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Relation Between Purity of an Open Qubit Dynamics and Its Initial Correlation with an Environment

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Abstract Purity as a quantifier of an impact of environment on an open quantum system is studied for a qubit dephasingly interacting with its environment. We analyze how time evolution of the purity depends on initial states of the composite system both in the case of infinite and finite environments. It is shown that for a certain class of initial preparations, the purity of an evolving qubit state initially correlated with infinite environment can be greater than in the case of uncorrelated qubit-environment initial preparations. We identify a class of initial states leading to such desired outcome.

Keywords Purity · Open system · Initial correlations

1 Introduction

An unavoidable presence of environment is one of the best known obstructions for effective implementation of quantum information processing [1]. According to the common wisdom the state ρ of an open quantum system becomes *mixed* due to decoherence-induced information loss. The simplest quantifier of this effect is the *purity* $\mathcal{P}[\rho]$ of a quantum state ρ defined as [2]

$$\mathcal{P}[\rho] = \mathrm{Tr}\left(\rho^2\right) \tag{1}$$

which takes its maximal value $\mathcal{P} = 1$ for pure states $\rho^2 = \rho$. When the system interacts with the infinite environment (environment in the thermodynamic sense), purity is a non-increasing function of time: open quantum systems cannot become spontaneously purified,

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i.e. open systems cannot become less mixed than initially. If the system of interest and its environment are initially in an uncorrelated state (i.e. when it is a product state) then the reduced dynamics of an open quantum system is described in terms of completely positive maps which are also contractive with respect to some distance measures. In the case of finite-dimensional Hilbert spaces, when the completely positive map is a quantum dynamical semigroup, the purity is a monotonically decreasing function of time if and only if it is unital (preserves the identity operator) [3]. In the infinite dimensional case, unitality is only sufficient [3]. If the initial state of the system and environment is correlated, the reduced dynamics need not be completely positive and the contractivity may fail [4]. However, results of the work [3] do not concern such systems and therefore studies are needed to include the case of open systems initially correlated with its environment. Effects induced by initial system-environment correlations have been studied in various context [5–16]. First experiments on initial system-environment correlations were reported in Refs. [17, 18].

The initial mixedness (i.e. how much the initial state is far from being pure) can originate both from imprecise state preparation and from system-environment correlations (entanglement). However, there is a quantitative difference: in the case of system-environment initial entanglement the information is stored not only in the state of the system but also is shared with its environment. Contrary to the case of imprecise preparation, the information is only partially lost and affects the reduced dynamics of an open system [1, 19]. Our aim is to provide an evidence that the *origin* of initial mixedness affects the loss of purity. We focus our attention on pure decoherence models for which reduced dynamics is exactly known for all values of model parameters [20]. Therefore we can consider the regime of weak and strong system-environment interaction in the presence of large initial correlations of the composite system. It is know that in such regimes, in general, the reduced dynamics exhibits non-Markovian evolution in which there is a flow of information from the environment back to the open system, indicating the occurrence of memory effects and strong dependence of evolution on initial conditions [21, 22]. We show that there are initial system-environment correlated states for which the purity of the reduced open system can be greater than in the case of initially uncorrelated system-environment states. Our results are complementary to those obtained in Ref. [23], where the damped Jaynes-Cummings model has been used as a composite system. Here, we consider both infinite and finite environments coupled to an open system.

The layout of the paper is as follows: in Sect. 2, we recall the dephasing model of decoherence for a qubit coupled to its environment. Next, in Sect. 3, we consider time evolution of the qubit purity in the case of an infinite bosonic environment with a continuum of bath modes and modelled by a bosonic field. In Sect. 4, we analyze the purity evolution in the case of a finite environment consisting of one boson. Section 5 provides a summary and conclusions.

2 Dephasing Models

We consider a two-level system (qubit, spin) S coupled to its outer environment E. We assume a pure dephasing interaction between the qubit and environment, and neglect the energy dissipation process of the qubit. This assumption is reasonable in the case when the phase coherence decays much faster than the energy. An example is a qubit constructed with a quantum dot in semiconductor devices, where the environment is a phonon system [24].

The system can be modeled by the following Hamiltonian [20, 25, 26]

$$H = \varepsilon S^z \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + S^z \otimes H_I, \tag{2}$$

where S^z is the z-component of the spin-1/2 operator and is represented by the diagonal 2×2 matrix $S^z = \mathrm{diag}[1, -1]$ of elements 1 and -1. The parameter ε is the qubit energy splitting, $\mathbb{I}_S$ and $\mathbb{I}_E$ are identity operators (matrices) in corresponding Hilbert spaces of the qubit S and the environment E, respectively. Finally, H_E is the Hamiltonian of the environment, H_I is the operator in the Hilbert space of the environment and characterizes the qubit-environment interaction.

