Ying *Journal of Inequalities and Applications* 2012, **2012**:142
http://www.journalofinequalitiesandapplications.com/content/2012/1/142

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Higher-order symmetric duality for a class of multiobjective fractional programming problems

Gao Ying

Correspondence: gaoyingimu@163.com
Department of Mathematics, Chongqing Normal University, Chongqing 400047, China

Abstract

In this paper, a pair of nondifferentiable multiobjective fractional programming problems is formulated. For a differentiable function, we introduce the definition of higher-order (F, α, ρ, d)-convexity, which extends some kinds of generalized convexity, such as second order F-convexity and higher-order F -convexity. Under the higher-order (F, α, ρ, d)-convexity assumptions, we prove the higher-order weak, higher-order strong and higher-order converse duality theorems.

Mathematics Subject Classification (2010) 90C29; 90C30; 90C46.

Keywords: Higher-order symmetric duality, multiobjective fractional programming, higher-order (F, α, ρ, d)-convexity.

Introduction

Symmetric duality in nonlinear programming in which the dual of the dual is the primal was introduced by Dorn [1]. The notion of symmetric duality was developed significantly by Dantzig et al. [2], and the Wolfe dual models presented in [2]. Mond [3] presented a slightly different pair of symmetric dual nonlinear programs and obtained more generalized duality results than that of Dantzig et al. [2]. Mond and Weir [4] then gave another pair of symmetric dual nonlinear programs in which a weaker convexity assumption was imposed on involved functions. Later, Mond and Weir [5], Weir and Mond [6] as well as Gulati et al. [7] generalized single objective symmetric duality to multiobjective case.

Chandra et al. [8] first formulated a pair of symmetric dual fractional programs with certain convexity hypothesis. Pandey [9] introduced second-order η-invex function for multiobjective fractional programming problem and established weak and strong duality theorems. Yang et al. [10] discussed a class of nondifferentiable multiobjective fractional programming problems, and proved duality theorems under the assumptions of invex (pseudoinvex, pseudoincave) functions. Higher-order duality in nonlinear programs have been studied by some researchers. Mangasarian [11] formulated a class of higher-order dual problems for the nonlinear programming problem by introducing twice differentiable functions. Mond and Zhang [12] obtained duality results for various higher-order dual programming problems under higher-order invexity assumptions. Under invexity-type conditions, such as higher-order type I, higher-order pseudo-type I, and higher-order quasi-type I conditions, Mishra and Rueda [13] gave various duality results. Recently, Chen [14] also discussed the duality theorems under higher-order F-convexity (F-pseudo-convexity, F-quasi-convexity) for a pair of multiobjective nondifferentiable program. But, up to now, there is not sufficient literatures dealing with higher-order fractional symmetric duality.

In this paper, we first formulate a pair of nondifferentiable multiobjective fractional pro-gramming problems. For a differentiable function h: $R^n \times R^n \to R$, we introduce the definition of higher-order (F, α, ρ, d)-convexity, which extends some kinds of generalized convexity, such as second order F-convexity in [15] and higher-order F -convexity in [14]. Under the higher-order (F, α, ρ, d)- convexity assumptions, we prove the higher-order weak, higher-order strong and higher-order converse duality theorems.

Preliminaries

Let R^n be the n-dimensional Euclidean space and let R^n_+ be its non-negative orthant. The following conventions for vectors in R^n will be used:

$$\begin{aligned} &x < y \quad \text{if and only if} \quad y - x \in \operatorname{int} R^n; \\ &x \leq y \quad \text{if and only if} \quad y - x \in R^n_+ \backslash \{0\}; \\ &x \leqq y \quad \text{if and only if} \quad y - x \in R^n_+; \\ &x \nleq y \quad \text{is the negation of} \quad x \leq y. \end{aligned}$$

For a real-valued twice differentiable function $h(x, y)$ defined on an open set in $R^n \times R^m$, denote by $\nabla_x h(\bar{x}, \bar{y})$ the gradient vector of h with respect to x at $(\bar{x}, \bar{y})$, $\nabla_{xx} h(\bar{x}, \bar{y})$ the hessian matrix with respect to x at $(\bar{x}, \bar{y})$. Similarly, $\nabla_y h(\bar{x}, \bar{y})$, $\nabla_{xy} h(\bar{x}, \bar{y})$ and $\nabla_{yy} h(\bar{x}, \bar{y})$ are also defined.

Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x|C) = \max\{x^T y : y \in C\}.$$

A support function, being convex and everywhere finite, has a subdifferential, that is, there exists a $z \in R^n$ such that

$$s(y|C) \geqq s(x|C) + z^T (y - x), \ \forall x \in C.$$

The subdifferential of $s(x|C)$ is given by

$$\partial s(x|C) = \{z \in C \ : \ z^T x = s(x|C)\}.$$

For a convex set $D \subset R^n$, the normal cone to D at a point $x \in D$ is defined by

$$N_D(x) = \{y \in R^n \ : \ y^T (z - x) \leqq 0, \ \forall z \in D\}.$$

When C is a compact convex set, $y \in N_C(x)$ if and only if $s(y|C) = x^T y$, or equivalently, $x \in \partial s(y|C)$.

