Probab. Theory Relat. Fields (2010) 147:349–378
DOI 10.1007/s00440-009-0249-z

On large deviations for the parabolic Anderson model

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Received: 3 November 2006 / Revised: 10 February 2009 / Published online: 22 October 2009

Abstract The focus of this article is on the different behavior of large deviations of random functionals associated with the parabolic Anderson model above the mean versus large deviations below the mean. The functionals we treat are the solution $u(x, t)$ to the spatially discrete parabolic Anderson model and a functional A_n which is used in analyzing the a.s. Lyapunov exponent for $u(x, t)$. Both satisfy a "law of large numbers", with $\lim_{t\to\infty} \frac{1}{t} \log u(x, t) = \lambda(\kappa)$ and $\lim_{n\to\infty} \frac{A_n}{n} = \alpha$. We then think of αn and $\lambda(\kappa)t$ as being the mean of the respective quantities A_n and $\log u(t, x)$. Typically, the large deviations for such functionals exhibits a strong asymmetry; large deviations above the mean take on a different order of magnitude from large deviations below the mean. We develop robust techniques to quantify and explain the differences.

Keywords Parabolic Anderson model · FKG inequality · Large deviations · Random media

Mathematics Subject Classification (2000) Primary 60F10 · 60K37; Secondary 60K35

Research of the authors supported in part by a grant from NSF 0706198 and SFNS 3510767. D. Gauthier wishes to acknowledge the support of CNRS during his stay in Paris 2008.

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0 Introduction

The parabolic Anderson model has long been of interest to physicists and mathematicians. It presents a physically relevant model for real world phenomena such as transport of electrons in crystals with impurities and the temperature on the surface of the sun to name just a couple of examples, see [3] and [14]. On the other hand, its analysis has provided mathematical challenges. Thus studying the behavior of solutions to the parabolic Anderson equation is both physically relevant and often requires a variety of new mathematical ideas. In this paper, we apply techniques developed by the authors in [1] to deal with large deviations for point to point first passage percolation. We now describe the model treated in this paper. Introduce, on a probability space $(\Omega, \mathcal{F}, P)$, a field $\{W_x : x \in \mathbb{Z}^d\}$ of i.i.d. one-dimensional Brownian motions. Let Δ denote the discrete Laplacian and let κ be a positive constant. Then consider the behavior of the solution to

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) &= \kappa \Delta u(x,t) + u(x,t)\partial W_x, \quad x \in \mathbb{Z}^d,\ t > 0, \\ u(x,0) &\equiv 1, \end{aligned} \tag{1}$$

where ∂ denotes the Stratonovich differential. The solution $u(x,t)$ (with the stochastic integrals appropriately interpreted) has the Feynman Kac representation:

$$u(x,t) = E_x\left[e^{\int_0^t dW_{X(t-s)}(s)}\right]$$

where $\{X(s),\, s \geq 0,\, P_x\}$ is the continuous time pure jump Markov process on $\mathbb{Z}^d$ with infinitesimal generator $\kappa\Delta$, independent of $\{W_x : x \in \mathbb{Z}^d\}$, see e.g. [3]. It is known that the (random in $\{W_x : x \in \mathbb{Z}^d\}$) solution satisfies

$$\lim_{t\to\infty} \frac{\log u(x,t)}{t} = \lambda(\kappa) > 0, \quad P\ a.s.,$$

for nonrandom $\lambda(\kappa)$, see [2,4,9,15] and [17] for more details. This raises the question of the large deviations regimes for these solutions: for $\epsilon > 0$, how rapidly do the probabilities of events $P(\frac{\log\, u(x,t)}{t} > \lambda(\kappa) + \epsilon)$ and $P(\frac{\log\, u(x,t)}{t} < \lambda(\kappa) - \epsilon)$ tend to zero as t tends to infinity? Similar questions for other models were considered in [6,8] and [12].

In [4], it was shown that for some $c(\epsilon) > 0$,

$$P\left(\frac{\log\, u(x,t)}{t} > \lambda(\kappa) + \epsilon\right) \leq e^{-c(\epsilon)t}, \quad \text{as } t \to \infty,$$

but the probabilities of small values of $u(x,t)$ were not considered. We remark here that the random variables $\{u(x,t),\, x \in \mathbb{Z}^d\}$ are identically distributed due to the fact that the field $\{W_x : x \in \mathbb{Z}^d\}$ is iid, but these random variables are correlated. Also, the above result and all results in this article for the solution to (1) remain true if the initial condition is replaced by a nontrivial positive function.

A closely related question (see [2] or [4]) concerns an additive functional associated to the Brownian field $\{W_x : x \in \mathbb{Z}^d\}$ on $(\Omega, \mathcal{F}, P)$ which we now more cleary define. Throughout most of this paper (the exception being the remark following the theorems in this section), we shall consider the space of right continuous, left limit paths which have only jumps of size one. Define, for $0 \leq a < b$ and $A, B \subset \mathbb{Z}^d$ and $k \in \mathbb{Z}_+ \equiv \{m \in \mathbb{Z} : m \geq 0\}$, the subspace of this space of paths defined by

$$\Gamma_{a,b,k}(A, B) = \left\{\gamma : [a, b] \to \mathbb{Z}^d : \gamma(a) \in A, \gamma(b) \in B, N(\gamma, [a, b]) = k\right\}, \tag{2}$$

where $N(\gamma, [a, b])$ is the number of jumps of γ in the interval $[a, b]$. If $A = \{x\}$ and $B = \mathbb{Z}^d$, we use the more compact notation

$$\Gamma^x_{a,b,k} = \Gamma_{a,b,k}(\{x\}, \mathbb{Z}^d) \tag{3}$$

when $[a, b] = [0, t]$ we write $N(\gamma, t) = N(\gamma, [0, t])$. Similarly, we simplify the notation in the special case $a = 0$, $b = n$, $A = \{0\}$, $B = \mathbb{Z}^d$ by writing

$$\Gamma_{n,k} = \Gamma_{0,n,k}(\{0\}, \mathbb{Z}^d).$$

Then we define, for $\gamma \in \Gamma_{a,b,k}(A, B)$, the value of γ as

$$V(\gamma) = \int_a^b dW_{\gamma(s)}(s). \tag{4}$$

Crucial to the asymptotic behavior of the random field $u(x, t)$ is the functional

$$A_{n,k} = \sup_{\gamma \in \Gamma_{n,k}} V(\gamma). \tag{5}$$

It is known that (see [13])

$$\frac{A_{n,n}}{n} \to \alpha > 0, \quad P - a.s. \tag{6}$$

for a nonrandom constant α. Scaling considerations give the limiting behaviour of $A_{n,[n\theta]}$ where $[\,\cdot\,]$ denotes, as usual, the integer part. Again the question is raised as to the behaviour, as n tends to infinity of the probabilities $P(\frac{A_{n,n}}{n} > \alpha + \epsilon)$ and $P(\frac{A_{n,n}}{n} < \alpha - \epsilon)$. Our results concerning these probabilities are the following.

Theorem 0.1 *For $A_{n,n}$ as defined above and for $\epsilon > 0$, the lower large deviation satisfies*

$$
\begin{aligned}
-\infty &< \underline{\lim}_{n\to\infty} \frac{1}{n^{d+1}} \log P\left(\frac{A_{n,n}}{n} \le \alpha - \epsilon\right) \\
&\le \overline{\lim}_{n\to\infty} \frac{1}{n^{d+1}} \log P\left(\frac{A_{n,n}}{n} \le \alpha - \epsilon\right) \\
&< 0,
\end{aligned} \tag{7}
$$

whereas for the upper large deviations,

$$
\begin{aligned}
-\infty &< \underline{\lim}_{n\to\infty} \frac{1}{n} \log P\left(\frac{A_{n,n}}{n} \ge \alpha + \epsilon\right) \\
&\le \overline{\lim}_{n\to\infty} \frac{1}{n} \log P\left(\frac{A_{n,n}}{n} \ge \alpha + \epsilon\right) \\
&< 0.
\end{aligned} \tag{8}
$$

And for $u(0, t)$ the solution of the parabolic Anderson model described above, we have

Theorem 0.2 *For each $\epsilon > 0$ for the lower large deviations,*

$$
\begin{aligned}
-\infty &< \underline{\lim}_{t\to\infty} \frac{1}{t^{d+1}} \log P\left(\frac{\log u(0,t)}{t} \le \lambda(\kappa) - \epsilon\right) \\
&\le \overline{\lim}_{t\to\infty} \frac{1}{t^{d+1}} \log P\left(\frac{\log u(0,t)}{t} \le \lambda(\kappa) - \epsilon\right) \\
&< 0,
\end{aligned} \tag{9}
$$

and for the upper large deviations,

$$
\begin{aligned}
-\infty &< \underline{\lim}_{t\to\infty} \frac{1}{t} \log P\left(\frac{\log u(0,t)}{t} \ge \lambda(\kappa) + \epsilon\right) \\
&\le \overline{\lim}_{t\to\infty} \frac{1}{t} \log P\left(\frac{\log u(0,t)}{t} \ge \lambda(\kappa) + \epsilon\right) \\
&< 0.
\end{aligned} \tag{10}
$$

Remark 0.1 The difference in large deviation rates for lower deviations and upper deviations can be explained qualitatively as follows. Since $A_{n,n}$ involves a supremum of an additive functional of a random medium, an upper deviation of the form $\{A_{n,n} \ge (\alpha+\epsilon)n\}$ can occur when there is a single path γ for which $V(\gamma) \ge (\alpha+\epsilon)n$. However, in order for a lower deviation of the form $\{A_{n,n} \le (\alpha - \epsilon)n\}$ to occur one must have for every path γ it holds that $V(\gamma) \le (\alpha - \epsilon)n$. Therefore, the upper deviation can occur when a small region in the random medium is deviant whereas a lower deviation can occur only when the entire medium is deviant. Part of the proof for the lower deviation involves decomposing space into disjoint channels which are space-time rectangles. These channels are large enough so that with probability at least $1 - e^{-cn}$, there is a path γ of duration of order n lying entirely in the channel and having $V(\gamma) \geq (\alpha - \epsilon)n$. On the other hand, the channels are small enough so that there are on the order of n^d of them. Since the channels are disjoint, events depending on the field in different channels are independent. As a result, the probability that no path γ satisfies $V(\gamma) \geq (\alpha - \epsilon)n$ is on the order of $(e^{-cn})^{n^d} = e^{-cn^{d+1}}$. This is the argument for times in an interval $[\delta n, n]$. For small times, i.e. on the interval $[0, \delta n]$, a different approach is used. The media can be "bad" (in the sense that the value of paths is low) near the starting point with relatively high probability. By taking paths which quickly leave a neighborhood of the origin, we can with relatively high probability arrive at regions where the media is not "bad". This is quantified in the section on *Small Times*.

