Brezin-Gross-Witten model as "pure gauge" limit of Selberg integrals

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ABSTRACT: The AGT relation identifies the Nekrasov functions for various $\mathcal{N} = 2$ SUSY gauge theories with the 2d conformal blocks, which possess explicit Dotsenko-Fateev matrix model (β-ensemble) representations the latter being polylinear combinations of Selberg integrals. The "pure gauge" limit of these matrix models is, however, a non-trivial multiscaling large-N limit, which requires a separate investigation. We show that in this pure gauge limit the Selberg integrals turn into averages in a Brezin-Gross-Witten (BGW) model. Thus, the Nekrasov function for pure SU(2) theory acquires a form very much reminiscent of the AMM decomposition formula for some model X into a pair of the BGW models. At the same time, X, which still has to be found, is the pure gauge limit of the elliptic Selberg integral. Presumably, it is again a BGW model, only in the Dijkgraaf-Vafa double cut phase.

Contents

1 Introduction

The pure gauge limit. The AGT relation [1]–[70] is an explicit formulation of duality between the $2d$ and $4d$ descriptions of conformal $6d$ theory of self-dual 2-forms, compactified on a Riemann surface [71]. The theory of the corresponding $M5$-brane is long known to be related to integrability theory [72–75], but explicit route from integrability to the AGT relation still remains a mystery.

A promising approach to origins of the AGT relations is through their reformulation as relations between matrix models [76–79] and Seiberg-Witten theory [80–83], see [84, 85] for a concise review of this idea (which is a new application of the topological recursion [86–89]–[98]).

Surprisingly or not, despite obvious conceptual advantages of such an approach, some simple properties of original AGT relations are not so easy to describe in the matrix-model reformulation. A typical example is the "pure gauge" limit (PGL), where the dimensional transmutation takes place and the conformal invariance gets broken. In the matrix model formulation, this corresponds to a non-trivial double-scaling large N limit of the relevant matrix models, which will be the subject of the present paper.

In this paper, we concentrate on the simple case of pure SU(2) Nekrasov function, which (in the context of the AGT relations) arises as the PGL of either the 4-point conformal block $B^{(0)}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; \Delta, c|x)$ on sphere [9, 10] or the 1-point conformal block

$B^{(1)}(\Delta_{\text{ext}}; \Delta, c|q)$ on torus (where $q = e^{2\pi i\tau}$ and τ is the modulus) [34, 70]:

$$B_*(\Delta|\Lambda) = \lim_{\substack{\Delta_1,\Delta_2,\Delta_3,\Delta_4\to\infty, \ x\to 0 \\ x(\Delta_2-\Delta_1)(\Delta_3-\Delta_4)\equiv\Lambda^4}} \left[B^{(0)}(\Delta_1, \Delta_2, \Delta_3, \Delta_4; \Delta, c|x) \right] =$$

$$= \lim_{\substack{\Delta_{\text{ext}}\to\infty, \ q\to 0 \\ q\Delta_{\text{ext}}^2\equiv\Lambda^4}} \left[B^{(1)}(\Delta_{\text{ext}}; \Delta, c|q) \right] \qquad (1.1)$$

Our aim is to study this pure gauge limit at the level of matrix models.

PGL in the matrix model formulation. Fortunately, matrix model (i.e. the Dotsenko-Fateev-like β-ensemble) representations are already known both for $B^{(0)}$ [18–24, 70] and for $B^{(1)}$ [64, 65]. The first one is represented [53] as an AMM decomposition [90, 91, 98] into two spherical Selberg integrals

$$B^{(0)}(x) = \exp\left(-2\beta \sum_{k=1}^{\infty} \frac{x^k}{k} \left[\frac{v_+}{2\beta} + \frac{\partial}{\partial t_k^+} \right] \left[\frac{v_-}{2\beta} + \frac{\partial}{\partial t_k^-} \right] \right)$$
$$Z_S^{(0)}\left(u_+, v_+, N_+\big|t^+\right) Z_S^{(0)}\left(u_-, v_-, N_-\big|t^-\right)\Big|_{t=0} \qquad (1.2)$$

while the second one is the single, but elliptic (toric) Selberg integral

$$B^{(1)}(q) = \prod_{p=1}^{\infty} (1 - q^p)^{\nu} \ Z_S^{(1)}\left(A, N\big|q\right) \qquad (1.3)$$

Then, eqs. (1.1), (1.2) and (1.3) imply that

$$B_*(\Delta|\Lambda) = \exp\left(-2\beta \sum_{k=1}^{\infty} \frac{\Lambda^{4k}}{k} \frac{\partial^2}{\partial t_k^+ \partial t_k^-} \right) Z_{*+}^{(0)}\left(t^+\right) Z_{*-}^{(0)}\left(t^-\right)\Big|_{t=0} = Z_*^{(1)}(A|\Lambda) \qquad (1.4)$$

and this is the straightforward matrix model formulation of eq. (1.1). The partition functions $Z_{*\pm}^{(0)}$ and $Z_*^{(1)}$ denote here what we call the PGL of the corresponding Selberg integrals. To describe this PGL, explicit expressions for the Selberg integrals are needed. Note that the second equality in (1.4) automatically provides an AMM decomposition formula for $Z_*^{(1)}$ into a pair of the $Z_*^{(0)}$ models, even before explicit expressions are discussed.

The Selberg integrals. Explicitly, the Selberg integrals have a form of the eigenvalue β-ensembles

$$Z_S^{(0)}\left(u, v, N\big|t\right) = \frac{1}{S_0} \int_0^1 dz_1 \ldots dz_N \prod_{i<j}^N (z_i - z_j)^{2\beta} \prod_{i=1}^N z_i^u (z_i - 1)^v e^{\beta \sum_{k=1}^{\infty} t_k z_i^k} \qquad (1.5)$$

$$Z_S^{(1)}\left(A, N\big|q\right) = \frac{1}{S_1} \int_0^{2\pi} dz_1 \ldots dz_N \prod_{i<j}^N \theta^*\left(z_i - z_j\big|q\right)^{2\beta} \prod_{i=1}^N \theta^*\left(z_i\big|q\right)^{-\beta N} e^{IAz_i} \qquad (1.6)$$

where $I = \sqrt{-1}$ and $\theta^*(z|q) = \sin(z/2) - q\sin(3z/2) + \ldots$ is the normalized odd theta-function on torus [64, 65]. The normalization constants S_0, S_1 are needed to satisfy the requirements $Z_S^{(0)}(t = 0) = Z_S^{(1)}(q = 0) = 1$ implied by the conditions $B^{(0)}(x = 0) = B^{(1)}(q = 0) = 1$ for properly normalized conformal blocks. It is these β-ensembles that we will use to study the pure gauge limit.

