^k\}$ defined in (4.2). We are interested in estimating how accurate the point \tilde{w}^k is to a solution point w^* of $\text{VI}(\Omega, F, \theta)$. The main result is proved in Theorem 4.1. To prove this main result, we require two lemmas. The first key lemma provides a lower bound on specially constructed functional in terms of a quadratic term involving the matrix Q defined in (2.11).

Lemma 4.1 *For given $w^k \in \Omega$, let w^{k+1} be generated by FSR-SADMM (4.1a)-(4.1d) and \tilde{w}^k be defined in (4.2). Then we have $\tilde{w} \in \Omega$ and*

$$\theta(w) - \theta(\tilde{w}^k) + (w - \tilde{w}^k)^T F(\tilde{w}^k) \geq (w - \tilde{w}^k)^T Q(w^k - \tilde{w}^k), \quad \forall w \in \Omega, \tag{4.4}$$

where Q is defined in (2.11).

Proof From the first-order optimality condition of z_1 -subproblem in (4.1a), for any $z_1 \in \mathbb{R}^{N_h}$, we obtain

$$\theta_1(z_1) - \theta_1(z_1^{k+1}) + (z_1 - z_1^{k+1})^T \{-B_1^T \lambda^k + \mu B_1^T (B_1 z_1^{k+1} + B_2 z_2^k) + R(z_1^{k+1} - z_1^k)\} \geq 0. \tag{4.5}$$

By the definition of \tilde{z}_1^k and $\tilde{\lambda}^k$ in (4.2), the above inequality can be written as

$$\theta_1(z_1) - \theta_1(\tilde{z}_1^k) + (z_1 - \tilde{z}_1^k)^T \{-B_1^T \tilde{\lambda}^k + R(\tilde{z}_1^k - z_1^k)\} \geq 0. \tag{4.6}$$

Similarly, from the first-order optimization condition of z_2 -subproblem in (4.1c), we achieve

$$\begin{aligned}
 &\theta_2(z_2) - \theta_2(z_2^{k+1}) + (z_2 - z_2^{k+1})^T \{-B_2^T \lambda^{k+\frac{1}{2}} + \mu B_2^T (B_1 z_1^{k+1} + B_2 z_2^{k+1})\} \\
 &\geq 0, \quad \forall z_2 \in \mathbb{R}^{2N_h}.
 \end{aligned} \tag{4.7}$$

From the definitions of $\lambda^{k+\frac{1}{2}}$, \tilde{z}_2^k and $\tilde{\lambda}^k$, we get

$$\begin{aligned}
 &\lambda^{k+\frac{1}{2}} - \mu (B_1 z_1^{k+1} + B_2 z_2^{k+1}) \\
 &= \lambda^k - r\mu (B_1 z_1^{k+1} + B_2 z_2^k) - \mu (B_1 z_1^{k+1} + B_2 z_2^k) - \mu B_2 (z_2^{k+1} - z_2^k)
 \end{aligned}$$

$$\begin{aligned}
 &= \lambda^k - \mu(B_1 z_1^{k+1} + B_2 z_2^k) - r(\lambda^k - \tilde{\lambda}^k) - \mu B_2(\tilde{z}_2^k - z_2^k) \\
 &= \tilde{\lambda}^k - r(\lambda^k - \tilde{\lambda}^k) - \mu B_2(\tilde{z}_2^k - z_2^k).
 \end{aligned}
 \tag{4.8}$$

Substituting this into (4.7) and considering the definition of \tilde{z}_2^k in (4.2), we have

$$\theta_2(z_2) - \theta_2(\tilde{z}_2^k) + (z_2 - \tilde{z}_2^k)^\top \{-B_2^\top \tilde{\lambda}^k + r B_2^\top (\lambda^k - \tilde{\lambda}^k) + \mu B_2^\top B_2 (\tilde{z}_2^k - z_2^k)\} \geq 0.
 \tag{4.9}$$

In addition, as follows from (4.2) again, we have

$$(B_1 \tilde{z}_1^k + B_2 \tilde{z}_2^k) - B_2(\tilde{z}_2^k - z_2^k) + \frac{1}{\mu}(\tilde{\lambda}^k - \lambda^k) = 0.
 \tag{4.10}$$

