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Evolutionary p-Laplacian with convection and reaction under dynamic boundary condition

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Abstract

We study the global existence and blow-up phenomenon of solutions to an evolutionary p-Laplacian with convection and reaction under dynamic boundary condition.

Keywords: dynamic boundary condition; blow up; p-Laplacian; reaction-diffusion

1 Introduction

In this paper, we are concerned with the following evolutionary p-Laplacian under dynamic boundary condition:

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \vec{g}(u) \cdot \nabla u + f(u), \quad x \in \Omega, t > 0, \tag{1.1}$$

$$\sigma u_t + |\nabla u|^{p-2}\nabla u \cdot \nu = 0, \quad x \in \partial\Omega, t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.3}$$

where $p > 1$, $\vec{g} : \mathbb{R} \to \mathbb{R}^N$, $f : \mathbb{R} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, and $\nu : \partial\Omega \to \mathbb{R}^N$ is the outer unit normal vector.

The quasilinear parabolic problems with dynamic boundary conditions of type (1.1)-(1.3) arise in numerous areas such as heat conduction, chemical reactor theory, colloid chemistry and population growth, see [1, 2] and the references therein. Many reaction-diffusion equations under dynamic boundary conditions have been considered in the past years. An early study of problem (1.1)-(1.3) with $p = 2$ and $\vec{g} = \vec{0}$ was carried out by Below and Mailly [3] who showed a complete result about the blow-up phenomena as well as the lower and upper bounds for the blow-up time. Moreover, some of the techniques were also applied to the porous medium equation with reaction. Later on, for the evolutionary p-Laplacian with $p \geq 2N/(N + 2)$, where N is the dimension of the domain, Gal and Warma [4] considered the following equation without convection:

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + f(u) = g(x), \quad x \in \Omega, t > 0,$$

coupled with dynamic boundary conditions. The well-posedness and the existence of a global attractor results were established. More recently, Mailly and Rault [2] studied the nonlinear convection problem (1.1)-(1.3) with $p = 2$ and proved the global existence and blow-up phenomena of local solutions. For other results about the solvability of quasilinear parabolic equations with dynamic boundary conditions, we refer the readers to [5–7], *etc.*

Throughout this paper, we suppose that the dissipativity condition holds

$$\sigma > 0, \quad \sigma \in C^1\big(\partial\Omega \times [0, +\infty)\big), \tag{1.4}$$

and the functions in problem (1.1)-(1.3) are smooth

$$f \in C^1(\mathbb{R}), \qquad f(s) \geq 0 \quad \text{for } s \geq 0, \qquad \vec{g} \in C^1\big(\mathbb{R}, \mathbb{R}^N\big), \tag{1.5}$$

the initial data is non-negative and satisfies

$$u_0 \geq 0, \quad u_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega). \tag{1.6}$$

In Section 2 we develop the comparison principle for a regularized problem and the local existence of weak and strong solutions of problem (1.1)-(1.3). In Section 3 we derive the global existence of the strong solutions, while in Section 4 we prove the blow-up phenomenon of strong solutions by formulating a family of radially symmetric lower solutions.

2 Comparison principle and local existence

In this section, we use the regularization method and compactness theorems to prove the local existence of the solutions to problem (1.1)-(1.3).

Consider the following regularized problem:

$$\frac{\partial u}{\partial t} = \text{div}\left(\left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u\right) - \vec{g}(u_+) \cdot \nabla u + f_M(u), \quad x \in \Omega, t > 0, \tag{2.1}$$

$$\sigma u_t + \left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u \cdot \nu = 0, \quad x \in \partial\Omega, t > 0, \tag{2.2}$$

$$u(x, 0) = u_{0,n}(x), \quad x \in \Omega, \tag{2.3}$$

where $f_M(s) = \min\{f_+(s), M\}$, $s_+ = \max\{s, 0\}$, $M > 0$, $n \in \mathbb{Z}^+$, $u_{0,n} \in C^\infty(\overline{\Omega})$ satisfies

$$\inf_\Omega u_0 \leq u_{0,n} \leq \sup_\Omega u_0, \qquad \|u_{0,n}\|_{W^{1,p}} \leq 2\|u_0\|_{W^{1,p}}, \qquad \lim_{n\to\infty} \|u_{0,n} - u_0\|_{W^{1,p}} = 0.$$

Since $f, \vec{g} \in C^1$, we can verify that $f_M, \vec{g}(s_+)$ are locally Lipschitz continuous.

Hereafter, we suppose that the regularized problem (2.1)-(2.3) has a classical solution $u_{n,M} \in C^{2,1}(\overline{\Omega} \times [0, T_{n,M}))$ with the maximal existence time $0 < T_{n,M} \leq +\infty$. Let $Q_T = \overline{\Omega} \times$

(0, T) for $T > 0$ and define

$$F_{n,M}[u] = F_{n,M}(u, \nabla u) = \operatorname{div}\left(\left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u\right) - \vec{g}(u_+) \cdot \nabla u + f_M(u),$$

$$B_n[u] = \sigma u_t + \left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u \cdot \nu.$$

Notice that, in the dynamic boundary condition, $B_n[u]$ is nonlinear with respect to ∇u. First, we need the following comparison principles which are simple variations of the comparison principles in [8].

Lemma 2.1 *Let* $u, v \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ *satisfying*

$$u_t - F_{n,M}[u] > v_t - F_{n,M}[v], \quad (x, t) \in Q_T,$$

$$B_n[u] > B_n[v], \quad (x, t) \in \partial\Omega \times (0, T),$$

$$u(x, 0) > v(x, 0), \quad x \in \overline{\Omega}.$$

Then

$$u(x, t) > v(x, t), \quad (x, t) \in Q_T.$$

Proof Suppose that there exists $(x_0, t_0) \in Q_T$ such that $u(x_0, t_0) \le v(x_0, t_0)$. Let

$$t^* = \sup\left\{\tau \in (0, T); u(x, t) > v(x, t), \forall (x, t) \in Q_\tau\right\}.$$