Parameters of Selberg integrals.   To study the PGL in terms of the Selberg integrals, one needs to describe clearly the values of their parameters. In the spherical case [18–24, 70], they are given by

$$N_+ = \frac{\alpha_1 - \alpha_2 + \alpha - \epsilon_1 - \epsilon_2}{\epsilon_1}, \qquad u_+ = 2\frac{\alpha_1 - \epsilon_1 - \epsilon_2}{\epsilon_2}, \qquad v_+ = -2\frac{\alpha_2}{\epsilon_2} \qquad (1.7)$$

$$N_- = \frac{\alpha_4 - \alpha_3 - \alpha}{\epsilon_1}, \qquad u_- = 2\frac{\alpha_4 - \epsilon_1 - \epsilon_2}{\epsilon_2}, \qquad v_- = -2\frac{\alpha_3}{\epsilon_2} \qquad (1.8)$$

and, in the toric case [64, 65], they are given by

$$N = -\frac{\alpha_{\text{ext}}}{\epsilon_1}, \quad A = \frac{2\alpha + \epsilon_1 + \epsilon_2}{\epsilon_2} \equiv \frac{2a}{\epsilon_2}, \quad \nu = 3\Delta_{\text{ext}} + 3N - 1 \qquad (1.9)$$

where the α-parameters are related to the initial Δ-parameters (conformal dimensions) via

$$\Delta(\alpha) = \frac{\alpha(\epsilon_1 + \epsilon_2 - \alpha)}{\epsilon_1 \epsilon_2} \qquad (1.10)$$

and

$$c = 1 - 6\left(\frac{\epsilon_1 + \epsilon_2}{\epsilon_1 \epsilon_2}\right)^2 \qquad (1.11)$$

is the central charge. Here it is convenient to write all these formulas in terms of Nekrasov's parameters ϵ_1, ϵ_2, which are in one-to-one correspondence with the matrix model parameters β (the power of the Vandermonde determinants) and $g_s = g$ (the "string" coupling constant, aka the genus expansion parameter):

$$\epsilon_1 = -g\sqrt{\beta}, \quad \epsilon_2 = g/\sqrt{\beta} \qquad (1.12)$$

or vice versa

$$g^2 = -\epsilon_1 \epsilon_2, \quad \beta = -\frac{\epsilon_1}{\epsilon_2} \qquad (1.13)$$

This completes the list of relations between the parameters, and allows one to look at the PGL of the Selberg integrals. This limit is simple in terms of the external dimensions:

$$\alpha_i \longrightarrow \infty, \qquad\qquad \alpha_i^4 x = \text{fixed} = \Lambda^4 \qquad (1.14)$$
$$\alpha_{\text{ext}} \longrightarrow \infty, \qquad\qquad \alpha_{\text{ext}}^4 q = \text{fixed} = \Lambda^4$$

while in terms of the matrix model parameters it gets more sophisticated. We will now describe the limit in terms of N, u, v (for the spherical Selberg models) and of N, A (for the elliptic model).

PGL of Selberg integrals.   Relations (1.7)–(1.8) imply that, in the PGL, the parameters u, v, N of the spherical Selberg integrals all tend to infinity. However, the same relations indicate that a particular combination of parameters, that is, $u + v + 2\beta N$ remains finite in the PGL, since it does not depend on the external dimensions. We find it most convenient to parametrize this combination by a single variable n

$$\text{PGL}(u + v + 2\beta N) = \beta n + \beta - 1 \qquad (1.15)$$

which is equal to

$$n_\pm = \pm \frac{2a}{\epsilon_1}, \quad a = \alpha - \frac{\epsilon_1 + \epsilon_2}{2} \tag{1.16}$$

for our "+" and "-" Selberg models. Consequently, the PGL for the spherical Selberg model looks like a non-trivial double scaling limit, where the parameters u, v, N tend to infinity in a particular way:

$$Z_*^{(0)}(n|t) = \lim_{\substack{u,v,N \to \infty \\ u+v+2\beta N \equiv \beta n + \beta - 1}} Z_S^{(0)}\left(u, v, N \,\Big|\, \frac{t_k}{(uN + \beta N^2)^k}\right) \tag{1.17}$$

Note that the additional rescaling the time-variables $t_k \mapsto t_k(uN + \beta N^2)^{-k}$ is necessary to suppress a growth of correlators in the model: only with variables defined in this way, the partition function has a finite PGL. In particular, only with such a rescaling of variables the decomposition formula (1.2) remains non-trivial in the PGL (1.14) and, moreover, turns into formula (1.4), i.e.

$$Z_{*\pm}^{(0)}(t^\pm) = Z_*^{(0)}\big(n_\pm \big| t^\pm\big) \tag{1.18}$$

Similarly, in the toric case, relations (1.9) imply that the PGL for the toric Selberg integral is

$$Z_*^{(1)}(A|\Lambda) = \lim_{\substack{N \to \infty, \ q \to 0 \\ q\beta^2 N^4 \equiv \Lambda^4}} Z_S^{(1)}\big(A, N \big| q\big) \tag{1.19}$$

where, since no time-variables are introduced, no additional rescalings are required. As one can see, the PGL's of the Selberg models are quite sophisticated: it is by no means transparent that eqs. (1.17) or (1.19) do at all have a finite limit. However, as we shall see below, they do, and the main problem is to give some constructive description of this limit. This paper is devoted to finding a (at least, partial) solution to this problem.

PGL of spherical Selberg: BGW model. As the first (simplest) part of solution to this problem, in this paper we demonstrate that $Z_*^{(0)}$, the PGL of the Selberg partition function $Z_S^{(0)}$ is actually the partition function of the (β-deformed) celebrated BGW model [98–101] of size n and in the character phase [101]:

$$Z_*^{(0)}\big(n|t\big) = Z_{\mathrm{BGWc}}\Big(n|t_k = \mathrm{tr}\,\Psi^k/k\Big) = \frac{1}{\mathrm{Vol}_\beta(n)} \int_{n \times n} [dU]_\beta e^{\mathrm{tr}\,U^+ + \mathrm{tr}\,\Psi U} = \tag{1.20}$$

$$= 1 + \frac{1}{\beta n + 1 - \beta} t_1 + \frac{\beta n + 2 - 2\beta}{(\beta n + 1 - \beta)(\beta n + 2 - \beta)(\beta n + 1 - 2\beta)} t_1^2$$

$$- \frac{1}{(\beta n + 1 - \beta)(\beta n + 2 - \beta)(\beta n + 1 - 2\beta)} t_2 + \ldots$$

where the integral over U is the β-deformed unitary integral, and $\mathrm{Vol}_\beta(n)$ is the β-deformed volume of the unitary group. As usual for the BGW model, the time-variables are identified with traces of the external field powers $t_k = \mathrm{tr}\,\Psi^k/k$, and this brings us directly to the topic

of β-ensembles with external fields, which is somewhat underinvestigated and not exhaustively covered in the existing (physical) literature. The point is that the β-deformations are usually defined for integrals of eigenvalues only. Whereas the notion of trace (of determinant, etc.) in (1.20) remains well-defined as combinations of the eigenvalues, the treatment of the external field term $\mathrm{tr}\,U\Psi$ in (1.20) deserves some comments. Actually one needs only the U-integrals (averages) of such quantities, they will be defined in section 2 with the help of a β-ensemble version of the Harish-Chandra-Itzykson-Zuber integral.

In this paper we prove eq. (1.20) in two independent, but complementary ways. After the β-unitary BGW model is defined in section 2, in section 3 we demonstrate that its Jack expansion (the β-ensemble counterpart of the character expansion) coincides with the PGL of the Jack expansion for the spherical Selberg model. This is just an algebraic exercise, which still may be not too much transparent. A more conceptual way may be the method of section 4, where one instead takes the PGL of the Virasoro constraints (Ward identities) for the spherical Selberg model, and then shows that they coincide with the known Virasoro constraints [98, 101] for the character phase of the BGW β-ensemble.