Combining (4.6), (4.9), and (4.10), we obtain

$$\begin{aligned}
 \theta(u) - \theta(\tilde{u}^k) + \begin{pmatrix} z_1 - \tilde{z}_1^k \\ z_2 - \tilde{z}_2^k \\ \lambda - \tilde{\lambda}^k \end{pmatrix}^\top &\left\{ \begin{pmatrix} -B_1^\top \tilde{\lambda}^k \\ -B_2^\top \tilde{\lambda}^k \\ B_1 \tilde{z}_1^k + B_2 \tilde{z}_2^k \end{pmatrix} - \begin{pmatrix} R(z_1^k - \tilde{z}_1^k) \\ \mu B_2^\top B_2(z_2^k - \tilde{z}_2^k) - r B_2^\top (\lambda^k - \tilde{\lambda}^k) \\ -B_2(z_2^k - \tilde{z}_2^k) + \frac{1}{\mu}(\lambda^k - \tilde{\lambda}^k) \end{pmatrix} \right\} \\
 &\geq 0.
 \end{aligned}
 \tag{4.11}$$

By using the notation of Q in (2.11), and w and F in (2.8), the compact form of the above inequality is exactly (4.4). □

In the next lemma, we further analyze the right-hand side of the inequality (4.4) and reformulate it as the sum of some quadratic terms. This new form is more convenient for our further analysis.

Lemma 4.2 *For given $w^k \in \Omega$, let w^{k+1} be generated by FSR-SADMM (4.1a)-(4.1d) and \tilde{w}^k be defined in (4.2). Then for any $w \in \Omega$, we get*

$$(w - \tilde{w}^k)^\top Q(w^k - \tilde{w}^k) = \frac{1}{2}(\|w - w^{k+1}\|_H^2 - \|w - w^k\|_H^2) + \frac{1}{2}\|w^k - \tilde{w}^k\|_G^2.
 \tag{4.12}$$

Proof By using $Q = HM$ and $M(w^k - \tilde{w}^k) = (w^k - w^{k+1})$ (see (4.3)), we have

$$(w - \tilde{w}^k)^\top Q(w^k - \tilde{w}^k) = (w - \tilde{w}^k)^\top HM(w^k - \tilde{w}^k) = (w - \tilde{w}^k)^\top H(w^k - w^{k+1}).
 \tag{4.13}$$

For any vectors $a, b, c, d \in \mathbb{R}^{N_h}$ and a matrix $H \in \mathbb{R}^{n \times n}$, it follows that

$$(a - b)^\top H(c - d) = \frac{1}{2}(\|a - d\|_H^2 - \|a - c\|_H^2) + \frac{1}{2}(\|c - b\|_H^2 - \|d - b\|_H^2).
 \tag{4.14}$$

Applying the above identity with $a = w$, $b = \tilde{w}^k$, $c = w^k$, and $d = w^{k+1}$ gives

$$\begin{aligned}
 (w - \tilde{w}^k)^\top H(w^k - w^{k+1}) &= \frac{1}{2}(\|w - w^{k+1}\|_H^2 - \|w - w^k\|_H^2) \\
 &\quad + \frac{1}{2}(\|w^k - \tilde{w}^k\|_H^2 - \|w^{k+1} - \tilde{w}^k\|_H^2).
 \end{aligned}
 \tag{4.15}$$

Rearranging the last term in the above identity by using (2.12), (2.13), and (4.3), we obtain

$$\begin{aligned}
 \|w^k - \tilde{w}^k\|_H^2 - \|w^{k+1} - \tilde{w}^k\|_H^2 &= \|w^k - \tilde{w}^k\|_H^2 - \|(w^k - \tilde{w}^k) - (w^k - w^{k+1})\|_H^2 \\
 &\stackrel{(4.3)}{=} \|w^k - \tilde{w}^k\|_H^2 - \|(w^k - \tilde{w}^k) - M(w^k - \tilde{w}^k)\|_H^2 \\
 &= 2(w^k - \tilde{w}^k)^\top HM(w^k - \tilde{w}^k) \\
 &\quad - (w^k - \tilde{w}^k)^\top M^\top HM(w^k - \tilde{w}^k) \\
 &\stackrel{(2.12)}{=} (w^k - \tilde{w}^k)^\top (Q^\top + Q - M^\top HM)(w^k - \tilde{w}^k) \\
 &\stackrel{(2.13)}{=} \|w^k - \tilde{w}^k\|_G^2.
 \end{aligned}$$

Substituting it in (4.15) and considering (4.13), we immediately obtain the assertion (4.12). The proof is complete. \square

Now, we try to find a lower bound in terms of the discrepancy between $\|w - w^{k+1}\|_H^2$ and $\|w - w^k\|_H^2$ for any $w \in \Omega$.