For our purposes, it is more convenient to rewrite the spin-bosons system (2) in the so-called block operator matrix representation [41], in which the Hamiltonian (2) has a block-diagonal structure:

$$H = \mathrm{diag}[H_+, H_-], \qquad H_\pm = H_E \pm H_I \pm \varepsilon \mathbb{I}_E, \tag{3}$$

where the Hamiltonians $H_\pm$ act only on the environment Hilbert space.

2.1 Initial Pure States

We assume that at the initial time $t = 0$ the total (isolated) system $S + E$ is in a pure state characterized by the wave function $|\Psi(0)\rangle$ in the form

$$\big|\Psi(0)\big\rangle = b_+|1\rangle \otimes |\Omega_1\rangle + b_-|-1\rangle \otimes |\Omega_2\rangle. \tag{4}$$

The states $|1\rangle$ and $|-1\rangle$ denote the excited and ground state of the qubit S, respectively. The complex numbers b_+ and b_- are chosen in such a way that the condition $|b_+|^2 + |b_-|^2 = 1$ is satisfied. The normalized states $|\Omega_1\rangle$ and $|\Omega_2\rangle$ are environment states. We will consider two different initial states:

(A) If $|\Omega_1\rangle\langle\Omega_1| \neq |\Omega_2\rangle\langle\Omega_2|$, the initial state is the correlated (entangled) state. The smaller the overlap $\langle\Omega_1|\Omega_2\rangle$ is the 'more entangled' the initial state is. It is maximally entangled provided that the states $|\Omega_1\rangle$ and $|\Omega_2\rangle$ are orthogonal and $b_\pm$ are nonzero.

(B) If $|\Omega_1\rangle\langle\Omega_1| = |\Omega_2\rangle\langle\Omega_2|$, the qubit and its environment are initially in an uncorrelated (product) state

$$\big|\Psi(0)\big\rangle = \big(b_+|1\rangle + b_-|-1\rangle\big) \otimes |\Omega_1\rangle. \tag{5}$$

The initial wave function (4) of the isolated system $S + E$ evolves unitarily according to the Hamiltonian (2) and reads

$$\big|\Psi(t)\big\rangle = b_+|1\rangle \otimes \big|\psi_+(t)\big\rangle + b_-|-1\rangle \otimes \big|\psi_-(t)\big\rangle, \tag{6}$$

where

$$\begin{aligned} \big|\psi_+(t)\big\rangle &= \exp(-iH_+t)|\Omega_1\rangle, \\ \big|\psi_-(t)\big\rangle &= \exp(-iH_-t)|\Omega_2\rangle. \end{aligned} \tag{7}$$

The state of the total system is a pure state and the corresponding density matrix reads $\varrho_1(t) = |\Psi(t)\rangle\langle\Psi(t)|$. In turn, the partial trace Tr_E over the environment degrees of freedom yields the reduced density matrix $\rho_1(t) = \mathrm{Tr}_E\varrho_1(t)$ of the qubit. In the qubit base $\{|1\rangle, |-1\rangle\}$, it takes the matrix form

$$\rho_1(t) = \begin{pmatrix} |b_+|^2 & b_+b_-^*A_1(t) \\ b_+^*b_-A_1^*(t) & |b_-|^2 \end{pmatrix}, \tag{8}$$

where the decoherence factor $A_1(t) = \langle \psi_-(t)|\psi_+(t)\rangle$ is a scalar product of two wave functions in the environment Hilbert space. At initial time, $A_1(0) = \langle \Omega_2|\Omega_1\rangle$. The qubit is in a pure state for $|A_1(t)|^2 = 1$ and it is completely decohered when $|A_1(t)|^2 = 0$.

2.2 Initial mixed states

Now, let us consider the next class of initial states of the total system, namely, mixed states in the form

$$\varrho_2(0) = \rho_1(0) \otimes |\Omega_3\rangle\langle\Omega_3|, \tag{9}$$

where $\rho_1(0)$ is the qubit state (8) at $t = 0$ and $|\Omega_3\rangle$ is an initial normalized state of the environment. It means that although initial states (4) and (9) of the total systems are different, the reduced (with respect to E) *initial states of the qubit are the same* in both cases. From the relation

$$\rho_2(t) = \mathrm{Tr}_E\left\{\mathrm{e}^{-iHt}\rho_1(0) \otimes |\Omega_3\rangle\langle\Omega_3|\mathrm{e}^{-iHt}\right\} \tag{10}$$

we obtain the reduced dynamics of the qubit. It can be expressed in the matrix form as:

$$\rho_2(t) = \begin{pmatrix} |b_+|^2 & b_+b_-^* A_2(t) \\ b_+^* b_- A_2^*(t) & |b_-|^2 \end{pmatrix}, \tag{11}$$

where the decoherence function $A_2(t) = \langle \Omega_2|\Omega_1\rangle\langle \Phi_-(t)|\Phi_+(t)\rangle$ and

$$\begin{aligned} \big|\Phi_+(t)\big\rangle &= \exp(-iH_+t)|\Omega_3\rangle, \\ \big|\Phi_-(t)\big\rangle &= \exp(-iH_-t)|\Omega_3\rangle. \end{aligned} \tag{12}$$

Let us emphasis that although $\rho_2(0) = \rho_1(0)$ (with $\rho_1(0)$ given in Eq. (8)) the origin of initial 'mixedness' is in both cases different. For $\rho_2(t)$ it is due to imperfect initial preparation whereas for $\rho_1(t)$ initial information is shared (via entanglement) by qubit and its environment.

