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Strong convergence theorems for equilibrium problems and weak Bregman relatively nonexpansive mappings in Banach spaces

Ravi P Agarwal[1,2], Jia-Wei Chen[3*] and Yeol Je Cho[4*]

*Correspondence:
mathchen@yahoo.cn;
yjcho@gnu.ac.kr
[3]College of Sciences, Northwest A&F University, Yangling, Xianyang, Shaanxi 712100, China
[4]Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju, 660-701, Korea
Full list of author information is available at the end of the article

Abstract

In this paper, a shrinking projection algorithm based on the prediction correction method for equilibrium problems and weak Bregman relatively nonexpansive mappings is introduced and investigated in Banach spaces, and then the strong convergence of the sequence generated by the proposed algorithm is derived under some suitable assumptions. These results are new and develop some recent results in this field.

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1 Introduction

In this paper, without other specifications, let R be the set of real numbers, C be a nonempty, closed and convex subset of a real reflexive Banach space E with the dual space E^*. The norm and the dual pair between E^* and E are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. Let $f : E \to R \cup \{+\infty\}$ be a proper convex and lower semicontinuous function. Denote the domain of f by $\operatorname{dom} f$, *i.e.*, $\operatorname{dom} f = \{x \in E : f(x) < +\infty\}$. The *Fenchel conjugate* of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(\xi) = \sup\{\langle\xi, x\rangle - f(x) : x \in E\}$. Let $T : C \to C$ be a nonlinear mapping. Denote by $F(T) = \{x \in C : Tx = x\}$ the set of fixed points of T. A mapping T is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$.

In 1994, Blum and Oettli [1] firstly studied the equilibrium problem: finding $\bar{x} \in C$ such that

$$H(\bar{x}, y) \ge 0, \quad \forall y \in C, \tag{1.1}$$

where $H : C \times C \to R$ is functional. Denote the set of solutions of the problem (1.1) by $\mathrm{EP}(H)$. Since then, various equilibrium problems have been investigated. It is well known that equilibrium problems and their generalizations have been important tools for solving problems arising in the fields of linear or nonlinear programming, variational inequalities, complementary problems, optimization problems, fixed point problems and have been widely applied to physics, structural analysis, management science and economics *etc.* One of the most important and interesting topics in the theory of equilibria is to develop efficient and implementable algorithms for solving equilibrium problems and their generalizations (see, *e.g.*, [2–8] and the references therein). Since the equilibrium problems have very close connections with both the fixed point problems and the variational inequalities problems, finding the common elements of these problems has drawn many people's attention and has become one of the hot topics in the related fields in the past few years (see, *e.g.*, [9–16] and the references therein).

In 1967, Bregman [17] discovered an elegant and effective technique for using of the so-called Bregman distance function D_f (see, Section 2, Definition 2.1) in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique has been applied in various ways in order to design and analyze not only iterative algorithms for solving feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria, for computing fixed points of nonlinear mappings and so on (see, *e.g.*, [18–24] and the references therein). In 2005, Butnariu and Resmerita [25] presented Bregman-type iterative algorithms and studied the convergence of the Bregman-type iterative method of solving some nonlinear operator equations.

Recently, by using the Bregman projection, Reich and Sabach [26] presented the following algorithms for finding common zeroes of maximal monotone operators $A_i : E \to 2^{E^*}$ $(i = 1, 2, \ldots, N)$ in a reflexive Banach space E, respectively:

$$\begin{cases} x_0 \in E, \\ y_n^i = \operatorname{Res}_{\lambda_n^i}^f (x_n + e_n^i), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \le D_f(z, x_n + e_n^i)\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^f x_0, \quad \forall n \ge 0 \end{cases}$$

and

$$\begin{cases} x_0 \in E, \\ \eta_n^i = \xi_n^i + \frac{1}{\lambda_n^i}(\nabla f(y_n^i) - \nabla f(x_n)), \quad \xi_n^i \in A_i y_n^i, \\ \omega_n^i = \nabla f^*(\lambda_n^i \eta_n^i + \nabla f(x_n)), \\ C_n^i = \{z \in E : D_f(z, y_n^i) \le D_f(z, \omega_n^i)\}, \\ C_n = \bigcap_{i=1}^N C_n^i, \\ Q_n = \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \le 0\}, \\ x_{n+1} = \operatorname{proj}_{C_n \cap Q_n}^f x_0, \quad \forall n \ge 0, \end{cases}$$

where $\{\lambda_n^i\}_{i=1}^N \subseteq (0, +\infty)$, $\{e_n\}_{i=1}^N$ is an error sequence in E with $e_n^i \to 0$ and proj_C^f is the Bregman projection with respect to f from E onto a closed and convex subset C. Further, under some suitable conditions, they obtained two strong convergence theorems of maximal monotone operators in a reflexive Banach space. Reich and Sabach [7] also studied the convergence of two iterative algorithms for finitely many Bregman strongly nonexpansive operators in a Banach space. In [15], Reich and Sabach proposed the following algorithms for finding common fixed points of finitely many Bregman firmly nonexpansive operators

$T_i : C \to C$ $(i = 1, 2, \dots, N)$ in a reflexive Banach space E if $\bigcap_{i=1}^N F(T_i) \neq \emptyset$:

$$\begin{cases} x_0 \in E, \\ Q_0^i = E, \quad i = 1, 2, \dots, N, \\ y_n^i = T_i(x_n + e_n^i), \\ Q_{n+1}^i = \{z \in Q_n^i : \langle \nabla f(x_n + e_n^i) - \nabla f(y_n^i), z - y_n^i \rangle \leq 0\}, \\ Q_n = \bigcap_{i=1}^N Q_n^i, \\ x_{n+1} = \operatorname{proj}_{Q_{n+1}}^f x_0, \quad \forall n \geq 0. \end{cases} \tag{1.2}$$

Under some suitable conditions, they proved that the sequence $\{x_n\}$ generated by (1.2) converges strongly to $\bigcap_{i=1}^N F(T_i)$ and applied the result to the solution of convex feasibility and equilibrium problems.