Theorem 4.1 *For given $w^k \in \Omega$, let w^{k+1} be generated by FSR-SADMM (4.1a)-(4.1d) and \tilde{w}^k be defined in (4.2). Then for any $w \in \Omega$, we have*

$$\theta(\tilde{u}^k) - \theta(u) + (\tilde{w}^k - w)^\top F(w) \leq \frac{1}{2}(\|w - w^k\|_H^2 - \|w - w^{k+1}\|_H^2) - \frac{1}{2}\|w^k - \tilde{w}^k\|_G^2. \tag{4.16}$$

Proof Note that F is monotone, we obtain

$$(\tilde{w}^k - w)^\top F(w) \leq (\tilde{w}^k - w)^\top F(\tilde{w}^k). \tag{4.17}$$

We have from the above inequality and (4.4)

$$\theta(\tilde{u}^k) - \theta(u) + (\tilde{w}^k - w)^\top F(w) \leq -(w - \tilde{w}^k)^\top Q(w^k - \tilde{w}^k), \quad \forall w \in \Omega. \tag{4.18}$$

The assertion (4.16) holds immediately from (4.12) and (4.18). The proof is complete. \square

The next lemma demonstrates the contraction property of the sequence $(w^k)_{k=0}^\infty$ generated by FSR-SADMM (4.1a)-(4.1d).

Lemma 4.3 *Let $(w^k)_{k=0}^\infty$ be the sequence generated by the FSR-SADMM (4.1a)-(4.1d) and $\{\tilde{w}^k\}$ be defined in (4.2). Then for any $w^* \in \Omega^*$, we have*

$$\|w^{k+1} - w^*\|_H^2 \leq \|w^k - w^*\|_H^2 - \|w^k - \tilde{w}^k\|_G^2. \tag{4.19}$$

Proof Setting $w = w^*$ in (4.16), we have

$$\begin{aligned}
 \|w^k - w^*\|_H^2 - \|w^{k+1} - w^*\|_H^2 &\geq \|w^k - \tilde{w}^k\|_G^2 + 2[\theta(\tilde{u}^k) - \theta(u^*) + (\tilde{w}^k - w^*)^\top F(w^*)] \\
 &\geq \|w^k - \tilde{w}^k\|_G^2,
 \end{aligned} \tag{4.20}$$

where the last inequality follows from the fact that $w^* \in \Omega^*$ (see (2.6)). The proof is complete. \square

Recall that for the case $0 < r < 1, 0 < s < 1$, the matrix G is positive definite, while when $0 < r < 1, s = 1$, G may not be positive definite. We need further investigate the term $\|w^k - \tilde{w}^k\|_G^2$. For the case $0 < r < 1, s = 1$, we have

$$\|w^k - \tilde{w}^k\|_G^2 = \|z_1^k - \tilde{z}_1^k\|_R^2 + \frac{1-r}{\mu} \|\lambda^k - \tilde{\lambda}^k\|^2. \tag{4.21}$$

Notice that

$$\begin{aligned} \lambda^{k+1} &\stackrel{(4.1d)}{=} \lambda^{k+\frac{1}{2}} - \mu(B_1 z_1^{k+1} + B_2 z_2^{k+1}) \\ &\stackrel{(4.2)}{=} \lambda^k - r(\lambda^k - \tilde{\lambda}^k) - \mu(B_1 z_1^{k+1} + B_2 z_2^{k+1}) \\ &= \lambda^k - r(\lambda^k - \tilde{\lambda}^k) - \mu(B_1 z_1^{k+1} + B_2 z_2^k) - \mu B_2(z_2^{k+1} - z_2^k) \\ &= \lambda^k - (1+r)(\lambda^k - \tilde{\lambda}^k) - \mu B_2(z_2^{k+1} - z_2^k). \end{aligned}$$

Thus, we have

$$\lambda^k - \tilde{\lambda}^k = \frac{1}{1+r}(\lambda^k - \lambda^{k+1}) + \frac{1}{1+r} \mu B_2(z_2^k - z_2^{k+1}). \tag{4.22}$$

Then we get

$$\begin{aligned} \|\lambda^k - \tilde{\lambda}^k\|^2 &= \frac{1}{(1+r)^2} \|(\lambda^k - \lambda^{k+1}) + \mu B_2(z_2^k - z_2^{k+1})\|^2 \\ &= \frac{1}{(1+r)^2} \|(\lambda^k - \lambda^{k+1})\|^2 + \frac{\mu^2}{(1+r)^2} \|B_2(z_2^k - z_2^{k+1})\|^2 \\ &\quad + \frac{2\mu}{(1+r)^2} (\lambda^k - \lambda^{k+1})^\top B_2(z_2^k - z_2^{k+1}). \end{aligned} \tag{4.23}$$

