Adequate subgroups II

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Abstract   The notion of adequate subgroups was introduced by Thorne (J Inst Math
Jussieu. arXiv:1107.5989, to appear). It is a weakening of the notion of big subgroup
used by Wiles and Taylor in proving automorphy lifting theorems for certain Galois
representations. Using this idea, Thorne was able to prove some new lifting theorems.
It was shown in Guralnick et al. (J Inst Math Jussieu. arXiv:1107.5993, to appear)
that certain groups were adequate. One of the key aspects was the question of whether
the span of the semisimple elements in the group is the full endomorphism ring of an
absolutely irreducible module. We show that this is the case in prime characteristic p
for p-solvable groups as long the dimension is not divisible by p. We also observe that
the condition holds for certain infinite groups. Finally, we present the first examples
showing that this condition need not hold and give a negative answer to a question of
Richard Taylor.

1 Introduction

Let k be a field of characteristic p and let V be a finite dimensional vector space over
k. Let $\rho : G \to \mathrm{GL}(V)$ be an absolutely irreducible representation. Following [12],

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we say (G, V) is *adequate* if the following conditions hold (we rephrase the conditions slightly):

(1) $H^1(G, k) = 0$;
(2) p does not divide dim V;
(3) $H^1(G, V \otimes V^*) = 0$; and
(4)   End(V) is spanned by the elements $\rho(g)$ with $\rho(g)$ semisimple.

If G is a finite group of order prime to p (or G is an algebraic or Lie group in characteristic zero), then it is well known that (G, V) is adequate. In this case, condition (4) is often referred to as Burnside's lemma. It is a trivial consequence of the Artin–Wedderburn Theorem.

These conditions are a weakening of the conditions used by Wiles and Taylor in studying the automorphic lifts of certain Galois representations. Thorne [12] generalized various results assuming these hypotheses. We refer the reader to [12] for more references and details.

In particular, it was shown in [6, Theorem 9] that:

Theorem 1.1 *Let k be a field of characteristic p and G a finite group. Let V be an absolutely irreducible faithful kG-module. If $p \geq 2 \dim V + 2$, then (G, V) is adequate.*

The proof depends on the classification of finite simple groups. The main ingredients include a result of the author [5] that reduces to the problem to the case that the subgroup of G generated by elements of order p is a central product of quasisimple finite groups of Lie type in characteristic p, a result of Serre [10] about complete reducibility of tensor products and results on the representation theory of the groups of Lie type in the natural characteristic [9].

In this note, we consider (4) and show that this holds under some conditions (none of these results depend upon the classification of finite simple groups). We say that (G, V) is *weakly adequate* if (4) holds.

Recall that a finite group is called p-solvable if every composition factor of G either has order p or order prime to p. It is known (cf. [8, Theorem B]) that if G is p-solvable and V is an absolutely irreducible G-module in characteristic p, then G contains an absolutely irreducible p'-subgroup, whence Burnside's lemma immediately implies:

Theorem 1.2 *Let G be a p-solvable subgroup, k a field of characteristic p and V an absolutely irreducible kG-module. If p does not divide* dim V*, then (G, V) is weakly adequate.*

This allows us to answer in the affirmative a question of R. Taylor for p-solvable groups.

Corollary 1.3 *Let G be a p-solvable subgroup, k a field of characteristic p and V an absolutely irreducible kG-module. If (G, V) satisfies conditions (1), (2) and (3) above, then (G, V) is adequate.*

Recall that a kG-module V is called primitive if G preserves no nontrivial direct sum decomposition of V.

We can also show:

Theorem 1.4 *Let G be a p-solvable subgroup, k a field of characteristic p and V an absolutely irreducible kG-module. If V is primitive, then (G, V) is weakly adequate.*

Note that if dim V is a multiple of p, then no p'-subgroup can act irreducibly. We also can obtain some results for possibly infinite groups.

Theorem 1.5 *Let k be algebraically closed of characteristic p. Let V be finite dimensional over k. Let Γ be an irreducible subgroup of $\mathrm{GL}(V)$ with Zariski closure G. Let G^0 be the connected component of G and $\Gamma^0 = G^0 \cap \Gamma$. Assume that either:*

(1) *$[\Gamma : \Gamma^0]$ is not a multiple of p; or*
(2) *dim V is not a multiple of p and G/G^0 is p-solvable.*
(3) *V is primitive and G/G^0 is p-solvable.*

Then (G, V) is weakly adequate.