Very recently, Chen *et al.* [27] introduced the concept of weak Bregman relatively nonexpansive mappings in a reflexive Banach space and gave an example to illustrate the existence of a weak Bregman relatively nonexpansive mapping and the difference between a weak Bregman relatively nonexpansive mapping and a Bregman relatively nonexpansive mapping. They also proved the strong convergence of the sequences generated by the constructed algorithms with errors for finding a fixed point of weak Bregman relatively nonexpansive mappings and Bregman relatively nonexpansive mappings under some suitable conditions.

This paper is devoted to investigating the shrinking projection algorithm based on the prediction correction method for finding a common element of solutions to the equilibrium problem (1.1) and fixed points to weak Bregman relatively nonexpansive mappings in Banach spaces, and then the strong convergence of the sequence generated by the proposed algorithm is derived under some suitable assumptions.

2 Preliminaries

Let $T : C \to C$ be a nonlinear mapping. A point $\omega \in C$ is called an *asymptotic fixed point* of T (see [28]) if C contains a sequence $\{x_n\}$ which converges weakly to ω such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. A point $\omega \in C$ is called a *strong asymptotic fixed point* of T (see [28]) if C contains a sequence $\{x_n\}$ which converges strongly to ω such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. We denote the sets of asymptotic fixed points and strong asymptotic fixed points of T by $\hat{F}(T)$ and $\tilde{F}(T)$, respectively.

Let $\{x_n\}$ be a sequence in E; we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$. For any $x \in \operatorname{int}(\operatorname{dom} f)$, the *right-hand derivative* of f at x in the direction $y \in E$ is defined by

$$f'(x, y) := \lim_{t \searrow 0} \frac{f(x + ty) - f(x)}{t}.$$

f is called *Gâteaux differentiable* at x if, for all $y \in E$, $\lim_{t\searrow 0} \frac{f(x+ty)-f(x)}{t}$ exists. In this case, $f'(x, y)$ coincides with $\nabla f(x)$, the value of the gradient of f at x. f is called *Gâteaux differentiable* if it is Gâteaux differentiable for any $x \in \operatorname{int}(\operatorname{dom} f)$. f is called *Fréchet differentiable* at x if this limit is attained uniformly for $\|y\| = 1$. We say that f is *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

The Legendre function $f : E \to (-\infty, +\infty]$ is defined in [18]. From [18], if E is a reflexive Banach space, then f is the Legendre function if and only if it satisfies the following conditions (L1) and (L2):

(L1) The interior of the domain of f, $\operatorname{int}(\operatorname{dom} f)$, is nonempty, f is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f)$ and $\operatorname{dom} f = \operatorname{int}(\operatorname{dom} f)$;

(L2) The interior of the domain of f^*, $\operatorname{int}(\operatorname{dom} f^*)$, is nonempty, f^* is Gâteaux differentiable on $\operatorname{int}(\operatorname{dom} f^*)$ and $\operatorname{dom} f^* = \operatorname{int}(\operatorname{dom} f^*)$.

Since E is reflexive, we know that $(\nabla f)^{-1} = \nabla f^*$ (see [29]). This, by (L1) and (L2), implies the following equalities:

$$\nabla f = (\nabla f^*)^{-1}, \qquad \operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}(\operatorname{dom} f^*)$$

and

$$\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}(\operatorname{dom} f).$$

By Bauschke *et al.* [18, Theorem 5.4], the conditions (L1) and (L2) also yield that the functions f and f^* are strictly convex on the interior of their respective domains. From now on we assume that the convex function $f : E \to (-\infty, +\infty]$ is Legendre.

We first recall some definitions and lemmas which are needed in our main results.

Assumption 2.1 Let C be a nonempty, closed convex subset of a uniformly convex and uniformly smooth Banach space E, and let $H : C \times C \to R$ satisfy the following conditions (C1)-(C4):

(C1) $H(x,x) = 0$ for all $x \in C$;

(C2) H is monotone, *i.e.*, $H(x,y) + H(y,x) \leq 0$ for all $x, y \in C$;

(C3) for all $x, y, z \in C$,

$$\limsup_{t \to 0^+} H(tz + (1-t)x, y) \leq H(x,y);$$

(C4) for all $x \in C$, $H(x, \cdot)$ is convex and lower semicontinuous.

Definition 2.1 [3, 17] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The function $D_f : \operatorname{dom} f \times \operatorname{int}(\operatorname{dom} f) \to [0, +\infty)$ defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$$

is called *the Bregman distance* with respect to f.