Remark 0.2 Using subadditivity arguments, it can be shown that

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(A_{n,n} > (\alpha + \epsilon)n\right) \in (-\infty, 0)$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log P\left(\frac{\log\ u(x,t)}{t} > \lambda(\kappa) + \epsilon\right) \in (-\infty, 0).$$

Remark 0.3 Theorem 0.2 implies the frequency of points $x \in \mathbb{Z}^d$ for which $u(x,t) \leq e^{(\lambda(\kappa)-\epsilon)t}$ are exceedingly rare. Indeed, suppose $c > 0$ satisfies $P(u(x,t) \leq e^{(\lambda(\kappa)-\epsilon)t}) \leq e^{-ct^{d+1}}$, $t \to \infty$. Now if $L(t)$ satisfies

$$\lim_{t\to\infty} L(t)^d e^{-ct^{d+1}} = 0,$$

and $Q_t = \{||x|| \leq L(t) : u(x,t) \leq e^{(\lambda(\kappa)-\epsilon)t}\}$ then

$$\begin{aligned} P(|Q_t| \geq 1) &\leq c_d L(t)^d P\left(u(0,t) \leq e^{(\lambda(\kappa)-\epsilon)t}\right) \\ &\leq c_d L(t)^d e^{-ct^{d+1}}. \end{aligned} \tag{11}$$

Thus in a box $\Lambda_L = \{x : ||x|| \leq L\}$ with $L = L(t) = o(e^{c^{1/d}t^{1+1/d}})$ there will with high probability be no $x \in \Lambda_L$ for which $u(x,t) \leq e^{(\lambda(\kappa)-\epsilon)t}$.

Remark 0.4 It may be illuminating to compare Theorem 0.1 with an analogous result in our previous work [1]. There we considered the graph $G = (\Xi, E)$ where

$$\Xi = \{(x,n) \in \mathbb{Z} \times \mathbb{Z}_+ : |x| + n \equiv 0 \ \ mod\, 2\}$$

and E denotes the set of directed nearest neighbour edges from vertices (x,n) to vertices of the form $(x \pm 1, n+1)$. On the edge set there is a field $\{X_e : e \in E\}$ of i.i.d. $\mathcal{N}(0,1)$ random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. Set

$$\Xi_n = \Xi \cap (\mathbb{Z} \times \{n\}) \tag{12}$$

and define the set of connected paths in G

$$\aleph_n = \{\gamma : \gamma(0) = 0, \gamma(n) \in \Xi_n\}. \tag{13}$$

If $e \in E$ is an edge along γ write $e \in \gamma$. Then define

$$Z_n = \sup_{\gamma \in \aleph_n} \sum_{e \in \gamma} X_e, \tag{14}$$

which is the "point-to-plane" supremum. Again, simple subadditive considerations lead us to the conclusion that

$$\lim_{n\to\infty} \frac{Z_n}{n} = \mu, \; P \; a.s. \text{ for } \mu \text{ nonrandom.}$$

It was shown in [1] (see also [6–8,10,11] and [12] for related work) that

$$\lim_{n\to\infty} \frac{1}{n} \log P\left(\frac{Z_n}{n} > \mu + \epsilon\right) \in (-\infty, 0).$$

However the deviations below for a Gaussian field satisfy

$$\begin{aligned} -\infty &< \liminf_{n\to\infty} \frac{log\, n}{n^2} \log P\left(\frac{Z_n}{n} < \mu - \epsilon\right) \\ &< \limsup_{n\to\infty} \frac{log\, n}{n^2} \log P\left(\frac{Z_n}{n} < \mu - \epsilon\right) < 0. \end{aligned} \tag{15}$$

At first one might think the upper large deviation rates for Z_n and A_n should be the same since they are both the supremum of a Gaussian field and the graph structures are relatively similar. However, the case of Z_n is analogous to a discrete time random walk. A path γ in Ξ is obliged to "use" a definite proportion of the random variables close to the origin in the graph (Ξ, E). On the other hand, A_n arises in a case analogous to a continuous time random walk. For A_n, the candidate paths for the supremum have the possibility of "quick escape" by passing through "bad regions" near the origin in $\mathbb{Z} \times \mathbb{Z}_+$. That is, there are candidate paths which have a lot of jumps in a short, initial time interval. This accounts for the presence of $\log n$ in the case of large deviations for Z_n and its absence for those of A_n. That is, the field near the origin plays a crucial role in the order of lower large deviations.

The proof of our results split naturally into a consideration of the field near the starting point and a consideration of its behavior at some remove from the starting point. The field "near the starting point" is indexed by a relatively smaller number of sites. As a result, the probability of a uniform aberration in the field near the starting point causing a lower deviation is relatively high compared to aberrations in the field far from the starting point required to cause a similar lower deviation. Accordingly, the remainder of the paper is split into two sections labelled *Small Times* and *Large Times* in which we present the proofs of our theorems. The authors gratefully acknowledge a patient and very helpful referee.

1 Large times

We first prove some propositions which deal with the value of the relevant functionals over portions of the random field $\{W_x : x \in \mathbb{Z}^d\}$ for points x far from the starting point. Recalling the definition of $\Gamma^x_{\delta n,n,k}$ at (3), define

$$M_{\delta 2^n,2^n,k(1-\delta)2^n} = \Big\{x \in [-\delta 2^n, \delta 2^n]^d : \exists \gamma \in \Gamma^x_{\delta 2^n,2^n,k(1-\delta)2^n}, V(\gamma) \geq \left(\alpha - \frac{\epsilon}{5}\right)\sqrt{k}(1-\delta)2^n\Big\}, \tag{16}$$

where α is the constant introduced at (6). We take a moment to explain the appearance of $\sqrt{k}$ in (16). This is due to Brownian scaling. Taking a path $\gamma \in \Gamma_{a,b,k(b-a)}$, the path $\tilde{\gamma} \in \Gamma_{ka,kb,k(b-a)}$ defined by $\tilde{\gamma}(r) = \gamma(\frac{r}{k})$ satisfies

$$\begin{aligned} V(\gamma) &= \int_a^b dW_{\gamma(s)}(s) \\ &= \int_{ka}^{kb} dW_{\gamma(\frac{r}{k})}\left(\frac{r}{k}\right) \\ &\overset{\mathcal{D}}{=} \frac{1}{\sqrt{k}} \int_{ka}^{kb} dW_{\tilde{\gamma}(r)}(r), \end{aligned} \tag{17}$$

where $\overset{\mathcal{D}}{=}$ denotes equality in distribution. Accordingly, $V(\gamma) \overset{\mathcal{D}}{=} \frac{1}{\sqrt{k}} V(\tilde{\gamma})$, so the existence of a $\tilde{\gamma} \in \Gamma_{ka,kb,k(b-a)}$ such that

$$V(\tilde{\gamma}) \geq \left(\alpha - \frac{\epsilon}{5}\right) k(b-a)$$

implies the scaled version $\gamma \in \Gamma_{a,b,k(b-a)}$ defined by $\gamma(r) = \tilde{\gamma}(kr)$ satisfies, at least in law,

$$V(\gamma) \geq \left(\alpha - \frac{\epsilon}{5}\right) \sqrt{k}(b-a).$$

Our first result says that a majority of points in the box $[-\delta 2^n, \delta 2^n]^d$ are the starting points of paths defined on the time interval $[\delta 2^n, 2^n]$ with a "good" value. Namely, if we denote by $|C|$ the cardinality of a set C, then

Proposition 1.1 *Given any $a > 0$, for all $\epsilon > 0$, $\delta > 0$ there exists a constant $c(\epsilon, \delta) > 0$, so that*

$$P\left(|M_{\delta 2^n,2^n,a(1-\delta)2^n}| \geq \frac{9}{10}(2\delta 2^n)^d\right) \geq 1 - e^{-c(\epsilon,\delta)(\sqrt{a}2^n)^{d+1}}. \tag{18}$$

for all n sufficiently large.

Remark 1.1 By elementary scaling considerations it is only necessary to consider $a = 1$.

This immediately gives

Corollary 1.1 *If $B \subset \mathbf{R}^+$ is a finite set then with $c(\epsilon, \delta)$ as in Proposition 1.1*

$$P\left(\forall a \in B,\ |M_{\delta 2^n,2^n,a(1-\delta)2^n}| \geq \frac{9}{10}(2\delta 2^n)^d\right) \geq 1 - |B|e^{-c(\epsilon,\delta)(\sqrt{a^*}2^n)^{d+1}} \tag{19}$$

for n sufficiently large, where a^* denotes the smallest element of B.

Denote for $x\epsilon\mathbb{Z}^d$ and $M > 0$,

$$u_M(x, (1-\delta)t) = E_x\left[e^{\int_0^{(1-\delta)t} dW_{X((1-\delta)t-s)}(s)} 1_{\{N(X,t)\leq Mt\}}\right]. \tag{20}$$

Then similarly, we also have,

Proposition 1.2 *For $M, \epsilon, \delta > 0$ there exists a positive constant $c(M, \epsilon, \delta)$ so that for all t sufficiently large,*

$$P\left(|\{x \in [-\delta t, \delta t]^d : u_M(x, (1-\delta)t) \geq e^{(\lambda(\kappa)-\epsilon)(1-\delta)t}\}| \geq \frac{9}{10}(2\delta t)^d\right)$$
$$\geq 1 - e^{-c(M,\epsilon,\delta)t^{d+1}}. \tag{21}$$

The reader might have noted that in Proposition 1.1 only the media between times $\delta 2^n$ and 2^n is involved. In Proposition 1.2 the media between times 0 and $(1-\delta)t$ comes into play. These are equivalent in distribution, (when $t = \delta 2^n$), by stationarity in time of the time increments of the Gaussian field $\{W_x : x \in \mathbb{Z}^d\}$. The form of Proposition 1.2 is suited to an application of the Markov property in the course of the proof of Theorem 0.2. The proof of these propositions follows the pattern of proof of Proposition 2.1 from [1]. The idea is to establish the existence of disjoint channels which with high probability contain paths with values near their predicted asymptotic value.

2 An FKG argument

In this section, we use the FKG inequality to establish a key lemma used to prove Proposition 1.1. The proof of Proposition 1.1 appears in the next section. Here, we are concerned with behavior between the times $\delta 2^n$ and 2^n for n large. We refer to these as large times. In the next section, we deal with behavior between time 0 and $\delta 2^n$. To begin we define

Definition 2.1 For a path $\gamma \in \Gamma_{a,b,k}(A, B)$ and a subset $C \subset \mathbb{Z}^d$, we say $\gamma \subset C$ if $\gamma(s) \in C$ for all $a \leq s \leq b$. The set of such paths will be denoted by $\Gamma_{a,b,k}(A, B, C)$. If $A = \{x\}$, then write $\Gamma^x_{a,b,k}(B, C) = \Gamma_{a,b,k}(\{x\}, B, C)$. We shall write $\Gamma^x_{a,b,k}(C) = \Gamma^x_{a,b,k}(\{x\}, \mathbb{Z}^d, C)$. Then $\Gamma^x_{a,b,k} = \Gamma^x_{a,b,k}(\mathbb{Z}^d)$ is consistent with the notation established at (3).