The only condition that is difficult to check for adequacy in the previous results is Condition (3). We do improve Theorem 1.1 for p-solvable groups. We first observe a result from [5].

Theorem 1.6 *Let k be a field of characteristic p. Let G be a finite subgroup of $\mathrm{GL}_n(k) = \mathrm{GL}(V)$. Assume that V is a completely reducible kG-module. If $p > n$ and is not a Fermat prime or $p > n + 1$, then G has no composition factors of order p.*

It is not difficult to extend this to the case of Zariski closed subgroups. Also, the complete reducibility hypothesis can be relaxed—all we need to assume is that G has no nontrivial normal subgroup consisting of unipotent elements. This result is not explicitly stated in [5] there but it is proved there. The result does depend upon the classification of finite simple groups (however, for p-solvable groups, it does not).

It now easily follows that if G is p-solvable and V is a completely reducible kG-module of small dimension, then G is in fact a p'-group and this gives:

Theorem 1.7 *Let k be an algebraically closed field of characteristic p. Let G be a p-solvable group. Let V be an irreducible kG-module. Then (G, V) is adequate if*:

(1) *$p > \dim V$ with p not a Fermat prime; or*
(2) *$p > \dim V + 1$.*

On the other hand, we present an infinite family of examples of imprimitive absolutely irreducible G-modules in characteristic p with dim V a multiple of p (including cases where G is p-solvable) with (G, V) not weakly adequate. These are generalizations of examples of Capdeboscq and Guralnick.

In order for this construction to give such examples where p does not divide dim V, we were led to prove the following result in [4]:

Theorem 1.8 *Let p be a prime. There exists a finite simple group G with a nontrivial Sylow p-subgroup P such that some coset of P contains no p'-elements.*

Thompson [11] verified this for $p = 2$ in response to a question of Paige.

Using a variation of the Theorem 1.8, we show that for any prime p, Taylor's question fails [i.e. (1), (2) and (3) do not necessarily imply (4)].

Note that another way to produce examples with (G, V) not weakly adequate is to find absolutely irreducible G-modules in characteristic p such that $(\dim V)^2$ is larger than the number of p'-elements in G. These examples are not so easy come by. The only primitive example we know is with $G = {}^2F_4(2)'$ (the Tits group) and V the irreducible module of dimension 2048 in characteristic 2. The number of elements of odd order in G is $3{,}290{,}625 < (2048)^2$. So (G, V) is not weakly adequate. It is easy to see that V is a primitive module (since G contains no proper subgroups of index dividing 2048).

This suggests the following variant of the problem:

Question 1.9 *Let G be a quasisimple finite group and p a prime. Classify all absolutely irreducible G-modules in characteristic p such that the number of p'-elements in G is less than* $(\dim V)^2$.

In particular, (G, V) cannot be weakly adequate. We suspect that there are very few such examples.

The paper is organized as follows. In the next section, we discuss p-solvable groups and prove Theorems 1.2 and 1.4. In the following sections we prove Theorems 1.5, 1.6 and 1.7. In the last section, we consider necessary conditions for induced modules to be weakly adequate. This allows us to construct many examples that are not weakly adequate including some whose dimension is not a multiple of the characteristic. In particular, this allows us to give a negative answer to Taylor's question.

2 p-Solvable groups

We prove Theorems 1.2 and 1.4. As noted above, the first result follows by [8, Theorem B] (see also [3]). We sketch an elementary proof of a slight generalization of what we require.

We first prove a lemma about tensor products. The first statement is well known.

Lemma 2.1 *Let G be a group with a normal subgroup N. Let k be an algebraically closed field. Let $V = U \otimes_k W$ be a finite dimensional kG-module where U and W are irreducible kG-modules. Assume that N acts irreducibly on U and trivially on W.*

(1) *V is an irreducible kG-module; and*
(2) *If N consists of semisimple elements and (G, W) is weakly adequate, then (G, V) is weakly adequate.*

Proof We prove both statements simultaneously. By assumption, $\mathrm{End}(U) \otimes kI$ is the linear span of the images of N in $\mathrm{GL}(U) \otimes kI$.