Formula (1.20) fully defines the middle part of formula (4). It still remains to explain how the time derivatives can be taken in the external field BGW model: the simplest possibility is provided by a Fourier-like transform in the β-character calculus, which is reminded in section 2.

PGL of elliptic Selberg: double BGW model. The second, harder part of the solution, taking the PGL of the elliptic Selberg model is not completely finalized in the present paper. The corresponding partition function $Z_*^{(1)}(A|\Lambda)$ should be given by some β-ensemble with the partition function

$$Z_*^{(1)}(A|\Lambda) = 1 + \frac{2\beta\Lambda^4}{A^2 - (\beta-1)^2} + \frac{\beta^2\Lambda^8\left(2A^2 + \beta - 8(\beta-1)^2\right)}{\left(A^2 - (\beta-1)^2\right)\left(A^2 - (2\beta-1)^2\right)\left(A^2 - (\beta-2)^2\right)} + \ldots \quad (1.21)$$

Note that the problem of finding the PGL of the elliptic Selberg model is just the same as finding the PGL $B_*(\Delta|\Lambda)$ of the conformal block/Nekrasov function. Having in mind the general context of the problem and lessons from the (successful) solution in the spherical case, one can go further in several directions.

One way to solve this problem is to attack it directly from the elliptic side, by trying to take the Inozemtsev limit [102, 103] of the Ward identities for the elliptic Selberg model $Z_*^{(1)}$. This remains to be done.

Another way is to make a direct educated guess for what the β-ensemble in question should be. Such an attempt has actually been made long ago in [104]. The conjecture was that $Z_*^{(1)}$ is again a BGW model, but this time in another phase: the double-cut DV phase, i.e.

$$Z_*^{(1)} \overset{\text{presumably}}{=} Z_{\mathrm{BGW}}^{\mathrm{DV}_2} \quad (1.22)$$

This suggestion can be checked, for example, by a derivation of AMM decomposition of $Z_{\mathrm{BGW}}^{\mathrm{DV}_2}$ and comparison with the second equality in (1.4). This also remains to be done.

The third possibility is to apply the character calculus of [105–109], either to the elliptic Selberg integral itself or to the decomposition formula (1.4). This is what we do in the present paper, in section 5 below. Applying the character calculus to the decomposition formula (1.4), one obtains a double β-unitary-ensemble

$$Z_*^{(1)}(A|\Lambda) = \int \frac{[dU]_\beta}{\mathrm{Vol}_\beta\,(n_+)} \int \frac{[d\tilde{U}]_\beta}{\mathrm{Vol}_\beta\,(n_-)}$$
$$Z_{\mathrm{BGWc}}\left(m_+ \Big| s_k\right)\, Z_{\mathrm{BGWc}}\left(m_- \Big| \tilde{s}_k\right) \det\left(1 - \Lambda^4 U^+ \otimes \tilde{U}^+\right)^{2\beta} \quad (1.23)$$

where $ks_k = \mathrm{tr}\,U^k, k\tilde{s}_k = \mathrm{tr}\,\tilde{U}^k$ and $m_\pm = n_\pm + (\beta-1)/\beta$. Thus, Z_{BGW}'s in the integrand are actually functions of U and $\tilde{U}$, while their conjugates $U^\dagger$ and $\tilde{U}^\dagger$ enter through the mixing (intertwining) determinant. Note that the β-unitary integrals in (1.23) have sizes $n_\pm = \pm 2a/\epsilon_1$, while the BGW models in the integrand have sizes $m_\pm = \pm 2\alpha/\epsilon_1$. As explained in section 5, this shift of sizes can be seen as a natural property of the Fourier transformation for the β-ensembles.

Eq. (1.23) or, at least, the first terms of its Λ-expansion (1.21) can be checked in practice by substituting (1.20) into (1.23) and taking the remaining averages

$$\int\limits_{n\times n} f(U)[dU]_\beta \equiv \int\limits_{-\pi}^{\pi} d\phi_1 \ldots \int\limits_{-\pi}^{\pi} d\phi_n f\left(e^{I\phi_1}, \ldots, e^{I\phi_n}\right) \prod_{a<b}^{n} \left| e^{I\phi_a} - e^{I\phi_b}\right|^{2\beta} \quad (1.24)$$

which is valid when $f(U)$ is an invariant function (depends only on eigenvalues or, what is the same, on traces of powers of U). The averages, necessary to reproduce the first two orders in Λ, are

$$\int\limits_{n\times n} \frac{[dU]_\beta}{\mathrm{Vol}_\beta\,(n)} \mathrm{tr}\,U \mathrm{tr}\,(U^+) = \frac{n}{\beta n + \beta - 1},$$
$$\int\limits_{n\times n} \frac{[dU]_\beta}{\mathrm{Vol}_\beta\,(n)} \mathrm{tr}\,U^2 \mathrm{tr}\,(U^+)^2 = \frac{2n(\beta^2 n^2 - \beta^2 + \beta - 1)}{(\beta n + \beta)(\beta n - 1)(\beta n + \beta - 1)} \quad (1.25)$$
$$\int\limits_{n\times n} \frac{[dU]_\beta}{\mathrm{Vol}_\beta\,(n)} \mathrm{tr}\,U \mathrm{tr}\,U \mathrm{tr}\,(U^+)^2 = \int\limits_{n\times n} \frac{[dU]_\beta}{\mathrm{Vol}_\beta\,(n)} \mathrm{tr}\,U^2 \mathrm{tr}\,(U^+) \mathrm{tr}\,(U^+) =$$
$$= \frac{2n(\beta - 1)}{(\beta n + \beta)(\beta n - 1)(\beta n + \beta - 1)} \quad (1.26)$$
$$\int\limits_{n\times n} \frac{[dU]_\beta}{\mathrm{Vol}_\beta\,(n)} (\mathrm{tr}\,U)^2 (\mathrm{tr}\,U^+)^2 = \frac{2n(\beta n^2 - 1)}{(\beta n + \beta)(\beta n - 1)(\beta n + \beta - 1)} \quad (1.27)$$

Using them, one easily verifies that (1.23) reproduces (1.21), so that operationally the model is well-defined and gives correct results. However, there are many conceptual questions left, in particular, the relation of the model (1.23) to the Inozemtsev limit of the elliptic Selberg model and, most importantly, to the double-cut BGW integrals. If the DV conjecture (1.22) is true, the model (1.23) should probably be interpreted as an integral representation of $Z_{\mathrm{BGW}}^{\mathrm{DV}_2}$. We do not consider this topic in the present paper, this will be done elsewhere.

2 Comments on the definition of β-deformed BGW model

Generalization of unitary integrals to $\beta \neq 1$ deserves comments. The problem is to give some concrete definition for the β-deformed unitary integral of the form

$$\int\limits_{n \times n} f(U)[dU]_\beta \tag{2.1}$$

which for $\beta = 1$ is the well-defined integral over the compact Lie group U(n) with the Haar invariant measure $[dU]_{\beta=1}$. We are unaware of any similar group theory definition for *generic* β (occasionally, such a definition exists for $\beta = 1/2$ and $\beta = 2$ in terms of the groups $O(n)$ and Sp(n), respectively). Instead, various other, more or less natural definitions can be suggested.