Remark 2.1 [15] The Bregman distance has the following properties:

(1) the *three point identity*, for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom} f)$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle;$$

(2) the *four point identity*, for any $y, \omega \in \operatorname{dom} f$ and $x, z \in \operatorname{int}(\operatorname{dom} f)$,

$$D_f(y,x) - D_f(y,z) - D_f(\omega,x) + D_f(\omega,z) = \langle \nabla f(z) - \nabla f(x), y - \omega \rangle.$$

Definition 2.2 [17] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman projection of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty, closed and convex set $C \subset \operatorname{dom} f$ is the necessarily unique vector $\operatorname{proj}_C^f(x) \in C$ satisfying the following:

$$D_f(\operatorname{proj}_C^f(x), x) = \inf\{D_f(y,x) : y \in C\}.$$

Remark 2.2

(1) If E is a Hilbert space and $f(y) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then the Bregman projection $\operatorname{proj}_C^f(x)$ is reduced to the metric projection of x onto C;

(2) If E is a smooth Banach space and $f(y) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then the Bregman projection $\operatorname{proj}_C^f(x)$ is reduced to the generalized projection $\Pi_C(x)$ (see [11, 28]), which is defined by

$$\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x),$$

where $\phi(y,x) = \|y\|^2 - 2\langle y, J(x)\rangle + \|x\|^2$, J is the normalized duality mapping from E to 2^{E^*}.

Definition 2.3 [21, 26, 27] Let C be a nonempty, closed and convex set of $\operatorname{dom} f$. The operator $T : C \to \operatorname{int}(\operatorname{dom} f)$ with $F(T) \neq \emptyset$ is called:

(1) *quasi-Bregman nonexpansive* if

$$D_f(u, Tx) \le D_f(u, x), \quad \forall x \in C, u \in F(T);$$

(2) *Bregman relatively nonexpansive* if $\hat{F}(T) = F(T)$ and

$$D_f(u, Tx) \le D_f(u, x), \quad \forall x \in C, u \in F(T);$$

(3) *Bregman firmly nonexpansive* if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \le \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle, \quad \forall x, y \in C,$$

or, equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \le D_f(Tx, y) + D_f(Ty, x), \quad \forall x, y \in C;$$

(4) *a weak Bregman relatively nonexpansive mapping* with $F(T) \neq \emptyset$ if $\tilde{F}(T) = F(T)$ and

$$D_f(u, Tx) \le D_f(u, x), \quad \forall x \in C, u \in F(T).$$

Definition 2.4 [4] Let $H : C \times C \to R$ be functional. The *resolvent* of H is the operator $\operatorname{Res}_H^f : E \to 2^C$ defined by

$$\operatorname{Res}_H^f(x) = \{z \in C : H(z, y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \ge 0, \forall y \in C\}.$$

Definition 2.5 [21] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. f is called:

(1) *totally convex* at $x \in \operatorname{int}(\operatorname{dom} f)$ if its modulus of total convexity at x, that is, the function $\nu_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in \operatorname{dom} f, \|y - x\| = t\},$$

is positive whenever $t > 0$;
(2) *totally convex* if it is totally convex at every point $x \in \operatorname{int}(\operatorname{dom} f)$;
(3) *totally convex on bounded sets* if $\nu_f(B,t)$ is positive for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty) \to [0, +\infty)$ defined by

$$\nu_f(B,t) := \inf\{\nu_f(x,t) : x \in B \cap \operatorname{dom} f\}.$$

Definition 2.6 [21, 26] The function $f : E \to (-\infty, +\infty]$ is called:
(1) *cofinite* if $\operatorname{dom} f^* = E^*$;
(2) *coercive* if $\lim_{\|x\| \to +\infty}(f(x)/\|x\|) = +\infty$;
(3) *sequentially consistent* if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}$ is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Lemma 2.1 [26, Proposition 2.3] *If $f : E \to (-\infty, +\infty]$ is Fréchet differentiable and totally convex, then f is cofinite.*

Lemma 2.2 [25, Theorem 2.10] *Let $f : E \to (-\infty, +\infty]$ be a convex function whose domain contains at least two points. Then the following statements hold*:
(1) *f is sequentially consistent if and only if it is totally convex on bounded sets*;
(2) *If f is lower semicontinuous, then f is sequentially consistent if and only if it is uniformly convex on bounded sets*;
(3) *If f is uniformly strictly convex on bounded sets, then it is sequentially consistent and the converse implication holds when f is lower semicontinuous, Fréchet differentiable on its domain and the Fréchet derivative ∇f is uniformly continuous on bounded sets.*

Lemma 2.3 [30, Proposition 2.1] *Let $f : E \to R$ be uniformly Fréchet differentiable and bounded on bounded subsets of E. Then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 2.4 [26, Lemma 3.1] *Let $f : E \to R$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.5 [26, Proposition 2.2] *Let $f : E \to R$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and let C be a nonempty, closed convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n, x_0) \le D_f(\operatorname{proj}_C^f(x_0), x_0)$ for any $n \in N$, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\operatorname{proj}_C^f(x_0)$.*

Lemma 2.6 [27, Proposition 2.17] *Let $f : E \to (-\infty, +\infty]$ be the Legendre function. Let C be a nonempty, closed convex subset of* $\operatorname{int}(\operatorname{dom} f)$ *and $T : C \to C$ be a quasi-Bregman nonexpansive mapping with respect to f. Then $F(T)$ is closed and convex.*