Since W is kG-irreducible, we can choose elements $g_i \in G$ such that g_i acts as $a_i \otimes b_i \in \mathrm{GL}(U) \otimes \mathrm{GL}(W)$ where the b_i form a basis for $\mathrm{End}(W)$. If (G, W) is weakly adequate, we can furthermore assume that the g_i are semisimple elements.

Thus, the images of the elements Ng_i span $\mathrm{End}(U) \otimes \mathrm{End}(W) = \mathrm{End}(V)$. This shows that V is an irreducible kG-module and that (G, V) is weakly adequate if N consists of semisimple elements and the g_i are semisimple (because then Ng_i consists of semisimple elements). □

Note that in (2) above, (G, V) weakly adequate implies that (G, W) is weakly adequate.

If p is a prime dividing $|G|$, a subgroup H is called a p-complement if p does not divide $|H|$ but $[G : H]$ is a power of p. It is an easy exercise to see that the following holds (just choose a minimal normal subgroup and apply the Schur–Zassenhaus result):

Lemma 2.2 *Let G be a p-solvable group. Any p'-subgroup of G is contained in a p-complement and all p-complements are conjugate.*

We state the next result for irreducible groups rather than absolutely irreducible groups. Most results in the literature assume the latter.

Lemma 2.3 *Let G be a p-solvable group, k a field of characteristic p and V an irreducible kG-module. Let $F = \mathrm{End}_G(V)$. Assume that p does not divide $\dim_F V$. Then a p'-complement H of G acts irreducibly on V and $F = \mathrm{End}_H(V)$.*

Proof First suppose that $k = F$ (i.e. V is absolutely irreducible). So we may assume that k is algebraically closed. We may also assume that $O_p(G) = 1$ (since this acts trivially on V). Let N be a minimal normal subgroup of G. So N is a p'-group. First suppose that N does not act homogeneously on V (i.e. N has at least two nonisomorphic simple submodules on V). Then we can write $V = \oplus_{i=1}^{t} V_i$, where the V_i are the homogeneous components of N. Let S be the stabilizer of V_1. Since $\dim V = t \dim V_1$, p is prime to both $\dim V_1$ and t. Let K be a p-complement in S and $H \geq K$ a p-complement of G. By induction, K is irreducible on V_1. Since $G = SH$ (since $[G : S] = t$ is prime to p), H acts transitively on the set of V_i.

Let W be a nonzero H-submodule of V. Since $N \leq H$, $W = \oplus(W \cap V_i)$ and since H is transitive on the V_i, we see that $W \cap V_1 \neq 0$. Since K acts irreducibly on V_1, $V_1 \leq W$ and since H is transitive on the V_i, $W = V$, whence the result.

Suppose that N acts homogeneously. It follows (cf. [2, Theorem 51.7]) that (passing to a p'-central cover if necessary), $V \cong U \otimes_k W$ where U, W are irreducible kG-modules with N irreducible on U and trivial on W. If H is a p-complement, then by induction, U and W are irreducible kH-modules. By Lemma 2.1, this implies that H acts irreducibly on V.

Now suppose that k is not F. Since G is finite, we can assume that k is a finite field. We can view V as an absolutely irreducible FG-module, By the proof above, V is absolutely irreducible as an FH-module. Thus, $F = \mathrm{End}_H(V)$. Since V is a semisimple kH-module (by Maschke's theorem) with endomorphism ring a field, V is an irreducible kH-module. □

Of course, if p does divide $\dim_F V$, then V cannot possibly be irreducible restricted to H, since the dimension of any absolutely irreducible H-module in characteristic p divides $|H|$. Isaacs [8] proves much more than we do above and in particular studies the restriction of V to H in all cases. These ideas are related to the Fong–Swan theorem: every absolutely irreducible G-module is the reduction of a characteristic zero module.

Theorem 1.2 now follows by Burnside's lemma. Theorem 1.4 now follows from the following observation:

Lemma 2.4 *Let G be a p-solvable group with k algebraically closed of characteristic p. If V is a primitive kG-module, then p does not divide* dim V.