Then $t^* \in (0, t_0] \subset (0, T)$ and $\min_{\overline{Q}_{t^*}}\{u - v\} = 0$. Thus, $u - v$ attains its minimum 0 at some point (x^*, t^*) with $x^* \in \overline{\Omega}$. If $x^* \in \Omega$, then

$$u = v, \qquad u_t \le v_t, \qquad \nabla u = \nabla v, \qquad D^2 u \ge D^2 v \quad \text{at } (x^*, t^*),$$

which contradicts $u_t - F_{n,M}[u] > v_t - F_{n,M}[v]$. If $x^* \in \partial\Omega$, then

$$u_t \le v_t, \qquad \frac{\partial u}{\partial \nu} \le \frac{\partial v}{\partial \nu}, \qquad \frac{\partial u}{\partial \mu_i} = \frac{\partial v}{\partial \mu_i} \quad \text{at } (x^*, t^*),$$

where $\frac{\partial}{\partial \mu_i}, i = 1, 2, \ldots, N-1$, are the tangential derivatives in the local coordinates at (x^*, t^*). We can verify that

$$\left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u \cdot \nu = \left(\frac{1}{n} + \sum_{i=1}^{N-1}\left|\frac{\partial u}{\partial \mu_i}\right|^2 + \left|\frac{\partial u}{\partial \nu}\right|^2\right)^{\frac{p-2}{2}} \frac{\partial u}{\partial \nu},$$

which is increasing with respect to $\frac{\partial u}{\partial \nu}$ since $p > 1$. Therefore, $B_n[u] \le B_n[v]$. We arrive at another contradiction. $\square$

Using Lemma 2.1, we can prove the following comparison principle, which is similar to Theorem 2.2 in [8], but without the global one-side Lipschitz condition.

Lemma 2.2 *Let* $u, v \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ *satisfying*

$$u_t - F_{n,M}[u] \geq v_t - F_{n,M}[v], \quad (x,t) \in Q_T,$$

$$B_n[u] \geq B_n[v], \quad (x,t) \in \partial\Omega \times (0,T),$$

$$u(x,0) \geq v(x,0), \quad x \in \overline{\Omega}.$$

Then

$$u(x,t) \geq v(x,t), \quad (x,t) \in Q_T.$$

Proof For any given $T > 0$, $\varepsilon > 0$, since $u, v \in C(\overline{\Omega} \times [0,T])$, by the continuities and $u(x,0) \geq v(x,0)$, there exists $\delta = \delta_\varepsilon > 0$ such that

$$u(x,t) > v(x,t) - \varepsilon, \quad x \in \overline{\Omega}, t \in (0,\delta].$$

Notice that $v \in C^{2,1}(\overline{\Omega} \times [\delta, T-\varepsilon])$, $v_+(x,t) \in [0, \max_{\overline{Q}_T} v]$, and $\vec{g} \in C^1([0, \max_{\overline{Q}_T} v])$. There exists a constant $K > 0$ such that

$$\left| \vec{g}\big((v-s)_+\big) \cdot \nabla v - \vec{g}(v_+) \cdot \nabla v \right| \leq \sup_{\overline{\Omega} \times [\delta, T-\varepsilon]} |\nabla v| \cdot \left| \vec{g}\big((v-s)_+\big) - \vec{g}(v_+) \right| \leq \frac{K}{2}s,$$

$$\left| f_M(v-s) - f_M(v) \right| \leq \frac{K}{2}s, \quad s \geq 0.$$

Define $\varphi = v - \varepsilon e^{(K+1)(t-\delta)}$. Thus,

$$\varphi_t - F_{n,M}[\varphi] \leq v_t - (K+1)\varepsilon e^{(K+1)(t-\delta)} - F_{n,M}[v] + K\varepsilon e^{(K+1)(t-\delta)}$$

$$< v_t - F_{n,M}[v] \leq u_t - F_{n,M}[u], \quad (x,t) \in \Omega \times [\delta, T-\varepsilon],$$

$$B_n[\varphi] = B_n[v] - (K+1)\sigma \varepsilon e^{(K+1)(t-\delta)} < B_n[v] \leq B_n[u], \quad (x,t) \in \partial\Omega \times (\delta, T-\varepsilon),$$

$$\varphi(x,\delta) = v(x,\delta) - \varepsilon < u(x,\delta), \quad x \in \overline{\Omega}.$$

Lemma 2.1 implies $u(x,t) \geq \varphi(x,t)$ for $(x,t) \in \Omega \times [\delta_\varepsilon, T-\varepsilon]$. Therefore, $u(x,t) \geq \min\{v(x,t) - \varepsilon, v(x,t) - \varepsilon e^{(K+1)(t-\delta_\varepsilon)}\}$ for $(x,t) \in \Omega \times (0, T-\varepsilon)$. By the arbitrariness of $\varepsilon > 0$, we deduce $u(x,t) \geq v(x,t)$ for $(x,t) \in \Omega \times (0,T)$. □

Lemma 2.3 *There exists at most one classical solution of problem* (2.1)-(2.3).

Proof Lemma 2.2 yields the uniqueness of classical solutions of problem (2.1)-(2.3). □

Lemma 2.4 *The solution* $u_{n,M}$ *of problem* (2.1)-(2.3) *satisfies*

$$\inf_{\Omega} u_0 \leq u_{n,M}(x,t) \leq \sup_{\Omega} u_0 + Mt, \quad (x,t) \in \overline{\Omega} \times (0, T_{n,M}). \tag{2.4}$$

Thus, the maximal existence time $T_{n,M} = +\infty$.

Proof For any given $T \in (0, T_{n,M})$ and $\varepsilon > 0$, define $\underline{u}_\varepsilon = \inf_\Omega u_0 - \varepsilon - \varepsilon t$, $\overline{u}_\varepsilon = \sup_\Omega u_0 + \varepsilon + (M + \varepsilon)t$. Then

$$\frac{\partial u_{n,M}}{\partial t} - F_{n,M}[u_{n,M}] = 0 > -\varepsilon \geq \frac{\partial \underline{u}_\varepsilon}{\partial t} - F_{n,M}[\underline{u}_\varepsilon],$$

$$B_n[u_{n,M}] = 0 > -\sigma\varepsilon = B_n[\underline{u}_\varepsilon],$$

$$u_{n,M}(x,0) = u_{0,n}(x) > u_0(x) - \varepsilon \geq \underline{u}_\varepsilon.$$

Lemma 2.1 implies $u_{n,M} \geq \underline{u}_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $u_{n,M} \geq \inf_\Omega u_0$. The proof of $u_{n,M} \leq \sup_\Omega u_0 + Mt$ follows similarly. □