Lemma 2.7 [27, Lemma 2.18] *Let $f : E \to (-\infty, +\infty]$ be Gâteaux differentiable and proper convex lower semicontinuous. Then, for all $z \in E$,*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \le \sum_{i=1}^{N} t_i D_f(z, x_i),$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2.8 [25, Corollary 4.4] *Let $f : E \to (-\infty, +\infty]$ be Gâteaux differentiable and totally convex on* $\operatorname{int}(\operatorname{dom} f)$. *Let $x \in \operatorname{int}(\operatorname{dom} f)$ and $C \subset \operatorname{int}(\operatorname{dom} f)$ be a nonempty, closed convex set. If $\hat{x} \in C$, then the following statements are equivalent*:

1. *the vector $\hat{x}$ is the Bregman projection of x onto C with respect to f*;
2. *the vector $\hat{x}$ is the unique solution of the variational inequality*:

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \ge 0, \quad \forall y \in C;$$

3. *the vector $\hat{x}$ is the unique solution of the inequality*:

$$D_f(y, z) + D_f(z, x) \le D_f(y, x), \quad \forall y \in C.$$

Lemma 2.9 [7, Lemmas 1 and 2] *Let $f : E \to (-\infty, +\infty]$ be a coercive Legendre function. Let C be a nonempty, closed and convex subset of* $\operatorname{int}(\operatorname{dom} f)$. *Assume that $H : C \times C \to R$ satisfies Assumption* 2.1. *Then the following results hold*:

1. Res_H^f *is single-valued and* $\operatorname{dom}(\operatorname{Res}_H^f) = E$;
2. Res_H^f *is Bregman firmly nonexpansive*;
3. $\operatorname{EP}(H)$ *is a closed and convex subset of C and* $\operatorname{EP}(H) = F(\operatorname{Res}_H^f)$;
4. *for all $x \in E$ and for all $u \in F(\operatorname{Res}_H^f)$*,

$$D_f\big(u, \operatorname{Res}_H^f(x)\big) + D_f\big(\operatorname{Res}_H^f(x), x\big) \le D_f(u, x).$$

Lemma 2.10 [31, Proposition 5] *Let $f : E \to R$ be a Legendre function such that ∇f^* is bounded on bounded subsets of* $\operatorname{int} \operatorname{dom} f^*$. *Let $x \in E$. If $\{D_f(x, x_n)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

3 Main results

In this section, we will introduce a new shrinking projection algorithm based on the prediction correction method for finding a common element of solutions to the equilibrium problem (1.1) and fixed points to weak Bregman relatively nonexpansive mappings in Banach spaces, and then the strong convergence of the sequence generated by the proposed algorithm is proved under some suitable conditions.

Let $\{\alpha_n\}$ and $\{\beta_n\}$ be the sequences in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\liminf_{n\to\infty}(1 - \alpha_n)\beta_n > 0$. We propose the following shrinking projection algorithm based on the prediction correction method.

Algorithm 3.1 Step 1: Select an arbitrary starting point $x_0 \in C$, let $Q_0 = C$ and $C_0 = \{z \in C : D_f(z,u_0) \leq D_f(z,x_0)\}$.

Step 2: Given the current iterate x_n, calculate the next iterate as follows:

$$\begin{cases} z_n = \nabla f^*(\beta_n \nabla f(T(x_n)) + (1-\beta_n)\nabla f(x_n)), \\ y_n = \nabla f^*(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)), \\ u_n = \mathrm{Res}_H^f(y_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : D_f(z,u_n) \leq \alpha_n D_f(z,x_0) + (1-\alpha_n)D_f(z,x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \mathrm{proj}_{C_n \cap Q_n}^f x_0, \quad \forall n \geq 0. \end{cases} \tag{3.1}$$

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real reflexive Banach space E, $f : E \to R$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subset of E, and ∇f^* be bounded on bounded subsets of E^*. Let $H : C \times C \to R$ satisfy Assumption* 2.1 *and $T : C \to C$ be a weak Bregman relatively nonexpansive mapping such that* $\mathrm{EP}(H) \cap F(T) \neq \emptyset$. *Then the sequence $\{x_n\}$ generated by Algorithm* 3.1 *converges strongly to the point* $\mathrm{proj}^f_{\mathrm{EP}(H)\cap F(T)}(x_0)$, *where* $\mathrm{proj}^f_{\mathrm{EP}(H)\cap F(T)}(x_0)$ *is the Bregman projection of C onto* $\mathrm{EP}(H) \cap F(T)$.

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.1 *Assume that* $\mathrm{EP}(H) \cap F(T) \subseteq C_n \cap Q_n$ *for all $n \geq 0$. Then the sequence $\{x_n\}$ is bounded.*

Proof Since $\langle \nabla f(x_0) - \nabla f(x_n), v - x_n\rangle \leq 0$ for all $v \in Q_n$, it follows from Lemma 2.8 that $x_n = \mathrm{proj}^f_{Q_n}(x_0)$ and so, by $x_{n+1} = \mathrm{proj}^f_{C_n \cap Q_n}(x_0) \in Q_n$, we have

$$D_f(x_n,x_0) \leq D_f(x_{n+1},x_0). \tag{3.2}$$

Let $\omega \in \mathrm{EP}(H) \cap F(T)$. It follows from Lemma 2.8 that

$$D_f\big(\omega, \mathrm{proj}^f_{Q_n}(x_0)\big) + D_f\big(\mathrm{proj}^f_{Q_n}(x_0), x_0\big) \leq D_f(\omega,x_0)$$

and so

$$D_f(x_n,x_0) \leq D_f(\omega,x_0) - D_f(\omega,x_n) \leq D_f(\omega,x_0).$$

Therefore, $\{D_f(x_n,x_0)\}$ is bounded. Moreover, $\{x_n\}$ is bounded and so are $\{T(x_n)\}$, $\{y_n\}$, $\{z_n\}$. This completes the proof. □