Now, we treat the cross term in the above equation. For the previous iteration, we have

$$\theta_2(z_2) - \theta_2(z_2^k) + (z_2 - z_2^k)^\top \{-B_2^\top \lambda^{k-\frac{1}{2}} + \mu B_2^\top (B_1 z_1^k + B_2 z_2^k)\} \geq 0, \quad \forall z_2 \in \mathbb{R}^{2N_h}. \tag{4.24}$$

Reconsidering $s = 1$ in (4.1d) and applying in (4.24), we achieve

$$\theta_2(z_2) - \theta_2(z_2^k) + (z_2 - z_2^k)^\top \{-B_2^\top \lambda^k\} \geq 0, \quad \forall z_2 \in \mathbb{R}^{2N_h}. \tag{4.25}$$

Similarly, we get

$$\theta_2(z_2) - \theta_2(z_2^{k+1}) + (z_2 - z_2^{k+1})^\top \{-B_2^\top \lambda^{k+1}\} \geq 0, \quad \forall z_2 \in \mathbb{R}^{2N_h}. \tag{4.26}$$

Setting $z_2 = z_2^{k+1}$ and $z_2 = z_2^k$ in (4.25) and (4.26), respectively, and then adding them, we get

$$(\lambda^k - \lambda^{k+1})^\top B_2(z_2^k - z_2^{k+1}) \geq 0. \tag{4.27}$$

Combining with (4.21), (4.27), and (4.23), we obtain

$$\begin{aligned} \|w^k - \tilde{w}^k\|_G^2 &\geq \|z_1^k - z_1^{k+1}\|_R^2 + \frac{1-r}{(1+r)^2\mu} \|\lambda^k - \lambda^{k+1}\|^2 \\ &\quad + \frac{(1-r)\mu}{(1+r)^2} \|z_2^k - z_2^{k+1}\|_{B_2^T B_2}^2. \end{aligned} \tag{4.28}$$

Recall \tilde{w}^k defined by (4.2) and combining with (4.19), we have the following lemma, which is important for the proof of the convergence.

Lemma 4.4 *Let $(w^k)_{k=0}^\infty$ be the sequence generated by the FSR-SADMM (4.1a)-(4.1d) with $r \in (0, 1), s = 1$, and $\{\tilde{w}^k\}$ be defined in (4.2). Then we have*

$$\begin{aligned} \|w^{k+1} - w^*\|_H^2 &\leq \|w^k - w^*\|_H^2 - \left\{ \|z_1^k - z_1^k\|_R^2 + \frac{1-r}{(1+r)^2\mu} \|\lambda^k - \lambda^{k+1}\|^2 \right. \\ &\quad \left. + \frac{(1-r)\mu}{(1+r)^2} \|z_2^k - z_2^k\|_{B_2^T B_2}^2 \right\}. \end{aligned} \tag{4.29}$$

With the above lemmas, we are now ready to establish the global convergence of FSR-SADMM for solving $\text{VI}(\Omega, F, \theta)$.

Theorem 4.2 *The sequence $(w^k)_{k=0}^\infty$ generated by FSR-SADMM (4.1a)-(4.1d) converges to some w^∞ that is a solution of $\text{VI}(\Omega, F, \theta)$.*

Proof (I) For the case $r \in (0, 1), s \in (0, 1)$. Summing (4.19) over $k = 0, \dots, \infty$ yields

$$\sum_{k=0}^\infty \|w^k - \tilde{w}^k\|_G^2 \leq \|w^0 - w^*\|_H^2, \tag{4.30}$$

which implies that

$$\lim_{k \rightarrow \infty} \|w^k - \tilde{w}^k\|_G = 0. \tag{4.31}$$

Recall the sequence $\{w^k\}$ is bounded (see Lemma 4.3), we see that the sequence $\{\tilde{w}^k\}$ is also bounded, and it has at least one cluster point. Let w^∞ be a cluster of $\{\tilde{w}^k\}$ and the subsequence \tilde{w}^{k_j} converge to w^∞ . Combining (4.4) and (4.31), we get

$$\theta(u) - \theta(\tilde{w}^{k_j}) + (w - \tilde{w}^{k_j})^T F(\tilde{w}^{k_j}) \geq 0, \quad \forall w \in \Omega. \tag{4.32}$$