Lemma 2.5 *For $M_1 \geq M_2 > 0$, there holds*

$$u_{n,M_1} \geq u_{n,M_2}, \quad x \in \Omega, t > 0.$$

Proof For any given $T > 0$, we see that $u_{n,M_1}, u_{n,M_2} \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ and $f_{M_1}(s) \geq f_{M_2}(s)$ for $s \in \mathbb{R}$. Thus,

$$\frac{\partial u_{n,M_1}}{\partial t} - F_{n,M_2}[u_{n,M_1}] \geq \frac{\partial u_{n,M_1}}{\partial t} - F_{n,M_1}[u_{n,M_1}] = 0 = \frac{\partial u_{n,M_2}}{\partial t} - F_{n,M_2}[u_{n,M_2}].$$

Using Lemma 2.2, we complete this proof. □

Lemma 2.6 *There exist constants $\delta_0, M_0 > 0$ independent of n, M such that*

$$\sup_{n,M} u_{n,M}(x,t) \leq \sup_\Omega u_0 + M_0\delta_0, \quad x \in \overline{\Omega}, t \in (0, \delta_0].$$

Proof Let $\underline{u}_0 = \inf_\Omega u_0$ and $\overline{u}_0 = \sup_\Omega u_0$. Set $M_0 = 2\max_{s \in \{\underline{u}_0, \overline{u}_0\}} f(s) + 1$ and define

$$h(t) = \max_{\underline{u}_0 \leq s \leq \overline{u}_0 + M_0 t} |f(s)|.$$

Since $f \in C^1([\underline{u}_0, \overline{u}_0 + M_0])$, we see that h is Lipschitz continuous on $[0,1]$ and $h(0) = \max_{s \in \{\underline{u}_0, \overline{u}_0\}} f(s) < M_0$. Thus, there exists a constant $0 < \delta_0 < 1$ such that $h(t) < M_0$ for all $t \in [0, \delta_0]$. By Lemma 2.4, $u_{n,M_0} \in [\underline{u}_0, \overline{u}_0 + M_0 t]$. Therefore,

$$f(u_{n,M_0}(x,t)) \leq h(t) < M_0, \quad x \in \overline{\Omega}, t \in [0, \delta_0], \tag{2.5}$$

and

$$f_{M_0}(u_{n,M_0}(x,t)) = \min\{f(u_{n,M_0}(x,t)), M_0\} = f(u_{n,M_0}(x,t)), \quad (x,t) \in \overline{\Omega} \times (0, \delta_0].$$

If $M' \leq M_0$, Lemma 2.5 implies

$$u_{n,M'} \leq u_{n,M_0} \leq \sup_\Omega u_0 + M_0\delta_0, \quad (x,t) \in \overline{\Omega} \times (0, \delta_0].$$

If $M' > M_0$, since $u_{n,M'} \in C(\overline{Q}_{\delta_0})$ and $u_{n,M'}(x,0) = u_{0,n}(x) \in [\underline{u}_0, \overline{u}_0]$, we have

$$f(u_{n,M'}(x,0)) \leq h(0) < M_0, \quad x \in \overline{\Omega},$$

and there exists a constant $\delta_{M'} > 0$ such that

$$f\left(u_{n,M'}(x,t)\right) < M_0, \quad (x,t) \in \overline{\Omega} \times (0, \delta_{M'}]. \tag{2.6}$$

Thus,

$$f_{M_0}\left(u_{n,M'}(x,t)\right) = f\left(u_{n,M'}(x,t)\right) = f_{M'}\left(u_{n,M'}(x,t)\right), \quad (x,t) \in \overline{\Omega} \times (0, \delta_{M'}].$$

We see that u_{n,M_0}, $u_{n,M'}$ are two classical solutions of problem (2.1)-(2.3) with $M = M_0$ on $\overline{\Omega} \times (0, \delta_{M'}]$. According to the uniqueness, Lemma 2.3, we have

$$u_{n,M_0}(x,t) = u_{n,M'}(x,t), \quad (x,t) \in \overline{\Omega} \times (0, \delta_{M'}].$$

By the continuity of $u_{n,M'}(x,t)$ and inequality (2.5), we can take $\delta_{M'} = \delta_0$ in inequality (2.6). Then we have

$$u_{n,M'}(x,t) = u_{n,M_0}(x,t) \le \sup_{\Omega} u_0 + M_0 \delta_0, \quad (x,t) \in \overline{\Omega} \times (0, \delta_0].$$

We arrive at a locally uniform bound of $u_{n,M}$. □

Remark Lemma 2.4 shows that $u_{n,M} \le \sup_{\Omega} u_0 + Mt$. However, the family $\{\sup_{\Omega} u_0 + Mt\}_{M>0}$ is not uniformly bounded on any interval $(0, \delta]$, $\delta > 0$. Lemma 2.6 provides the locally uniform bound of $u_{n,M}$.

Next, we derive some estimates on the solution $u_{n,M}$.

Lemma 2.7 *Suppose that σ does not depend on time. For any given $T > 0$, $M > 0$, there exists a constant $C = C(M, T)$ independent of n such that*

$$\int_{\Omega} u_{n,M}^2(x,t)\,dx, \qquad \int_{\partial\Omega} \sigma u_{n,M}^2(x,t)\,dS, \qquad \int_{Q_T} |\nabla u_{n,M}|^p\,dx\,dt \le C, \quad t \in (0,T).$$

Moreover, if $T = \delta_0$ (the constant in Lemma 2.6), then the constant $C = C(\delta_0)$ is independent of n, M.

