Int J Game Theory (2010) 39:185–189
DOI 10.1007/s00182-009-0216-z

ORIGINAL PAPER

A note on the nucleolus for 2-convex TU games

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Accepted: 1 December 2009 / Published online: 22 December 2009

Abstract For 2-convex n-person cooperative TU games, the nucleolus is determined as some type of constrained equal award rule. Its proof is based on Maschler, Peleg, and Shapley's geometrical characterization for the intersection of the prekernel with the core. Pairwise bargaining ranges within the core are required to be in equilibrium. This system of non-linear equations is solved and its unique solution agrees with the nucleolus.

Keywords Cooperative game · 2-convex n-person game · Core · Nucleolus

Mathematics Subject Classification (2000) Primary 91A12

1 Introduction and notions

Fix the player set N and its power set $\mathcal{P}(N) = \{S|S \subseteq N\}$ consisting of all the subsets of N (including the empty set $\emptyset$). A *cooperative transferable utility (TU) game* is given by the so-called *characteristic function* $v : \mathcal{P}(N) \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. That is, the TU game v assigns to each coalition $S \subseteq N$ its *worth* $v(S)$ amounting the (monetary) benefits achieved by cooperation among the members of S. The *marginal benefit* b_i^v of player i in the game v is defined by $b_i^v = v(N) - v(N\backslash\{i\})$ for all $i \in N$. Associated with the game v there is the so-called *gap function* $g^v : \mathcal{P}(N) \to \mathbb{R}$ such

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that, for every coalition S, its gap $g^v(S)$ represents the surplus of the marginal benefits of its members over its worth, i.e., $g^v(S) = \sum_{k\in S} b_k^v - v(S)$ for all $S \subseteq N$, where $g^v(\emptyset) = 0$. A payoff vector $\vec{x} = (x_k)_{k\in N} \in \mathbb{R}^N$ is said to belong to the *core* $C(v)$ if it satisfies, besides the efficiency constraint $\sum_{k\in N} x_k = v(N)$, the group rationality constraints $\sum_{k\in S} x_k \geq v(S)$ for all $S \subseteq N$, $S \neq \emptyset$. It is simple to observe that the marginal benefit of any player is an upper bound for core allocations in that $x_i \leq b_i^v$ for all $i \in N$, all $\vec{x} \in C(v)$.

Definition 1.1 An n-person game v is said to be 1-*convex* if its corresponding non-negative gap function g^v attains its minimum at the grand coalition N, i.e.,

$$g^v(S) \geq g^v(N) \geq 0 \qquad \text{for all } S \subseteq N,\ S \neq \emptyset \tag{1.1}$$

In terms of the characteristic function v, (1.1) requires that $v(N) \geq v(S) + \sum_{k\in N\setminus S} b_k^v$ for every non-trivial coalition. In words, concerning the division problem, the worth $v(N)$ is sufficiently large to meet the coalitional demand amounting its worth $v(S)$, as well as the desirable marginal benefit for any nonmember of S. The theory on 1-convex n-person games has been well developed (Driessen 1988). The key feature of 1-convex n-person games is the geometrically regular structure of its core, composed as the convex hull of n extreme points of which all the coordinates, except one, agree with the marginal benefits of all, but one, players. Moreover, the center of gravity of the core turns out to coincide with the so-called *nucleolus* of the 1-convex game. So, the payoff to player i according to the nucleolus of 1-convex n-person games equals $b_i^v - \frac{g^v(N)}{n}$ for all $i \in N$. Particularly, the nucleolus on the class of 1-convex n-person games satisfies the mathematically attractive additivity property.

For any payoff vector $\vec{x} \in \mathbb{R}^N$ satisfying $\sum_{k\in N} x_k = v(N)$ as well as $x_i \leq b_i^v$ for all $i \in N$, it is simple to observe the validity of the core constraint $\sum_{k\in S} x_k \geq v(S)$ whenever the gap of S weakly majorizes the gap of N, i.e., $g^v(S) \geq g^v(N)$. Consequently, for 1-convex n-person games v, the following core equivalence holds:

$$\vec{x} \in C(v) \quad \text{if and only if} \quad \sum_{k\in N} x_k = v(N), \qquad x_i \leq b_i^v \quad \text{for all } i \in N \tag{1.2}$$