Taking $j \rightarrow \infty$ in the left term of the above inequality yields

$$\theta(u) - \theta(w^\infty) + (w - w^\infty)^T F(w^\infty) \geq 0, \quad \forall w \in \Omega, \tag{4.33}$$

which implies that $w^\infty \in \Omega^*$. From $\lim_{k \rightarrow \infty} \|w^k - \tilde{w}^k\|_G = 0$, we can deduce $\lim_{k \rightarrow \infty} \|w^k - w^k\|_H = 0$. Recall $\{\tilde{w}^{k_j}\} \rightarrow w^\infty$, thus for any given $\epsilon > 0$, there exists an integer l such that

$$\|w^{k_l} - \tilde{w}^{k_l}\|_H \leq \frac{\epsilon}{2} \quad \text{and} \quad \|\tilde{w}^{k_l} - w^\infty\|_H \leq \frac{\epsilon}{2}. \tag{4.34}$$

Thus, for any $k \geq k_l$, it follows from the above two inequalities and (4.19) that

$$\|w^k - w^\infty\|_H \leq \|w^{k_l} - w^\infty\|_H \leq \|w^{k_l} - \tilde{w}^{k_l}\|_H + \|\tilde{w}^{k_l} - w^\infty\|_H < \epsilon. \tag{4.35}$$

This implies that the sequence $\{w^k\}$ converges to $w^\infty \in \Omega^*$. This completes the proof.

(II) For the case $r \in (0, 1)$, $s = 1$. Summing (4.29) over $k = 0, \dots, \infty$ yields

$$\begin{aligned} & \sum_{k=0}^{\infty} \|z_1^k - \tilde{z}_1^k\|_R^2 + \frac{(1-r)\mu}{(1+r)^2} \sum_{k=0}^{\infty} \|z_2^k - \tilde{z}_2^k\|_{B_2^1 B_2}^2 + \frac{1-r}{(1+r)^2 \mu} \sum_{k=0}^{\infty} \|\lambda^k - \lambda^{k+1}\|^2 \\ & \leq \|w^0 - w^*\|_H^2, \end{aligned} \tag{4.36}$$

which implies that^d

$$\lim_{k \rightarrow \infty} \|z_1^k - \tilde{z}_1^k\|_R = 0, \quad \lim_{k \rightarrow \infty} \|z_2^k - \tilde{z}_2^k\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|\lambda^k - \tilde{\lambda}^k\| = 0. \tag{4.37}$$

Recall the bounded sequence $\{w^k\}$ again, we see that the sequence $\{\tilde{w}^k\}$ is also bounded, and it has at least one cluster point. The following proof for the assertion is similar and omitted. □

5 Numerical results

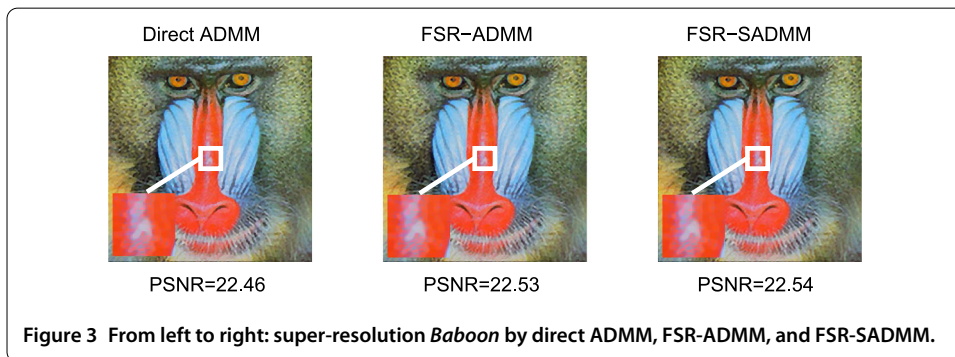
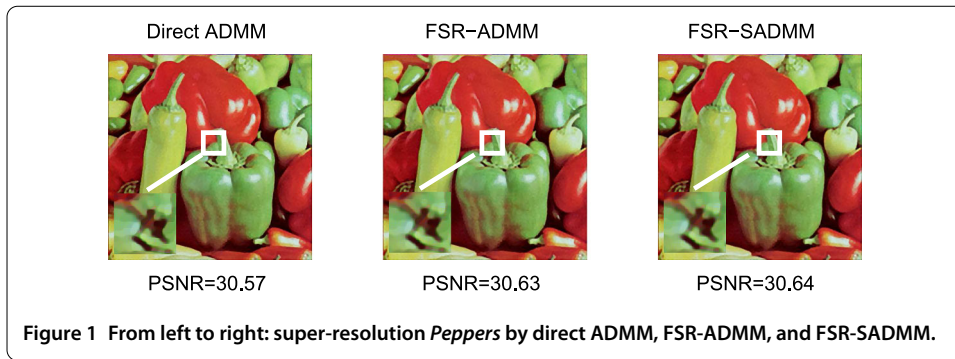
In this section, we study the performance of the FSR-SADMM for solving (1.2). Our codes were written in MATLAB R2014a. In addition, all of the experiments were performed on a laptop with an Intel Core 2 Duo CPU at 2.2 GHz and 2 GB memory.^e