Lemma 2.1 *Given $k > 0, \epsilon > 0$ and $l > 0$ let*

$$A_{l,k}(\epsilon) = \left\{\forall x \in [0,l)^d \cap \mathbb{Z}^d,\ \exists \gamma \in \Gamma^x_{0,l,kl}([0,l)^d),\ V(\gamma) \geq (\alpha - \epsilon)\sqrt{k}\, l\right\}.$$

Then for any $\epsilon > 0$ for all l sufficiently large (depending on ϵ,)

$$P(A_{l,kl}(\epsilon)) \geq 1 - \frac{\epsilon}{2(\alpha+3)}.$$

Proof For notational convenience, we restrict our exposition to $d = 1$. We only need prove the lemma with $k = 1$, the case of arbitrary k follows by the scaling argument preceding Proposition 1.1. Given $\alpha > \epsilon > 0$, set $\epsilon' = \frac{\epsilon}{\alpha+3}$. Then there exists l_0 so that for $l' \geq l_0$, (for notational convenience, we will suppose that l' and l'/ϵ' are even integers)

$$P\left(\exists \gamma \in \Gamma^0_{0,l',l'},\ V(\gamma) \geq (\alpha - \epsilon')l'\right) \geq 1 - \left(\frac{\epsilon'^3}{100}\right)^2.$$

Consider the equally probable, decreasing events

$$\begin{aligned} A_1 &= \{\nexists\, \gamma \in \Gamma^0_{0,l',l'},\ \gamma(l') \in \mathbb{Z}_+,\ V(\gamma) \geq (\alpha - \epsilon')l'\} \\ A_2 &= \{\nexists\, \gamma \in \Gamma^0_{0,l',l'},\ \gamma(l') \in \mathbb{Z}_-,\ V(\gamma) \geq (\alpha - \epsilon')l'\}, \end{aligned}$$

where $\mathbb{Z}_- \equiv \{m \in \mathbb{Z} : m \leq 0\}$. From the FKG inequality applied to these events we have

$$\begin{aligned} \left(\frac{\epsilon'^3}{100}\right)^2 &\geq P(A_1 \cap A_2) \\ &\geq P(A_1)P(A_2), \text{ FKG} \\ &= P(A_1)^2 \end{aligned}$$

and consequently,

$$P(A_1^c) = P(A_2^c) \geq 1 - \frac{\epsilon'^3}{100}. \tag{22}$$

Notice that the γ under consideration in either A_1^c or A_2^c must satisfy $\gamma \subset [-l', l']$ since such a γ starts at 0 and has only l' steps of size 1. We now concatenate paths. First find, with high probability, a path $\gamma_1 \in \Gamma^0_{0,l',l'}$ for which $V(\gamma_1) \geq (\alpha - \epsilon')l'$ by selecting a path as prescribed in A_1. This path satisfies both $\gamma_1 \subset [-l', l']$ and $\gamma_1(l') \geq 0$. Treat $(\gamma_1(l'), l')$ as the new origin, and continue by selecting a path $\gamma_2 \in \Gamma^{\gamma_1(l')}_{l',2l',l'}$ in an appropriately shifted version of A_2 (require $\gamma_2(2l') \leq \gamma_1(l')$.) By stationarity of the medium, the shifted versions of A_1^c and A_2^c also satisfy (22). Note this path stays in $[-2l', 2l']$ and has $\gamma_2(2l') \in [-l', \gamma_1(l')] \subset [-l', l']$ and $V(\gamma_2) \geq (\alpha - \epsilon')l'$. Repeating this procedure $\frac{2}{\epsilon'}$ times of going back and forth to stay in $[-2l', 2l']$ and concatenating the resulting paths gives a path $\gamma \in \Gamma^0_{0,\frac{2l'}{\epsilon'},\frac{2l'}{\epsilon'}}([-2l', 2l'])$ with $V(\gamma) \geq (\alpha - \epsilon')\frac{2l'}{\epsilon'}$. By the independence of the field $\{W_x : x \in \mathbb{Z}\}$ and (22), we have for all $l' \geq l_0$,

$$P\left(\exists \gamma \in \Gamma^0_{0,\frac{2l'}{\epsilon'},\frac{2l'}{\epsilon'}}\left([-2l', 2l']\right),\ V(\gamma) \geq (\alpha - \epsilon')\frac{2l'}{\epsilon'}\right) \geq 1 - \frac{\epsilon'^2}{50}.$$

This, by translation invariance and independence of the field $\{W_x : x \in \mathbb{Z}\}$, yields that with probability at least $1 - \frac{\epsilon'}{10}$, for all of the (no more than) $\frac{4}{\epsilon'}$ points

$$y \in \left[-\frac{l'}{\epsilon'} + l', \frac{l'}{\epsilon'} - l'\right] \cap \mathbb{Z}l'$$

there is a path $\gamma_y^1 \in \Gamma^y_{2l',\frac{2l'}{\epsilon'}+2l',\frac{2l'}{\epsilon'}}([y - 2l', y + 2l'])$ with

(i) $|\gamma_y^1(i) - y| \leq 2l',\ \ \forall\, i \in [0, \frac{2l'}{\epsilon'}]$
(ii) $V(\gamma_y^1) \geq (\alpha - \epsilon')\frac{2l'}{\epsilon'}$.

That is if, we define the event A by conditions (i) and (ii) holding for every $y \in [-\frac{l'}{\epsilon'} + l', \frac{l'}{\epsilon'} - l'] \cap \mathbb{Z}l'$ then

$$P(A) \geq 1 - \frac{\epsilon'}{10}. \tag{23}$$

Now to each $x \in [-\frac{l'}{\epsilon'} - l', \frac{l'}{\epsilon'} + l'] \cap \mathbb{Z}$, we associate a

$$y \in \left[-\tfrac{l'}{\epsilon'} + l', \tfrac{l'}{\epsilon'} - l'\right] \cap \mathbb{Z}l',$$

with $|y - x| \leq 2l'$. For each such x and its associated point y, pick (arbitrarily) a path $\gamma_x^2 \in \Gamma^x_{0,2l',2l'}([-\frac{l'}{\epsilon'} - l', \frac{l'}{\epsilon'} + l'])$ with $\gamma_x^2(2l') = y$. For these x denote the concatenation of γ_x^2 and γ_y^1 by γ_x. We then have

$$V(\gamma_x) \geq \left(\alpha - \epsilon'\right)\frac{2l'}{\epsilon'} + Z$$

where $Z = \min_x V(\gamma_x^2)$ and each path satisfies, $\gamma_x \subset [-\frac{l'}{\epsilon'} - l', \frac{l'}{\epsilon'} + l']$. If $Z \geq -4l'$, then since $\epsilon' = \frac{\epsilon}{\alpha+3}$,

$$
\begin{aligned}
V(\gamma_x) &\geq \left(\alpha - \epsilon'\right) \frac{2l'}{\epsilon'} - 4l' \\
&= \left(\alpha - 3\epsilon'\right) \frac{2l'}{\epsilon'} \\
&= \left(\alpha - \frac{3\epsilon}{\alpha+3}\right) \frac{2l'}{\epsilon'} \\
&= \left(\alpha - \epsilon + \frac{\alpha\epsilon}{\alpha+3}\right) \frac{2l'}{\epsilon'} \\
&= (\alpha - \epsilon) \left(\frac{2l'}{\epsilon'} + \frac{\alpha\epsilon}{(\alpha+3)(\alpha-\epsilon)} \frac{2l'}{\epsilon'}\right) \\
&\geq (\alpha - \epsilon) \left(\frac{2l'}{\epsilon'} + 2l'\right).
\end{aligned} \tag{24}
$$

Thus, on the event $A \cap \{Z \geq -4l'\}$,

$$
V(\gamma_x) \geq (\alpha - \epsilon) \left(\frac{2l'}{\epsilon'} + 2l'\right).
$$

But the random variables $V(\gamma_x^2)$, for $x \in [-\frac{l'}{\epsilon'} - l', \frac{l'}{\epsilon'} + l'] \cap \mathbb{Z}$, are centered Gaussian random variables with variance $2l'$. Thus, for some $c > 0$,

$$
\begin{aligned}
P\left(Z \geq -4l'\right) &= P\left(\min_{x \in \left[-\frac{l'}{\epsilon'} - l', \frac{l'}{\epsilon'} + l'\right] \cap \mathbb{Z}} V(\gamma_x^2) \geq -4l'\right) \\
&= 1 - P\left(V(\gamma_x^2) \leq -4l', \text{for some } x \in \left[-\frac{l'}{\epsilon'} - l', \frac{l'}{\epsilon'} + l'\right] \cap \mathbb{Z}\right) \\
&\geq 1 - \frac{2l'}{\epsilon'} P\left(V(\gamma_x^2) \leq -4l'\right) \\
&= 1 - \frac{2l'}{\epsilon'} \frac{1}{\sqrt{2\pi 2l'}} \int_{-\infty}^{-4l'} e^{-\frac{y^2}{4l'}} dy \\
&= 1 - \frac{2l'}{\epsilon'} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{8l'}} e^{-\frac{y^2}{2}} dy \\
&\geq 1 - c\frac{\sqrt{l'}}{\epsilon'} e^{-4l'}, \quad \text{for some positive constant } c.
\end{aligned} \tag{25}
$$

Now shift the interval $[-\frac{l'}{\epsilon'} - l', \frac{l'}{\epsilon'} + l']$ to the right by $\frac{l'}{\epsilon'} + l'$ and set $l = \frac{2l'}{\epsilon'} + 2l'$. Then we see from (23) and (25) and the independence of the events A and $\{Z \geq -4l'\}$

that

$$P(A_{l,1}(\epsilon)) \geq \left(1 - \frac{\epsilon'}{10}\right)\left(1 - c\frac{\sqrt{l'}}{\epsilon'}e^{-4l'}\right).$$

By taking l' large enough we get

$$P(A_{l,1}(\epsilon)) \geq \left(1 - \frac{\epsilon}{2(\alpha+3)}\right).$$

As remarked earlier, the case of general k follows from scaling. □

We need a (crude) bound on the lower tail of the distribution of the random variable

$$\min_{x\in[0,l]} \max_{\gamma\in\Gamma^x_{0,l,kl}([0,l)^d)} V(\gamma) = Y_{l,k}.$$

If $Y_{l,k}$ can take on big negative values with high probability, then there would also potentially be a high probability that $V(\gamma) \leq (\alpha - \epsilon)2^n$ on a path γ of length 2^n. The following will suffice to show such an outcome is unlikely.