Properties of the purity of qubit states evolving according to the reduced dynamics (8) and (11) are analyzed in this paper. For dephasing-type density matrices (8) or (11) the purity takes a form

$$\mathcal{P}\big[\rho_i(t)\big] = |b_+|^4 + |b_-|^4 + 2|b_+|^2|b_-|^2\big|A_i(t)\big|^2, \quad i = 1, 2. \tag{13}$$

Time evolution of the purity depends on the initial state of the total system and (of course) on interaction with environment via the decoherence function $A_i(t)$. The aim of our paper is to determine the difference between purities $\mathcal{P}[\rho_i(t)]$ of the two types of reduced dynamics.

3 Infinite Environment

In this section, we consider the case when the environment is infinite with a continuum of bath modes and modelled by a bosonic field. Then, in Eq. (2), the environment parts of the Hamiltonian assume the form (with $\hbar = 1$)

$$H_E = \int_0^\infty d\omega h(\omega) a^\dagger(\omega) a(\omega), \tag{14}$$

$$H_I = \int_0^\infty d\omega\big[g^*(\omega)a(\omega) + g(\omega)a^\dagger(\omega)\big], \tag{15}$$

where the operators $a^\dagger(\omega)$ and $a(\omega)$ are the bosonic creation and annihilation operators, respectively. The real-valued function $h(\omega)$ characterizes the energy spectrum of the environment. The coupling is described by the function $g(\omega)$ and its complex conjugate $g^*(\omega)$.

To determine the environment vectors in Eqs. (4) and (9), we can propose any vectors from the corresponding Hilbert space but for convenience we choose the following example:

$$\begin{aligned} |\Omega_1\rangle &= |\Omega_3\rangle = |\Omega_0\rangle, \\ |\Omega_2\rangle &= C_\lambda^{-1}\big[(1-\lambda)|\Omega_0\rangle + \lambda|\Omega_f\rangle\big], \end{aligned} \tag{16}$$

where $|\Omega_0\rangle$ is the ground (vacuum) state of the environment and $|\Omega_2\rangle$ is the linear combination of the vacuum state $|\Omega_0\rangle$ and the coherent state $|\Omega_f\rangle$ defined by the relation

$$|\Omega_f\rangle = \exp(i\phi)D(f)|\Omega_0\rangle. \tag{17}$$

It is determined by the phase ϕ and the square-integrable function f. The Weyl displacement operator reads [29]

$$D(f) = \exp\left\{\int_0^\infty d\omega\big[f(\omega)a^\dagger(\omega) - f^*(\omega)a(\omega)\big]\right\}. \tag{18}$$

The constant C_λ normalizes the state (4) and is given by the expression

$$C_\lambda^2 = (1-\lambda)^2 + \lambda^2 + 2\lambda(1-\lambda)\,\mathrm{Re}\langle\Omega_0|\Omega_f\rangle, \tag{19}$$

where the real number $\lambda \in [0, 1]$ and Re is a real part of the scalar product $\langle\Omega_0|\Omega_f\rangle$ of two states in the environment Hilbert space.

There is no special reason for our choice of the superposition of vacuum and coherent state in the environment, apart from the fact that it provides the simple instructive example. The correlated initial state (4) with (16) differs from the one used in Refs. [27, 28] due to the phase ϕ which affects considerably results of this paper. The decoherence functions $A_1(t)$ and $A_2(t)$, which occur in the reduced density operators (8) and (11), take the form

$$A_1(t) = B_\lambda(t), \qquad A_2(t) = B_\lambda(0)B_0(t), \tag{20}$$

where [27, 28]

$$B_\lambda(t) = C_\lambda^{-1}e^{-2i\varepsilon t - r(t)}\big[1 - \lambda + \lambda e^{-2i\Omega(t)+s(t)}\big] \tag{21}$$

and

$$r(t) = 4\int_0^\infty d\omega g_h^2(\omega)\big\{1 - \cos\big[h(\omega)t\big]\big\}, \tag{22}$$

$$s(t) = 2\int_0^\infty d\omega g_h(\omega)f(\omega)\big\{\cos\phi - \cos\big[h(\omega)t - \phi\big]\big\} - \frac{1}{2}\int_0^\infty d\omega f^2(\omega), \tag{23}$$

where $g_h(\omega) = g(\omega)/h(\omega)$ and

$$\Omega(t) = \int_0^\infty d\omega g_h(\omega)f(\omega)\big\{\sin\big[h(\omega)t + \phi\big] - \sin\phi\big\}. \tag{24}$$

We have assumed in Eqs. (22)–(24) that the functions $h(\omega)$, $g(\omega)$ and $f(\omega)$ are real valued.