Lemma 3.2 *Assume that* $\mathrm{EP}(H) \cap F(T) \subseteq C_n \cap Q_n$ *for all $n \geq 0$. Then the sequence $\{x_n\}$ is a Cauchy sequence.*

Proof By the proof of Lemma 3.1, we know that $\{D_f(x_n,x_0)\}$ is bounded. It follows from (3.2) that $\lim_{n\to\infty} D_f(x_n,x_0)$ exists. From $x_m \in Q_{m-1} \subseteq Q_n$ for all $m > n$ and Lemma 2.8, one has

$$D_f\big(x_m, \operatorname{proj}^f_{Q_n}(x_0)\big) + D_f\big(\operatorname{proj}^f_{Q_n}(x_0), x_0\big) \le D_f(x_m, x_0)$$

and so $D_f(x_m, x_n) \le D_f(x_m, x_0) - D_f(x_n, x_0)$. Therefore, we have

$$\lim_{n\to\infty} D_f(x_m, x_n) \le \lim_{n\to\infty} \big(D_f(x_m, x_0) - D_f(x_n, x_0)\big) = 0. \tag{3.3}$$

Since f is totally convex on bounded subsets of E, by Definition 2.6, Lemma 2.2 and (3.3), we obtain

$$\lim_{n\to\infty} \|x_m - x_n\| = 0. \tag{3.4}$$

Thus $\{x_n\}$ is a Cauchy sequence and so $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. This completes the proof. □

Lemma 3.3 *Assume that* $\mathrm{EP}(H) \cap F(T) \subseteq C_n \cap Q_n$ *for all* $n \ge 0$. *Then the sequence* $\{x_n\}$ *converges strongly to a point in* $\mathrm{EP}(H) \cap F(T)$.

Proof From Lemma 3.2, the sequence $\{x_n\}$ is a Cauchy sequence. Without loss of generality, let $x_n \to \hat{\omega} \in C$. Since f is uniformly Fréchet differentiable on bounded subsets of E, it follows from Lemma 2.2 that ∇f is norm-to-norm uniformly continuous on bounded subsets of E. Hence, by (3.4), we have

$$\lim_{n\to\infty} \big\|\nabla f(x_m) - \nabla f(x_n)\big\| = 0$$

and so

$$\lim_{n\to\infty} \big\|\nabla f(x_{n+1}) - \nabla f(x_n)\big\| = 0. \tag{3.5}$$

Since $x_{n+1} \in C_n$, we have

$$D_f(x_{n+1}, u_n) \le \alpha_n D_f(x_{n+1}, x_0) + (1 - \alpha_n) D_f(x_{n+1}, x_n).$$

It follows from $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$ that $\{D_f(x_{n+1}, u_n)\}$ is bounded and

$$\lim_{n\to\infty} D_f(x_{n+1}, u_n) = 0.$$

By Lemma 2.10, $\{u_n\}$ is bounded. Hence, $\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0$ and so

$$\lim_{n\to\infty} \big\|\nabla f(x_{n+1}) - \nabla f(u_n)\big\| = 0. \tag{3.6}$$

Taking into account $\|x_n - u_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|$, we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0$$

and so $u_n \to \hat{\omega}$ as $n \to \infty$. For any $\omega \in \mathrm{EP}(H) \cap F(T)$, from Lemma 2.9, we get

$$\begin{aligned}
&D_f(u_n, y_n) \\
&\quad \le D_f(\omega, y_n) - D_f(\omega, u_n) \\
&\quad = D_f\bigl(\omega, \nabla f^*\bigl(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)\bigr)\bigr) - D_f(\omega, u_n) \\
&\quad \le \alpha_n D_f(\omega, x_0) + (1-\alpha_n) D_f\bigl(\omega, \nabla f^*\bigl(\beta_n \nabla f\bigl(T(x_n)\bigr) + (1-\beta_n)\nabla f(x_n)\bigr)\bigr) - D_f(\omega, u_n) \\
&\quad \le \alpha_n D_f(\omega, x_0) + (1-\alpha_n) D_f(\omega, x_n) - D_f(\omega, u_n) \\
&\quad = \alpha_n\bigl[D_f(\omega, x_0) - D_f(\omega, x_n)\bigr] + D_f(\omega, x_n) - D_f(\omega, u_n).
\end{aligned}$$

By the three point identity of the Bregman distance, one has

$$\begin{aligned}
&D_f(\omega, x_n) - D_f(\omega, u_n) \\
&\quad = -D_f(x_n, u_n) + \bigl\langle \nabla f(u_n) - \nabla f(x_n), \omega - x_n \bigr\rangle \\
&\quad \le -f(x_n) + f(u_n) + \bigl\langle \nabla f(u_n), x_n - u_n \bigr\rangle + \bigl\langle \nabla f(u_n) - \nabla f(x_n), \omega - x_n \bigr\rangle.
\end{aligned}$$