Definition 1.2 An n-person game v is said to be 2-*convex* if on the one hand, the gap of the grand coalition N is weakly majorized by the gap of every multi-person coalition S, and on the other, the *concavity* of the gap function g^v with respect to the sequential formation of the grand coalition N by individuals up to size 1, whereas the remaining $n-1$ players merge as one syndicate to complete the sequential formation of N, i.e.,

$$g^v(S) \geq g^v(N) \qquad \text{for all } S \subseteq N \text{ with } |S| \geq 2, \text{ and} \tag{1.3}$$

$$g^v(\{j\}) \geq g^v(N) - g^v(\{i\}) \geq 0 \qquad \text{for all } i, j \in N,\ i \neq j, \text{ or equivalently,} \tag{1.4}$$

$$g^v(\{j\}) + g^v(\{i\}) \geq g^v(N) \geq g^v(\{i\}) \qquad \text{for every pair } i, j \in N \text{ of players.} \tag{1.5}$$

In view of (1.3), for 2-convex n-person games v, the following core equivalence holds:

$$\vec{x} \in C(v) \quad \text{if and only if} \quad \sum_{k \in N} x_k = v(N), \quad v(\{i\}) \leq x_i \leq b_i^v \quad \text{for all } i \in N \tag{1.6}$$

Alternatively, for 2-convex n-person games, its core coincides with a so-called *core catcher* associated with appropriately chosen lower- and upper core bounds. Our main goal is to exploit the core equivalence (1.6) in order to determine the nucleolus based on bargaining ranges within the core.

Example 1.3 Consider the zero-normalized 3-person game $\langle \{1, 2, 3\}, v \rangle$ of which the characteristic function is given by $v(\{1, 2\}) = 6$, $v(\{1, 3\}) = 7$, $v(\{2, 3\}) = 8$, and $v(N)$ not yet specified.

In case the worth $v(N)$ is small enough, for instance $v(N) = 12$, then the marginal benefit vector $\vec{b}^v = (4, 5, 6)$, and so, its gap function g^v is given by $g^v(\{i\}) = 4, 5, 6$ for $i = 1, 2, 3$, respectively, whereas $g^v(S) = 3$ otherwise. By (1.1), the 3-person game v is 1-convex, but fails to be 2-convex, and its core is the convex hull of the three vertices $(1, 5, 6)$, $(4, 2, 6)$, $(4, 5, 3)$. Further, the nucleolus coincides with the center $(3, 4, 5)$ of gravity of the core.

In case the worth $v(N)$ is large enough, say $v(N) = 15$, then $\vec{b}^v = (7, 8, 9)$, and so, $g^v(\{i\}) = 7, 8, 9$ for $i = 1, 2, 3$, respectively, whereas $g^v(S) = 9$ otherwise. By (1.5), the 3-person game v is 2-convex, but fails to be 1-convex, and its core is the convex hull of the five vertices $(7, 0, 8)$, $(6, 0, 9)$, $(0, 6, 9)$, $(0, 8, 7)$, $(7, 8, 0)$ (the latter with geometric multiplicity 2).

In summary, the 3-person game v turns out to be 1-convex iff $10.5 \leq v(N) \leq 13$ and moreover, to be 2-convex iff $v(N) \geq 15$. Appealing examples of 1-convex games are discovered, like the *library game* together with a suitably chosen basis (Driessen et al. 2005) as well as the *co-insurance game* (Driessen et al. 2009). It is still an outstanding challenge to search for appealing examples of 2-convex games.

2 The nucleolus of 2-convex n-person games

The main purpose is to apply the geometric characterization for the intersection of the prekernel with the core as introduced by Maschler et al. (1979). In view of the core equivalence (1.6) for 2-convex games, the largest amount that can be transferred from player i to another player j with respect to a given core allocation $\vec{x} \in C(v)$ while remaining in the core of the game is either player's i-th decrease amounting $x_i - v(\{i\})$, or player's j-th increase amounting $b_j^v - x_j$, whichever is less. Hence, the largest transfer from i to j equals $\delta_{ij}^v(\vec{x}) = \min\left[x_i - v(\{i\}), \quad b_j^v - x_j\right]$. We are looking for core allocations $\vec{x}$ satisfying the equilibrium condition $\delta_{ij}^v(\vec{x}) = \delta_{ji}^v(\vec{x})$ for every pair $i, j \in N$ of players.