The initial state (4) of the composite system depends on three parameters: (1) on the complex number b_+ or b_- (remember that the condition $|b_+|^2+|b_-|^2=1$ is satisfied); (2) on the real number $\lambda\in[0,1]$ and (3) on the phase ϕ of the coherent state. Moreover, it depends on the function $f(\omega)$. The degree of the initial correlation of the qubit and environment can be controlled by the above four quantities. For a fixed coherent state and fixed number b_+, the parameter $\lambda\in[0,1]$ controls the strength of initial correlations of the qubit with environment. For $\lambda=0$ the qubit and the environment are initially uncorrelated while for $\lambda=1$ the correlation is most prominent for the assumed class of initial states.

Without loss of generality, we can assume that the energy spectrum $h(\omega)$ is an increasing function of ω. Then, by changing the integration variable as $\tilde{\omega}=h(\omega)$, we obtain expressions for $r(t)$, $s(t)$ and $\Omega(t)$ which in form are similar to (22)–(24) but with new redefined functions g_h and f. It formally corresponds to the case $h(\omega)=\omega$. Such a change of the integration variable is convenient because we can apply the Fourier transform theory for their analysis. In particular, by applying the Riemann-Lebesgue lemma [30], we can evaluate the long-time limit ($t\to\infty$) of the functions (22)–(24). When the initial state is uncorrelated ($\lambda=0$) then the decoherence function $B_\lambda(t)$ does not depend on $s(t)$ and $\Omega(t)$. The function $r(t)$ increases from zero to finite or infinite value (in dependence on the form of $g_h(\omega)$). When the initial state is correlated ($\lambda\neq 0$) then in the long time limit the behavior of the decoherence function $B_\lambda(\infty)$ depends on

$$r(\infty)-s(\infty)=\int_0^\infty d\omega\left[4g_h^2(\omega)-2g_h(\omega)f(\omega)\cos\phi+\frac{1}{2}f^2(\omega)\right]>0. \tag{25}$$

The integrand is a quadratic form of two variables g_h and f. One can easily check that this form is positive definite and therefore (25) is positive for any g_h, f and ϕ. It means that the purity (13) in both cases of uncorrelated and correlated initial states is a decreasing function of time.

For detailed analysis of purity properties, we still have to specify two quantities: the spectral density $g_h(\omega)=g(\omega)/\omega$ for the linear energy spectrum $h(\omega)=\omega$ and the function $f(\omega)$ which determines the coherent state. The spectral density function $g_h(\omega)$ describes the influence of environment modes at different frequency scales and for continuum environment is taken as some continuous function of frequency. In literature, there are many examples of such a function, see e.g. [31]. With this study we restrict ourselves to the case in which this function assumes the explicit form [32]

$$g_h^2(\omega)=\alpha\omega^{\mu+1}\exp(-\omega/\omega_c), \tag{26}$$

where $\alpha>0$ is the qubit-environment coupling constant, ω_c is a cut-off frequency and $\mu>-1$ is the "ohmicity" parameter: the case $-1<\mu<0$ corresponds to the sub-ohmic, $\mu=0$ to the ohmic and $\mu>0$ to super-ohmic environments, respectively. To avoid mathematical inconsistencies we limit our considerations to super-ohmic systems $\mu>0$ [33, 34].

The function $f(\omega)$ determining the coherent state $|\Omega_f\rangle$ is taken in the form

$$f^2(\omega)=\gamma\omega^{\nu+1}\exp(-\omega/\omega_c). \tag{27}$$

This form is convenient because we are able to calculate analytically the integrals in Eqs. (23) and (24) but one can take any square-integrable function. As a result one gets

$$r(t)=4\alpha\Gamma(\mu)\omega_c^\mu\left\{1-\frac{\cos[\mu\arctan(\omega_c t)]}{(1+\omega_c^2t^2)^{\mu/2}}\right\}, \tag{28}$$

$$s(t) = 2\big[L_1 - L_2(t)\big]\cos\phi - L_3(t)\sin\phi - \frac{1}{2}\gamma\Gamma(\nu)\omega_c^{\nu}, \tag{29}$$

$$\Omega(t) = \big[L_1 + L_2(t)\big]\sin\phi + L_3(t)\cos\phi, \tag{30}$$

where

$$L_1 = \sqrt{\alpha\gamma}\,\Gamma\left(\frac{\mu+\nu}{2}\right)\omega_c^{\frac{\mu+\nu}{2}}, \tag{31}$$

$$L_2(t) = L_1\frac{\cos[\frac{\mu+\nu}{2}\arctan(\omega_c t)]}{(1+\omega_c^2 t^2)^{\frac{\mu+\nu}{4}}}, \tag{32}$$

$$L_3(t) = L_1\frac{\sin[\frac{\mu+\nu}{2}\arctan(\omega_c t)]}{(1+\omega_c^2 t^2)^{\frac{\mu+\nu}{2}}} \tag{33}$$

and $\Gamma(z)$ is the Euler gamma function.