Consider the following multiobjective programming problem (P):

$$\text{Minimize} \quad f(x) \quad \text{subject to } g(x) \leqq 0, \quad x \in X,$$

where f: $R^n \to R^m$, g: $R^n \to R^l$ and $X \subset R^n$. Denote by S the set of feasible solutions of (P).

Definition 2.1. (a) A feasible solution x_0 is said to be an efficient solution of (P) if there is no other $x \in S$ such that $f(x) \leq f(x_0)$.

(b) A feasible solution x_0 is said to be a properly efficient solution of (P) if it is an efficient solution of (P), and there exists a real number $M > 0$ such that for all $i \in \{1, ..., m\}$, $x \in S$, and $f_i(x) < f_i(x_0)$,

$$f_i(x_0) - f_i(x) \leqq M(f_j(x) - f_j(x_0))$$

for some $j \in \{1, ..., m\}$ such that $f_j(x) > f_j(x_0)$.

Definition 2.2. A functional F: $X \times X \times R^n \to R$ (where $X \subset R^n$) is sublinear in its third component if for all $(x, u) \in X \times X$,

$$F(x, u; a_1 + a_2) \leqq F(x, u; a_1) + F(x, u; a_2) \text{for all } a_1, a_2 \in R^n;$$
$$F(x, u; \alpha a) = \alpha F(x, u; a) \text{ for all } \alpha \in R_+ \text{ and for all } a \in R^n.$$

For convenience, we write $F_{x,u}(a) = F(x, u, a)$.

We now introduce higher-order (F, α, ρ, d)-convex function. Where, F: $X \times X \times R^n \to R$ is a sublinear functional, α: $X \times X \to R_+ \setminus \{0\}$, $\rho \in R$ and d: $X \times X \to R$. Let Φ: $X \to R$ and h: $X \times R^n \to R$ be differentiable real valued functions.

Definition 2.3. Φ is said to be higher-order (F, α, ρ, d)-convex at $u \in X$ with respect to h if, $\forall (x, p) \in X \times R^n$,

$$\Phi(x) - \Phi(u) \geqq F_{x,u}(\alpha(\nabla_x \Phi(u) + \nabla_p h(u, p))) + h(u, p) - p^T \nabla_p h(u, p) + \rho d^2(x, u).$$

Remark 2.1. (1) When $\alpha = 1$, and $\rho = 0$ or $d = 0$, the higher-order (F, α, ρ, d)-convexity reduces to higher-order F-convexity in [14].

(2) When $\alpha = 1$, $\rho = 0$ or $d = 0$, and $h(u,p) = \frac{1}{2}p^T \nabla_{xx}\Phi(u)p$, the higher-order (F, α, ρ, d)-convexity reduces to second order F-convexity in [15].

we now give an example of higher-order (F, α, ρ, d)-convex function with respect to $h(u, p)$, which is not higher-order F -convex and second order F-convex.

Example 2.1. Let $X \subset R$, $X = \{x: x \geqq 1\}$, f: $X \to R$, F: $X \times X \times R \to R$, h: $X \times R \to R$ and d: $X \times X \to R$ given as follows

$$f(x) = x + \frac{2}{x+1}, \; F_{x,u}(a) = |a|(x-u)^2, \; h(u,p) = \frac{p}{u+1}, \; d(x, u) = x - u.$$

And let $u = 1$, $\rho = -1$, $\alpha = \frac{3}{4}$. Then for all $(x, p) \in X \times R$

$$f(x) - f(u) = \frac{x^2 - x}{x+1} \geqq F_{x,u}\left(\frac{3}{4}(\nabla_x f(u) + \nabla_p h(u,p))\right)$$
$$+ h(u,p) - p^T \nabla_p h(u,p) - d^2(x, u) = -\frac{1}{4}(x-1)^2.$$

This implies $f(x)$ is a higher-order (F, α, ρ, d)-convex function with respect to h at u. But when we let $x = 2$, $p = 3$ and $x = 6$, $p = 3$ respectively, we have

$$f(2) - f(1) = \frac{2}{3} < F_{x,u}(\nabla_x f(u) + \nabla_p h(u,p)) + h(u,p) - p^T \nabla_p h(u,p) = \frac{3}{4},$$
$$f(6) - f(1) = \frac{30}{7} < F_{x,u}(\nabla_x f(u) + \nabla_{xx} f(u)) - \frac{1}{2} p^T \nabla_{xx} f(u) p = \frac{66}{4}.$$