Proof We write $u = u_{n,M}$ in this proof for the sake of convenience. Since $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$, multiplying equation (2.1) by u and integrating by parts over Q_τ, $\tau \in (0, T]$, we have

$$\int_{Q_\tau} u u_t\,dx\,dt + \int_{Q_\tau} \left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}} |\nabla u|^2\,dx\,dt$$

$$- \int_0^\tau \int_{\partial\Omega} u \left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u \cdot \nu\,dS\,dt$$

$$= - \int_{Q_\tau} \left(\vec{g}(u_+) \cdot \nabla u\right) u\,dx\,dt + \int_{Q_\tau} f_M(u) u\,dx\,dt.$$

Using the dynamic boundary condition (2.2), we conclude

$$\int_{Q_\tau} u u_t \, dx \, dt + \int_{Q_\tau} \left(\frac{1}{n} + |\nabla u|^2 \right)^{\frac{p-2}{2}} |\nabla u|^2 \, dx \, dt + \int_0^\tau \int_{\partial\Omega} \sigma u u_t \, dS \, dt$$

$$= -\int_{Q_\tau} (\vec{g}(u_+) \cdot \nabla u) u \, dx \, dt + \int_{Q_\tau} f_M(u) u \, dx \, dt.$$

That is,

$$\frac{1}{2} \int_\Omega u^2(x, \tau) \, dx + \int_{Q_\tau} \left(\frac{1}{n} + |\nabla u|^2 \right)^{\frac{p-2}{2}} |\nabla u|^2 \, dx \, dt + \frac{1}{2} \int_{\partial\Omega} \sigma u^2(x, \tau) \, dS$$

$$= \frac{1}{2} \int_\Omega u_{0,n}^2(x) \, dx + \frac{1}{2} \int_{\partial\Omega} \sigma u_{0,n}^2(x) \, dS - \int_{Q_\tau} (\vec{g}(u_+) \cdot \nabla u) u \, dx \, dt + \int_{Q_\tau} f_M(u) u \, dx \, dt.$$

Notice that $u_{0,n} \le \sup_\Omega u_0$,

$$\left| \int_{Q_\tau} (\vec{g}(u_+) \cdot \nabla u) u \, dx \, dt \right| \le \frac{1}{4} \int_{Q_\tau} |\nabla u|^p \, dx \, dt + C \int_{Q_\tau} \left(|\vec{g}(u_+)| u \right)^{\frac{p}{p-1}} dx \, dt,$$

and

$$|\nabla u|^p \le \left(\frac{1}{n} + |\nabla u|^2 \right)^{\frac{p-2}{2}} |\nabla u|^2, \quad p \ge 2,$$

$$|\nabla u|^p \le 2^{\frac{2-p}{2}} \left(\frac{1}{n} + |\nabla u|^2 \right)^{\frac{p-2}{2}} |\nabla u|^2 + \left(\frac{1}{n} \right)^{\frac{p}{2}}$$

$$\le 2 \left(\frac{1}{n} + |\nabla u|^2 \right)^{\frac{p-2}{2}} |\nabla u|^2 + 1, \quad 1 < p < 2.$$

Lemma 2.4 implies $|u| \le \sup_\Omega u_0 + MT$ for $(x, t) \in Q_T$. Therefore,

$$\int_\Omega u^2(x, t) \, dx, \qquad \int_{\partial\Omega} \sigma u^2(x, t) \, dS, \qquad \int_{Q_T} |\nabla u|^p \, dx \, dt \le C(M, T).$$

If $T = \delta_0$, Lemma 2.6 shows $|u| \le \sup_\Omega u_0 + M_0 \delta_0$ for $(x, t) \in Q_{\delta_0}$, which is a uniform bound independent of n, M. □

Lemma 2.8 *Suppose that σ does not depend on time and $p \ge 2$. For any given $T > 0$, $M > 0$, there exists a constant $C = C(M, T)$ independent of n such that*

$$\int_\Omega |\nabla u_{n,M}|^p \, dx, \qquad \int_{Q_T} \left| \frac{\partial u_{n,M}}{\partial t} \right|^2 \, dx \, dt, \qquad \int_0^T \int_\Omega \sigma \left| \frac{\partial u_{n,M}}{\partial t} \right|^2 \, dS \, dt \le C.$$

Moreover, if $T = \delta_0$ (the constant in Lemma 2.6), then the constant $C = C(\delta_0)$ is independent of n, M.

Proof We write $u = u_{n,M}$ in this proof for the sake of convenience. Since f_M, $g(s_+)$ are Lipschitz continuous, the classical regularity results in [9] imply that $u_t \in C^{1,0}(\overline{\Omega} \times (0, T))$.

Multiplying equation (2.1) by u_t and integrating over Q_τ, we have

$$\int_{Q_\tau} u_t^2 \, dx \, dt + \int_{Q_\tau} \left(\frac{1}{n} + |\nabla u|^2 \right)^{\frac{p-2}{2}} \nabla u \cdot \nabla u_t \, dx \, dt + \int_0^\tau \int_{\partial\Omega} \sigma u_t^2 \, dS \, dt$$

$$= -\int_{Q_\tau} \left(\vec{g}(u_+) \cdot \nabla u \right) u_t \, dx \, dt + \int_{Q_\tau} f_M(u) u_t \, dx \, dt.$$

Next, we show that

$$\int_{Q_\tau} \left(\frac{1}{n} + |\nabla u|^2 \right)^{\frac{p-2}{2}} \nabla u \cdot \nabla u_t \, dx \, dt$$

$$= \int_{Q_\tau} \frac{1}{2} \frac{\partial}{\partial t} \int_0^{|\nabla u(x,t)|^2} \left(s + \frac{1}{n} \right)^{\frac{p-2}{2}} ds \, dx \, dt$$

$$= \int_{Q_\tau} \frac{1}{p} \frac{\partial}{\partial t} \left(\left(\frac{1}{n} + |\nabla u(x,t)|^2 \right)^{\frac{p}{2}} - \left(\frac{1}{n} \right)^{\frac{p}{2}} \right) dx \, dt$$

$$= \frac{1}{p} \int_\Omega \left(\frac{1}{n} + |\nabla u(x,\tau)|^2 \right)^{\frac{p}{2}} dx - \frac{1}{p} \int_\Omega \left(\frac{1}{n} + |\nabla u_{0,n}(x)|^2 \right)^{\frac{p}{2}} dx$$

$$\geq \frac{1}{p} \int_\Omega |\nabla u(x,\tau)|^p \, dx - \frac{1}{p} 2^{\frac{p}{2}} \int_\Omega |\nabla u_{0,n}|^p \, dx - \frac{1}{p} 2^{\frac{p}{2}} |\Omega|.$$

Young's inequality yields

$$\left| -\int_{Q_\tau} \left(\vec{g}(u_+) \cdot \nabla u \right) u_t \, dx \, dt \right| \leq \frac{1}{4} \int_{Q_\tau} u_t^2 \, dx \, dt + \int_{Q_\tau} |\vec{g}(u_+)|^2 |\nabla u|^2 \, dx \, dt$$

$$\leq \frac{1}{4} \int_{Q_\tau} u_t^2 \, dx \, dt + C(M,T) \int_{Q_\tau} |\nabla u|^p \, dx \, dt, \quad p \geq 2,$$

$$\left| \int_{Q_\tau} f_M(u) u_t \, dx \, dt \right| \leq \frac{1}{4} \int_{Q_\tau} u_t^2 \, dx \, dt + \int_{Q_\tau} f_M^2(u) \, dx \, dt.$$

We conclude the estimate. □

Now, we define the following two types of weak solutions of problem (1.1)-(1.3).