Experiments were conducted on three different text images: *Peppers*, *Lena*, and *Baboon*. Images are all with $256 \times 256 \times 3$ RGB images. Color images were processed using the illuminate channel only, as in [9]. Precisely, the RGB images were transformed into YUV coordinates and the color channels (Cb, Cr) were upsampled using bicubic interpolation. The results were quantitatively evaluated by using the standard peak signal-to-noise ratio (PSNR) [17], defined as

$$\text{PSNR} = 10 \log_{10} \frac{256^2 L^2}{\|x - \hat{x}\|^2}, \tag{5.1}$$

where x and \hat{x} are the original and reconstructed images, L denotes the maximum intensity value in x . We compare direct ADMM, FSR-ADMM and FSR-SADMM under the stop criterion such as $\frac{\|x^k - x^{k-1}\|}{\|x^{k-1}\|} < \{1e^{-6}, 1e^{-5}, 1e^{-4}\}$.

Figures 1-3 compare the PSNR outputs of our FSR-SADMM with direct ADMM and FSR-ADMM, while Figure 4 depicts the comparison of total computational time of the three methods. Direct ADMM splits the super-resolution imaging problem into three subproblems by using two dual multipliers while FSR-ADMM reduces the computational complexity of each iteration by efficiently reducing the order of computation complexity of their subproblems from $\mathcal{O}(N_h^3)$ to $\mathcal{O}(N_h \log N_h)$. Similarly, our FSR-SADMM has the computation complexity of $\mathcal{O}(N_h \log N_h)$ and shares much better convergence performance. In our experiments, we set $\mu = 0.005$ for all three methods. In all figures, we list the PSNR below the reconstructed image corresponding to compared methods. As the reconstructed images shown in Figures 1-3, there are not noticeable differences in the texture



of the *Peppers*, the wrinkles on the eyes of *Lena*, and also hair on *Baboon's* nose. FSR-SADMM can only get a little higher PSNR rather than Direct ADMM and FSR-ADMM for various tolerance. However, the results shown in Figure 4 indicate that, compared with FSR-ADMM, our FSR-SADMM saves roughly 45% computational time with high precision Tolerance = 10^{-6} and can still save roughly 20% computational time under low precision Tolerance = 10^{-4} . On the whole average, 30% computational time can be reduced, which efficiently accelerates the super-resolution imaging process.

We also test the influence of relaxation parameter r for FSR-SADMM. We fix $s = 1$, $\mu = 0.005$ and choose different values of r in the interval $[0.3, 1]$ (more specifically, we choose $r = \{0.3, 0.5, 0.7, 0.9, 1\}$). For comparison purposes, we also plot for FSR-ADMM with $\mu = 0.005$. As shown in Figure 5, we see that the relaxation parameter r works well for a wide range of values. In particular, when $r \geq 0.9$, the PSNR cannot be improved and even decrease in the first 20 iterations, while the effectiveness of the improvement of