Lemma 2.2 *There exists $c > 0,\ l_0$ and r_0 so that for all $l \geq l_0$*

$$P\left(Y_{l,k} \leq -r\,l\right) \leq l^d e^{-crl}, \quad \textit{for all } r > r_0. \tag{26}$$

Proof For each $x \in [0,l]^d$ consider the constant path $\gamma_x(s) \equiv x$, and the random variable

$$V(\gamma_x) = \int_0^l dW_{\gamma_x(s)}(s) = W_x(l) - W_x(0).$$

Then for $r > r_0$,

$$\begin{aligned} P\left(V(\gamma_x) \leq -rl\right) &= P\left(-r_0V(\gamma_x) \geq r_0rl\right) \\ &\leq e^{-r_0rl}E[e^{-r_0V(\gamma_x)}] \\ &= e^{-r_0rl+\frac{lr_0^2}{2}} \\ &\leq e^{-\frac{r_0}{2}rl}. \end{aligned}$$

Thus

$$P\left(\min_{x\in[0,l]^d} \max_{\gamma\in\Gamma^x_{0,l,kl}([0,l)^d)} V(\gamma) \leq -r\,l\right) \leq l^d e^{-crl}.$$

□

3 Proof of Propositions 1.1 and 1.2

In the proof of Proposition 1.1, we need

Definition 3.1 For $(i, r) \in \{0, 1, \ldots, \frac{2^n}{l} - 1\} \times \{0, 1, \ldots, \frac{2^n}{l} - 1\}$ say $C_l(i, r) \equiv [li, l(i+1)) \times \{rl\}$ is $\mathbf{k-good}$ if for all $x \in [li, l(i+1)) \cap \mathbb{Z}$, there exists a path $\gamma_x \in \Gamma^x_{rl,(r+1)l,kl}([li, l(i+1)))$ such that $V(\gamma_x) \geq (\alpha - \frac{\epsilon}{100})\sqrt{k}\, l$.

Although the definition is stated for the case $d = 1$, it can be readily adapted to the case $d \geq 1$.

Proof of Proposition 1.1 We continue to give proof in the case $d = 1$. The strategy of the proof is to use Lemma 2.1 to show there are on the order of 2^n channels of width l starting at time $\delta 2^n$ and ending at time 2^n. There is a high probability that each channel contains a path with value not less than $(\alpha - \epsilon)(1 - \delta)2^n$. Then we exploit the independence of the field in different channels.

Recall that given $\epsilon \in (0, \alpha)$, by Lemma 2.1, we can fix l so large that

$$P(A_{l,k}(\epsilon)) > 1 - \frac{\epsilon}{2(\alpha + 3)}. \tag{27}$$

Thus

$$P(C_l(i, r) \text{ is } \mathbf{k-good}) > 1 - \frac{\epsilon}{200(\alpha + 3)}.$$

Suppose also that $\delta > 0$ is given. Without loss of generality we take $\delta 2^n$ and 2^n to be multiples of l.

Denote the set of **k-good** intervals by $\mathcal{G}_k$ and set

$$\psi(i, r) = 1_{\mathcal{G}_k}([li, l(i+1)) \times \{rl\}).$$

By (27) and the fact that the field $\{W_x : x \in \mathbb{Z}^d\}$ is i.i.d. Brownian, the $\psi(i, r),\ (i, r) \in \{0, 1, \ldots, 2^n/l - 1\} \times \{0, 1, \ldots, 2^n/l - 1\}$ are i.i.d. Bernoulli random variables with parameter exceeding $1 - \frac{\epsilon}{200(\alpha+3)}$. Now for each such (i, r), define

$$Y_{(i,r)} = \min_{x:(x,rl)\in C_l(i,r)} \ \max_{\gamma\in\Gamma^x_{rl,r(l+1),kl}([li,l(i+1))} V(\gamma). \tag{28}$$

Then we have that the random variables $Y_{(i,r)}$ are i.i.d. with lower tail behaviour governed by (26) in Lemma 2.2. We now partition

$$[-\delta 2^n, \delta 2^n] \times \{[\delta 2^n]\} = \bigcup_{m=1}^{R} C_m \times \{[\delta 2^n]\}$$

into $R = \frac{\delta 2^{n+1}}{l}$ disjoint subintervals with $C_m = [-\delta 2^n + (m-1)l, -\delta 2^n + ml),\ m = 1, 2, \ldots, R$ of side length l. (In the case of arbitrary dimensions, d, we have

$R = (\frac{\delta 2^{n+1}}{l})^d$. This is the source of the dimensional dependence in the lower large deviation rates.) Also partition the channels, $C_m \times [\delta 2^n, 2^n)$, as follows

$$C_m \times [\delta 2^n, 2^n) = \bigcup_{j=1}^{\frac{(1-\delta)2^n}{l}} C_m \times [\delta 2^n + (j-1)l, \delta 2^n + jl)$$

into $\frac{(1-\delta)2^n}{l}$ disjoint squares, $C_m \times [\delta 2^n + (j-1)l, \delta 2^n + jl)$, of side length l. Abbreviate the notation by writing

$$R_{m,j} = C_m \times [\delta 2^n + (j-1)l, \delta 2^n + jl).$$

Fix $m \in \{1, 2, \ldots, R\}$ and let $Y_{m,j}$ be the, so-to-speak "worst good situation" random variables as in (28), for the squares $R_{m,j},\ j = 1, 2, \ldots, \frac{(1-\delta)2^n}{l}$. That is,

$$Y_{m,j} = \min_{x \in C_m} \max_{\gamma \in \Gamma^x_{\delta 2^n + (j-1)l, \delta 2^n + jl, kl}(C_m)} V(\gamma). \tag{29}$$

Notice that the $Y_{m,j},\ j = 1, 2, \ldots, \frac{(1-\delta)2^n}{l}$ are independent. For c_1 a small constant, set

$$A(c_1, m) = \left\{ \exists J \subseteq \left\{1, 2, \ldots, \frac{(1-\delta)2^n}{l}\right\},\ |J| \right.$$
$$\left. \leq \frac{c_1(1-\delta)2^n}{l}, \sum_{j \in J} Y_{m,j} \leq -\frac{\epsilon}{10}\sqrt{k}\,(1-\delta)2^n \right\}. \tag{30}$$

In words, lower deviations arise from the event $A(c_1, m)$. We now establish some control over the number of sub-blocks $R_{m,j},\ j = 1, 2, \ldots, \frac{(1-\delta)2^n}{l}$ in a channel, $C_m \times [\delta 2^n, 2^n]$, which are "bad" as specified in the definition of the event $A(c_1, m)$.

Lemma 3.1 *There exists $c > 0$ so that for c_1 small and n, l sufficiently large,*

$$P(A(c_1, m)) \leq e^{-c\epsilon(1-\delta)2^n}. \tag{31}$$

Proof First note that

$$\binom{(1-\delta)\frac{2^n}{l}}{c_1 \frac{2^n}{l}} \leq e^{(1-\delta)I(c_1/(1-\delta))\frac{2^n}{l}}$$

for $I(\theta) = -\theta \log \theta - (1-\theta)\log(1-\theta)$, by large deviations for binomial random variables. This bounds the number of subsets J under consideration.

Let J be a subset as described in (30). Using Lemma 2.2 and the constants c and r_0 there and Chebychev bounds, for $c > c' > 0$ not depending on ϵ, L or n, we have

$$\begin{aligned}
P\left(\sum_{j\in J} Y_{l,j} \le -\frac{\epsilon}{10}\sqrt{k}\,(1-\delta)2^n\right) &\le e^{-c'\frac{\epsilon}{10}\sqrt{k}\,(1-\delta)2^n}\left(E\left[e^{-c'Y}\right]\right)^{|J|}\\
&= \left(E\left[e^{-c'Y}; Y \ge -r_0\sqrt{k}\,l\right]\right.\\
&\quad \left.+ E\left[e^{-c'Y}; Y \le -r_0\sqrt{k}\,l\right]\right)^{|J|}\\
&\quad \times e^{-c'\frac{\epsilon}{10}\sqrt{k}\,(1-\delta)2^n}\\
&\le e^{-c'\frac{\epsilon}{10}\sqrt{k}\,(1-\delta)2^n}\left(e^{c'r_0\sqrt{k}\,l} + c_2 l e^{(c'+c)l}\right)^{\frac{c_1(1-\delta)2^n}{l}}\\
&\le e^{-(c\epsilon/20)(1-\delta)2^n}, \quad \text{if } c_1 \text{ is small enough.} \qquad (32)
\end{aligned}$$

Thus $P(A(c_1,m)) \le e^{-(I(c_1/(1-\delta))\frac{1}{l} - c\epsilon/20)(1-\delta)2^n}$. Again by taking l to be large we obtain the desired bound. □

Now returning to the proof of Proposition 1.1 consider, with fixed m,

$$V_m = \sum_{j=1}^{\frac{(1-\delta)2^n}{l}} 1_{\mathcal{G}_k}(C_m \times \{\delta 2^n + (j-1)l\}).$$

We have that

$$\begin{aligned}
&V_m \text{ is stochastically larger than a binomial random variable}, X_m,\\
&\qquad \text{with parameters } \frac{(1-\delta)2^n}{l} \text{ and } 1 - \frac{\epsilon}{200(\alpha+3)}. \qquad (33)
\end{aligned}$$

On the event

$$B \equiv \left\{V_m \ge \left(\frac{1-c_1}{l}\right)(1-\delta)2^n\right\} \cap A(c_1,m)^c,$$

for each point $x \in C_m$, there exists a path $\gamma_x \in \Gamma^x_{\delta 2^n, 2^n, k(1-\delta)2^n}(C_m)$ which is constructed by concatenating paths with values exceeding $(\alpha - \epsilon/100)\sqrt{k}\,l$ through the $\mathbf{k}-\mathbf{good}$ squares $R_{m,j}$ and on $A(c_1,m)^c$, the total value for paths in the squares which aren't $\mathbf{k}-\mathbf{good}$ can not be less than $-\frac{\epsilon}{10}\sqrt{k}\,(1-\delta)2^n$, so we find that on B, the value of such a concatenated path satisfies,

$$\begin{aligned}
V(\gamma_x) &\ge V_m\left(\alpha - \frac{\epsilon}{100}\right)\sqrt{k}l - \frac{\epsilon}{10}\sqrt{k}\,(1-\delta)2^n\\
&\ge \left((1-c_1)(\alpha - \frac{\epsilon}{100}) - \frac{\epsilon}{10}\right)\sqrt{k}\,(1-\delta)2^n\\
&\ge (\alpha - \epsilon/5)\sqrt{k}\,(1-\delta)2^n, \quad \text{for } c_1 \text{ small enough.} \qquad (34)
\end{aligned}$$