Hence, f is neither a higher-order F-convex function nor a second order F-convex function. From now on, suppose that the sublinear functional F satisfies the following condition:

$$F_{x,y}(a) + a^T y \geqq 0, \quad \forall a \in R_+^n. \tag{1}$$

Higher-order symmetric duality

In the section, we consider the following multiobjective fractional symmetric dual problems: **(MFP)** Minimize $L(x, y, p) = (L_1(x, y, p_1), ..., L_k(x, y, p_k))^T$ subject to

$$\sum_{i=1}^{k} \lambda_i \left[(\nabla_y f_i(x, y) - z_i + \nabla_{p_i} H_i(x, y, p_i)) \right.$$
$$\left. - L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i)) \right] \leqq 0,$$
$$y^T \sum_{i=1}^{k} \lambda_i \left[(\nabla_y f_i(x, y) - z_i + \nabla_{p_i} H_i(x, y, p_i)) \right.$$
$$\left. - L_i(x, y, p_i)(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i)) \right] \geqq 0,$$
$$\lambda > 0, \quad \lambda^T e = 1, \quad z_i \in D_i, \quad r_i \in F_i, \quad i = 1 \ldots, k.$$

(MFD) Maximize $M(u, v, q) = (M_1(u, v, q_1),..., M_k(u, v, q_k))^T$ subject to

$$\sum_{i=1}^{k} \lambda_i \left[(\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \Phi_i(u, v, q_i)) \right.$$
$$\left. - M_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \Psi_i(u, v, q_i)) \right] \geqq 0,$$
$$u^T \sum_{i=1}^{k} \lambda_i \left[(\nabla_x f_i(u, v) + w_i + \nabla_{q_i} \Phi_i(u, v, q_i)) \right.$$
$$\left. - M_i(u, v, q_i)(\nabla_x g_i(u, v) - t_i + \nabla_{q_i} \Psi_i(u, v, q_i)) \right] \leqq 0,$$
$$\lambda > 0, \quad \lambda^T e = 1, \quad w_i \in C_i, \quad t_i \in E_i, \quad i = 1 \ldots, k.$$

where

$$L_i(x, y, p_i) = \frac{f_i(x, y) + s(x|C_i) - y^T z_i + H_i(x, y, p_i) - p_i^T \nabla_{p_i} H_i(x, y, p_i)}{g_i(x, y) - s(x|E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_{p_i} G_i(x, y, p_i)},$$
$$M_i(u, v, q_i) = \frac{f_i(u, v) - s(v|D_i) + u^T w_i + \Phi_i(u, v, q_i) - q_i^T \nabla_{q_i} \Phi_i(u, v, q_i)}{g_i(u, v) + s(v|F_i) - u^T t_i + \Psi_i(u, v, q_i) - q_i^T \nabla_{q_i} \Psi_i(u, v, q_i)},$$

f_i: $R_n \times R_m \to R$; g_i: $R^n \times R^m \to R$; H_i, G_i: $R^n \times R^m \to R$ and Φ_i, Ψ_i: $R_n \times R_m \times R_n \to R$ are twice differentiable functions for all $i = 1 ..., k$. C_i, E_i are compact convex sets in R^n, and D_i, F_i are compact convex sets in R^m, $i = 1, ..., k$. $e = (1, ..., 1)^T \in R^k$. $p_i \in R^m$, $q_i \in R^n$, $i = 1, ..., k$, $p = (p_1, ..., p_k)$, $q = (q_1, ..., q_k)$. It is assumed that in the feasible regions the numerators are nonnegative and denominators are positive.

We let $S = (S_1, ..., S_k)^T$, $W = (W_1, ..., W_k)^T \in R^k$. Then we can express the programs (MFP) and (MFD) equivalently as:

(MFP)$_{\mathbf{S}}$ Minimize S subject to

$$\begin{aligned} &(f_i(x, y) + s(x|C_i) - y^T z_i + H_i(x, y, p_i) - p_i^T \nabla_{p_i} H_i(x, y, p_i)) \\ &\quad - S_i(g_i(x, y) - s(x|E_i) + y^T r_i + G_i(x, y, p_i) - p_i^T \nabla_{p_i} G_i(x, y, p_i)) = 0, \ i = 1, \ldots, k, \end{aligned} \tag{2}$$

$$\begin{aligned} &\sum_{i=1}^{k} \lambda_i \left[(\nabla_y f_i(x, y) - z_i + \nabla_{p_i} H_i(x, y, p_i)) \right. \\ &\quad \left. - S_i(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i)) \right] \leqq 0, \end{aligned} \tag{3}$$

$$\begin{aligned}&y^T\sum_{i=1}^{k}\lambda_i\left[(\nabla_y f_i(x,\ y)-z_i+\nabla_{p_i}H_i(x,\ y,p_i))\right.\\&\quad\left.-S_i(\nabla_y g_i(x,\ y)+r_i+\nabla_{p_i}G_i(x,\ y,p_i))\right]\geqq 0,\\&\lambda>0,\quad \lambda^Te=1,\quad z_i\in D_i,\quad r_i\in F_i,\quad i=1\dots,\ k.\end{aligned}\tag{4}$$