Proof As above, we may assume that $O_p(G) = 1$. Let N be a minimal normal non-central subgroup of G. Then N is a p'-group and acts homogeneously on V. If N acts irreducibly, then dim V divides $|N|$ and the result holds. Otherwise, $V = U \otimes_k W$ where U and W are primitive kG-modules, whence the result follows by induction on dimension. □

We now give an example to show that conditions (1), (2) and (4) do not guarantee that condition (3) holds (even for solvable groups).

Let $r \neq p$ be an odd prime. Let R be an extraspecial r-group of exponent r and order r^{1+2a}. Let s be a prime distinct from p and r. Let S be an s-group with a faithful absolutely irreducible $\mathbb{F}_p S$ module W. Let X be an irreducible $\mathbb{F}_p S$-submodule of the semisimple module $W \otimes W^*$. Set $K = XS$, a semidirect product. We can choose a sufficiently large so that K embeds in $\mathrm{Sp}(2a, r)$ and so K acts as a group of auto-morphisms of R. Then $RK \leq R\mathrm{Sp}(2a, r)$ has an irreducible module U over k of dimension p^a. Set $V = U \otimes_k W$ (where we extend scalars and view W over k). Then $V \otimes V^* \cong (U \otimes U^*) \otimes (W \otimes W^*)$. Note that $V = V^R \oplus [R, V]$. and $V^R \cong W \otimes W^*$. Thus, $H^1(G, V \otimes V^*) = H^1(G/R, W \otimes W^*) \cong \mathrm{Hom}_S(X, W \otimes W^*) \neq 0$.

3 Infinite groups

Let k be an algebraically closed field of characteristic $p \geq 0$. Let Γ be an absolutely irreducible subgroup of $\mathrm{GL}_d(k) = \mathrm{GL}(V)$. Let G be the Zariski closure of Γ. Let G^0 denote the connected component of 1 in G. Set $\Gamma^0 = \Gamma \cap G^0$. Note that $G = \Gamma G^0$, whence $G/G^0 \cong \Gamma/\Gamma^0$.

We first note:

Lemma 3.1 *Let G be a reductive algebraic group over k (i.e. G^0 is reductive). Let $g_i \in G, 1 \leq i \leq r$ be such that the order of $g_i G^0/G^0$ is not a multiple of p. Then $X := \{x \in G^0| g_i x$ is semisimple$\}$ contains a Zariski open dense subset of G^0.*

Proof Since the intersection of finitely many open dense sets is open and dense, it suffices to proves this for $r = 1$. A straightforward argument reduces this to the case that G^0 is a simple algebraic group and g_1 is either inner or is in the coset of a graph automorphism. If g_1 is inner, the result follows since the set of regular semisimple elements is open and dense. If g_1 is a graph automorphism, the same is true—see [7, Lemma 6.8]. □

Applying this to the Zariski closure of Γ, we immediately obtain:

Corollary 3.2 *Let g_i be a finite set of elements of Γ such that none of the orders of $g_i \Gamma^0$ in Γ/Γ^0 are a multiple of p. Then $X := \{x \in \Gamma^0| g_i x$ is semisimple$\}$ is Zariski dense in G^0.*

In particular, this implies:

Corollary 3.3 *If k is algebraically closed of characteristic 0 and V is an irreducible finite dimensional $k\Gamma$-module, then (Γ, V) is weakly adequate.*

Lemma 3.4 *Suppose that $V = U \otimes_k W$ where U and W are irreducible finite dimensional $k\Gamma$-modules and that Γ^0 acts irreducibly on U and trivially on W. If (Γ, W) is weakly adequate, then (Γ, V) is weakly adequate.*

Proof If (Γ, W) is weakly adequate, then we can choose finitely many $g_i \in \Gamma$ semisimple with $g_i = a_i \otimes b_i \in \mathrm{GL}(U) \otimes GL(W)$ where the span of the b_i is $\mathrm{End}(W)$. Let X be the subset of Γ^0 consisting of all elements x such that $g_i x$ is semisimple for all g_i (take $g_1 = 1$). By Corollary 3.2, X is Zariski dense in G^0. Thus, the linear span of X is Zariski dense in the linear span of G^0 which is precisely $\mathrm{End}(U) \otimes kI$. Thus, $\cup_{g_i} X$ consists of semisimple elements and contains a basis for $\mathrm{End}(V) = \mathrm{End}(U) \otimes \mathrm{End}(W)$. □

We now prove Theorem 1.5.