Define the event

$$D_m = \left\{\forall x \in C_m,\, \exists \gamma_x \in \Gamma^x_{\delta 2^n, 2^n, k(1-\delta)2^n}(C_m),\, V(\gamma_x) \geq (\alpha - \epsilon/5)\sqrt{k}\,(1-\delta)2^n\right\}. \quad (35)$$

Then it follows from Lemma 3.1 and (33) that for l large enough, there is a constant $c(\epsilon) > 0$,

$$\begin{aligned} P\left(D_m\right) &= P\left(A(c_1, m)^c \bigcap \{V_m \geq \left(\frac{1-c_1}{l}\right)(1-\delta)2^n\}\right) \\ &= P(A(c_1, m)^c) + P\left(V_m \geq \left(\frac{1-c_1}{l}\right)(1-\delta)2^n\right) - 1 \\ &\geq \left(1 - e^{-c\epsilon(1-\delta)2^n}\right) + P\left(X_m \geq \left(\frac{1-c_1}{l}\right)(1-\delta)2^n\right) - 1 \\ &\geq \left(1 - e^{-c\epsilon(1-\delta)2^n}\right) + \left(1 - e^{I(c_1 - \frac{\epsilon}{200(\alpha+3)})(1-\delta)2^n/l}\right) - 1 \\ &\geq 1 - e^{-c(\epsilon)2^n}. \end{aligned} \quad (36)$$

Now the events $D_m,\ m = 1, 2, \ldots, \frac{\delta 2^n}{l}$, are independent so $\sum_{m=1}^R 1_{D_m^c}$ is stochastically bounded by a Bernoulli random variable with parameters R and $e^{-c(\epsilon)2^n}$. Thus, with $p = e^{-c(\epsilon)2^n}$ and $\Phi_p(\theta) = \theta \log \frac{\theta}{p} + (1-\theta)\log\frac{1-\theta}{1-p}$ and recalling $R = \frac{\delta 2^n}{l}$, we have by large deviations

$$\begin{aligned} P\left(\sum_{m=1}^R I_{D_m^c} \geq \frac{R}{10}\right) &\leq e^{-\left(\Phi_p(\frac{1}{10}) + o(1)\right)R} \\ &= \left[\left(\frac{1}{10p}\right)^{\frac{1}{10}}\left(\frac{9}{10(1-p)}\right)^{\frac{9}{10}}\right]^{-R} p^{\frac{R}{10}}(1-p)^{\frac{9R}{10}} e^{o(1)R} \\ &= \left[\left(\frac{1}{10}\right)^{\frac{1}{10}}\left(\frac{9}{10}\right)^{\frac{9}{10}}\right]^{-\frac{\delta 2^n}{l}} e^{-c(\epsilon)\frac{\delta 2^{2n}}{10l}}\left(1 - e^{-c(\epsilon)2^n}\right)^{\frac{9\delta 2^n}{10l}} e^{o(1)\frac{\delta 2^n}{l}} \\ &\leq e^{-c(\epsilon)2^{2n}}, \quad \text{for a new value of the constant } c(\epsilon). \end{aligned} \quad (37)$$

But the discussion above shows that $\{\sum_{m=1}^R I_{D_m^c} \leq \frac{R}{10}\}$ is a subset of

$$\begin{aligned} \Big\{ &|x \in [-\delta 2^n, \delta 2^n] : \exists \gamma_x \in \Gamma^x_{\delta 2^n, 2^n, k(1-\delta)2^n},\, V(\gamma_x) \\ &\geq (\alpha - \epsilon/5)\sqrt{k}\,(1-\delta)2^n| \geq \frac{9}{10}\delta 2^{n+1}\Big\} \end{aligned} \quad (38)$$

and Proposition 1.1 is proven for $d = 1$. The case of arbitrary d follows on noting that instead of "$R = \frac{\delta 2^n}{2l}$ disjoint intervals with $C_m,\ m = 1, 2, \ldots, R$ of side length l," we would have $R = (\frac{\delta 2^n}{2l})^d$ disjoint d-dimensional cubes with $C_m,\ m = 1, 2, \ldots, R$ of side length l. This changes the bound in (37) from $e^{-c(\epsilon)2^{2n}}$ to $e^{-c(\epsilon)2^{(d+1)n}}$. This gives the result in $d > 1$ dimensions and Proposition 1.1 is proven. □

We now sketch the proof of Proposition 1.2 drawing on results in [4] using the following steps. First we note that, since we are dealing with a fixed time t, we may consider

$$u'(0,t) = E_0[e^{\int_0^t dW_{X(s)}(s)}]$$

which is equal to $u(0,t)$ in distribution but which has better subadditive properties.

Secondly, we note that for all $\epsilon > 0$, with probability tending to 1 as ℓ tends to infinity

$$u'(0,\ell) \geq e^{(\lambda(\kappa)-\epsilon/10^6)\ell}$$

and then argue that for k not depending on ℓ.

$$P\left(E_0[e^{\int_0^\ell dW_{X(s)}(s)} I_{\{N(X,\ell)\leq k\ell\}}] \geq e^{(\lambda(\kappa)-\epsilon/10^5)\ell}\right) \geq 1-\epsilon/10^5$$

if ℓ is sufficiently large.

We then note that since there are only a polynomial number (in ℓ) of final positions, $X(l)$, for a path starting at 0 making at most $k\ell$ jumps, we have

$$P\left(\max_x E_0[e^{\int_0^\ell dW_{X(s)}(s)} I_{\{X(\ell)=x\}} I_{\{N(X,\ell)\leq k\ell\}}] \geq e^{(\lambda(\kappa)-\epsilon/10^4)\ell}\right) \geq 1-\ \epsilon^4/10^6$$

provided ℓ is large enough.

From this, in the one dimensional case for simplicity, we can use the FKG inequality to deduce that

$$P\left(\max_{x\geq 0} E_0[e^{\int_0^\ell dW_{X(s)}(s)} I_{\{X(\ell)=x\}} I_{\{N(X,\ell)\leq k\ell\}}] \geq e^{(\lambda(\kappa)-\epsilon/10^4)\ell}\right) \geq 1-2\epsilon^2/10^2.$$

and

$$P\left(\max_{x\leq 0} E_0[e^{\int_0^\ell dW_{X(s)}(s)} I_{\{X(\ell)=x\}} I_{\{N(X,\ell)\leq k\ell\}}] \geq e^{(\lambda(\kappa)-\epsilon/10^4)\ell}\right) \geq 1-2\epsilon^2/10^2.$$

for ℓ large.

Next we use the fact that for any $x = x_0, \ldots, x_{\frac{1}{\epsilon}}$ with probability exceeding $1 - \epsilon/10^4$, we have

$$
\begin{aligned}
&E_{x_0}[e^{\int_0^{\ell/\epsilon} dW_{X(s)(s)}} I_{\{|X(s)| \le 2k\ell,\, 0 \le s \le \ell/\epsilon\}}] \\
&\quad \ge \max_{x_1,\ldots,x_{1/\epsilon}} \prod_{i=1}^{1/\epsilon} E_{x_{i-1}}[e^{\int_0^{\ell} dW_{X(s)}((i-1)\ell+s)} I_{\{X(\ell)=x_i\}} I_{\{N(X,\ell)\le k,\ell\}}] \\
&\quad \ge e^{(\lambda(\kappa)-\epsilon/10^4)\ell/\epsilon}
\end{aligned}
$$

where the maximum is taken over $x_1, x_2, \ldots$ with $|x_i|, |x_i - x_{i-1}| \le k\ell,\ i = 1 \ldots 1/\epsilon$.

Given this we arrive at the following analogue of Lemma 2.1.

Lemma 3.2 *Fix* $\epsilon > 0$. *Then*

$$
\begin{aligned}
&\lim_{L\to\infty} P\Bigg(\forall x \in [0, L)^d, \sup_z E^x[e^{\int_0^L dW_{X(s)}(s)} I_{\{X(s)\in[0,L)^d, 0\le s\le L,\, X(L)=z\}}] \\
&\qquad \ge e^{(\lambda(\kappa)-\epsilon)L}\Bigg) = 1. \qquad (39)
\end{aligned}
$$

We can prove an analogue of Lemma 2.2 after which the path to Proposition 1.2 is entirely analogous to the proof of Proposition 1.1 given Lemmas 2.1 and 2.2.

This concludes the proof of Proposition 1.2.

4 Small times

Next we examine the influence of the field $\{W_x : x \in \mathbb{Z}^d\}$ for points x near the starting point 0 and small times. We seek paths that leave a neighborhood of the origin in a hurry so that they can avoid a potentially bad realization of the media $\{W_x : x \in \mathbb{Z}^d\}$ near the starting point. For the upper bound to the lower large deviation probabilities, $\overline{\lim}_{n\to\infty} \frac{1}{n^{d+1}} \log P(\frac{A_{n,n}}{n} \le \alpha - \epsilon) < 0$, we consider a basic path space

$$
\Gamma_{\delta 2^n} = \left\{\gamma : [0, \delta 2^n] \to \mathbb{Z}^d,\ \gamma(0) = 0\right\} \qquad (40)
$$

whose paths are right continuous with left limits and again all jumps of size one. Recall that for $\gamma \in \Gamma_{\delta 2^n}$ and for $I \subseteq [0, \delta 2^n]$,

$$
N(\gamma, I) = |\{s \in I : |\gamma(s) - \gamma(s-)| = 1\}|,
$$

denotes the number of jumps of γ on the interval I. Define, for a positive function $f(k, n)$ to be fixed later but having values not exceeding 1, the values a_k by $a_0 = 0$ and $a_{k+1} = a_k + 2^k f(k, n)$ and intervals

$$
I_k = [a_k, a_{k+1}) = [a_k, a_k + 2^k f(k, n)).
$$

Define subintervals of I_k by

$$I_{i,k} = [a_k + (i-1) f(k,n), a_k + i f(k,n)), \quad i = 1, 2, \ldots, 2^k. \tag{41}$$

Denote the interior of an interval I by I^o. We will restrict attention to paths in

$$\Gamma_{\delta 2^n, f} = \left\{ \gamma \in \Gamma_{\delta 2^n} : N(\gamma, I_{i,k}) = N(\gamma, I^o_{i,k}) \leq 1, k \geq 0, \quad i = 1, \ldots, 2^k \right\}. \tag{42}$$

We define, for $||x||_\infty \leq \frac{2^k}{4d}$, the random variables

$$Z^k_{x,i} = \sup\{W_x(t) - W_x(s) \colon s, t \in I_{i,k},\ s > t\},$$

which are i.i.d.