$(\mathbf{MFD})_{\mathbf{W}}$ Maximize W subject to

$$\begin{aligned}&(f_i(u,\ v)-s(v|D_i)+u^Tw_i+\Phi_i(u,\ v,\ q_i)-q_i^T\nabla_{q_i}\Phi_i(u,\ v,\ q_i))\\&\quad-W_i(g_i(u,\ v)+s(v|F_i)-u^Tt_i+\Psi_i(u,\ v,\ q_i)-q_i^T\nabla_{q_i}\Psi_i(u,\ v,\ q_i))=0,\quad i=1,\ \dots,\ k,\end{aligned}\tag{5}$$

$$\begin{aligned}&\sum_{i=1}^{k}\lambda_i\left[(\nabla_x f_i(u,\ v)+w_i+\nabla_{q_i}\Phi_i(u,\ v,\ q_i))\right.\\&\quad\left.-W_i(\nabla_x g_i(u,\ v)-t_i+\nabla_{q_i}\Psi_i(u,\ v,\ q_i))\right]\geqq 0,\end{aligned}\tag{6}$$

$$\begin{aligned}&u^T\sum_{i=1}^{k}\lambda_i\left[(\nabla_x f_i(u,\ v)+w_i+\nabla_{q_i}\Phi_i(u,\ v,\ q_i))\right.\\&\quad\left.-W_i(\nabla_x g_i(u,\ v)-t_i+\nabla_{q_i}\Psi_i(u,\ v,\ q_i))\right]\leqq 0,\\&\lambda>0,\quad \lambda^Te=1,\quad w_i\in C_i,\quad t_i\in E_i,\quad i=1\dots,\ k.\end{aligned}\tag{7}$$

Now we can prove weak, strong and converse duality theorems for $(\mathrm{MFP})_S$ and $(\mathrm{MFD})_W$, but equally apply to (MFP) and (MFD).

Theorem 3.1 (*Weak duality*). Let $(x, y, S, z_1, ..., z_k, r_1, ..., r_k, \lambda, p)$ be feasible for $(\mathrm{MFD})_S$ and let $(u, v, W, w_1, ..., w_k, t_1 ..., t_k, \lambda, q)$ be feasible for $(\mathrm{MFD})_W$. Let $\forall i \in \{1, ..., k\}$, $f_i(., v) + (.)^T w_i$ be higher-order (F, α, ρ_i, d_i)-convex at u with respect to $\Phi_i(u, v, q_i)$, - $(g_i(., v) - (.)^T t_i)$ be higher-order (F, α, ρ, d_i)-convex at u with respect to $-\Psi_i(u, v, q_i)$, - $(f_i(x, .) - (.)^T z_i)$ be higher-order $(K,\bar{\alpha},\bar{\rho}_i,\bar{d}_i)$-convex at y with respect to $-H_i(x, y, p_i)$, $g_i(x, .) + (.)^T r_i$ be higher-order $(K,\bar{\alpha},\bar{\rho}_i,\bar{d}_i)$-convex at y with respect to $G_i(x, y, p_i)$, where sublinear functional F: $R^n \times R^n \times R^n \to R$ and K: $R^m \times R^m \times R^m \to R$ satisfy the condition (1). If the following conditions hold:

$$g_i(x,\ v)+v^Tr_i-s(x|E_i)>0, i=1,\dots,k,\tag{8}$$

$$\sum_{i=1}^{k}\lambda_i((1+W_i)\rho_i d_i^2(x,\ u)+(1+S_i)\bar{\rho}_i\bar{d}_i^2(v,\ y))\geqq 0.\tag{9}$$

Then $S \nleq W$.

Proof. Since $(u, v, W, w_1, ..., w_k, t_1 ..., t_k, \lambda, q)$ is feasible for $(\mathrm{MFD})_W$, from (6), (7) and F satisfies condition (1), it follows that

$$F_{x,u}\left(\sum_{i=1}^{k}\lambda_i[(\nabla_x f_i(u,\ v)+w_i+\nabla_{q_i}\Phi_i(u,\ v,\ q_i))-W_i(\nabla_x g_i(u,\ v)-t_i+\nabla_{q_i}\Psi_i(u,\ v,\ q_i))]\right)\geqq 0.\tag{10}$$