The simplest case is when $f(U)$ is an invariant function, i.e. it depends only on the traces of powers of U or, equivalently, only on the eigenvalues of U. In this case, the most natural definition of β-generalization, motivated by consideration of the above-mentioned three group integrals (U(n) for $\beta = 1$, $O(n)$ for $\beta = 2$ and Sp(n) for $\beta = 1/2$) is the following eigenvalue integral:

$$\int\limits_{n \times n} f(U)[dU]_\beta \equiv \int\limits_{-\pi}^{\pi} d\phi_1 \ldots \int\limits_{-\pi}^{\pi} d\phi_n \ f\left(\mathrm{diag}\left[e^{I\phi_1}, \ldots, e^{I\phi_n}\right]\right) \prod_{a<b}^{n} \left|e^{I\phi_a} - e^{I\phi_b}\right|^{2\beta} \tag{2.2}$$

i.e. the role of the β-deformation is just to have the power 2β of the Van-der-monde determinant. It is this definition which is most commonly recalled when the words "β-ensemble" are mentioned.

However, for the purpose of present paper this definition *is not enough*. The case when the integrand $f(U)$ depends only on the eigenvalues, does *not* cover *all* the *eigenvalue models* [76–79, 113–116]. In particular, the main object of the present paper, which we use to describe the PGL of the Selberg integrals, has an integrand which is *not* a function of the eigenvalues of U only, it involves an "external field" matrix Ψ in the following way

$$Z_{\mathrm{BGW}}(\Psi) = \int\limits_{n \times n} \frac{[dU]_\beta}{\mathrm{Vol}_\beta(n)} e^{\mathrm{tr}\, U^\dagger + \mathrm{tr}\, \Psi U} \tag{2.3}$$

so that definition (2.2) is not applicable. Note that for $\beta = 1$ the integral $Z_{\mathrm{BGW}}(\Psi)$ obviously depends only on the eigenvalues of the matrix Ψ. We can quite naturally assume the same property to hold for all β. After that, the two options still remain: to consider the integral as a function of traces of *positive* powers $kt_k = \mathrm{tr}\, \Psi^k$ or of *negative* powers $k\tau_k = \mathrm{tr}\, \Psi^{-k}$. As is well-known in the theory of the ordinary ($\beta = 1$) BGW model, these two choices lead to different results, commonly known as the *character phase* $Z_{\mathrm{BGWc}}(t)$ and the *Kontsevich phase* $Z_{\mathrm{BGWk}}(\tau)$, respectively [101]. In this paper we are interested in the definition of the first one, $Z_{\mathrm{BGWc}}(t)$.

Our definition refers to the Harish-Chandra-Itzykson-Zuber (HCIZ) integral over the unitary matrix V, $Z_{\mathrm{IZ}}(t,s) = \int\limits_{n \times n} \frac{[dV]_\beta}{\mathrm{Vol}_\beta(n)} e^{\mathrm{tr}\, \Psi V U V^\dagger}$ which is a function of $kt_k = \mathrm{tr}\, \Psi^k$ and

$ks_k = \operatorname{tr} U^k$. To see this at $\beta = 1$, it is enough to diagonalize the matrices, $\Psi = V_\Psi \Psi_d V_\Psi^\dagger$ and $U = V_U U_d V_U^\dagger$ and use invariance of the Haar measure $[dV]$ to change $V \longrightarrow V_\Psi^\dagger V V_U$. Then

$$Z_{\mathrm{IZ}}(t,s) = \int\limits_{n \times n} \frac{[dV]_\beta}{\operatorname{Vol}_\beta(n)} e^{\operatorname{tr} \Psi V U V^\dagger} = \int\limits_{n \times n} \frac{[dV]_\beta}{\operatorname{Vol}_\beta(n)} e^{\sum_{i,j} \Psi_i U_j |V_{ij}|^2} \tag{2.4}$$

is indeed a function of the eigenvalues $\{\Psi_i\}$ and $\{U_j\}$ only, and thus of t and s (the symmetry under permutation of the eigenvalues is obvious). Once Z_{IZ} is defined, one can write

$$Z_{\mathrm{BGWc}}(t) = \int\limits_{n \times n} \frac{[dU]_\beta}{\operatorname{Vol}_\beta(n)} Z_{\mathrm{IZ}}(t,s) e^{tr U^\dagger} \tag{2.5}$$

which is now a function of the positive time-variables t. Moreover, the integrand also depends only on the eigenvalues of the integration matrix-variable U, so that definition (2.2) is applicable.

Thus, to define $Z_{\mathrm{BGWc}}(t)$, it suffices to define the Itzykson-Zuber integral (2.4) in some independent way. Such a possibility is provided by the character calculus [105–109]: for $\beta = 1$, the IZ integral can be defined as an expansion

$$Z_{\mathrm{IZ}}(t,s) = \sum_R \frac{\chi_R(t_k = \delta_{k,1})}{\chi_R(t_k = n/k)} \chi_R(t) \chi_R(s), \quad \beta = 1 \tag{2.6}$$

where χ_R are the characters. Thus, for arbitrary β one can naturally *define*

$$Z_{\mathrm{IZ}}(t,s) = \sum_R \frac{j_R(t_k = \delta_{k,1})}{\beta^{|R|} j_R(t_k = n/k)} j_R(t) j_R(s), \quad \forall \beta \tag{2.7}$$

where j_R are the well-known β-characters, actually, the properly normalized Jack polynomials [110, 111], reviewed in detail in appendix 1 of the present paper. The values of characters in the special points $t_k = \delta_{k,1}$ and $t_k = n/k$ are standardly denoted d_R and D_R correspondingly. Note that the n-dependence in formula (2.7) emerge only due to the n-dependent quantity $j_R(t_k = n/k)$. Note also that the manifest β-factor in (2.7) is due to the chosen coefficient in the exponential of the IZ integral: it would disappear if one puts the coefficient β in front of the trace in (2.4).

Eqs. (2.5) and (2.7) provide our *constructive* definition of the BGW partition function in the character phase. It should be emphasized, that *in this way the β-deformation of any unitary eigenvalue model with no more than one external field in character phase can be defined*. This is enough for our purposes here, but not enough in principle: to study the β-deformations of the Kontsevich phases or the β-deformations of the models with multiple external fields (known as non-eigenvalue models), some other definitions have to be invented. Hopefully, there exists a unifying framework for the β-deformations of *any* matrix model, not necessarily of the eigenvalue type. This framework remains to be discovered.

Note that definition (2.7) already appeared in mathematical literature, see, for example, [112]. It goes without saying that it properly defines the HCIZ integrals not only for unitary ($\beta = 1$), but also for orthogonal ($\beta = 1/2$) and symplectic ($\beta = 2$) matrices.

Given such a definition of Z_{BGWc} β-ensemble, it is possible to study if it indeed coincides with the PGL $Z_*^{(0)}$ of the spherical Selberg β-ensemble. The next two sections are devoted to two ways of proving this. The first method makes a direct use of properties of the β-characters j_R, while the second method relies upon Ward identities for the β-ensembles.

3 Eq. (1.20) via Jack expansion

To check the equality $Z_*^{(0)} = Z_{\mathrm{BGWc}}$, we expand the both quantities in the basis of Jack polynomials.