The following follows simply from the reflection principle bounds on the maximum and minimum of Brownian motion on an interval

Lemma 4.1 *For all $t > 0$,*

$$P\left(Z^k_{x,i} > t\right) \leq 4\left(1 - \Phi\left(\frac{t}{2\sqrt{f(k,n)}}\right)\right)$$

where Φ is the standard normal distribution function.

Remark 4.1 This is in fact a poor upper bound, see e.g. [16].

Consequently, for

$$W^k_{x,i} = \left(\sqrt{f(k,n)}\right)^{-1} Z^k_{x,i}$$

and M sufficiently large and all k, n, $W^k_{x,i} I_{W^k_{x,i} \geq M}$ is stochastically less than $V I_{V \geq M}$ for V a $\mathcal{N}(0, 8)$ random variable. Thus, we obtain

Lemma 4.2 *There exists finite K so that for c sufficiently small and all k, n, if $t \geq K 2^{k-n}$ then*

$$P\left(\sum_{||x||_\infty \leq \frac{2^k}{4d}, i=1,\ldots,2^k} 2^{-dk} W^k_{x,i} \geq 2^n t\right) \leq e^{-c 2^{2n-k(d-1)} t^2}.$$

Proof Using independence and Chebychev's inequality we have for $\lambda > 0$,

$$P\left(\sum_{|x|_\infty \le \frac{2^k}{4d}, i=1,\dots,2^k} 2^{-dk} W^k_{x,i} \ge 2^n t\right) \le e^{-\lambda 2^n t} E\left[\Pi_{||x||_\infty \le \frac{2^k}{4d}, i=1,\dots,2^k} e^{\lambda 2^{-dk} W^k_{x,i}}\right]$$
$$= e^{-\lambda 2^n t} E\left[e^{\lambda 2^{-dk} W^k_{x,1}}\right]^{(2^{k+1}+1)^{d+1}}$$
$$\le e^{-\lambda 2^n t + c'\lambda^2 2^{-2dk}(2^{k+1}+1)^{d+1}}, \text{ by Lemma 4.1}$$
$$\le e^{-c 2^{2n-k(d-1)} t^2}, \tag{43}$$

with a constant c, which depends only on d, by taking the optimal choice of

$$\lambda = 2^n t \frac{2^{2dk}}{2c'(2^{k+1}+1)^{d+1}}.$$

□

Remark 4.2 The appearance of $4d$ in the spatial restriction $||x||_\infty \le \frac{2^k}{4d}$ is due to our soon to be introduced random selection of path procedure. We will look at paths that begin in the box $\{x \in \mathbb{Z}^d : ||x||_\infty \le \frac{2^k}{4d}\}$ and finish in $\{x \in \mathbb{Z}^d : ||x||_\infty \le \frac{2^{k+1}}{4d}\}$ and take at most 2^k steps. The factor $4d$ insures the path can begin and end at arbitrary points in this box.

The following lemma is clear.

Lemma 4.3 *For any $\gamma \in \Gamma_{\delta 2^n, f}$, with $\sup_{s\in I_k} ||\gamma(s)||_\infty \le \frac{2^k}{4d}$, we have*

$$\int_{I_k} dW_{\gamma(s)}(s) \ge -\sum_{i=1}^{2^k}\left(Z^k_{\gamma(a_k+if(k,n)),i} + Z^k_{\gamma(a_k+(i-1)f(k,n)),i}\right)$$

We now specify a random selection procedure for paths $\gamma^k : I_k \to \mathbb{Z}^d$ as follows:

1. Pick $\gamma^k(a_k)$ uniformly on $\left\{x \in \mathbb{Z}^d : ||x||_\infty \le \frac{2^k}{4d}\right\}$.
2. Pick (independently of $\gamma^k(a_k)$) $\gamma^k(a_{k+1})$ uniformly on $\left\{x \in \mathbb{Z}^d : ||x||_\infty \le \frac{2^{k+1}}{4d}\right\}$.
3. Pick $\gamma^k(a_k + if(k,n)),\ i = 1, 2, \dots, 2^k$, as follows.

Let $n_j = ||\gamma^k(a_{k+1})_j - \gamma^k(a_k)_j||,\ j = 1, 2, \dots, d$, and let $e_1, \dots, e_d$ denote the unit basis vectors in $\mathbb{Z}^d$. For $1 \le i \le n_1$,

$$\gamma^k(a_k + if(k,n)) - \gamma^k(a_k + (i-1)f(k,n)) = e_1 \operatorname{sgn}(\gamma^k(a_{k+1})_1 - \gamma^k(a_k)_1),$$

for $n_1 + 1 \le i \le n_1 + n_2$,

$$\gamma^k(a_k + if(k,n)) - \gamma^k(a_k + (i-1)f(k,n)) = e_2 \operatorname{sgn}(\gamma^k(a_{k+1})_2 - \gamma^k(a_k)_2)$$

etc.

for $\sum_1^d n_i + 1 \leq i \leq 2^k$,

$$\gamma^k(a_k + if(k,n)) = \gamma^k(a_k + (i-1)f(k,n)).$$

Let $\tilde{P}$ denote the probability measure involved in this random selection of path which is independent of the field $\{W_x : x \in \mathbb{Z}^d\}$. The selection procedure yields the following result.

Lemma 4.4 *There is a constant $c = c_d$ so that $\forall (x,i) \in \{u : ||u||_\infty \leq \frac{2^{k+1}}{4d}\} \times \{0, 1, 2, \ldots, 2^k\}$*

$$\tilde{P}\left(x = \gamma^k(a_k + if(k,n))\right) \leq \frac{c_d}{2^{dk}}.$$

Proof The case $i = 0$ is clear so we take $i \geq 1$. Let $u = \gamma^k(a_k)$ and $v = \gamma^k(a_{k+1})$ be the initial and final points of γ^k. Let $\vec{n} = (n_1, \ldots, n_d)$ be the vector defined above by specifying $n_j = ||\gamma^k(a_{k+1})_j - \gamma^k(a_k)_j||_\infty$. For a given i, there are $d+1$ possibilities for the position of i relative to the sum of the n_j. Namely, for $x = \gamma^k(a_k + if(k,n))$, one of these two possibilities occurs:

$$\begin{aligned} &(1)\ \sum_{j=1}^{d} n_j < i \\ &(2)\ \text{for some } 1 \leq m \leq d,\ \sum_{j=1}^{m-1} n_j < i \leq \sum_{j=1}^{m} n_j \text{ with} \end{aligned} \tag{44}$$

$\sum_{j=1}^0$ taken to be 0.

The event $\{\sum_{j=1}^d n_j < i\}$ is contained in the event $\{v = x\}$, which is an event of probability bounded by $c2^{-kd}$ for some c.

On the other hand, for the event $\{\sum_{j=1}^{m-1} n_j < i \leq \sum_{j=1}^m n_j\}$ we must have that for $\ell = 1, 2, \ldots, m-1$, $x_\ell = v_\ell$ and for $\ell = m, m+1, \ldots, d$, $x_\ell = u_\ell$. In this case, we say that v, u are m-compatible with (x, i). Given $n_1, n_2, \ldots, n_{m-1}$, for all choices of n_m there are at most two choices of v_m, u_m so that $x = \gamma^k(a_n + if(k,n))$ with

$$\sum_{j=1}^{m-1} n_j < i \leq \sum_{j=1}^{m} n_j.$$

Thus we can choose the number of v, u that are m-compatible with (x, i) as follows

(1) choose u_1 and hence n_1 (there are at most 2^k choices)

(2) choose u_2 and hence n_2 (there are at most 2^k choices)

.

.

$(m-1)$ choose u_{m-1} and hence n_{m-1} (there are at most 2^k choices)
(m) choose n_m and $\mathrm{sgn}(u_m - v_m)$ (there are at most 2^k choices)
$(m+1)$ choose v_{m+1} and hence n_{m+1} (there are at most 2^k choices)
.
.
(3) choose v_d and hence n_d (there are at most 2^k choices).

Therefore, the lemma follows. □

An immediate consequence is:

Corollary 4.1 *For c_d from Lemma 4.4 and γ^k selected as above*

$$\tilde{E}\left[\sum_{i=1}^{2^k} Z^k_{\gamma^k(a_k+if(k,n)),i} + \sum_{i=1}^{2^k} Z^k_{\gamma^k(a_k+(i-1)f(k,n)),i}\right] \le 2c_d\, 2^{-dk} \sum_{||x||_\infty \le \frac{2^k}{4d}, 1\le i\le 2^k} Z^k_{x,i}$$

We now set

$$f(k,n) = (n-k)^{-3} 2^{-(d-1)(n-k)}, \quad 0 \le k \le n + \log\delta$$

and with $t_{n,k} = \left(\sqrt{f(k,n)(n-k)^3}\right)^{-1}$, define the event $A(k,n)$, $0 \le k \le n + \log\delta$ by

$$A(k,n) = \left\{2^{-dk} \sum_{||x||_\infty \le \frac{2^k}{4d}, 1\le i\le 2^k} W^k_{x,i} \le 2^n t_{n,k}\right\}.$$

Then by Lemma 4.2,

Corollary 4.2 *For n sufficiently large,*

$$P\left(\cap_{k=0}^{n+\log\delta} A(k,n)\right) \ge 1 - e^{-c2^{n(d+1)}}.$$

Furthermore, on $\cap_{k=0}^{n+\log\delta} A(k,n)$ we have for any $0 \le k \le n + \log\delta$,

$$\tilde{E}\left[\sum_{i=1}^{2^k} Z^k_{\gamma(a_k+if(k,n)),i} + Z^k_{\gamma(a_n+(i+1)f(k,n)),i}\right] \le 2c_d\sqrt{f(k,n)}\, 2^{-dk}$$
$$\times \sum_{||x||_\infty \le \frac{2^k}{4d}, 1\le i\le 2^k} W^k_{x,i} 2c_d\sqrt{f(k,n)} 2^n t_{n,k} \le 2c_d\, (n-k)^{-3/2}\, 2^n$$

and so we have

Corollary 4.3 *There exists positive constant C_d such that on the event $\cap_{k=0}^{n+\log\delta} A(k,n)$, there exists a path from 0 to x, call it γ_x, for $\frac{9}{10}$ of $x \in \{u : ||u||_\infty \leq \frac{2^n\delta}{4d}\}$ such that both*

(i)

$$\int_0^{a_{n+\log\delta}} dB_{\gamma_x(s)} \geq -C_d\, 2^n \left(\log\left(\frac{1}{\delta}\right)\right)^{-1/2}$$

and

(ii) $N(\gamma_x,\ [0, a_{n+\log\delta}]) \leq 2^n\delta.$

Proof From Lemma 4.3.