Using the convexity assumptions of $f_i(., v) + (.)^T w_i$ and $-(g_i(., v) - (.)^T t_i)$ at u, we have

$$\begin{aligned}
&f_i(x, v) + x^T w_i - f_i(u, v) - u^T w_i \\
&\quad \geqq F_{x,u}(\alpha(\nabla_x f_i(u, v) + w_i + \nabla_{q_i}\Phi_i(u, v, q_i))) + \Phi_i(u, v, q_i) - q_i^T \nabla_{q_i}\Phi_i(u, v, q_i) + \rho_i d_i^2(x, u), \\
&-g_i(x, v) + x^T t_i + g_i(u, v) - u^T t_i \\
&\quad \geqq F_{x,u}(\alpha(-\nabla_x g_i(u, v) + t_i - \nabla_{q_i}\Psi_i(u, v, q_i))) - \Psi_i(u, v, q_i) + q_i^T \nabla_{q_i}\Phi_i(u, v, q_i) + \rho_i d_i^2(x, u).
\end{aligned}$$

Since F is a sublinear functional and $\lambda > 0$, $W \geqq 0$, $\alpha > 0$, from (10) and the above two inequalities, we have

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i (f_i(x, v) + x^T w_i - f_i(u, v) - u^T w_i - \Phi_i(u, v, q_i) + q_i^T \nabla_{q_i}\Phi_i(u, v, q_i)) \\
&\quad + \sum_{i=1}^{k} \lambda_i W_i (g_i(u, v) + v^T r_i - u^T t_i + \Psi_i(u, v, q_i) - q_i^T \nabla_{q_i}\Psi_i(u, v, q_i)) \qquad (11) \\
&\quad + \sum_{i=1}^{k} \lambda_i W_i (x^T t_i - g_i(x, v) - v^T r_i) \geqq \sum_{i=1}^{k} \lambda_i (1 + W_i)\rho_i d_i^2(x, u).
\end{aligned}$$

Since $v^T r_i \leqq s(v|F_i)$, from (5) and (11), we have

$$\sum_{i=1}^{k} \lambda_i [(f_i(x, v) + x^T w_i - s(v|D_i)) + W_i(x^T t_i - v^T r_i - g_i(x, v))] \geqq \sum_{i=1}^{k} \lambda_i (1 + W_i)\rho_i d_i^2(x, u). \qquad (12)$$

On the other hand, from (3), (4) and sublinear functional K satisfies condition (1), we obtain

$$\begin{aligned}
&K_{v,y}\Bigg(-\sum_{i=1}^{k} \lambda_i \left((\nabla_y f_i(x, y) - z_i + \nabla_{p_i} H_i(x, y, p_i)\right) \qquad (13) \\
&\quad -S_i(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i)))\Bigg) \geqq 0.
\end{aligned}$$

Using the convexity assumptions of $-f_i(x, .) + (.)^T z_i$ and $g_i(x, .) + (.)^T r_i$ at y, we have

$$\begin{aligned}
-f_i(x, v) + v^T z_i + f_i(x, y) - y^T z_i &\geqq K_{v,y}(\bar{\alpha}(-\nabla_y f_i(x, y) + z_i - \nabla_{p_i} H_i(x, y, p_i))) \\
&\quad - H_i(x, y, p_i) + p_i^T \nabla_{p_i} H_i(x, y, p_i) + \bar{\rho}_i \bar{d}_i^2(v, y), \\
g_i(x, v) + v^T r_i - g_i(x, y) - y^T r_i &\geqq K_{v,y}(\bar{\alpha}(\nabla_y g_i(x, y) + r_i + \nabla_{p_i} G_i(x, y, p_i))) \\
&\quad + G_i(x, y, p_i) - p_i^T \nabla_{p_i} G_i(x, y, p_i) + \bar{\rho}_i \bar{d}_i^2(v, y).
\end{aligned}$$

Since K is a sublinear functional, and $\lambda > 0$, $S \geqq 0$, $\bar{\alpha} > 0$, from (13) and the above two inequalities, it holds

$$\begin{aligned}
&\sum_{i=1}^{k} \lambda_i (-f_i(x, v) + v^T z_i + f_i(x, y) - y^T z_i + H_i(x, y, p_i) - p_i^T \nabla_{p_i} H_i(x, y, p_i)) \\
&\quad + \sum_{i=1}^{k} \lambda_i S_i (-g_i(x, y) + x^T t_i - y^T r_i - G_i(x, y, p_i) + p_i^T \nabla_{p_i} G_i(x, y, p_i)) \qquad (14) \\
&\quad + \sum_{i=1}^{k} \lambda_i S_i (g_i(x, v) + v^T r_i - x^T t_i) \geqq \sum_{i=1}^{k} \lambda_i (1 + S_i)\bar{\rho}_i \bar{d}_i^2(v, y).
\end{aligned}$$

Since $x^T t_i \leqq s(x|E_i)$, from (2) and (14) we have

$$\sum_{i=1}^{k} \lambda_i[(-f_i(x, v) + v^T z_i - s(x|C_i)) + S_i(g_i(x, v) + v^T r_i - x^T t_i)] \geqq \sum_{i=1}^{k} \lambda_i(1+S_i)\bar{\rho}_i \bar{d}_i^2(v, y).$$