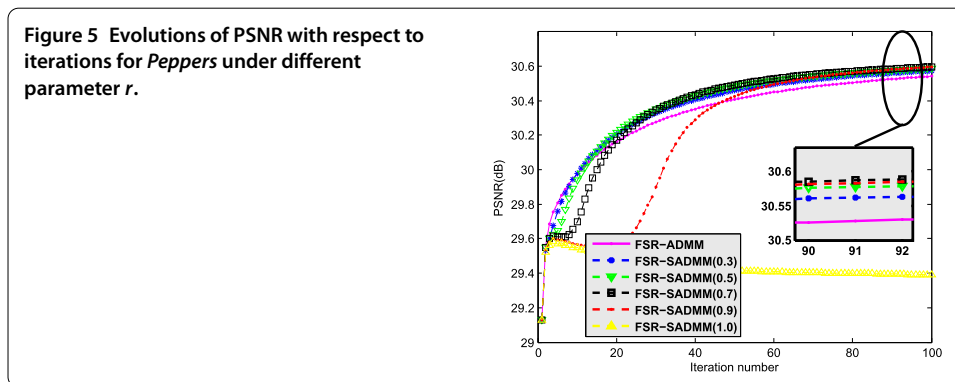
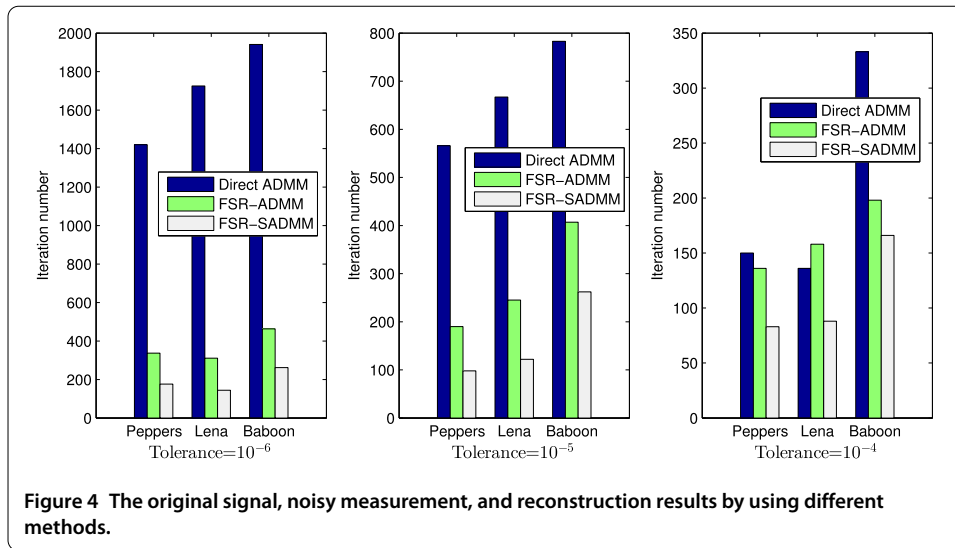


Table 1 The performance of different SADMM-type algorithms in terms of PSNR and SSIM

Tolerance	Images	<i>Peppers</i>			<i>Lena</i>			<i>Baboon</i>		
		PSNR (dB)	Time (sec.)	Iter.	PSNR (dB)	Time (sec.)	Iter.	PSNR (dB)	Time (sec.)	Iter.
10^{-6}	ADMM	30.6	205	1,420	30.9	246	1,725	22.5	274	1,941
	FSR-ADMM	30.6	71.7	337	31.2	65.9	311	22.5	98	463
	FSR-SADMM	30.6	37.6	176	31.2	31.1	144	22.5	56.3	262
10^{-5}	ADMM	30.6	80.7	566	31	94.3	667	22.5	111	783
	FSR-ADMM	30.6	40.9	190	31.2	51.5	245	22.5	85.9	407
	FSR-SADMM	30.6	21.1	98	31.2	26.4	122	22.5	56.4	262
10^{-4}	ADMM	30.6	21	150	31.2	19.8	136	22.6	46.8	333
	FSR-ADMM	30.6	29.1	136	31.2	33.1	158	22.6	42.2	198
	FSR-SADMM	30.6	17.6	83	31.2	19.1	88	22.6	35.9	166

The **best** and **second best** for each algorithm are indicated by **red** and **blue**, respectively, with respect to $r = 0.6$.

PSNR is not obvious especially when r is less than or equal to 0.3. As such, some offending values for the interval $[0.7, 0.9]$ are preferred.

From Table 1, our proposed FSR-SADMM outperforms ADMM and FSR-ADMM for all the tolerance. Especially, FSR-SADMM costs much less computing time and has a shorter iteration number for tolerance with higher precision such as 10^{-6} .

6 Conclusions

For solving the SISR problem (1.2), we proposed a new algorithm based on the strictly contractive semiproximal Peaceman-Rachford splitting method in this paper. The global convergence of our algorithm is established. Then the computational results indicate that our algorithm achieves better performance compared with start-of-the-art methods including direct ADMM and FSR-ADMM. More specifically with Tolerance = 10^{-6} , our algorithm can save computing time about 80% and 45% compared with direct ADMM and FSR-ADMM, respectively.