Definition 2.1 A function $u \in L^p((0,T); W^{1,p}(\Omega))$ is called a local weak solution of problem (1.1)-(1.3) if the integral equality

$$-\int_\Omega u_0 \varphi \, dx - \int_{Q_T} u \varphi_t \, dx \, dt + \int_{Q_T} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx \, dt - \int_0^T \int_{\partial\Omega} u(\sigma\varphi)_t \, dS \, dt$$

$$= -\int_{Q_T} \left(\vec{g}(u) \cdot \nabla u \right) \varphi \, dx \, dt + \int_{Q_T} f(u) \varphi \, dx \, dt \tag{2.7}$$

holds for any $\varphi \in C^\infty(\overline{Q}_T)$ that satisfies $\varphi(x,T) = 0$ for $x \in \overline{\Omega}$, $\varphi(x,0) = 0$ for $x \in \partial\Omega$.

Definition 2.2 A function $u \in L^p((0,T); W^{1,p}(\Omega))$ is called a local strong solution of problem (1.1)-(1.3) if $u_t \in L^2(Q_T)$, u is the a.e. limit function of a subsequence $\{u_{n_k,M_k}\}$ of classical solutions to the regularized problem (2.1)-(2.3), and the integral equality (2.7) holds for any $\varphi \in C^\infty(\overline{Q}_T)$ that satisfies $\varphi(x,T) = 0$ for $x \in \overline{\Omega}$, $\varphi(x,0) = 0$ for $x \in \partial\Omega$.

Theorem 2.1 *Suppose that σ does not depend on time. Problem (1.1)-(1.3) admits at least one local weak solution.*

Proof For any $T > 0$, $\varphi \in C^\infty(\overline{Q}_T)$ that satisfies $\varphi(x, T) = 0$ for $x \in \overline{\Omega}$, $\varphi(x, 0) = 0$ for $x \in \partial\Omega$, multiplying (2.1) by φ, integrating over Q_T, we have

$$
\int_{Q_T} \frac{\partial u_{n,M}}{\partial t} \varphi \, dx \, dt + \int_{Q_T} \left(\frac{1}{n} + |\nabla u_{n,M}|^2 \right)^{\frac{p-2}{2}} \nabla u_{n,M} \cdot \nabla\varphi \, dx \, dt
$$

$$
- \int_0^T \int_{\partial\Omega} \left(\frac{1}{n} + |\nabla u_{n,M}|^2 \right)^{\frac{p-2}{2}} \nabla u_{n,M} \cdot \nu\varphi \, dS \, dt
$$

$$
= - \int_{Q_T} \left(\vec{g}\left((u_{n,M})_+ \right) \cdot \nabla u_{n,M} \right) \varphi \, dx \, dt + \int_{Q_T} f_M(u_{n,M}) \varphi \, dx \, dt.
$$

By the dynamic boundary condition (2.2), we obtain

$$
- \int_0^T \int_{\partial\Omega} \left(\frac{1}{n} + |\nabla u_{n,M}|^2 \right)^{\frac{p-2}{2}} \nabla u_{n,M} \cdot \nu\varphi \, dS \, dt
$$

$$
= \int_0^T \int_{\partial\Omega} \sigma \frac{\partial u_{n,M}}{\partial t} \varphi \, dS \, dt = - \int_0^T \int_{\partial\Omega} u_{n,M} (\sigma\varphi)_t \, dS \, dt.
$$

Thus,

$$
- \int_\Omega u_{0,n}\varphi \, dx - \int_{Q_T} u_{n,M}\varphi_t \, dx \, dt + \int_{Q_T} \left(\frac{1}{n} + |\nabla u_{n,M}|^2 \right)^{\frac{p-2}{2}} \nabla u_{n,M} \cdot \nabla\varphi \, dx \, dt
$$

$$
= \int_0^T \int_{\partial\Omega} u_{n,M} (\sigma\varphi)_t \, dS \, dt - \int_{Q_T} \left(\vec{g}(u_{n,M}) \cdot \nabla u_{n,M} \right) \varphi \, dx \, dt + \int_{Q_T} f_M(u_{n,M}) \varphi \, dx \, dt.
$$

By the uniform estimates in Lemma 2.6 and Lemma 2.7, there exist a subsequence $\{u_{n_k,M_k}\}$ ($n_k \to \infty$, $M_k \to \infty$, as $k \to \infty$) and a function $u \in L^p((0, \delta_0); W^{1,p}(\Omega))$ such that u_{n_k,M_k} converges weakly to u in $L^2(Q_{\delta_0})$, $\nabla u_{n_k,M_k}$ converges weakly to ∇u in $L^p(Q_{\delta_0})$, and u_{n_k,M_k} converges weakly to u in $L^p(\partial\Omega \times (0, \delta_0))$ in the sense of trace. Hence the above integral equality converges to (2.7) for $T = \delta_0$ and u is a local weak solution to problem (1.1)-(1.3). □

Theorem 2.2 *Suppose that σ does not depend on time and $p \geq 2$. Problem (1.1)-(1.3) admits at least one local strong solution.*

Proof By the uniform estimates in Lemma 2.6, Lemma 2.7, and Lemma 2.8, the norms $\|u_{n,M}\|_{H^1(Q_{\delta_0})}$, $\|\nabla u_{n,M}\|_{L^p(Q_{\delta_0})}$ are uniformly bounded. There exist a subsequence $\{u_{n_k,M_k}\}$ ($n_k \to \infty$, $M_k \to \infty$, as $k \to \infty$) and a function $u \in L^p((0, \delta_0); W^{1,p}(\Omega))$, $u_t \in L^2(Q_{\delta_0})$ such that $\{u_{n_k,M_k}\}$ converges weakly to u in $H^1(Q_{\delta_0})$, $\nabla u_{n_k,M_k}$ converges weakly to ∇u in $L^p(Q_{\delta_0})$. Hence $u_{n_k,M_k} \to u$ almost everywhere and the integral equality (2.7) holds. □