Adding the above inequality and (12), we get

$$\sum_{i=1}^{k} \lambda_i(v^T z_i - s(v|D_i) + x^T w_i - s(x|C_i)) + \sum_{i=1}^{k} \lambda_i(S_i - W_i)(g_i(x, v) + v^T r_i - x^T t_i)$$
$$\geqq \sum_{i=1}^{k} \lambda_i(\rho_i d_i^2(x, u)(1+W_i) + \bar{\rho}_i \bar{d}_i^2(v, y)(1+S_i)).$$

Since $\lambda_i > 0$, $v^T z_i - s(v|D_i) + x^T w_i - s(x|C_i) \leqq 0$, $i = 1, ..., k$, by (9) it yields

$$\sum_{i=1}^{k} \lambda_i(S_i - W_i)(g_i(x, v) + v^T r_i - x^T t_i) \geqq 0.$$

By assumptions (8), we have $g_i(x, v)+v^T r_i - x^T t_i > 0$, $i = 1, ..., k$. Since $\lambda > 0$, it follows that $S \nleq W$. □

Theorem 3.2 (*Strong duality*). Let $(\bar{x}, \bar{y}, \bar{S}, \bar{z}_1, \ldots, \bar{z}_k, \bar{r}_1, \ldots, \bar{r}_k, \bar{\lambda}, \bar{p})$ be a properly efficient solution of $(\mathrm{MFP})_S$, and fix $\lambda = \bar{\lambda}$ in $(\mathrm{MFD})_W$. Suppose that

(*a*) $\nabla_x H_i(\bar{x}, \bar{y}, 0) = \nabla_x G_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_{q_i} \Phi_i(\bar{x}, \bar{y}, 0) = \nabla_{q_i} \Psi_i(\bar{x}, \bar{y}, 0) = 0$, $H_i(\bar{x}, \bar{y}, 0) = G_i(\bar{x}, \bar{y}, 0) = 0$, $\Phi_i(\bar{x}, \bar{y}, 0) = \Psi_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_y H_i(\bar{x}, \bar{y}, 0) = \nabla_y G_i(\bar{x}, \bar{y}, 0) = 0$, $\nabla_{p_i} H_i(\bar{x}, \bar{y}, 0) = \nabla_{p_i} G_i(\bar{x}, \bar{y}, 0) = 0, i = 1, \ldots, k$.

(*b*) For all $i \in \{1, ..., k\}$,

$$f_i(\bar{x}, \bar{y}) + s(\bar{x}|C_i) - \bar{y}^T \bar{z}_i + H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{p}_i^T \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) > 0.$$

(*c*) (i) $\nabla_{p_i p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_{p_i p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i) \neq 0$ for $\bar{p}_i = 0$, $i = 1, ..., k$ and $\nabla_{p_i p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_{p_i p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)$ is nonsingular for all $i = 1, ..., k$,

(ii) $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{S}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ is positive definite and $\bar{p}_i^T((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) \geqq 0$ for all $i = 1, ..., k$, or $\sum_{i=1}^{k} \bar{\lambda}_i(\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{S}_i \nabla_{yy} g_i(\bar{x}, \bar{y}))$ is negative definite and $\bar{p}_i^T((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) \leqq 0$ for all $i = 1, ..., k$.

(iii) $\{\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i)) : i = 1, \ldots, k\}$ is linearly independent.

Then $\bar{p} = 0$, and there exist $\bar{w}_i \in C_i$ and $\bar{t}_i \in E_i$, $i = 1, ..., k$ such that $(\bar{x}, \bar{y}, \bar{S}, \bar{w}_1, ..., \bar{w}_k, \bar{t}_1, \ldots, \bar{t}_k, \bar{\lambda}, \bar{q} = 0)$ is a feasible solution of $(\mathrm{MFD})_W$. Furthermore, if the hypotheses in Theorem 3.1 are satisfied, then $(\bar{x}, \bar{y}, \bar{S}, \bar{w}_1, ..., \bar{w}_k, \bar{t}_1, \ldots, \bar{t}_k, \bar{\lambda}, \bar{q} = 0)$ is a properly efficient solution of $(\mathrm{MFD})_W$, and the two objective values are equal.