$$\int_0^{a_{n+\log\delta}} dB_{\gamma^k(s)} \geq -\sum_{k=0}^{n+\log\delta}\sum_{i=1}^{2^k}\left(Z^k_{\gamma^k(a_k+if(k,n)),i} + Z^k_{\gamma^k(a_k+(i-1)f(k,n)),i}\right). \quad (45)$$

By Corollary 4.1,

$$\tilde{E}\left[\sum_{i=1}^{2^k}\left(Z^k_{\gamma^k(a_k+if(k,n)),i} + Z^k_{\gamma^k(a_k+(i-1)f(k,n)),i}\right)\right] = 2c_d\, 2^{-dk}\sum_{i=1}^{2^k}\sum_{||x||_\infty\leq\frac{2^k}{4d}} Z^k_{x,i}.$$

So, by Chebychev,

$$\tilde{P}\left(\sum_{i=1}^{2^k}\left(Z^k_{\gamma^k(a_k+if(k,n)),i} + Z^k_{\gamma^k(a_k+(i-1)f(k,n)),i}\right) > 200\, c_d\, 2^{-dk}\sum_{i=1}^{2^k}\sum_{|x|_\infty\leq\frac{2^k}{4d}} Z^k_{x,i}\right) \leq \frac{1}{100}. \quad (46)$$

Consequently, for $\frac{9}{10}$ of the points x in $\{u : ||u||_\infty \leq \frac{2^k}{4d}\}$ there is a path $\gamma_{x,y} : I_k \to \mathbb{Z}^d$ for $\frac{9}{10}$ of the points y in $\{u : ||u||_\infty \leq \frac{2^k}{4d}\}$ for which $\gamma_{x,y}(a_k) = x,\ \gamma_{x,y}(a_{k+1}) = y$, $\sup_{s\in I_k} ||\gamma_{x,y}(s)||_\infty \leq \frac{2^k}{4d}$ and

$$\sum_{i=1}^{2^k}\left(Z^k_{\gamma_{x,y}(a_k+if(k,n)),i} + Z^k_{\gamma_{x,y}(a_k+(i-1)f(k,n)),i}\right) \leq 100\, c_d\, 2^{-dk}\sum_{i=1}^{2^k}\sum_{||x||_\infty\leq\frac{2^k}{4d}} Z^k_{x,i}. \quad (47)$$

By an induction argument we conclude that for $\frac{9}{10}$ of the points $x \in \{u : ||u||_\infty \leq \frac{\delta 2^n}{4d}\}$ we can select successive pairs (x_k, y_k), $k = 1, 2, \ldots, n + \log \delta$ with $x_{k+1} = y_k$ and concatenating the resulting paths, $\gamma_k \equiv \gamma_{x_k, y_k}$ to obtain a path $\gamma \in \Gamma_{\delta 2^n, f}$ satisfying $\gamma(\delta 2^n) = x$, $\sup_{s \in [0, \delta 2^n]} ||\gamma(s)||_\infty \leq \delta 2^n$, $N(\gamma, [0, \delta 2^n]) \leq \delta 2^n$ and

$$\sum_{k=0}^{n+\log\delta} \sum_{i=1}^{2^k} \left(Z^k_{\gamma(a_k + i f(k,n)), i} + Z^k_{\gamma(a_k + (i-1) f(k,n)), i} \right)$$
$$\leq 200\, c_d\, 2^{-dk} \sum_{k=0}^{n+\log\delta} \sum_{||x||_\infty \leq \frac{2^k}{4d}} Z^k_{x,i}. \tag{48}$$

Thus, by (45) and (48) we have

$$\int_0^{a_{n+\log\delta}} dB_{\gamma^k(s)} \geq - \sum_{k=0}^{n+\log\delta} \sum_{i=1}^{2^k} \left(Z^k_{\gamma^k(a_k + i f(k,n)), i} + Z^k_{\gamma^k(a_k + (i-1) f(k,n)), i} \right)$$
$$\geq -100 c_d \sum_{k=1}^{n+\log\delta} \sum_{i=1}^{2^k} \sum_{||x||_\infty \leq \frac{2^k}{4d}} 2^{-dk} Z^k_{x,i}. \tag{49}$$

Observe that on $\cap_{k=0}^{n-\log(\frac{1}{\delta})} A(k,n)$

$$200 c_d \sum_{k=0}^{n+\log\delta} \sum_{i=1}^{2^k} \sum_{||x||_\infty \leq \frac{2^k}{4d}} 2^{-dk} Z^k_{x,i} \leq 200 c_d \sum_{k=0}^{n+\log\delta} \left(\sqrt{n-k}\right)^{-3/2} 2^n$$
$$\leq C_d \left(\log \frac{1}{\delta}\right)^{-1/2} 2^n \tag{50}$$

for a universal C_d. By (49) and (50), on the event $\cap_{k=0}^{n-\log\frac{1}{\delta}} A(k,n)$

$$\int_0^{a_{n+\log\delta}} dB_{\gamma^k(s)} \geq C_d \left(\log \frac{1}{\delta}\right)^{-1/2} 2^n.$$

This concludes the proof. $\square$

The developments to this point will now be used to prove Theorem 0.1.

Proof We first prove $\overline{\lim_{n\to\infty}} \frac{1}{n^{d+1}} \log P(A_{n,n} < (\alpha - \epsilon) n) < 0$. By Corollarys 4.2, 4.3 and Proposition 1.1 we can, with probability at least $1 - 3e^{-c 2^{n(d+1)}}$ select a common point x in the intersection of the two sets involved in those results which occupy $\frac{9}{10}$ of the points in $[-\delta 2^n, \delta 2^n]^d$ together with a path $\gamma \in \Gamma_{0,2^n,2^n}$ such that

(i)

$$\gamma(t) = x,\ a_{n+\log\delta} \le t \le \delta 2^n,$$

(ii)

$$\int\limits_0^{a_{n+\log\delta}} dB_{\gamma_x(s)}(s) \ge -\frac{C_d 2^n}{(\log\frac{1}{\delta})^{1/2}},$$

(iii)

$$B_x(\delta 2^n) - B_x(a_{n+\log\delta}) \ge -\frac{\epsilon}{100}(\delta 2^n - a_{n+\log\delta}),$$

(iv)

$$\int\limits_{\delta 2^n}^{2^n} dB_{\gamma_x(s)}(s) \ge (\alpha-\epsilon)(1-\delta)2^n.$$

Thus, with probability at least $1 - 3e^{-c2^{n(d+1)}}$ we find a $\gamma \in \Gamma_{0,2^n,2^n}$ for which

$$\begin{aligned} V(\gamma) &\ge -\frac{C_d 2^n}{(\log(\frac{1}{\delta}))^{1/2}} - \frac{\epsilon}{100}(\delta 2^n - a_{n+\log\delta}) + (\alpha-\epsilon)(1-\delta)2^n \\ &\ge (\alpha - 2\epsilon)2^n. \end{aligned} \tag{51}$$

That proves the upper bound. We now prove the lower bound

$$\underline{\lim}_{n\to\infty}\frac{1}{n^{d+1}}\log P\left(A_{n,n} < (\alpha-\epsilon)n\right) > -\infty. \tag{52}$$

For the Brownian field, $\{W_x : x \in \mathbb{Z}^d\}$ there exists a natural decomposition

$$W_x(t) = B_x^n(t) + \frac{t}{n}W_x(n),\ 0 \le t \le n,$$

where $\{B_x^n : x \in \mathbb{Z}^d\}$ are independent Brownian bridges over the time interval $[0, n]$ independent of the Gaussian field $\{W_x(n) : x \in \mathbb{Z}^d\}$. In the following we use this decomposition $\Lambda_n = \{x \in \mathbb{Z}^d : ||x||_\infty \le n\}$. Now we have for a vector $\underline{u} \in \mathbb{R}^{(2n+1)^d}$, $|\Lambda_n| = (2n+1)^d$, the function $\omega \to F(\underline{u}, \omega)$ is measurable with respect to $\sigma\{B_x^n : ||x||_\infty \le n\}$ given by $F(\underline{u}, \omega) = A_{n,n}$ for the trajectories

$$W_x^{\underline{u}}(t) = B_x^n(t) + \frac{t}{n}\underline{u}(x),$$

that is

$$F(\underline{u},\omega) = \sup_{\gamma\in\Gamma^0_{0,n,n}} \int_0^n dW^{\underline{u}}_{\gamma(t)}(t).$$

So of course $F(W_\cdot(n),\omega)$ gives the original functional $A_{n,n}$ where we take $W_\cdot(n)$ to be the vector $(W_x(n))_{||x||_\infty\le n}$. We have the following obvious bound:

$$\|F(\underline{u},\omega) - F(\underline{u}',\omega)\|_\infty \;\le\; \|\underline{u}-\underline{u}'\|_\infty. \tag{53}$$

Also, for each ω, $\underline{u}\to F(\underline{u},\omega)$ is a nondecreasing function of the components of $\underline{u}$. Monotonicity of F by components of $\underline{u}$ and (53) give that, with probability at least 1/2, for large n

$$F(\underline{n}^{2/3},\omega)\le(\alpha+\epsilon)n$$

where $\underline{n}^{2/3}$ is the vector each of whose components equals $n^{2/3}$. Now define the event

$$A_\epsilon(n) = \{W_x(n)\le -2\epsilon n, \forall x\in\Lambda_n\}\cap\{F(\underline{n}^{2/3},\omega)\le(\alpha+\epsilon)n\}. \tag{54}$$

Then we have easily that for n large

- $P(A_\epsilon(n))\ge\frac{1}{2}e^{-\epsilon^2 n^{d+1}c_d}$ for a positive constant c_d not depending on n and
- on the event $A_\epsilon(n)$, $A_{n,n}\le F(\underline{n}^{2/3},w) - 2\epsilon n - n^{2/3}\le(\alpha-\epsilon)n$.