Proof First suppose that p does not divide $[\Gamma : \Gamma^0]$. It follows by Corollary 3.2 that the set of semisimple elements of Γ contain a Zariski dense subset of G. Thus, the linear span of the semisimple elements of Γ is Zariski dense in the linear span of G. Since linear spaces are closed, it follows that the two sets have the same linear span, whence the result.

Next suppose that p does not divide d and G/G^0 is p-solvable. Let H/G^0 be a p-complement in G/G^0. The exact same proof as in the previous section shows that H is irreducible on V. Thus, $\Gamma \cap H$ (which is Zariski dense in H) is also irreducible on V. Now apply (1) to $\Gamma \cap H$.

Finally consider (3). Since V is primitive, Γ^0 acts homogeneously on V. Thus, $V = U \otimes_k W$, where U and W are irreducible $k\Gamma$-modules, Γ acts irreducibly on U and trivially on W. Since Γ/Γ^0 is p-solvable and Γ^0 is trivial on W, (Γ, W) is weakly adequate by Theorem 1.4. Now apply Lemma 3.4. □

4 Composition factors

We first prove Theorem 1.6. As we noted this is essentially in [5]. We sketch the proof indicating in particular how the classification is not required for the case of p-solvable groups.

Theorem 4.1 *Let G be a completely reducible finite subgroup of $\mathrm{GL}_n(k) = \mathrm{GL}(V)$ with k a field of characteristic p. If $H^1(G, k) \neq 0$, then either $n \geq p$ or p is a Fermat prime and $n = p - 1$.*

Proof If $p \leq 3$, then all we are asserting is that $n \geq 2$ and the result is clear. So assume that $p \geq 5$ and $p > n$ with $H^1(G, k) \neq 0$.

Let N be the normal subgroup generated by elements of order p. Then $H^1 G, k)$ embeds into $H^1(N, k)$ and so we may assume that $N = G$. Let A be a minimal normal noncentral subgroup of G. We consider four cases:

Case 1 A is an elementary abelian r-group for some prime $r \neq p$.

Then G permutes the weight spaces of A and since G is generated by elements of order p, some element of order p does not centralize A, whence it must have an orbit of size p and so $n \geq p$.

Case 2 A is of symplectic type (i.e $A/Z(A)$ is elementary abelian of order r^{2a} for some prime $r \neq p$ with $Z(A)$ of order r if r is odd or of order 2 or 4 if $r = 2$; moreover, A has exponent r is r is odd and has exponent 4 if $r = 2$).

Again, some element g of order p acts nontrivial on A. Thus, g embeds in $\mathrm{Sp}(2a, r)$, whence $p \leq r^a + 1$ with equality if and only $r = 2$ and p is a Fermat prime. Since the minimal faithful representation of A in characteristic p is r^a, the result follows.

Case 3 A is a central quotient of a direct product of quasisimple subgroups and p does not divide $|A|$.

Again some element g of order p acts nontrivially on A. If g does not preserve each quasisimple factor of A, then there are least p such factors and we easily see that $n \geq 2p$. So g normalizes each factor of A. Thus, A is quasisimple. We can assume that A acts homogeneously (and nontrivially) on V (otherwise, we may assume that g permutes the homogeneous factors and so there would be at least p of them, whence $n \geq 2p$). Since p does not divide $|A|$, it follows by Sylow's theorem, that g will normalize a Sylow r-subgroup of A for each prime r dividing $|A|$. Thus, g will act nontrivially on some Sylow r-subgroup of G and the result follows from cases 1 and 2.

Case 4 A is a central quotient of a direct product of quasisimple subgroups and p does divide $|A|$.