The right hand side. For Z_{BGWc}, eqs. (2.5) and (2.7) imply that

$$Z_{\mathrm{BGWc}}(t) = \int Z_{\mathrm{IZ}}(t,s) e^{trU^\dagger} [dU]_\beta = \sum_R \frac{j_R(t)}{[\beta n]_R} \int j_R(s) e^{trU^\dagger} [dU]_\beta \tag{3.1}$$

where $[\ldots]_R$ is a convenient notation for the ratio of two particular β-characters,

$$[\beta n]_R = \frac{\beta^{|R|} \chi_R(t_k = n/k)}{\chi_R(t_k = \delta_{k,1})} \tag{3.2}$$

To deal with the integral at the r.h.s., one needs just two properties of the β-characters: the completeness

$$\exp\left(\beta \sum_k k t_k \tilde{t}_k\right) = \sum_R j_R(t) j_R(\tilde{t}) \tag{3.3}$$

and the Haar orthogonality

$$\int j_R(s) j_{\tilde{R}}(s^{-1}) [dU]_\beta = \delta_{R,\tilde{R}} \frac{[\beta n]_R}{[\beta n + 1 - \beta]_R} \tag{3.4}$$

where n is the size of U. The first identity (with $k t_k = \mathrm{tr}\,(U^\dagger)^k, k\tilde{t}_k = \delta_{k,1}/\beta$) implies that

$$e^{\mathrm{tr}\,U^\dagger} = \sum_R \beta^{-|R|} j_R(s^{-1}) j_R(\delta_{k,1}) \tag{3.5}$$

and the second property allows one to perform the integration over U:

$$Z_{\mathrm{BGWc}}(t) = \int Z_{\mathrm{IZ}}(t,s) e^{trU^\dagger} [dU]_\beta = \sum_R \sum_{\tilde{R}} \frac{j_R(t) j_{\tilde{R}}(\delta_{k,1})}{[\beta n]_R} \int j_R(s) j_{\tilde{R}}(s^{-1}) [dU]_\beta$$

$$= \sum_R \frac{j_R(\delta_{k,1})}{[\beta n + 1 - \beta]_R} j_R(t) \tag{3.6}$$

The expansion obtained

$$Z_{\mathrm{BGWc}}(t) = \sum_R \frac{j_R(\delta_{k,1})}{[\beta n + 1 - \beta]_R} j_R(t) \tag{3.7}$$

is of course a β-deformation of the well-known character expansion [105–109]

$$Z_{\mathrm{BGWc}}(t) = \sum_R \frac{\chi_R(\delta_{k,1})}{[n]_R} \chi_R(t), \qquad \beta = 1 \tag{3.8}$$

but not quite a naive one (because of the non-trivial $(1-\beta)$ shift in the denominator). Note that the matrix sizes enter the character expansions only through the explicit factors $[\beta n + 1 - \beta]_R$, the β-deformations of $[n]_R$. Let us now do find a similar expansion for the Selberg partition function.

The left hand side. The Jack expansion for the Selberg β-ensemble

$$Z_S^{(0)}(u,v,N|t) = \frac{1}{S_0} \int\limits_0^1 dz_1 \dots dz_N \prod_{i<j}^N (z_i - z_j)^{2\beta} \prod_{i=1}^N z_i^u (z_i - 1)^v e^{\beta \sum_{k=1}^\infty t_k z_i^k} \tag{3.9}$$

using the same completeness condition (3.3) can be written as

$$Z_S^{(0)}(u,v,N|t) = \sum_R \left\langle j_R \right\rangle j_R(t) \tag{3.10}$$

where the coefficients $\left\langle j_R \right\rangle$ are averages of the Jack polynomials of the z-variables,

$$\left\langle j_R \right\rangle = \frac{1}{S_0} \int\limits_0^1 dz_1 \dots dz_N \; j_R \left(t_k = \sum_i z_i^k / k \right) \prod_{i<j}^N (z_i - z_j)^{2\beta} \prod_{i=1}^N z_i^u (z_i - 1)^v \tag{3.11}$$

Fortunately, the averages of Jack polynomials in the Selberg model are well-known to be simple quantities [117, 118]. They factorize nicely, and can be generally expressed by the Kadell formula

$$\left\langle j_R \right\rangle = j_R(\delta_{k,1}) \frac{[N\beta + 1]_R [u + (N-1)\beta + 2]_R}{[u + v + 2N\beta + 2 - 2\beta]_R} \tag{3.12}$$

Using the explicit product representation (see appendix 1)

$$[x]_R = \prod_{(i,j) \in R} \Big(x - \beta(i-1) + (j-1) \Big) \tag{3.13}$$

it becomes immediate to take the PGL:

$$\left\langle j_R \right\rangle_* = j_R(\delta_{k,1}) \frac{(uN + \beta N^2)^{|R|}}{[\beta n + 1 - \beta]_R} \tag{3.14}$$

As one can see, the correlators grow like $(uN + \beta N^2)^{|R|}$ in the PGL. This is actually the reason to introduce the corresponding rescaling of time-variables in the PGL of the partition function:

$$Z_*^{(0)}(n|t) = \lim_{\substack{u,v,N\to\infty \\ u+v+2\beta N \equiv \beta n + \beta - 1}} Z_S^{(0)} \left(u,v,N \Big| \frac{t_k}{(uN+\beta N^2)^k} \right) = \sum_R \left\langle j_R \right\rangle_* \frac{j_R(t)}{(uN+\beta N^2)^{|R|}} \tag{3.15}$$

Substituting here eq. (3.14), one finds

$$Z_*^{(0)}(n|t) = \sum_R \frac{j_R(\delta_{k,1})}{[\beta n + 1 - \beta]_R} j_R(t) \tag{3.16}$$

what precisely coincides with (3.7). In this way, the PGL limit of the Kadell formula reproduces the BGW model: we conclude that $Z_*^{(0)} = Z_{\text{BGWc}}$, since their Jack expansions are just the same. Let us now pass to the second method of proving this statement.

4   Eq. (1.20) via Virasoro constraints

Another way to see that $Z_*^{(0)} = Z_{\text{BGW}}$ is to study the PGL of the Virasoro constraints for the Selberg model. Just as any matrix model (or the β-ensemble), the Selberg model can be characterized by certain linear differential equations, which arise as a consequence of the reprarametrization invariance of the multiple integral (i.e. as the Ward identities). In the case of the Selberg model, these linear differential equations have the form

$$\left[\left(u + v + 2\beta N + (k+1)(1-\beta) \right) \frac{\partial}{\partial t_k} \right.$$
$$\left. + \beta \sum_m m t_m \frac{\partial}{\partial t_{k+m}} + \sum_{a+b=k} \frac{\partial^2}{\partial t_a \partial t_b} + v \sum_{h=1}^{k-1} \frac{\partial}{\partial t_h} \right] Z_S(t_1, t_2, \ldots) =$$
$$= \beta \left(N + \sum_{i=1}^{m-1} k_i + uN + N(N-1)\beta \right) Z_S(t_1, t_2, \ldots), \quad k > 0 \tag{4.1}$$

Their derivation is given in appendix 2. These equations completely determine the partition function of the Selberg model, therefore, one can take the PGL in these equations, not in the integral. In the PGL, all the three parameters u, v, N of the Selberg models tend to infinity in such a way that their combination $u + v + 2\beta N$ is held finite. In this limit, the Virasoro constraints get simplified (many terms can be thrown out) and turn into

$$\left[\left(u + v + 2\beta N - (k+1)(\beta - 1) \right) \frac{\partial}{\partial t_k} \right.$$
$$\left. + \beta \sum_m m t_m \frac{\partial}{\partial t_{k+m}} + \sum_{a=1}^{k-1} \frac{\partial^2}{\partial t_a \partial t_{k-a}} + \beta \delta_{k,1} \right] Z_*^{(0)}(t) = 0, \quad k > 0 \tag{4.2}$$

and these are precisely the Ward identities for integral (2.5). They can be found in [98, 101], of course, only for the most popular case of $\beta = 1$. Despite there are no doubts that eqs. (4.2) hold for arbitrary β, in principle, it would be nice to derive them directly from the definition of the BGW β-ensemble (in the present paper, the role of such a definition is played by the character calculus). This remains to be done.