As it was shown, the purity is a decreasing function of time for any functions h, g_h, f and ϕ. Therefore to reveal the role of initial qubit-environment correlations (entanglement) on purity evolution, we will analyze the difference $\mathcal{P}[\rho_1(t)] - \mathcal{P}[\rho_2(t)]$ between the purity $\mathcal{P}[\rho_1(t)]$ for the initially correlated qubit-environment state (8) and the purity $\mathcal{P}[\rho_2(t)]$ for the initially uncorrelated qubit-environment state (11). We fix all values of parameters except λ, which controls degree of correlations. From Eq. (13) it follows that sign of the purity difference is fully determined by sign of the difference

$$\Delta(t) = \big|A_1(t)\big|^2 - \big|A_2(t)\big|^2 = C_\lambda^{-2}e^{-2r(t)}\Pi(t), \tag{34}$$

where

$$\Pi(t) = \lambda^2\big[e^{2s(t)} - e^{2s(0)}\big] + 2\lambda(1-\lambda)\big[e^{s(t)}\cos 2\Omega(t) - e^{s(0)}\big]. \tag{35}$$

We address the issue of whether, and to which extent, the nonzero correlation parameter λ can influence purity of qubit states as time grows. Our results are shown in the upper panel of Fig. 1 for selected values of the correlation parameter λ and the phase $\phi = 0$ of the coherent bosonic field, cf. Eq. (17). In this case, increase of the correlation parameter λ results in higher purity in comparison to the case of an initial uncorrelated state. For the maximally entangled initial state (for $\lambda = 1$) the purity difference is highest. However, it is very sensitive to the relative phase ϕ between the ground state $|\Omega_0\rangle$ and the coherent state $|\Omega_f\rangle$ of the environment. It is demonstrated in the bottom panel of Fig. 1. The main conclusion from this figure is that the phase ϕ is crucial and can control the dynamics of purity states of the qubit and the sign of $\Delta(t)$ can be modified by a change of the phase. Let us notice that for the proper choice of ϕ in Eq. (17) one can relate the purity difference to the amount of initial system-environment entanglement of reduced dynamics.

4 Finite Environment

While studying open systems it is natural to think about *infinite* environment (with a continuum of bath modes) causing true *irreversibility* of various processes related e.g. to thermodynamical properties of open systems [35]. Nevertheless, states of such systems are

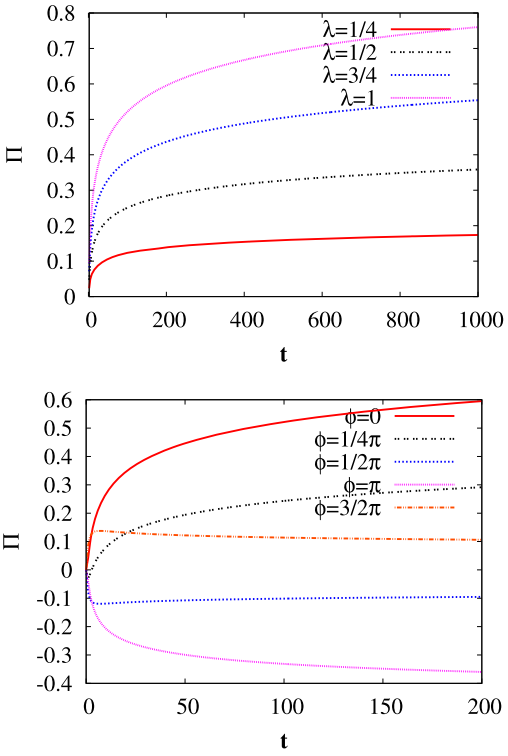

Fig. 1 Rescaled purity difference $\Pi(t)$ given by equation (35) as a function of time in units of $1/\omega_c$. *Top panel*: $\Pi(t)$ for selected values of the parameter λ characterizing initial correlations between the qubit and its environment and fixed phase $\phi = 0$. *Bottom panel*: $\Pi(t)$ for several values of the phase ϕ and fixed correlation parameter $\lambda = 1$. Remaining parameters are: $\epsilon = 1$, $\alpha = 0.1$, $\gamma = 0.01$, $\mu = \nu = 0.1$ (Color figure online)

rather awkward for an effective engineering. Much more convenient is to manipulate *finite* environments (with discrete bath modes) which can, to some extent and for certain time scales, mimic the properties of 'real' surroundings. From a finite environment we require the amount of the minimal energy which it can absorb to be finite. This property does not necessary indicate finiteness of the environment Hilbert space. As an example, we consider a qubit coupled to a single boson via the dephasing process (3) with