Since f is a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on a bounded subset of E, it follows from Lemma 2.3 that f is continuous on E and ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*. Therefore, we have

$$\begin{aligned}
&\lim_{n\to\infty} D_f(u_n, y_n) \\
&\quad \le \lim_{n\to\infty} \bigl\{\alpha_n\bigl[D_f(\omega, x_0) - D_f(\omega, x_n)\bigr] - f(x_n) + f(u_n) \\
&\qquad + \bigl\langle \nabla f(u_n), x_n - u_n \bigr\rangle + \bigl\langle \nabla f(u_n) - \nabla f(x_n), \omega - x_n \bigr\rangle\bigr\} \\
&\quad \le \lim_{n\to\infty} \bigl\{\alpha_n\bigl[D_f(\omega, x_0) - D_f(\omega, x_n)\bigr] - f(x_n) + f(u_n)\bigr\} \\
&\qquad + \lim_{n\to\infty} \bigl[\bigl\langle \nabla f(u_n), x_n - u_n \bigr\rangle + \bigl\langle \nabla f(u_n) - \nabla f(x_n), \omega - x_n \bigr\rangle\bigr] \\
&\quad = 0,
\end{aligned}$$

that is, $\lim_{n\to\infty} D_f(u_n, y_n) = 0$. Furthermore, one has $\lim_{n\to\infty} \|u_n - y_n\| = 0$ and thus

$$\lim_{n\to\infty} \bigl\|\nabla f(u_n) - \nabla f(y_n)\bigr\| = 0.$$

Since $u_n \to \hat{\omega}$ as $n \to \infty$, we have $y_n \to \hat{\omega}$ as $n \to \infty$. Further, in the light of $u_n = \mathrm{Res}_H^f(y_n)$ and Definition 2.4, it follows that, for each $y \in C$,

$$H(u_n, y) + \bigl\langle \nabla f(u_n) - \nabla f(y_n), y - u_n \bigr\rangle \ge 0$$

and hence, combining this with Assumption 2.1,

$$\bigl\langle \nabla f(u_n) - \nabla f(y_n), y - u_n \bigr\rangle \ge -H(u_n, y) \ge H(y, u_n).$$

Consequently, one can conclude that

$$\begin{aligned} H(y,\hat{\omega}) &\leq \liminf_{n\to\infty} H(y,u_n) \\ &\leq \liminf_{n\to\infty} \langle \nabla f(u_n) - \nabla f(y_n), y - u_n \rangle \\ &\leq \liminf_{n\to\infty} \| \nabla f(u_n) - \nabla f(y_n) \| \cdot \|y - u_n\| \\ &= 0. \end{aligned}$$

For any $y \in C$ and $t \in (0,1]$, let $y_t = ty + (1-t)\hat{\omega} \in C$. It follows from Assumption 2.1 that $H(y_t, \hat{\omega}) \leq 0$ and

$$0 = H(y_t, y_t) \leq tH(y_t, y) + (1-t)H(y_t, \hat{\omega}) \leq tH(y_t, y)$$

and so $H(y_t, y) \geq 0$. Moreover, one has

$$0 \leq \limsup_{t\to 0^+} H(y_t, y) = \limsup_{t\to 0^+} H\big(ty + (1-t)\hat{\omega}, y\big) \leq H(\hat{\omega}, y), \quad \forall y \in C,$$

which shows that $\hat{\omega} \in \mathrm{EP}(H)$.

Next, we prove that $\hat{\omega} \in F(T)$. Note that

$$\begin{aligned} &\| \nabla f(x_n) - \nabla f(y_n) \| \\ &\quad = \| \nabla f(x_n) - \nabla f\big(\nabla f^*\big(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)\big)\big) \| \\ &\quad = \| \nabla f(x_n) - \big(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)\big) \| \\ &\quad = \| \alpha_n\big(\nabla f(x_n) - \nabla f(x_0)\big) + (1-\alpha_n)\big(\nabla f(x_n) - \nabla f(z_n)\big) \| \\ &\quad = \| \alpha_n\big(\nabla f(x_n) - \nabla f(x_0)\big) + (1-\alpha_n)\big(\nabla f(x_n) - \nabla f\big(\nabla f^*\big(\beta_n \nabla f\big(T(x_n)\big) \\ &\qquad + (1-\beta_n)\nabla f(x_n)\big)\big)\big) \| \\ &\quad = \| \alpha_n\big(\nabla f(x_n) - \nabla f(x_0)\big) + (1-\alpha_n)\beta_n\big(\nabla f(x_n) - \nabla f\big(T(x_n)\big)\big) \| \\ &\quad \geq (1-\alpha_n)\beta_n \| \nabla f(x_n) - \nabla f\big(T(x_n)\big) \| - \alpha_n \| \nabla f(x_n) - \nabla f(x_0) \|. \end{aligned}$$

This implies that

$$(1-\alpha_n)\beta_n \| \nabla f(x_n) - \nabla f\big(T(x_n)\big) \| \leq \| \nabla f(x_n) - \nabla f(y_n) \| + \alpha_n \| \nabla f(x_n) - \nabla f(x_0) \|. \quad (3.7)$$

Letting $n \to \infty$ in (3.7), it follows from $\liminf_{n\to\infty}(1-\alpha_n)\beta_n > 0$ and $\lim_{n\to\infty} \alpha_n = 0$ that

$$\lim_{n\to\infty} \| \nabla f(x_n) - \nabla f\big(T(x_n)\big) \| = 0.$$

Moreover, we have that $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. This together with $x_n \to \hat{\omega}$ implies that $\hat{\omega} \in \tilde{F}(T)$. In view of $\tilde{F}(T) = F(T)$, one has $\hat{\omega} \in \mathrm{EP}(H) \cap F(T)$. Therefore, the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to a point $\hat{\omega}$ in $\mathrm{EP}(H) \cap F(T)$. This completes the proof. □

Now, we prove Theorem 3.1 by using lemmas.