5 Eq. (1.23) via Fourier transform

Definition (3.7) can be directly used to convert the action of the intertwining operator in (1.4) into a kind of a 2-matrix BGW model. This can be done via the Fourier transform, widely used in the character calculus [105–109]. The basic and most important Fourier relation has the form

$$j_R(t) = \frac{[\beta n + 1 - \beta]_R}{[\beta n]_R} \int\limits_{n \times n} \frac{[dU]_\beta}{\text{Vol}_\beta(n)} j_R(s) \exp\left(\beta \sum_k k t_k s_k\right) \tag{5.1}$$

where $k s_k = \text{tr}\, U^k$. This relation is a direct consequence of the completeness condition (3.3) and the Haar orthogonality (3.4): one substitutes (3.3) into the r.h.s. and calculates integral with the help of (3.4). Contracting both sides of this relation with the coefficients of the BGW expansion, one finds

$$Z_{\text{BGWc}}(n|t) = \sum_R \frac{j_R(\delta_{k,1})}{[\beta n + 1 - \beta]_R} j_R(t)$$

$$= \int\limits_{n \times n} \frac{[dU]_\beta}{\text{Vol}_\beta(n)} \sum_R \frac{j_R(\delta_{k,1})}{[\beta n]_R} j_R(s) \exp\left(\beta \sum_k k t_k s_k\right) \tag{5.2}$$

and the sum in the r.h.s. is the BGW partition function again, only of a different size:

$$Z_{\text{BGWc}}(n|t) = \int\limits_{n \times n} \frac{[dU]_\beta}{\text{Vol}_\beta(n)} Z_{\text{BGWc}}\left(n + \frac{\beta - 1}{\beta}\Big|s\right) \exp\left(\beta \sum_k k t_k s_k\right) \tag{5.3}$$

This relation can be understood as a Fourier transform for the BGW β-ensemble. It is important for two reasons.

First, it allows one to write t-derivatives of the partition function (correlators) as integrals: say,

$$\frac{\partial^m}{\partial t_{k_1} \dots \partial t_{k_m}} Z_{\text{BGWc}}(n|t)\Big|_{t=0} = \int\limits_{n \times n} \frac{[dU]_\beta}{\text{Vol}_\beta(n)} Z_{\text{BGWc}}\left(n + \frac{\beta - 1}{\beta}\Big|s\right) \text{tr}\, U^{k_1} \dots \text{tr}\, U^{k_m} \tag{5.4}$$

whereas without the Fourier transform one could only write

$$\frac{\partial^m}{\partial t_{k_1} \dots \partial t_{k_m}} Z_{\text{BGWc}}(n|t)\Big|_{t=0} = \frac{\partial^m}{\partial t_{k_1} \dots \partial t_{k_m}} \int\limits_{n \times n} \frac{[dU]_\beta}{\text{Vol}_\beta(n)} e^{\text{tr}\, U^+ + \text{tr}\, U\Psi}\Big|_{\Psi=0} \tag{5.5}$$

which is largely a symbolical notion: the integrand in the r.h.s. does not depend on the t-variables, only the integral does. Thus, the Fourier transform appears to be a convenient tool in this case.

Second, the Fourier transform allows one to convert the decomposition operator into an integral. Indeed, we have

$$Z_*^{(1)} = \exp\left(-2\beta \sum_{k=1}^\infty \frac{\Lambda^{4k}}{k} \frac{\partial^2}{\partial t_k^+ \partial t_k^-}\right) Z_{\text{BGWc}}(n_+|t^+) Z_{\text{BGWc}}(n_-|t^-)\Big|_{t=0} \tag{5.6}$$

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Substituting each of the BGW partition functions by its Fourier transform, one then finds

$$Z_*^{(1)} = \int \frac{[dU]_\beta}{\text{Vol}_\beta\,(n_+)} \int \frac{[d\tilde{U}]_\beta}{\text{Vol}_\beta\,(n_-)}$$

$$Z_{\text{BGWc}}\left(m_+\Big|s_k\right)\ Z_{\text{BGWc}}\left(m_-\Big|\tilde{s}_k\right) \det\left(1 - \Lambda^4 U^+ \otimes \tilde{U}^+\right)^{2\beta} \quad (5.7)$$

where $m_\pm = n_\pm + (\beta - 1)/\beta$ are the shifted sizes. Such a shift looks like a natural property of the Fourier transforms for the β-ensembles. For $\beta = 1$, the shift vanishes.

6 Conclusion

In this paper, we have obtained two BGW-model-representations of the Selberg models (β-ensembles), elementary constitutients of the conformal blocks, in the pure gauge limit. The spherical Selberg model after the PGL turned out to become the character phase of the BGW model, while the elliptic Selberg model after the PGL is converted into some other model X. In this paper we succeeded in rewriting X as a double BGW model, but we still expect some simpler representations for this model. A possible candidate on the role of such a simpler representation is the double-cut BGW model, suggested long ago by Dijkgraaf and Vafa [104]. If this is correct, then the double BGW model of the present paper is yet another integral representation for this double-cut model.

Acknowledgments

A Properties of Jack polynomials

The Jack polynomials form an important class of symmetric polynomials, which are often useful in calculations with arbitrary β-ensembles, not only of the Dotsenko-Fateev type. For convenience, in this appendix we list several basic formulas related to the Jack polynomials, which are well-known but scattered in the literature.