$$H_{\pm} = \omega a^{\dagger} a \pm g_0 \left(a + a^{\dagger}\right) \pm \varepsilon \mathbb{I}_E, \tag{36}$$

where g_0 is a coupling constant. Now, as the environment vectors in Eqs. (4) and (9), we choose

$$\begin{aligned} |\Omega_1\rangle &= |\Omega_3\rangle = |0\rangle, \\ |\Omega_2\rangle &= C_{\lambda}^{-1}\big[(1-\lambda)|0\rangle + \lambda|z\rangle\big]. \end{aligned} \tag{37}$$

The state $|0\rangle$ is a vacuum state (a ground state) of the boson and $|z\rangle$ is the coherent state for any complex number $z = |z|e^{i\phi}$. Initial coherent states are chosen because of their experimental accessibility both in optical [36–38] and microwave [39, 40] systems. The decoherence functions $A_1(t)$ and $A_2(t)$, which occur in the reduced density operators (8) and (11), take the form (20) but now with the modified function $B_{\lambda}(t)$ in the form

$$B_{\lambda}(t) = C_{\lambda}^{-1} e^{-2i\varepsilon t - R(t)} \big[1 - \lambda + \lambda e^{-2i\Lambda(t) + S(t)}\big], \tag{38}$$

where

$$R(t) = 4g^2[1 - \cos\omega t],$$
$$S(t) = 2g|z|\big[\cos\phi - \cos(\omega t - \phi)\big] - \frac{1}{2}|z|^2, \tag{39}$$
$$\Lambda(t) = g|z|\big[\sin(\omega t + \phi) - \sin\phi\big]$$

for $z = |z|e^{i\phi}$ and $g = g_0/\omega$ is the rescaled coupling constant.

The function $\Delta(t)$ has the similar form as in Eq. (34) and reads

$$\Delta(t) = \big|A_1(t)\big|^2 - \big|A_2(t)\big|^2 = C_\lambda^{-2} e^{-2R(t)} \Pi(t), \tag{40}$$

and

$$\Pi(t) = \lambda^2\big[e^{2S(t)} - e^{2S(0)}\big] + 2\lambda(1-\lambda)\big[e^{S(t)}\cos 2\Lambda(t) - e^{S(0)}\big]. \tag{41}$$

This formula has a similar structure to (35) for an infinite environment. Quantitative results obtained for finite systems presented in Fig. 2: (top panel) show that as long as $z = z^* = |z|e^{i\phi}$ is real (for the phase $\phi = 0$), one gets purity enhancement, with no regard to the value of $|z| > 0$. Let us notice that when the environment is finite, the characteristics of purity oscillate in time (as expected for a finite system) but the most pronounced difference occurs in the case of large correlations in the initial state of the qubit and its environment. The ordering of graphs with respect to λ is similar to that observed for infinite systems in Fig. 1. There is also an optimal amplitude $|z|$ for which the difference $\Pi(t)$ is largest. It is depicted in the middle panel of Fig. 2. The ordering of $\Pi(t)$ with respect to the correlation parameter λ, as shown in the upper panel, is not preserved in the case when z is complex and the phase $\phi \neq 0$, see the bottom panel of Fig. 2. As in the case of infinite environment, the phase ϕ can be a control parameter for the qubit states purity.

5 Summary

We have analyzed an impact of initial correlations between the open system (qubit) and its environment on the properties of purity of a qubit. With this work we have presented two models of the environment: the infinite one described by the bosonic field and the finite one consisting of a single boson. In both models we compare two types of evolution *different* with respect to the form of initial $S + E$ preparation but of *the same* reduced (with respect to the environment) state. The first type of preparation is defined by Eq. (4) and carries non-trivial qubit-environment correlations whereas the second one is the product state given by Eq. (9). As a natural reference we use a separable initial state. Correlated initial states of the total system $S + E$ are chosen in a specific form of a superposition of two qubit-environment states consisting of a qubit basis vectors tensorized with coherent states of environment (with vacuum as one of them). The formal structure of this states is identical for both infinite, Eq. (16), and finite, Eq. (37), environments. We have revealed that *the phase of the coherent environment state* can radically influence of the purity difference. The general conclusion is that by a proper choice of an initial state preparation of the qubit-infinite environment, one can arrive at 'initial entanglement-assisted' purity growth, i.e. there are initial system-environment correlated states for which the purity of the qubit can be always greater than in the case of initially uncorrelated qubit-environment states. We hope that our results, although provided for very specific models, can serve as a guideline for an effective design of quantum evolutions equipped with desired properties.