Remark For any given $M > 0$, by the estimates in Lemma 2.7 and Lemma 2.8, using the diagonal procedure, we can choose a subsequence $\{u_{n_k,M}\}$ ($\{n_k\}$ might depend on M) and a function u_M such that $u_{n_k,M}$ converges to u_M on Q_T for any $T > 0$ in the manner stated in the proof of Theorem 2.2. Furthermore, we can verify that u_M is the global strong solution

to the following equation:

$$\frac{\partial u}{\partial t} = \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) - \vec{g}(u) \cdot \nabla u + f_M(u)$$

coupled with the initial-boundary value conditions (1.2)-(1.3). Using the diagonal procedure again, we can choose a subsequence $\{n_k\}$ independent of M and then choose u_M such that $u_{n_k,M}$ converges to u_M for any $M \in \mathbb{Z}^+$ in the same manner. Lemma 2.5 implies $u_{n_k,M_1} \geq u_{n_k,M_2}$ for $M_1 \geq M_2$. Thus, $\{u_M\}_{M\in\mathbb{Z}^+}$ is monotone with respect to M. Define

$$T^* = \sup\left\{T > 0; \sup_{M\in\mathbb{Z}^+} \sup_{(x,t)\in\overline{\Omega}\times(0,T)} u_M(x,t) < \infty\right\},$$

and

$$u(x,t) = \sup_{M\in\mathbb{Z}^+} u_M(x,t), \quad (x,t) \in \overline{\Omega} \times \left(0, T^*\right).$$

Lemma 2.6 shows $T^* \geq \delta_0$. Similar to the proof of Theorem 2.1 and Theorem 2.2, we can prove that u is a strong solution to problem (1.1)-(1.3) with maximal existence time T^*.

3 Global existence

In this section, we study the global existence of local strong solutions to problem (1.1)-(1.3) defined in Section 2. We need to find an appropriate upper-solution to the regularized problem (2.1)-(2.3) which is independent of n, M and exists globally. If $p = 2$, the p-Laplacian is reduced to Laplacian, so we only consider $p > 2$ in this section.

Lemma 3.1 *Let* $\alpha = \frac{p-1}{p-2}, p > 2, K > 0, \eta \in C^1([0,+\infty))$. *For a fixed integer* $1 \leq j \leq N$, *define* $\underline{x}_j = \min_{\overline{\Omega}} x_j, \overline{x}_j = \max_{\overline{\Omega}} x_j$, *and*

$$U(x,t) = \frac{1}{\alpha}\left(Ke^{\eta(t)} + x_j - \overline{x}_j\right)^\alpha, \quad x \in \overline{\Omega}, t \geq 0.$$

Then U is an upper solution of the regularized problem (2.1)-(2.3) *provided*

$$Ke^{\eta(0)} + \underline{x}_j - \overline{x}_j \geq 1, \qquad \frac{1}{\alpha}\left(Ke^{\eta(0)} + \underline{x}_j - \overline{x}_j\right)^\alpha \geq \sup_\Omega u_0,$$

$$\eta'(t) \geq \alpha 2^p, \qquad \sigma(x,t)\eta'(t) \geq 2^p, \quad x \in \partial\Omega, t \geq 0,$$

and

$$g_j(s)s^{\frac{1}{p-1}} \geq f(s), \qquad s \geq \frac{1}{\alpha}\left(Ke^{\eta(0)} + \underline{x}_j - \overline{x}_j\right)^\alpha.$$

Proof By a simple computation, we have

$$\mathrm{div}\left(\left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}}\nabla u\right)$$

$$= \left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}}\Delta u + (p-2)\left(\frac{1}{n} + |\nabla u|^2\right)^{\frac{p-2}{2}-1}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}.$$

Notice that $\alpha > 1$ and $(\alpha - 1)(p - 1) = \alpha$. We show that

$$|\nabla U| = \frac{\partial U}{\partial x_j} = \left(Ke^{\eta(t)} + x_j - \overline{x}_j\right)^{\alpha-1} \geq 1,$$

$$U_t = \left(Ke^{\eta(t)} + x_j - \overline{x}_j\right)^{\alpha-1} Ke^{\eta(t)} \eta'(t) \geq \left(Ke^{\eta(t)} + x_j - \overline{x}_j\right)^{\alpha} \eta'(t),$$

$$\left| \left(\frac{1}{n} + |\nabla U|^2\right)^{\frac{p-2}{2}} \nabla U \cdot v \right| \leq 2^{\frac{p-2}{2}} |\nabla U|^{p-1} \leq 2^p \left(Ke^{\eta(t)} + x_j - \overline{x}_j\right)^{\alpha},$$

$$\text{div}\left(\left(\frac{1}{n} + |\nabla U|^2\right)^{\frac{p-2}{2}} \nabla U\right) \leq 2^{\frac{p-2}{2}} |\nabla U|^{p-2} \Delta U + (p - 2) \max\left\{2^{\frac{p-2}{2}-1}, 1\right\} |\nabla U|^{p-2} \frac{\partial^2 U}{\partial x_j^2}$$

$$\leq \alpha 2^p \left(Ke^{\eta(t)} + x_j - \overline{x}_j\right)^{\alpha-1}.$$

Thus,

$$B_n[U] = \sigma U_t - \left(\frac{1}{n} + |\nabla U|^2\right)^{\frac{p-2}{2}} \nabla U \cdot v \geq 0,$$

$$U_t - F_{n,M}[U] = U_t - \text{div}\left(\left(\frac{1}{n} + |\nabla U|^2\right)^{\frac{p-2}{2}} \nabla U\right) + \vec{g}(U) \cdot \nabla U - f(U)$$

$$\geq g_j(U)(\alpha U)^{\frac{1}{p-1}} - f(U) \geq 0, \quad x \in \Omega, t > 0,$$

and

$$U(x, 0) = \frac{1}{\alpha}\left(Ke^{\eta(0)} + x_j - \overline{x}_j\right)^{\alpha} \geq \sup_{\Omega} u_0 \geq u_{0,n}(x), \quad x \in \overline{\Omega}.$$

Lemma 2.2 implies that $U(x, t)$ is an upper solution of problem (2.1)-(2.3). □

Now we give some conditions on the functions f, g, and σ, which ensure the global existence of local solutions.