1. *Symmetric polynomials.* Symmetric polynomials $f(z_1, \ldots, z_N)$ of given degree $\deg f$ form a linear space of finite dimension, with the basis vectors labeled by the Young diagrams $Y = Y_1 \geq Y_2 \geq \ldots$. Frequently used bases are the following: the power sums $s_Y = \prod_i s_{Y_i}$, where $s_k(z) = \sum_i z_i^k$ are the Newton power sums; the elementary symmetric polynomials $e_Y = \prod_i e_{Y_i}$, where $e_k = $ the coefficient of x^k in $\prod_i(1 + xz_i)$;

the monomial functions m_Y = symmetrization of $\prod_i z_i^{Y_i}$. The transition matrices between these bases have the form

$$e_1 = m_1, \qquad\qquad e_{11} = m_2 + 2m_{11}, \qquad e_2 = m_{11},$$
$$e_{111} = m_3 + 3m_{21} + 6m_{111}, \qquad e_{21} = m_{21} + 3m_{111}, \qquad e_3 = m_{111}, \qquad \cdots$$
$$m_1 = s_1, \qquad\qquad m_{11} = s_{11}/2 - s_2/2, \qquad m_2 = s_2,$$
$$m_{111} = s_{111}/6 - s_{21}/2 + s_3/3, \quad m_{21} = s_{21} - s_3, \qquad m_3 = s_3, \qquad \cdots$$
$$s_1 = e_1, \qquad\qquad s_{11} = e_{11}, \qquad s_2 = e_{11} - 2e_2,$$
$$s_{111} = e_{111}, \qquad\qquad s_{21} = e_{111} - 2e_{21}, \qquad s_3 = e_{111} - 3e_{21} + 3e_3, \quad \cdots$$

2. *Jack polynomials as eigenfunctions.* The Jack polynomials J_Y are the eigenfunctions

$$\hat{W} J_Y = \sum_i Y_i \big(Y_i - 2\beta i - 1 \big) J_Y \tag{A.1}$$

of the $\hat{W}$-like operator

$$\hat{W} = \sum_{k=1}^{\infty} \Big(k(k-1) - \beta k(k+1) \Big) s_k \frac{\partial}{\partial s_k}$$
$$+ \sum_{k,m=1}^{\infty} \left(km s_{k+m} \frac{\partial^2}{\partial s_k \partial s_m} + \beta(k+m) s_k s_m \frac{\partial}{\partial s_{k+m}} \right) \tag{A.2}$$

normalized with a condition $J_Y = m_Y + \ldots$ (i.e. the coefficient in front of m_Y is equal to unity).

3. *Several first Jack polynomials.* Explicitly, they have the form

$$J_1(s_k) = s_1$$
$$J_2(s_k) = \frac{s_2 + \beta s_{11}}{\beta + 1}, \qquad\qquad J_{11}(s_k) = \frac{1}{2} s_{11} - s_2$$
$$J_3(s_k) = \frac{2s_3 + 3\beta s_{21} + \beta^2 s_{111}}{(\beta + 1)(\beta + 2)}, \qquad J_{21}(s_k) = \frac{(1 - \beta)s_{21} - s_3 + \beta s_{111}}{(\beta + 1)(\beta + 2)},$$
$$J_{111}(s_k) = \frac{1}{6} s_{111} - \frac{1}{2} s_{21} + \frac{1}{3} s_3$$

and so on.

4. *The Young factors and Young-Pochhammer symbol.* A set of quantities, frequently encountered in almost any application of the Jack polynomials, are the two "Young normalization factors"

$$P_Y = \prod_{(i,j) \in Y} \Big(\beta(Y_j' - i) + (Y_i - j) + \beta \Big) \tag{A.3}$$

$$Q_Y = \prod_{(i,j) \in Y} \Big(\beta(Y_j' - i) + (Y_i - j) + 1 \Big) \tag{A.4}$$

and the "Young-Pochhammer symbol"

$$[x]_Y = \prod_{(i,j)\in Y} \Big(x - \beta(i-1) + (j-1) \Big) \tag{A.5}$$

where Y is any Young diagram and Y' stands for its transposed diagram. Note that

$$P_Y = \frac{\beta^{|Y|}}{J_Y(s_k = \delta_{k,1})} \tag{A.6}$$

The Young factors provide the β-deformations of the product of hook lengths of Y, an important quantity in representation theory of the symmetric group. The Young-Pochhammer symbol is a generalization of the classical Pochhammer falling and rising factorials from the undergraduate calculus

$$(x)_n = x(x-1)\dots(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)} \tag{A.7}$$

$$(x)^{(n)} = x(x+1)\dots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{A.8}$$

which simply correspond to the row and coloumn diagrams Y, respectively.

5. *Relation between Young-Pochhammer symbol and Jack polynomials.* It has the form

$$J_Y\Big(s_k = N\Big) = \frac{[\beta N]_Y}{P_Y} \tag{A.9}$$

where N is an arbitrary number. Equivalently, this may be understood as a formula

$$[x]_Y = P_Y J_Y\Big(s_k = \frac{x}{\beta}\Big) \tag{A.10}$$

for the Young-Pochhammer symbol in terms of the Jack polynomials.

6. *Jack polynomials as orthogonal polynomials.* The Jack polynomials are orthogonal

$$\Big\langle J_A \Big| J_B \Big\rangle = \delta_{AB} \frac{Q_A}{P_A} \tag{A.11}$$

with respect to the combinatorial inner product

$$\Big\langle s_A \Big| s_B \Big\rangle \equiv \prod_j \frac{B_j}{\beta} \frac{\partial}{\partial s_{B_j}} \prod_i s_{A_i} \Big|_{p=0} \tag{A.12}$$

which is often called the *intersection product*. Note that it becomes normalized to unity at $\beta = 1$.

7. *Relation to the β-unitary integrals.* The Jack polynomials are also orthogonal w.r.t. the product

$$\Big\langle s_A \Big| s_B \Big\rangle_N \equiv \frac{\int s_A(U) s_B(U^+) [dU]_\beta}{\int 1 \, [dU]_\beta}$$

$$= \frac{\int\limits_{-\pi}^{\pi} d\phi_1 \dots \int\limits_{-\pi}^{\pi} d\phi_N \prod\limits_{a<b} \left| e^{i\phi_a} - e^{i\phi_b} \right|^{2\beta} \, s_A(e^{i\phi}) \, s_B(e^{-i\phi})}{\int\limits_{-\pi}^{\pi} d\phi_1 \dots \int\limits_{-\pi}^{\pi} d\phi_N \prod\limits_{a<b} \left| e^{i\phi_a} - e^{i\phi_b} \right|^{2\beta}} \tag{A.13}$$

which is the β-deformed unitary integral. The norm is

$$\left\langle J_A \middle| J_B \right\rangle_N = \frac{\int J_A(U) J_B(U^+)[dU]_\beta}{\int 1 \, [dU]_\beta} = \delta_{AB} \frac{Q_A}{P_A} \frac{[\beta N]_A}{[\beta N + \beta - 1]_A} \qquad \text{(A.14)}$$

Note that

$$\left\langle J_A \middle| J_B \right\rangle = \lim_{N \to \infty} \left\langle J_A \middle| J_B \right\rangle_N \qquad \text{(A.15)}$$

Note also that the product becomes normalized to unity at $\beta = 1$.