Define the vector $\vec{y} = (y_k)_{k \in N} \in \mathbb{R}^N$ by $y_k = b_k^v - x_k$ for all $k \in N$. Note that $\sum_{k \in N} y_k = g^v(N)$ and the equilibrium conditions may be rewritten by

$$\min\left[g^v(\{i\}) - y_i, y_j\right] = \min\left[g^v(\{j\}) - y_j, y_i\right] \text{ or equivalently,} \tag{2.1}$$

$$y_j + \min\left[g^v(\{i\}), y_i + y_j\right] = y_i + \min\left[g^v(\{j\}), y_i + y_j\right] \text{ for every pair of players.} \tag{2.2}$$

From (2.2), it follows that $y_j \geq y_i$ whenever $g^v(\{j\}) \geq g^v(\{i\})$. In fact, the system (2.1) of pairwise (non-linear) equations, together with the adapted efficiency constraint $\sum_{k \in N} y_k = g^v(N)$, is uniquely solvable (Driessen 1998, p. 47) and its unique solution is of the *parametric* form

$$y_k = \min\left[\lambda, \frac{g^v(\{k\})}{2}\right] \quad \text{and so,} \quad x_k = v(\{k\}) + \max\left[g^v(\{k\}) - \lambda, \frac{g^v(\{k\})}{2}\right] \tag{2.3}$$

for all $k \in N$, where the parameter $\lambda \in \mathbb{R}$ is determined by the efficiency constraints $\sum_{k \in N} y_k = g^v(N)$ and $\sum_{k \in N} x_k = v(N)$. The latter solution (2.3) applies only if $\frac{1}{2} \cdot \sum_{k \in N} g^v(\{k\}) \geq g^v(N)$, otherwise for all $k \in N$

$$y_k = \max\left[g^v(\{k\}) - \lambda, \frac{g^v(\{k\})}{2}\right] \quad \text{and so,} \quad x_k = v(\{k\}) + \min\left[\lambda, \frac{g^v(\{k\})}{2}\right] \tag{2.4}$$

Theorem 2.1 *The nucleolus of a* 2*-convex n-person game v is of the parametric form* (2.3) *or* (2.4)*, a so-called* constrained equal award rule*, incorporating the constraints amounting a half of the individual gaps $g^v(\{k\})$, $k \in N$. For instance, by (2.3), the payoff to any player i according to the nucleolus equals either the midpoint of its individual worth $v(\{i\})$ and its marginal benefit b_i^v, or its parametric shortage $b_i^v - \lambda$, whichever is more. By (2.4), its payoff equals either the same midpoint, or its parametric gain $v(\{i\}) + \lambda$, whichever is less.*

Remark 2.2 The non-void intersection of the two classes of 1-convex and 2-convex n-person games is fully characterized by identical individual gaps such that $g^v(\{k\}) = g^v(N)$ for all $k \in N$. In this setting, (2.3) applies, and the parameter λ is determined through the slightly adapted efficiency constraint

$$\sum_{k \in N} \min\left[\lambda, \frac{g^v(N)}{2}\right] = g^v(N). \qquad \text{Thus,} \qquad y_k = \lambda = \frac{g^v(N)}{n} \quad \text{and so,}$$

the nucleolus payoff equals $x_k = b_k^v - y_k = b_k^v - \frac{g^v(N)}{n}$ for all $k \in N$, which is in accordance with previous remarks involving the nucleolus payoff vector $\vec{x}$.

Remark 2.3 In view of the core equivalence (1.2) for 1-convex n-person games v, the largest transfer from player i to another player j, while remaining in the core of the game, is fully determined by player's j-th increase amounting $b_j^v - x_j$. That is, $\delta_{ij}^v(\vec{x}) = b_j^v - x_j$ for all $i, j \in N, i \neq j$. The equilibrium condition $\delta_{ij}^v(\vec{x}) = \delta_{ji}^v(\vec{x})$, or equivalently, the system of linear equations $b_j^v - x_j = b_i^v - x_i$ for every pair

$i, j \in N$ of players, is easily solved by the unique efficient payoff vector of which the coordinates are given by $b_k^v - \frac{g^v(N)}{n}$, $k \in N$.

Remark 2.4 In Quant et al. (2005), the authors study the so-called class of *compromise stable* games of which the core agrees with a certain core cover in the sense of (1.6) by replacing the weak lower bound $v(\{i\})$ by another sharp lower bound amounting $b_i^v - \min_{S \ni i} g^v(S)$. Their approach to determine the nucleolus of compromise stable games games is totally different and strongly based on the study of (convex) bankruptcy games (Quant et al. 2005, Theorem 4.2, pp. 497–498). Our geometrical approach to determine the nucleolus of compromise stable games applies once again, but is left to the reader. In fact, (2.1) applies once again, replacing $g^v(\{i\})$ by $\min_{S \ni i} g^v(S)$.

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