Proof of Theorem 3.1 From Lemmas 2.6 and 2.9, it follows that $\mathrm{EP}(H) \cap F(T)$ is a nonempty, closed and convex subset of E. Clearly, C_n and Q_n are closed and convex and so $C_n \cap Q_n$ are closed and convex for all $n \geq 0$.

Now, we show that $\mathrm{EP}(H) \cap F(T) \subseteq C_n \cap Q_n$ for all $n \geq 0$. Take $\omega \in \mathrm{EP}(H) \cap F(T)$ arbitrarily. Then

$$\begin{aligned} D_f(\omega, u_n) &= D_f\bigl(\omega, \mathrm{Res}^f_H(y_n)\bigr) \\ &\leq D_f(\omega, y_n) - D_f\bigl(\mathrm{Res}^f_H(y_n), y_n\bigr) \\ &\leq D_f(\omega, y_n) \\ &= D_f\bigl(\omega, \nabla f^*\bigl(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)\bigr)\bigr) \\ &\leq \alpha_n D_f(\omega, x_0) + (1-\alpha_n) D_f(\omega, z_n) \\ &= \alpha_n D_f(\omega, x_0) + (1-\alpha_n) D_f\bigl(\omega, \nabla f^*\bigl(\beta_n \nabla f\bigl(T(x_n)\bigr) + (1-\beta_n)\nabla f(x_n)\bigr)\bigr) \\ &\leq \alpha_n D_f(\omega, x_0) + (1-\alpha_n)\bigl[\beta_n D_f\bigl(\omega, T(x_n)\bigr) + (1-\beta_n) D_f(\omega, x_n)\bigr] \\ &\leq \alpha_n D_f(\omega, x_0) + (1-\alpha_n) D_f(\omega, x_n), \end{aligned}$$

which implies that $\omega \in C_n$ and so $\mathrm{EP}(H) \cap F(T) \subseteq C_n$ for all $n \geq 0$.

Next, we prove that $\mathrm{EP}(H) \cap F(T) \subseteq Q_n$ for all $n \geq 0$. Obviously, $\mathrm{EP}(H) \cap F(T) \subseteq Q_0$ ($Q_0 = C$). Suppose that $\mathrm{EP}(H) \cap F(T) \subseteq Q_k$ for all $k \geq 0$. In view of $x_{k+1} = \mathrm{proj}^f_{C_k \cap Q_k}(x_0)$, it follows from Lemma 2.8 that

$$\bigl\langle \nabla f(x_0) - \nabla f(x_{k+1}), x_{k+1} - v \bigr\rangle \geq 0, \quad \forall v \in C_k \cap Q_k.$$

Moreover, one has

$$\bigl\langle \nabla f(x_0) - \nabla f(x_{k+1}), x_{k+1} - \omega \bigr\rangle \geq 0, \quad \forall \omega \in \mathrm{EP}(H) \cap F(T)$$

and so, for each $\omega \in \mathrm{EP}(H) \cap F(T)$,

$$\bigl\langle \nabla f(x_0) - \nabla f(x_{k+1}), \omega - x_{k+1} \bigr\rangle \leq 0.$$

This implies that $\mathrm{EP}(H) \cap F(T) \subseteq Q_{k+1}$. To sum up, we have $\mathrm{EP}(H) \cap F(T) \subseteq Q_n$ and so

$$\mathrm{EP}(H) \cap F(T) \subseteq C_n \cap Q_n, \quad \forall n \geq 0.$$

This together with $\mathrm{EP}(H) \cap F(T) \neq \emptyset$ yields that $C_n \cap Q_n$ is a nonempty, closed convex subset of C for all $n \geq 0$. Thus $\{x_n\}$ is well defined and, from both Lemmas 3.2 and 3.3, the sequence $\{x_n\}$ is a Cauchy sequence and converges strongly to a point $\hat{\omega}$ of $\mathrm{EP}(H) \cap F(T)$.

Finally, we now prove that $\bar{\omega} = \mathrm{proj}^f_{\mathrm{EP}(H)\cap F(T)}(x_0)$. Since $\mathrm{proj}^f_{\mathrm{EP}(H)\cap F(T)}(x_0) \in \mathrm{EP}(H) \cap F(T)$, it follows from $x_{n+1} = \mathrm{proj}^f_{(C_n \cap Q_n)}(x_0)$ that

$$D_f(x_{n+1}, x_0) \leq D_f\bigl(\mathrm{proj}^f_{\mathrm{EP}(H)\cap F(T)}(x_0), x_0\bigr).$$

Thus, by Lemma 2.5, we have $x_n \to \mathrm{proj}^f_{\mathrm{EP}(H)\cap F(T)}(x_0)$ as $n \to \infty$. Therefore, the sequence $\{x_n\}$ converges strongly to the point $\mathrm{proj}^f_{\mathrm{EP}(H)\cap F(T)}(x_0)$. This completes the proof. □

Remark 3.1 (1) If $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then the weak Bregman relatively nonexpansive mapping is reduced to the weak relatively nonexpansive mapping defined by Su *et al.* [32], that is, T is called a *weak relatively nonexpansive mapping* if the following conditions are satisfied:

$$\tilde{F}(T) = F(T) \neq \emptyset, \qquad \phi(u, Tx) \leq \phi(u, x), \quad \forall x \in C, u \in F(T),$$

where $\phi(y,x) = \|y\|^2 - 2\langle y, J(x)\rangle + \|x\|^2$ for all $x, y \in E$ and J is the normalized duality mapping from E to 2^{E^*};

(2) If $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then Algorithm 3.1 is reduced to the following iterative algorithm.