8. *β-deformed Cauchy identity.* Another useful formula is the β-deformed Cauchy identity:

$$\exp\left(\sum_k \frac{\beta}{k} s_k \tilde{s}_k\right) = \sum_A \frac{P_A}{Q_A} J_A(s) J_A(\tilde{s}) \qquad \text{(A.16)}$$

9. *β-deformed IZ integral.* Yet another useful formula is the β-deformed analogue of the IZ integral:

$$\frac{\int e^{\text{tr} \, (XUYU^+)}[dU]_\beta}{\int 1 \, [dU]_\beta} = \sum_A \frac{1}{Q_A} \frac{J_A\left(s_k = \sum X_i^k\right) J_A\left(s_k = \sum Y_i^k\right)}{J_A\left(s_k = N\right)} \qquad \text{(A.17)}$$

10. *Carlsson shift identity.* A useful formula is the Carlsson identity:

$$\left\langle J_A\left(s_k - (-1)^k \frac{\beta - m - 1}{\beta}\right) \middle| J_B\left(s_k - (-1)^k \frac{m}{\beta}\right) \right\rangle = \prod_{(i,j) \in A} \left(m + \beta(A_j' - i) + (B_i - j) + 1\right)$$
$$\prod_{(i,j) \in B} \left(-m + \beta(B_j' - i) + (A_i - j) + \beta\right) \quad \text{(A.18)}$$

11. *Normalized Jack polynomials (β-characters).* The polynomials J_Y can be normalized as

$$J_Y = \sqrt{\frac{Q_Y}{P_Y}} j_Y \qquad \text{(A.19)}$$

These j_Y are orthonormal w.r.t. the combinatorial product:

$$\left\langle j_A \middle| j_B \right\rangle = \delta_{AB} \qquad \text{(A.20)}$$

In general, formulas in terms of j_Y are simpler than formulas in terms of J_Y. However, the polynomials j_Y themselves are more complicated and depend on β irrationally.

B Derivation of Ward identities for the Selberg model

The Ward identities can be most simply derived by requiring the vanishing of the integral of full derivative:

$$\int\limits_0^L dz_1 \ldots dz_N \sum_{a=1}^N \frac{\partial}{\partial z_a} z_a^{k_m+1} \left(\prod_{i<j}(z_i - z_j)^{2\beta} \prod_{i=1}^N z_i^u (z_i - L)^v \, s_{k_1} \ldots s_{k_{m-1}} \right) = 0,$$

$$s_k = \sum_i z_i^k, \quad k_m > 0 \quad \text{(B.1)}$$

where L is an auxiliary parameter introduced for the purpose of derivation of the Ward identities (we put $L = 1$ in the end). Whenever possible, we substitute the zero indices of the correlators with appropriate powers of N, in order not to work with the t_0 variable. Differentiating the expression in the brackets, and using for $k_m > 0$ the relations

$$\sum_{a=1}^N z_a^{k_m+1} \frac{\partial}{\partial z_a} \prod_{i<j}(z_i - z_j)^{2\beta} = \beta \left(-(k_m+1)s_{k_m} + \sum_{p=0}^{k_m} s_p s_{k_m-p} \right) \prod_{i<j}(z_i - z_j)^{2\beta} \quad \text{(B.2)}$$

$$\sum_{a=1}^N z_a^{k_m+1} \frac{\partial}{\partial z_a} \prod_i z_i^u = u s_{k_m} \prod_i z_i^u \quad \text{(B.3)}$$

$$\sum_{a=1}^N z_a^{k_m+1} \frac{\partial}{\partial z_a} \prod_i (z_i - L)^v = v \sum_{a=1}^N \frac{z_a^{k_m+1}}{z_a - L} \prod_i (z_i - L)^v$$

$$= \left(v \sum_{h=0}^{k_m} L^{k_m-h} s_h - L^{k_m+1} \partial_L \right) \prod_i (z_i - L)^v \quad \text{(B.4)}$$

$$\sum_{a=1}^N z_a^{k_m+1} \frac{\partial}{\partial z_a} s_l = l s_{k_m+l} \quad \text{(B.5)}$$

one finds

$$\left(u + (k_m+1)(1-\beta) \right) \tilde{C}_{k_1 \ldots k_m} + \sum_{i=1}^{m-1} k_i \tilde{C}_{k_1 \ldots k_i + k_m \ldots k_{m-1}} + \beta \sum_{p=0}^{k_m} \tilde{C}_{k_1 \ldots k_{m-1}, k_m - p, p}$$

$$+ v \sum_{h=0}^{k_m} L^{k_m-h} \tilde{C}_{k_1 \ldots, k_{m-1} h} - L^{k_m} \left(L \partial_L \right) \tilde{C}_{k_1 \ldots k_{m-1}} = 0 \quad \text{(B.6)}$$

where

$$\tilde{C}_{k_1 \ldots k_m}(N) = \int\limits_0^L dz_1 \ldots dz_N \prod_{i<j}(z_i - z_j)^{2\beta} \prod_{i=1}^N z_i^u (z_i - L)^v \, s_{k_1} \ldots s_{k_m} \quad \text{(B.7)}$$

are the Selberg integrals with an additional parameter L. Actually, from now on this parameter is not needed: it was only useful in derivation of the Ward identities. Using the

obvious homogeneity property

$$L\partial_L \tilde{C}_{k_1...k_m} = \left(N + \sum_{i=1}^{m-1} k_i + uN + vN + N(N-1)\beta\right) \tilde{C}_{k_1...k_m} \qquad (B.8)$$

and putting $L = 1$, one finds that the original integrals $C_{k_1...k_m}(N) = \tilde{C}_{k_1...k_m}(N)\big|_{L=1}$ satisfy the Ward identities

$$\left(u + v + 2\beta N + (k_m+1)(1-\beta)\right)C_{k_1...k_m} + \sum_{i=1}^{m-1} k_i C_{k_1...k_i+k_m...k_{m-1}} + \beta\sum_{p=1}^{k_m-1} C_{k_1...k_{m-1},k_m-p,p}$$

$$+ v\sum_{h=1}^{k_m-1} C_{k_1...k_{m-1},h} - \left(N + \sum_{i=1}^{m-1} k_i + uN + N(N-1)\beta\right)C_{k_1...k_{m-1}} = 0 \quad (B.9)$$

Note that we explicitly moved the two contributions $\beta N C_{k_1...k_m}$ (which arise at particular values of $p = 0$ and $p = k_m$ in the third term) into the first term. Similarly, we explicitly moved the contributions $vC_{k_1...k_m}$ and $vNC_{k_1...k_{m-1}}$ into the first and the last terms, respectively. All these trivial transformations are necessary to get rid of the zero indices in the correlators and, hence, of presence of the t_0 variable in the partition function.

Because of the obvious formula

$$\frac{1}{S}C_{k_1...k_m} = \left(\frac{1}{\beta}\frac{\partial}{\partial t_{k_1}}\right)\ldots\left(\frac{1}{\beta}\frac{\partial}{\partial t_{k_m}}\right)Z_S(t)\bigg|_{t=0} \qquad (B.10)$$

the same relations can be rewritten as differential equations known as generalized Virasoro constraints:

$$\left[\left(u + v + 2\beta N + (k+1)(1-\beta)\right)\frac{\partial}{\partial t_k} + \beta\sum_m mt_m\frac{\partial}{\partial t_{k+m}} + \sum_{a+b=k}\frac{\partial^2}{\partial t_a\partial t_b} + v\sum_{h=1}^{k-1}\frac{\partial}{\partial t_h}\right]$$

$$Z_S(t_1,t_2,\ldots) = \beta\left(N + \sum_{i=1}^{m-1} k_i + uN + N(N-1)\beta\right)Z_S(t_1,t_2,\ldots), \quad k > 0 \quad (B.11)$$

This completes the derivation of the Virasoro constraints for the Selberg model.

The trick (B.1) with insertion of a new dimensional parameter L does not work for the elliptic Selberg integral, as it does not work in the original Dotsenko-Fateev integrals: dimensionless parameters are present in the both cases. Still analogues of the Virasoro constraints in these both cases exist, they will be considered and analyzed elsewhere.

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