Proof. Since $(\bar{x}, \bar{y}, \bar{S}, \bar{z}_1, \ldots, \bar{z}_k, \bar{r}_1, \ldots, \bar{r}_k, \bar{\lambda}, \bar{p})$ is a properly efficient solution of $(\mathrm{MFP})_S$, by the Fritz John type necessary optimality conditions [16], there exist $\alpha \in R^k$, $\beta \in R^k$, $\gamma \in R^m$, $\delta \in R$, $\mu \in R^k$ and $\bar{w}_i \in R^n, \bar{t}_i \in R^n$, $i = 1, ..., k$ such that

$$\begin{aligned}&\sum_{i=1}^{k}\beta_i\big((\nabla_x f_i(\bar{x},\bar{y})+\bar{w}_i+\nabla_x H_i(\bar{x},\bar{y},\bar{p}_i))-\bar{S}_i(\nabla_x g_i(\bar{x},\bar{y})-\bar{t}_i+\nabla_x G_i(\bar{x},\bar{y},\bar{p}_i))\big)\\&\quad+(\gamma-\delta\bar{y})^T\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{yx}f_i(\bar{x},\bar{y})-\bar{S}_i\nabla_{yx}g_i(\bar{x},\bar{y}))\\&\quad+\sum_{i=1}^{k}(\nabla_{p_i x}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_{p_i x}G_i(\bar{x},\bar{y},\bar{p}_i))^T((\gamma-\delta\bar{y})\bar{\lambda}_i-\beta_i\bar{p}_i)=0,\end{aligned}\tag{15}$$

$$\begin{aligned}&\sum_{i=1}^{k}(\beta_i-\delta\bar{\lambda}_i)\big((\nabla_y f_i(\bar{x},\bar{y})-z_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))-\bar{S}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)\\&\quad+\sum_{i=1}^{k}\beta_i\big((\nabla_y H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_y G_i(\bar{x},\bar{y},\bar{p}_i))-(\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)\\&\quad+\sum_{i=1}^{k}\bar{\lambda}_i\big((\nabla_{yy}f_i(\bar{x},\bar{y})-\bar{S}_i\nabla_{yy}g_i(\bar{x},\bar{y}))^T(\gamma-\delta\bar{y})\big)\\&\quad+\sum_{i=1}^{k}(\nabla_{p_i y}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_{p_i y}G_i(\bar{x},\bar{y},\bar{p}_i))^T(-\beta_i\bar{p}_i+(\gamma-\delta\bar{y})\bar{\lambda}_i)=0,\end{aligned}\tag{16}$$

$$\begin{aligned}&\alpha_i-\beta_i(g_i(\bar{x},\bar{y})-s(\bar{x}|E_i)+\bar{y}^T\bar{r}_i+G_i(\bar{x},\bar{y},\bar{p}_i)-\bar{p}_i^T\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\\&\quad-(\gamma-\delta\bar{y})^T(\bar{\lambda}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i)))=0,\quad i=1,\dots,k,\end{aligned}\tag{17}$$

$$\begin{aligned}&(\gamma-\delta\bar{y})^T\big((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)\\&\quad-\mu_i=0,\quad i=1,\dots,k,\end{aligned}\tag{18}$$

$$(\bar{\lambda}_i(\gamma-\delta\bar{y})-\beta_i\bar{p}_i)^T(\nabla_{p_ip_i}H_i(\bar{x},\bar{y},\bar{p}_i)-\bar{S}_i\nabla_{p_ip_i}G_i(\bar{x},\bar{y},\bar{p}_i))=0,\quad i=1,\dots,k,\tag{19}$$

$$\beta_i\bar{y}+(\gamma-\delta\bar{y})\bar{\lambda}_i\in N_{D_i}(\bar{z}_i),\quad i=1,\dots,k,\tag{20}$$

$$\beta_i\bar{S}_i\bar{y}+\bar{\lambda}_i\bar{S}_i(\gamma-\delta\bar{y})\in N_{F_i}(\bar{r}_i),\quad i=1,\dots,k,\tag{21}$$

$$\begin{aligned}&\gamma^T\sum_{i=1}^{k}\bar{\lambda}_i\big((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))\\&\quad-\bar{S}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)=0,\end{aligned}\tag{22}$$

$$\begin{aligned}&\delta\bar{y}^T\sum_{i=1}^{k}\bar{\lambda}_i\big((\nabla_y f_i(\bar{x},\bar{y})-\bar{z}_i+\nabla_{p_i}H_i(\bar{x},\bar{y},\bar{p}_i))\\&\quad-\bar{S}_i(\nabla_y g_i(\bar{x},\bar{y})+\bar{r}_i+\nabla_{p_i}G_i(\bar{x},\bar{y},\bar{p}_i))\big)=0,\end{aligned}\tag{23}$$

$$\mu^T\bar{\lambda}=0,\tag{24}$$

$$\bar{w}_i\in C_i,\bar{t}_i\in E_i,\bar{x}^T\bar{t}_i=s(\bar{x}|E_i),\bar{x}^T\bar{w}_i=s(\bar{x}|C_i),\quad i=1,\dots,k,\tag{25}$$

$$(\alpha,\ \beta,\ \gamma,\ \delta,\ \mu) \neq 0, (\alpha,\ \gamma,\ \delta,\ \mu) \geqq 0. \tag{26}$$