Algorithm 3.2 Step 1: Select an arbitrary starting point $x_0 \in C$, let $Q_0 = C$ and $C_0 = \{z \in C : \phi(z, u_0) \leq \phi(z, x_0)\}$.

Step 2: Given the current iterate x_n, calculate the next iterate as follows:

$$\begin{cases} z_n = J^{-1}(\beta_n J(T(x_n)) + (1-\beta_n)J(x_n)), \\ y_n = J^{-1}(\alpha_n J(x_0) + (1-\alpha_n)J(z_n)), \\ u_n = \operatorname{Res}_H^f(y_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle J(x_0) - J(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0. \end{cases} \tag{3.8}$$

(3) Particularly, if $\mathrm{EP}(H) = C$, then Algorithm 3.2 is reduced to the following iterative algorithm.

Algorithm 3.3 Step 1: Select an arbitrary starting point $x_0 \in C$, let $Q_0 = C$ and $C_0 = \{z \in C : \phi(z, u_0) \leq \phi(z, x_0)\}$.

Step 2: Given the current iterate x_n, calculate the next iterate as follows:

$$\begin{cases} z_n = J^{-1}(\beta_n J(T(x_n)) + (1-\beta_n)J(x_n)), \\ y_n = J^{-1}(\alpha_n J(x_0) + (1-\alpha_n)J(z_n)), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \alpha_n \phi(z, x_0) + (1-\alpha_n)\phi(z, x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle J(x_0) - J(x_n), z - x_n\rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0. \end{cases} \tag{3.9}$$

(4) If $Tx = x$ for all $x \in C$, then, by Algorithm 3.3, we can get the following modified Mann iteration algorithm for the equilibrium problem (1.1).

Algorithm 3.4 Step 1: Select an arbitrary starting point $x_0 \in C$, let $Q_0 = C$ and $C_0 = \{z \in C : \phi(z, u_0) \leq \phi(z, x_0)\}$.

Step 2: Given the current iterate x_n, calculate the next iterate as follows:

$$\begin{cases} y_n = J^{-1}(\alpha_n J(x_0) + (1-\alpha_n)J(x_n)), \\ u_n = \operatorname{Res}_H^f(y_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z,u_n) \leq \alpha_n \phi(z,x_0) + (1-\alpha_n)\phi(z,x_n)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle J(x_0) - J(x_n), z - x_n \rangle \leq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad \forall n \geq 0. \end{cases} \tag{3.10}$$

If $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then, by Theorem 3.1 and Remark 3.1, the following results hold.

Corollary 3.1 *Let C be a nonempty, closed convex subset of a real reflexive Banach space E. Suppose that $H : C \times C \to R$ satisfies Assumption* 2.1 *and $T : C \to C$ is a weak relatively nonexpansive mapping such that* $\mathrm{EP}(H) \cap F(T) \neq \emptyset$. *Then the sequence $\{x_n\}$ generated by Algorithm* 3.2 *converges strongly to the point* $\Pi_{\mathrm{EP}(H)\cap F(T)}(x_0)$, *where* $\Pi_{\mathrm{EP}(H)\cap F(T)}(x_0)$ *is the generalized projection of C onto* $\mathrm{EP}(H) \cap F(T)$.

Corollary 3.2 *Let C be a nonempty, closed convex subset of a real reflexive Banach space E. Let $T : C \to C$ be a weak relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm* 3.3 *converges strongly to the point* $\Pi^f_{F(T)}(x_0)$, *where* $\Pi_{F(T)}(x_0)$ *is the generalized projection of C onto $F(T)$.*

Corollary 3.3 *Let C be a nonempty, closed convex subset of a real reflexive Banach space E. Suppose that $H : C \times C \to R$ satisfies Assumption* 2.1 *such that* $\mathrm{EP}(H) \neq \emptyset$. *Then the sequence $\{x_n\}$ generated by Algorithm* 3.4 *converges strongly to the point* $\Pi_{\mathrm{EP}(H)}(x_0)$, *where* $\Pi_{\mathrm{EP}(H)}(x_0)$ *is the generalized projection of C onto* $\mathrm{EP}(H)$.

Remark 3.2

(1) It is well known that any closed and firmly nonexpansive-type mapping (see [11, 33]) is a weak Bregman relatively nonexpansive mapping whenever $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$. If $\beta_n \equiv 1$ for all $n \geq 0$ and E is a uniformly convex and uniformly smooth Banach space, then Corollary 3.2 improves [11, Corollary 3.1];

(2) If $\alpha_n \equiv 0$ for all $n \geq 0$ and E is a uniformly convex and uniformly smooth Banach space, then Corollary 3.2 is reduced to [32, Theorem 3.1];

(3) If $\beta_n \equiv 1 - \beta'_n$ for all $n \geq 0$, $\beta'_n \in [0,1]$, $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$ and E is a uniformly convex and uniformly smooth Banach space, then Corollary 3.1 improves [11, Theorem 4.1].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

Author details
[1]Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX 78363, USA. [2]Department of Mathematics, Faculty of Sciences, King Abdulaziz University, Jeddah, 21589, Saudi Arabia. [3]College of Sciences, Northwest A&F University, Yangling, Xianyang, Shaanxi 712100, China. [4]Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju, 660-701, Korea.

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