Theorem 3.1 *Suppose* $p > 2$, $(\inf_{x \in \partial\Omega} \sigma(x, \cdot))^{-1} \in L^1_{\text{loc}}([0, +\infty))$, *there exist an integer* $1 \leq j \leq N$ *and a constant* $M > 1$ *such that*

$$g_j(s)s^{\frac{1}{p-1}} \geq f(s), \quad s \geq M.$$

Then the strong solution of problem (1.1)-(1.3) *is a global solution.*

Proof Take $K = \max\{1, (\alpha \sup_{\Omega} u_0)^{\frac{1}{\alpha}}, (\alpha M)^{\frac{1}{\alpha}}\} + \overline{x}_j - \underline{x}_j$, and define

$$\eta(t) = 2^p \int_0^t \left(\inf_{x \in \partial\Omega} \sigma(x, \tau)\right)^{-1} d\tau + \alpha 2^p t,$$

where α, $\overline{x}_j$, $\underline{x}_j$ are the constants defined in Lemma 3.1. Thus, $U(x, t) = \frac{1}{\alpha}(Ke^{\eta(t)} + x_j - \overline{x}_j)^{\alpha}$ is an upper solution to the regularized problem (2.1)-(2.3) for any $n \in \mathbb{Z}^+$ and $M > 0$. That

is, $u_{n,M}(x,t) \leq U(x,t)$ for $x \in \overline{\Omega}$ and $t \geq 0$. According to the definition of strong solution, we have $u(x,t) \leq U(x,t)$. Hence u does not blow up in finite time. $\qquad \square$

4 Blow-up

In this section, we investigate the blow-up phenomenon of problem (1.1)-(1.3). We need to construct a family of lower solutions of the regularized problem (2.1)-(2.3) whose supremum blows up in finite time.

Lemma 4.1 *Suppose that $p > 2$, Ω is a convex domain, and there exist constants $C_1, C_2 > 0$ such that*

$$f(s) \geq C_1 s^{p-1}, \qquad |\vec{g}(s)| \leq C_2 s^{p-2}, \quad s \geq 0.$$

Choose $x_0 \in \Omega$ with $B_r(x_0) \subset \Omega$, $r > 0$. For $A, B > 0$ and $\varphi_M \in C^1([0, +\infty))$, define

$$v_M(x,t) = (A - B|x - x_0|^2)\varphi_M(t), \quad x \in \overline{\Omega}, t \geq 0.$$

Then the function v_M is a lower solution of the regularized problem (2.1)-(2.3) provided

$$A \geq 2Bd^2, \qquad 2Bd\varphi_M(0) \geq 1, \qquad A\varphi_M(0) \leq \inf_\Omega u_0, \qquad C_1(A\varphi_M(t))^{p-1} \leq M,$$

$$\varphi_M' \geq 0, \qquad \sigma A\varphi_M' \leq (2Br)^{p-1}\delta\varphi_M^{p-1}, \qquad A\varphi_M' \leq K\varphi_M^{p-1},$$

where $d = \sup_\Omega |x - x_0|$, $\delta = \inf_{\partial\Omega} \frac{x - x_0}{|x - x_0|} \cdot \nu > 0$ (by the convexity of Ω), and

$$K = \frac{1}{2}C_1\left(\frac{1}{2}A\right)^{p-1} - 2^p(2B)^{p-1}d^{p-2}(N + p) - \left(\frac{1}{2}C_1\right)^{-(p-2)}C_2^{p-1}(2Bd)^{p-1}.$$

Proof Let $\rho(x) = |x - x_0|$. A direct calculation shows

$$\nabla v_M = -2B\varphi_M(x - x_0), \qquad \frac{\partial v_M}{\partial t} = (A - B\rho^2)\varphi_M'(t),$$

$$\left(\frac{1}{n} + |\nabla v_M|^2\right)^{\frac{p-2}{2}}\nabla v_M = -\left(\frac{1}{n} + (2B)^2\rho^2\varphi_M^2\right)^{\frac{p-2}{2}}2B\varphi_M(x - x_0),$$

$$\operatorname{div}\left(\left(\frac{1}{n} + |\nabla v_M|^2\right)^{\frac{p-2}{2}}\nabla v_M\right) = -\left(\frac{1}{n} + (2B)^2\rho^2\varphi_M^2\right)^{\frac{p-2}{2}}2NB\varphi_M$$

$$- (p-2)\left(\frac{1}{n} + (2B)^2\rho^2\varphi_M^2\right)^{\frac{p-2}{2}-1}(2B)^2\rho^2\varphi_M^2 2B\varphi_M$$

$$= -\left(\frac{1}{n} + (2B)^2\rho^2\varphi_M^2\right)^{\frac{p-2}{2}-1}2B\varphi_M$$

$$\cdot \left(\left(\frac{1}{n} + (2B)^2\rho^2\varphi_M^2\right)N + (p-2)(2B)^2\rho^2\varphi_M^2\right).$$

Thus, we have

$$\left|\operatorname{div}\left(\left(\frac{1}{n} + |\nabla v_M|^2\right)^{\frac{p-2}{2}} \nabla v_M\right)\right| \leq \left(\frac{1}{n} + (2B)^2 \rho^2 \varphi_M^2\right)^{\frac{p-2}{2}} (N + p - 1) 2B\varphi_M$$

$$\leq \left(1 + (2Bd)^2 \varphi_M^2\right)^{\frac{p-2}{2}} (N + p - 1) 2B\varphi_M$$

$$\leq 2^p (2B)^{p-1} d^{p-2} (N + p) \varphi_M^{p-1}, \quad x \in \overline{\Omega}, t \leq 0,$$

and

$$\left(\frac{1}{n} + |\nabla v_M|^2\right)^{\frac{p-2}{2}} \nabla v_M \cdot \nu = -\left(\frac{1}{n} + (2B)^2 \rho^2 \varphi_M^2\right)^{\frac{p-2}{2}} 2B\varphi_M \rho \frac{x - x_0}{|x - x_0|} \cdot \nu$$

$$\leq -\left(\frac{1}{n} + (2B)^2 \rho^2 \varphi_M^2\right)^{\frac{p-2}{2}} 2B\varphi_M \rho \delta$$

$$\leq -(2Br)^{p-1} \delta \varphi_M^{p-1}, \quad x \in \partial\Omega, t \geq 0.$$