Appendix: Proof of the equivalence of (3.2a) and (1.7a)

Proof From (3.2a) and the definitions B_1 and B_2 above (2.1), it is not difficult to verify

$$\begin{aligned}
 x^{k+1} &= \operatorname{argmin}_{x \in \mathbb{R}^{N_h}} \frac{1}{2} \|y - \Phi Hx\|_2^2 - \lambda^k (Ax - u^k) + \frac{\mu}{2} \|Ax - u^k\|_2^2 + \frac{\mu}{2} \|x - x^k\|_{\frac{1}{\tau} - A^T A}^2 \\
 &= \operatorname{argmin}_{x \in \mathbb{R}^{N_h}} \frac{1}{2} \|y - \Phi Hx\|_2^2 + \frac{\mu}{2} \left\| Ax - u^k - \frac{\lambda^k}{\mu} \right\|_2^2 + \frac{\mu}{2} \|x - x^k\|_{\frac{1}{\tau} - A^T A}^2 \\
 &= \operatorname{argmin}_{x \in \mathbb{R}^{N_h}} \frac{1}{2} \|y - \Phi Hx\|_2^2 + \frac{\mu}{2\tau} \|x - x^k\|_2^2 - \frac{\mu}{2} \|Ax - Ax^k\|_2^2 + \frac{\mu}{2} \left\| Ax - u^k - \frac{\lambda^k}{\mu} \right\|_2^2 \\
 &= \operatorname{argmin}_{x \in \mathbb{R}^{N_h}} \frac{1}{2} \|y - \Phi Hx\|_2^2 + \frac{\mu}{2\tau} \|x - x^k\|_2^2 + \mu x^T A^T \left(Ax - u^k - \frac{\lambda^k}{\mu} \right) \\
 &= \operatorname{argmin}_{x \in \mathbb{R}^{N_h}} \frac{1}{2} \|y - \Phi Hx\|_2^2 + \frac{\mu}{2\tau} \left\| x - x^k + \tau A^T \left(Ax^k - u^k - \frac{\lambda^k}{\mu} \right) \right\|_2^2 \\
 &= \left(H^T \Phi^T \Phi H + \frac{\mu}{\tau} I_{N_h} \right)^{-1} \left(H^T \Phi^T y + \frac{\mu}{\tau} x^k - \mu A^T \left(Ax^k - u^k - \frac{\lambda^k}{\mu} \right) \right).
 \end{aligned}$$

The above equation is exactly (1.7a); the proof is complete. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

BG proposed the main algorithm and drafted the manuscript. FS participated in the design of the method. YT participated in the design of the study and performed the experiment part. SX conceived of the study and helped to draft the manuscript. All authors read and approved the final manuscript.

Author details

- ¹College of Communications Engineering, PLA University of Science and Technology, Nanjing, 210007, P.R. China.
- ²College of Information Science and Engineering, Shandong Agricultural University, Tai'an, 271018, P.R. China.
- ³Department of Communication Engineering, Nanjing Institute of Technology, Nanjing, 211167, P.R. China.

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Endnotes

^a Note that TV model can be defined anisotropically or isotropically, *i.e.*,

$$\|x\|_{TV} := \begin{cases} |(\nabla_h x)| + |(\nabla_v x)| & \text{(anisotropic);} \\ \sqrt{\|\nabla_h x\|_2^2 + \|\nabla_v x\|_2^2} & \text{(isotropic).} \end{cases}$$

As our proposed method can be applied in the two different TV models in a similar way, we just consider the isotropical model and omit the anisotropical case.

^b The symmetric ADMM is also known as the Peaceman-Rachford splitting method [10].

- ^c Note that the proximal term $\frac{\mu}{2} \|z_1 - z_1^k\|_R^2$ plays an important role to linearize the term $-(\lambda^k)^T B_1 z_1 + \frac{\mu}{2} \|B_1 z_1 - B_2 z_2^k\|_2^2$ so as to avoid computing $(H^T \Phi^T \Phi H + \mu B_1^T B_1)^{-1}$ to find an analytical solution of (1.7a). In fact, many manuscripts using approximate linearization process [18], they first denote $h(x) = \frac{1}{2} \|B_1 z_1 - B_2 z_2^k - \lambda^k / \mu\|_2^2$, given the Lipschitz constant τ , and use the Taylor approximation $h(x) = \nabla h(z_1^k)(z_1 - z_1^k) + \frac{1}{2} \|z_1 - z_1^k\|_2^2$ to approximate the term $\mu \|B_1 z_1 - z_2^k - \lambda^k / \mu\|_2^2$, which increases the difficulty of convergence analysis and needs a specific customized stop criterion [19].
- ^d Note that in this paper $B_2^T B_2 = I_{2N_p}$, R is assumed to be semidefinite and the last term in (4.37) is deduced with the aid of (4.22).
- ^e The MATLAB code has been released on Github <https://github.com/gaobinggaobingao/SISR>.

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