Unfortunately, we do not have a proof without the classification (although we suspect there is one). We argue as in case 3. Now apply [5, Theorem B] to conclude that A is of Lie type in characteristic p. It follows that g must induce a field automorphism and this forces $n \geq 2p$ (one further possibility is that $A = J_1$ with $p = 11$, but then A has no outer automorphisms of odd order). □

Now Theorem 1.6 follows immediately (if there is a composition factor of order p, there will be a normal subgroup N of G with $H^1(N, k) \neq 0$ and N is still completely reducible).

An immediate corollary is:

Corollary 4.2 *Let G be a completely reducible p-solvable subgroup of $\mathrm{GL}_n(k) = \mathrm{GL}(V)$ with k a field of characteristic p. If p divides $|G|$, then either $n \geq p$ or $n = p - 1$ with p a Fermat prime.*

We now prove Theorem 1.7.

Proof Assume that $p > \dim V$ (or $p > \dim V + 1$ if p is a Fermat prime) and that G is an irreducible subgroup of $\mathrm{GL}(V)$ as in the hypotheses. By the corollary G is in fact a p'-group, whence (G, V) is adequate. □

5 Induced modules

Let k be an algebraically closed field of characteristic $p > 0$. Suppose that $V = \mathrm{Ind}_K^G(W)$. Let g_i be a set of coset representatives for the cosets of K in G. So we can write $V = W_1 \oplus \cdots \oplus W_m$ where $m = [G : K]$ and $W_i = g_i \otimes W$.

So $\mathrm{End}(V) = \oplus_{ij}\mathrm{Hom}(W_i, W_j)$. Let π_{ij} be the corresponding projection from $\mathrm{End}(V)$ to $\mathrm{Hom}(W_i, W_j)$. Note that the set of $g \in G$ such that $\pi_{1j}(g) \neq 0$ is $g_j K$. This observation yields:

Lemma 5.1 *If (G, V) is weakly adequate, then π_{1j} maps the set of p'-elements of $g_j K$ to a spanning set of $\mathrm{Hom}(W_1, W_j)$. In particular, if some coset $g_j K$ contains no p'-elements, then (G, V) is not weakly adequate.*

Using this criterion, we can produce many examples (G, V) which are not weakly adequate. Of course, we want V to be irreducible and we also want G to be generated by p'-elements.

Here is our first family of examples.

Let H be any finite group whose order is divisible by p with H generated by its p'-elements. Let r be a prime not equal to p and let A be an irreducible H-module such that H has a regular orbit on $\mathrm{Hom}(A, k^*)$ (this can be easily arranged—if r is sufficiently large, then any faithful irreducible module A will have this property). Set $G = AH$, a semidirect product.

Let W be a 1-dimensional kA-module with character $\lambda \in \mathrm{Hom}(A, k^*)$ so that λ is in a regular G-orbit. Set $V = W_A^G$. We note that V is an irreducible kG-module of dimension equal to $|H|$ (since V is a direct sum of 1-dimensional non-isomorphic kA-modules permuted transitively by H). Clearly, G is generated by its p'-elements. If $g \in G$ has order divisible by p the coset gA has no p'-elements, whence:

Theorem 5.2 (G, V) *is not weakly adequate.*

In particular, we can take $G = AS_3$ where A is elementary abelian of order 25 with $p = 3$ and $\dim V = 6$.

In fact, we can generalize these examples. Here is the setup:

(1) Let L and T be finite groups each generated by p'-elements.
(2) Let W be an absolutely irreducible faithful kL-module.
(3) Let T_1 be a subgroup of T of index t such that T_1 contains no nontrivial normal subgroup of T and such that some coset xT_1 of T_1 in T contains no p'-elements (eg, if T_1 is a proper subgroup of a Sylow p-subgroup P of T, then let $x \in P \backslash T_1$).

Set $G = L \wr T = NT$, where $N = L_1 \times \cdots \times L_m$ with $L_i \cong L$ and $m = [T_1 : T]$ Then G acts on $V := W_1 \oplus \cdots \oplus W_t$ where $W_i \cong W$ (L_i acts as L on W_i and trivially on W_j with $j \neq i$ and T permutes the W_i as it does the coset of T_1). We can also describe V as $\mathrm{Ind}_K^G(W_1)$ where $K = NT_1$ with L_1 acting on W_1 as L does on W and $(L_2 \times \cdots L_t)T_1$ acting trivially on W_1).