Young's inequality shows

$$\left|\vec{g}(v_M) \cdot \nabla v_M\right| \leq C_2 v_M^{p-2} |\nabla v_M| \leq \frac{1}{2} C_1 v_M^{p-1} + \left(\frac{1}{2} C_1\right)^{-(p-2)} C_2^{p-1} |\nabla v_M|^{p-1}.$$

We obtain

$$f_M(v_M) = \min\{M, f(v_M)\} \geq C_1 v_M^{p-1},$$

and

$$f_M(v_M) - \vec{g} \cdot \nabla v_M \geq \frac{1}{2} C_1 v_M^{p-1} - \left(\frac{1}{2} C_1\right)^{-(p-2)} C_2^{p-1} |\nabla v_M|^{p-1}$$

$$\geq \frac{1}{2} C_1 \left(\frac{1}{2} A\varphi_M\right)^{p-1} - \left(\frac{1}{2} C_1\right)^{-(p-2)} C_2^{p-1} (2Bd\varphi_M)^{p-1}.$$

Furthermore,

$$\frac{\partial v_M}{\partial t} - F_{n,M}[v_M] \leq A\varphi_M' - K\varphi_M^{p-1} \leq 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}^+,$$

$$B_n[v_M] = \sigma \frac{\partial v_M}{\partial t} + \left(\frac{1}{n} + |\nabla v_M|^2\right)^{\frac{p-2}{2}} \frac{\partial v_M}{\partial \nu}$$

$$\leq \sigma A\varphi_M' - (2Br)^{p-1} \delta \varphi_M^{p-1} \leq 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}^+,$$

$$v_M(x, 0) \leq A\varphi_M(0) \leq \inf_{\Omega} u_0 \leq u_{0,n}(x), \quad x \in \overline{\Omega}.$$

Lemma 2.2 implies that v_M is a lower solution of problem (2.1)-(2.3). □

Theorem 4.1 *Suppose that $p > 2$, Ω is a convex domain, $\sigma \in L^\infty(\partial\Omega \times \mathbb{R}^+)$, and there exist constants $C_1, C_2 > 0$ such that*

$$f(s) \geq C_1 s^{p-1}, \qquad \left|\vec{g}(s)\right| \leq C_2 s^{p-2}, \quad s \geq 0.$$

Then the strong solution of problem (1.1)-(1.3) *blows up in finite time provided that* $\inf_\Omega u_0$ *is sufficiently large.*

Proof Let x_0, r, d, δ, K be as defined in Lemma 4.1. Since $K = K(A, B)$ converges to $K(A, 0) = \frac{1}{2}C_1(\frac{1}{2}A)^{p-1} > 0$ as B tends to 0^+, we can choose $A, B > 0$ such that

$$C_1 A^{p-1} = 1, \qquad 2Bd^2 \leq A, \qquad K = K(A, B) > 0.$$

Then set

$$\varphi_M(0) = \frac{1}{2Bd} > 0, \qquad \overline{\sigma} = \sup_{\partial\Omega \times \mathbb{R}^+} \sigma, \qquad \gamma = \min\left\{\frac{K}{A}, \frac{(2Br)^{p-1}\delta}{\overline{\sigma}A}\right\}.$$

By Lemma 4.1, the function $v_M = (A - B|x - x_0|^2)\varphi_M(t)$ is a lower solution provided

$$\inf_\Omega u_0 \geq \frac{A}{2Bd}, \qquad \varphi_M(t) \leq M^{\frac{1}{p-1}}, \qquad 0 \leq \varphi'_M(t) \leq \gamma \varphi_M^{p-1}(t).$$

Define

$$\varphi_M(t) = \min\left\{\left(\varphi_M^{2-p}(0) - (p-2)\gamma t\right)_+^{-\frac{1}{p-2}}, M^{\frac{1}{p-1}}\right\}, \quad t \geq 0.$$

Although $\varphi_M(t)$ is not C^1 continuous, we can change the partial derivative $\frac{\partial}{\partial t}$ to the left-ward partial derivative $\frac{\partial}{\partial t^-}$ in the proof of Lemma 2.1, Lemma 2.2, and Lemma 4.1, then we conclude that v_M is a lower solution of problem (2.1)-(2.3). Hence $u_{n,M}(x, t) \geq v_M(x, t)$ for $(x, t) \in \overline{\Omega} \times \mathbb{R}^+$. By the definition of strong solution, we have

$$u(x, t) \geq \sup_{M \in \mathbb{Z}^+} v_M(x, t), \quad x \in \overline{\Omega}, t \in (0, T_0),$$

where $T_0 = \frac{(2Bd)^{p-2}}{(p-2)\gamma}$. Since v_M blows up at finite time T_0, the strong solution u must blow up at time $T^* \leq T_0$. □

References
1. Escher, J: Quasilinear parabolic systems with dynamical boundary conditions. Commun. Partial Differ. Equ. **18**, 1309-1364 (1993)
2. Mailly-Pincet, G, Rault, J-F: Nonlinear convection in reaction-diffusion equations under dynamical boundary conditions. Electron. J. Differ. Equ. **2013**, 10 (2013)
3. von Below, J, Mailly-Pincet, G: Blow up for reaction diffusion equation under dynamical boundary conditions. Commun. Partial Differ. Equ. **28**, 223-247 (2003)

4. Gal, CG, Warma, M: Well posedness and the global attractor of some quasilinear parabolic equations with nonlinear dynamic boundary conditions. Differ. Integral Equ. **23**, 327-358 (2010)

5. Andreu, F, Mazón, JM, Toledo, J: A degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions. Interfaces Free Bound. **8**, 447-479 (2006)

6. von Below, J, Mailly-Pincet, G: Blow up for some nonlinear parabolic problems with convection under dynamical boundary conditions. In: Discrete and Continuous Dynamical Systems, Dynamical Systems and Differential Equations, Proceedings of the 6th AIMS International Conference, Supplement, pp. 1031-1041 (2007)

7. Constantin, A, Escher, J: Global existence for fully parabolic boundary value problems. Nonlinear Differ. Equ. Appl. **13**, 91-118 (2006)

8. von Below, J, De Coster, C: A qualitative theory for parabolic problems under dynamical boundary conditions. J. Inequal. Appl. **5**, 467-486 (2000)

9. Ladyženskaya, OA, Solonnikov, VA, Uraltseva, NN: Linear and Quasilinear Equations of Parabolic Type. Transl. of Math. Monogr., vol. 23. Am. Math. Soc. , Providence (1968)