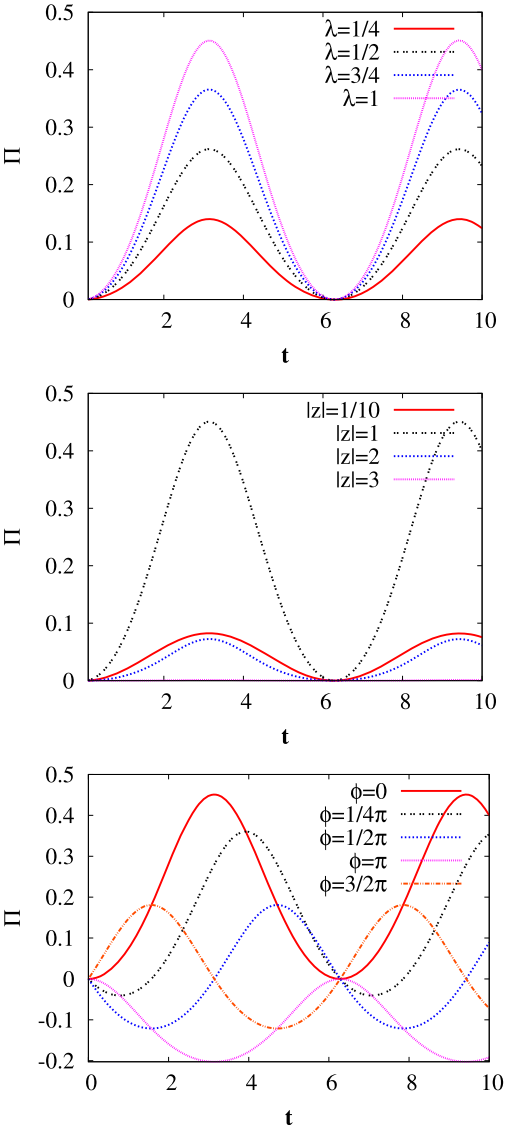

Fig. 2 (Color online) Rescaled purity difference $\Pi(t)$ given by equation (41) plotted as a function of time in units of $1/\omega$. *Top panel*: $\Pi(t)$ for various initial entanglement quantified by λ. The state $|z\rangle$ is fixed with $z = |z|e^{i\phi} = 1$ ($\phi = 0$). *Middle panel*: $\Pi(t)$ for selected values of $z = |z|$, fixed $\phi = 0$ and $\lambda = 1$. *Bottom panel*: $\Pi(t)$ for several values of the phase ϕ, fixed $|z| = 1$ and $\lambda = 1$. In all cases $g = 0.1$ and $\varepsilon = 1$

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References

1. Nielsen, M.A., Chuang, L.I.: Quantum Computation and Quantum Information, vol. 700. Cambridge University Press, Cambridge (2000)