Theorem 5.3 *With notation as above, V is a faithful irreducible kG-module of dimension equal to $m \dim W$, G is generated by p'-elements and (G, V) is not weakly adequate.*

Proof Since the W_i are nonisomorphic irreducible kN-modules and T permutes them transitively, V is irreducible. Since L and T are generated by p'-elements, so is G. Since T_1 contains no nontrivial normal subgroup of T, the kernel of this representation would be contained in N. Clearly N acts faithfully. Since the coset $xT_1 N$ contains no p'-elements, the result follows.   □

Using the result of [4] for any odd prime p, we can find a sufficiently large q with p exactly dividing $q - 1$ so that for $T = L_2(q)$ and T_1 a dihedral subgroup of order $2p$, we can find a $t \in T$ with tT_1 containing no p'-elements.

This allows us to give a negative answer to Richard Taylor's question.

Theorem 5.4 *Let k be an algebraically closed field of characteristic p. Let $T = L_2(q)$ and let T_1 be a subgroup of T isomorphic to a dihedral group of order $2p$ as above. Let L be a cyclic group order 2 and let W be the nontrivial 1-dimensional kL-module. Set $G = L \wr_{T_1} T$, $N = L \times \cdots \times L$ of order 2^m with $m = [T_1 : T]$. Let T_1 act trivially on W. Set $V = \mathrm{Ind}_K^G(W)$ where $K = NT_1$ Then*

(1) *V is an absolutely irreducible kG-module of dimension m (and so prime to p);*
(2) *G satisfies conditions (1), (2) and (3) of the introduction; and*
(3) *G is not adequate.*

Proof As we have seen above, the first condition holds and (G, V) is not weakly adequate by the construction Clearly p does not divide $m = \dim V$. By construction G is generated by p'-elements. So it remains to that show $H^1(G, V \otimes V^*) = 0$.

Set $U = V \otimes V^*$. Since N is a normal p'-group, it follows that $U = C_U(N) \oplus [N, U]$ where $C_U(N)$ are the fixed points of N on U and $[N, U]$ is the submodule generated by all nontrivial irreducible N-submodules. By the inflation restriction sequence, it follows that $H^1(G, [N, U]) = 0$. Note that $\dim C_U(N) = m$ and indeed $C_U(N)$ contains $U_1 := W \otimes W^*$ and the stabilizer of U_1 in G is NT_1. Thus, $C_U(N) \cong \mathrm{Ind}_K^G(k)$. So by Shapiro's lemma, $H^1(G, C_U(N)) \cong H^1(K, k) \cong H^1(T_1, k) = 0$. □

One can also produce examples showing that Taylor's question has a negative answer with $p = 2$ as well. For example, we can take $T = L_2(137)$ and $T_1 = A_4 \leq T$ and L cyclic of order 3 with W a 1-dimensional nontrivial L-module.

Here is a variation of Taylor's question:

Question 5.5 *Let V be an absolutely irreducible primitive kG-module. If (G, V) satisfies (1), (2) and (3) of the introduction, is (G, V) adequate?*

Now suppose that G is p-solvable. Let V be an irreducible kG-module. If N is a noncentral normal p'-subgroup of G that acts homogeneously, then as usual we can write $V = U \otimes_k W$. By Lemma 2.1 and the remark following it, (G, V) is weakly adequate if and only if $(G/N, W)$ is. Thus, if this is the case, the problem reduces to a smaller module. So we may assume that no noncentral normal p'-subgroup acts homogeneously. In this case, set $N = O_{p'}(G)$ (the largest normal p'-subgroup). Then $V = V_1 \oplus \cdots \oplus V_m$ with $m > 1$ where the V_i are the kN-components of V. Thus, $V = \mathrm{Ind}_K^G(V_1)$ where $N \leq K$. We ask:

Question 5.6 *If G is p-solvable and every coset gK of K contains a semisimple element, is (G, V) weakly adequate?*

If the answer is yes, then we have an essentially complete answer as to when an absolutely irreducible kG-module V is weakly adequate for G a p-solvable group.

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