This proves the lower bound (52) and the proof of (8) of Theorem 0.1 is complete. The proof of (7) is relatively easy. By Borell's inequality (see ([4] for a justification of its use in this context), we have $\lim_{n\to\infty}\frac{1}{n}E[A_{n,n}]=\alpha$ and there exist K, c' such that

$$P(A_{n,n}-E[A_{n,n}]\ge an)\le Ke^{-c'a^2n}.$$

Thus,

$$\begin{aligned}P(A_{n,n}\ge(\alpha+\epsilon)n) &= P(A_{n,n}-E[A_{n,n}]\ge(\alpha+\epsilon-E[A_{n,n}]/n)n)\\ &\le Ke^{-c'\epsilon^2 n},\end{aligned} \tag{55}$$

with some positive c'. This implies the upper half of (7). On the other hand, if one takes the path $\gamma(s)\equiv 0$, then we easily see that

$$\begin{aligned}P(A_{n,n} &\ge(\alpha+\epsilon)n)\ge P(W_0(n)\ge(\alpha+\epsilon)n)\\ &\ge e^{-cn},\end{aligned} \tag{56}$$

for some positive c, which gives the proof of the lower estimate in (7). This completes the proof of Theorem 0.1. □

We now use the previous results to deal with the large deviations for the solution to the parabolic Anderson model (1). We first remark that the distribution of $u(x,t)$ is independent of x. Also,

$$u(x,t) \stackrel{\mathcal{L}}{=} E_x\left[e^{\int_0^t dW_{X(s)}(s)}\right].$$

As remarked earlier, the right hand side is more convenient for applications of the results on $A_{n,n}$, (time is running in the same direction in X and in W.) Therefore in our arguments we shall use $E_0[e^{\int_0^t dW_{X(s)}(s)}]$ instead of $u(x,t)$. We now extend this approach to solutions of (1). The point is that the paths γ_x created in the proof of Theorem 0.1, are chosen randomly and uniformly: their law is thus absolutely continuous with respect to that of a random walk. The arguments for large deviations for $A_{n,n}$ tell us that outside of a set of probability $e^{-c(\epsilon,\delta)2^{n(d+1)}}$ for 9/10 of $x \in \Lambda_{\frac{2^n\delta}{4d}}$ there exists a sequence of values in $\mathbb{Z}^d$

$$\gamma_x(a_k + if(k,n)) \quad i = 1, 2, \ldots, 2^k \quad k = 0, 1, \ldots, n + \log\delta$$

with $|\gamma_x(a_k + (i+1)f(k,n)) - \gamma_x(a_k + if(k,n))|_1 \le 1$ for all i and k, so that for any curve $\gamma_x : [0, a_{n+\log\delta}] \to \mathbb{Z}^d$, if $\gamma_x(a_k + if(k,n)) = \gamma_x(a_k + (i+1)f(k,n))$ then for all $s \in [a_n + if(k,n), a_k + (i+1)f(k,n)]$,

$$\gamma_x(s) \equiv \gamma_x(a_n + if(k,n)),$$

and if

$$|\gamma_x(a_k + if(k,n)) - \gamma_x(a_n + (i+1)f(k,n))|_1 = 1$$

then γ_x makes exactly one jump on $[a_n + if(k,n), a_n + (i+1)f(k,n)]$, necessarily from $\gamma_x(a_k + if(k,n))$ to $\gamma_x(a_k + (i+1)f(k,n))$. Then

$$\int_0^{a_{n+\log\delta}} dB_{\gamma_x(s)}(s) \ge -C_d\left(\log\frac{1}{\delta}\right)^{-1/2} 2^n.$$

Now take random γ_x (for suitable x) according to the above recipe with (in the case of $\gamma_x(a_k + if(k,n)) \ne \gamma_x(a_k + (i+1)f(k,n))$ the jump time in the interval $[a_k + if(k,n), a_n + (i+1)f(k,n)]$ chosen uniformly on this interval independently of other jumps. The random walk probability of such a path is

$$p = \prod_{k=0}^{n+\log\delta} \prod_{j=1}^{2^k} (e^{-\kappa f(k,n)}\Theta_{kj} + e^{-\kappa f(k,n)}\frac{\kappa f(k,n)}{2d}(1 - \Theta_{kj}))$$

where $\Theta_{kj} = 1$ if and only if there is no jump on interval $[a_k + jf(k,n), a_k + (j+1) f(k,n)]$ and otherwise is equal to 0. Then, if δ is fixed small,

$$p = e^{-\kappa \sum_{k=0}^{n+\log\delta} 2^k f(k,n)} \prod_{k=0}^{n+\log\delta} \prod_{j=1}^{2^k} \left(\Theta_{kj} + (1-\Theta_{k,j}) \frac{\kappa f(k,n)}{2d} \right)$$
$$\geq e^{-\kappa 2^n n\delta} \prod_{k=0}^{n+\log\delta} e^{\log\left(\frac{\kappa f(k,n)}{2d}\right) 2^k} \geq e^{-2^n \epsilon/100}.$$

Thus, we have that outside the event $\cap_{k=1}^{n+\log\delta} A(k,n)$, for 9/10 of $x \in \Lambda_{\frac{2^n\delta}{4d}}$,

$$E_0 \left[e^{\int_0^{a_{n+\log\delta}} dW_{X(s)}(s)} \mid X(a_{n+\log\delta}) = x \right] \geq e^{-\frac{\epsilon}{100} - C_d \delta / \sqrt{\log(\frac{1}{\delta})} 2^n}.$$

Staying at x, obviously gives

$$E_x \left[e^{\int_{a_{n+\log\delta}}^{\delta 2^n} dW_{X(s)}(s)} \right] \geq E_x \left[e^{\int_{a_{n+\log\delta}}^{\delta 2^n} dW_{X(s)}(s)} 1_{N(X,t)=0} \right]$$

and for any $r > 0$,

$$\begin{aligned} &P \left(E_x \left[e^{\int_{a_{n+\log\delta}}^{\delta 2^n} dW_{X(s)}(s)} 1_{N(X,t)=0} \right] \geq e^{\epsilon(\delta 2^n - a_{n+\log\delta})} \right) \\ &\leq e^{-r\epsilon(\delta 2^n - a_{n+\log\delta})} \\ &\quad \times E_x \left[E \left[e^{r \int_{a_{n+\log\delta}}^{\delta 2^n} dW_{X(s)}(s)} \right] \right] \\ &= e^{(\frac{r^2}{2} - r\epsilon)(\delta 2^n - a_{n+\log\delta})} \\ &\leq e^{-\frac{\epsilon^2}{2}(\delta 2^n - a_{n+\log\delta})}. \end{aligned} \tag{57}$$

But applying Proposition 1.2, we have for 9/10 of $x \in \Lambda_{\frac{2^n\delta}{4d}}$,

$$E_x \left[e^{\int_{\delta 2^n}^{2^n} dW_{X(s)}(s)} \right] \geq e^{(\lambda(\kappa) - \epsilon/3)(2^n - \delta 2^n)}.$$

Thus, there is an x with

$$\begin{aligned} E_0 \left[e^{\int_0^{2^n} dW_{X(s)}(s)} \right] &\geq E_0 \left[e^{\int_0^{a_{n+\log\delta}} dW_{X(s)}(s)} \mid X(a_{n+\log\delta}) = x \right] \\ &\quad \times E_x \left[e^{\int_{a_{n+\log\delta}}^{\delta 2^n} dW_{X(s)}(s)} 1_{N(X,t)=0} \right] E_x \left[e^{\int_{\delta 2^n}^{2^n} dW_{X(s)})(s)} \right] \\ &\geq e^{-2^n \epsilon/100 - C_d \delta/(\log\frac{1}{\delta})^{1/2} 2^n + (\lambda(\kappa) - \epsilon/3)(2^n - a_n)} \\ &\geq e^{-(\lambda(\kappa) - \epsilon)2^n} \end{aligned}$$

provided δ is chosen small enough. We next show that the large deviation can be achieved in this order of probability. We can argue similarly to the above that $\forall m,\ \exists c_m < \infty$ so that for all large n

$$P\left(E_0\left[e^{\int_0^t dW_{X(s)}(s)} I_{\{|X(s)|_1 \le mt,\ \forall 0 \le s \le t\}}\right] \le e^{(\lambda(\kappa)-2\epsilon)t}\right) \ge e^{-c_m \epsilon^{2d} t^{(d+1)}}.$$

This will suffice since $\exists m_0$ so that

$$P\left(E_0\left[e^{\int_0^t dW_{X(s)}(s)} I_{\{\exists 0 \le s \le t : |X(s)| \ge m_0 t\}}\right] \le e^{(\lambda(\kappa)-2\epsilon)t}\right) \ge 1/2.$$

We now make appeal to the FKG inequality applied to the positively correlated events

$$A = \left\{E_0\left[e^{\int_0^t dW_{X(s)(s)}} I_{\{|X(s)|_1 \le mt,\ \forall 0 \le s \le t\}}\right] \le e^{(\lambda(\kappa)-2\epsilon)t}\right\}$$

and

$$B = \left\{E_0\left[e^{\int_0^t dW_{X(s)}(s)} I_{\{\exists 0 \le s \le t : |X(s)| \ge m_0 t\}}\right] \le e^{(\lambda(\kappa)-2\epsilon)t}\right\}$$

to conclude

$$P\left(u(t,0) \le 2e^{(\lambda(\kappa)-2\epsilon)t}\right) \ge \frac{1}{2} e^{-c_{m_0} \epsilon^{2d} t^{(d+1)}}$$

for t large. Of course, for t large,

$$2e^{(\lambda(\kappa)-2\epsilon)t} \le e^{(\lambda(\kappa)-\epsilon)t}$$

and we are done with the proof of (9).

Again the proof of (10) is easier. Observe that

$$E_0\left[e^{\int_0^t dW_{X(s)}(s)}\right] \ge e^{-\kappa t + W_0(t)}$$

and so there is a positive constant c such that

$$\begin{aligned} P\left(E_0\left[e^{\int_0^t dW_{X(s)}(s)}\right] \ge e^{(\lambda(\kappa)+\epsilon)t}\right) &\ge P\left(e^{-\kappa t + W_0(t)} \ge e^{(\lambda(\kappa)+\epsilon)t}\right) \\ &\ge e^{-ct}. \end{aligned} \tag{58}$$

Thus,

$$\varliminf_{t\to\infty} \frac{1}{t} P\left(\frac{\log u(0,t))}{t} \ge \lambda(\kappa) + \epsilon\right) > -\infty.$$

For the other direction, if $r > 0$,

$$
\begin{aligned}
P\left(E_0\left[e^{\int_0^t dW_{X(s)}(s)}\right] \geq e^{(\lambda+\epsilon)t}\right) &\leq e^{-r(\lambda(\kappa)+\epsilon)t} E\left[E_0\left[e^{r\int_0^t dW_{X(s)}(s)}\right]\right] \\
&= e^{-r(\lambda(\kappa)+\epsilon)t} E_0\left[E\left[e^{r\int_0^t dW_{X(s)}(s)}\right]\right] \\
&= e^{-(\lambda(\kappa)+\epsilon)rt+\frac{r^2}{2}t},
\end{aligned}
\tag{59}
$$

which gives, on selecting r small,

$$
\overline{\lim_{t\to\infty}} \frac{1}{t} P\left(\frac{\log u(0,t)}{t} \geq \lambda(\kappa) + \epsilon\right) < 0.
$$

That completes the proof of Theorem 0.2.

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