2. Zanardi, P., Lidar, D.A.: Purity and state fidelity of quantum channels. Phys. Rev. A **70**, 012315 (2004)
3. Lidar, D.A., Shabani, A., Alicki, R.: Conditions for strictly purity-decreasing quantum Markovian dynamics. Chem. Phys. **322**, 82 (2006)
4. Rodríguez-Rosario, C.A., Modi, K., Kuah, A., Shaji, A., Sudarshan, E.C.G.: Completely positive maps and classical correlations. J. Phys. A, Math. Theor. **41**, 205301 (2008)
5. Pechukas, P.: Reduced dynamics need not be completely positive. Phys. Rev. Lett. **73**, 1060 (1994)
6. Pechukas, P.: Comment on "Reduced dynamics need not be completely positive". Phys. Rev. Lett. **75**, 3021 (1995)
7. Stelmachovic, P., Buzek, V.: Dynamics of open quantum systems initially entangled with environment: beyond the Kraus representation. Phys. Rev. A **64**, 062106 (2001)
8. Boulant, N., Emerson, J., Havel, T.F., Cory, D.G., Furuta, S.: Incoherent noise and quantum information processing. J. Chem. Phys. **121**, 2955 (2004)
9. Jordan, T.F., Shaji, A., Sudarshan, E.C.G.: Dynamics of initially entangled open quantum systems. Phys. Rev. A **70**, 052110 (2004)
10. Smirne, A., Breuer, H.-P., Piilo, J., Vacchini, B.: Initial correlations in open-systems dynamics: the Jaynes-Cummings model. Phys. Rev. A **82**, 062114 (2010)
11. Tan, H.-T., Zhang, W.-M., Li, G.: Entangling two distant nanocavities via a waveguide. Phys. Rev. A **83**, 032102 (2011)
12. Ban, M., Kitajima, S., Shibata, F.: Qubit decoherence with an initial correlation. Phys. Lett. A **375**, 2283 (2011)
13. Ban, M., Kitajima, S., Shibata, F.: Trace distance in stochastic dephasing with initial correlation. Phys. Rev. A **84**, 042115 (2011)
14. Ban, M., Kitajima, S., Shibata, F.: Distance between qubit states with initial system-environment correlations. Int. J. Theor. Phys. **51**, 2419 (2012)
15. Morozov, V.G., Mathey, S., Röpke, G.: Decoherence in an exactly solvable qubit model with initial qubit-environment correlations. Phys. Rev. A **85**, 022101 (2012)
16. Laine, E.M., Breuer, H.-P., Piilo, J., Li, C.-F., Guo, G.-C.: Nonlocal memory effects in the dynamics of open quantum systems. Phys. Rev. Lett. **108**, 210402 (2012)
17. Li, C.-F., Tang, J.-S., Li, Y.-L., Guo, G.-C.: Experimentally witnessing the initial correlation between an open quantum system and its environment. Phys. Rev. A **83**, 064102 (2011)
18. Smirne, A., Brivio, D., Cialdi, S., Vacchini, B., Paris, M.G.A.: Experimental investigation of initial system-environment correlations via trace-distance evolution. Phys. Rev. A **84**, 032112 (2011)
19. Benatti, F., Floreanini, F.: Open quantum dynamics: complete positivity and entanglement. Int. J. Mod. Phys. B **19**, 3063 (2005)
20. Łuczka, J.: Spin in contact with thermostat: exact reduced dynamics. Physica A **167**, 919 (1990)
21. Laine, E.-M., Piilo, J., Breue, H.-P.: Measure for the non-Markovianity of quantum processes. Phys. Rev. A **81**, 062115 (2010)
22. Laine, E.-M., Piilo, J., Breuer, H.-P.: Witness for initial system-environment correlations in open-system dynamics. Europhys. Lett. **92**, 60010 (2010)
23. Rossatto, D.Z., Werlang, T., Castelano, L.K., Villas-Boas, C.J., Fanchini, F.F.: Purity as a witness for initial system-environment correlations in open-system dynamics. Phys. Rev. A **84**, 042113 (2011)
24. Ban, M.: Photon-echo technique for reducing the decoherence of a quantum bit. J. Mod. Opt. **45**, 2315 (1998)
25. Skinner, J.L., Hsu, D.: Pure dephasing of a two-level system. J. Phys. Chem. **90**, 4931 (1986)
26. Unruh, W.G.: Maintaining coherence in quantum computers. Phys. Rev. A **51**, 992 (1995)
27. Dajka, J., Łuczka, J., Hänggi, P.: Distance between quantum states in the presence of initial qubit-environment correlations: a comparative study. Phys. Rev. A **84**, 032120 (2011)
28. Dajka, J., Łuczka, J.: Distance growth of quantum states due to initial system-environment correlations. Phys. Rev. A **82**, 012341 (2010)
29. Brattelli, O., Robinson, D.W.: Operator Algebras and Quantum Statistical Mechanics, vol. 2, p. 517. Springer, Berlin (1997)
30. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces vol. 312. Princeton, University Press, Princeton (1971)
31. Mitra, K., Lobb, C.J., Sá de Melo, C.A.R.: Environmentally-induced Rabi oscillations and decoherence in superconducting phase qubits. Phys. Rev. B **79**, 132507 (2009)
32. Leggett, A.J., Chakravarty, S., Dorsey, A.T., Fisher, M.P.A., Garg, A., Zwerger, W.: Dynamics of the dissipative two-state system. Rev. Mod. Phys. **59**, 1 (1987)
33. Spohn, H.: Ground state(s) of the spin-boson Hamiltonian. Commun. Math. Phys. **123**, 277 (1989)
34. Alicki, R.: Pure decoherence in qauntum systems. Open Syst. Inf. Dyn. **11**, 53 (2004)
35. Spohn, H., Lebowitz, J.L.: Irreversible thermodynamics for quantum systems weakly coupled to thermal reservoirs. Adv. Chem. Phys. **38**, 109 (1978)

36. Loudon, R., Knight, P.L.: Squeezed light. J. Mod. Opt. **34**, 709 (1987)
37. Loudon, R.: The Quantum Theory of Light, vol. 448. Oxford University Press, Oxford (2000)
38. Walls, D.F., Milburn, G.: Quantum Optics, vol. 425. Springer, Berlin (1994)
39. Yurke, B., Kaminsky, P.G., Miller, R.E., Whittaker, E.A., Smith, A.D., Silver, A.H., Simon, R.W.: Observation of 4.2-K equilibrium-noise squeezing via a Josephson-parametric amplifier. Phys. Rev. Lett. **60**, 764 (1988)
40. Yurke, B., Corruccini, L.R., Kaminsky, P.G., Rupp, L.W., Smith, A.D., Silver, A.H., Simon, R.W., Whittaker, E.A.: Observation of parametric amplification and deamplification in a Josephson parametric amplifier. Phys. Rev. A **39**, 2519 (1989)
41. Gardas, B.: Exact solution of the Schrodinger equation with the spin-boson Hamiltonian. J. Phys. A, Math. Theor. **44**, 195301 (2011)