Since $\bar{\lambda} > 0$, and $\mu \geqq 0$, (24) implies $\mu = 0$. Consequently, (18) yields

$$\begin{aligned}&(\gamma - \delta\bar{y})^T \big((\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) \\ &\quad -\bar{S}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))\big) = 0,\ i = 1, \dots, k.\end{aligned} \tag{27}$$

By assumption (i) and (19), we have

$$\bar{\lambda}_i(\gamma - \delta\bar{y}) = \beta_i\bar{p}_i, \quad i = 1, ..., k. \tag{28}$$

Multiplying (16) $(\gamma - \delta\bar{y})$ by left, from (27) and (28) we have

$$\begin{aligned}&(\gamma - \delta\bar{y})^T \sum_{i=1}^k \beta_i((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i\nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i\nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) \\ &\quad +(\gamma - \delta\bar{y})^T \sum_{i=1}^k \bar{\lambda}_i(\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{S}_i\nabla_{yy} g_i(\bar{x}, \bar{y}))(\gamma - \delta\bar{y}) = 0.\end{aligned}$$

Since $\bar{\lambda} > 0$, from (28) and the above equation, we have

$$\begin{aligned}&\sum_{i=1}^k \frac{\beta_i^2}{\bar{\lambda}_i}\bar{p}_i^T((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i\nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i\nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) \\ &\quad +(\gamma - \delta\bar{y})^T \sum_{i=1}^k \bar{\lambda}_i(\nabla_{yy} f_i(\bar{x}, \bar{y}) - \bar{S}_i\nabla_{yy} g_i(\bar{x}, \bar{y}))(\gamma - \delta\bar{y}) = 0.\end{aligned}$$

Which by assumption (ii), we can obtain

$$\gamma - \delta\bar{y} = 0. \tag{29}$$

Using (29) in (28), we have $\beta_i\bar{p}_i = 0$, $i = 1, ..., k$. This implies that $\bar{p}_i = 0$ when $\beta_i \neq 0$, for all $i \in \{1, ..., k\}$. Hence, by assumption (1), we get

$$\sum_{i=1}^k \beta_i((\nabla_y H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i\nabla_y G_i(\bar{x}, \bar{y}, \bar{p}_i)) - (\nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i\nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0.$$

Combining this with (16), (28) and (29), it follows that

$$\sum_{i=1}^k (\beta_i - \delta\bar{\lambda}_i)(\nabla_y f_i(\bar{x}, \bar{y}) - \bar{z}_i + \nabla_{p_i} H_i(\bar{x}, \bar{y}, \bar{p}_i) - \bar{S}_i(\nabla_y g_i(\bar{x}, \bar{y}) + \bar{r}_i + \nabla_{p_i} G_i(\bar{x}, \bar{y}, \bar{p}_i))) = 0,$$

which by assumption (iii), it yields

$$\beta_i - \delta\bar{\lambda}_i = 0, \quad i = 1, \dots, k. \tag{30}$$

We claim that $\delta \neq 0$, otherwise, from (29) and (30) we get $\beta = 0$, $\gamma = 0$. Using (29) in (17), we get $\alpha = 0$. This contradicts with (26). Hence $\delta = 0$. Since $\bar{\lambda} > 0$, from (30) we get $\beta > 0$. Hence $\beta_i\bar{p}_i = 0$, $i = 1, ..., k$ implies $\bar{p}_i = 0$, $i = 1, ..., k$. Using (28), (29) and the fact $\bar{p}_i = 0$, $i = 1, ..., k$ in (15), by assumption (a), we get

$$\sum_{i=1}^k \beta_i((\nabla_x f_i(\bar{x}, \bar{y}) + \bar{w}_i) - \bar{S}_i(\nabla_x g_i(\bar{x}, \bar{y}) - \bar{t}_i)) = 0,$$

combining this with (30) and $\delta>0$, $\bar{\lambda}>0$, it holds

$$\sum_{i=1}^{k}\bar{\lambda}_i((\nabla_x f_i(\bar{x},\bar{y})+\bar{w}_i)-\bar{S}_i(\nabla_x g_i(\bar{x},\bar{y})-\bar{t}_i))=0, \tag{31}$$

which yields

$$\bar{x}^T\sum_{i=1}^{k}\bar{\lambda}_i((\nabla_x f_i(\bar{x},\bar{y})+\bar{w}_i)-\bar{S}_i(\nabla_x g_i(\bar{x},\bar{y})-\bar{t}_i))=0. \tag{32}$$

On the other hand, by assumption (a) and (2) we get

$$(f_i(\bar{x},\bar{y})+s(\bar{x}|C_i)-\bar{y}^T\bar{z}_i)-\bar{S}_i(g_i(\bar{x},\bar{y})-s(\bar{x}|E_i)+\bar{y}^T\bar{r}_i)=0,\quad i=1,\ldots,k. \tag{33}$$

Since $\beta>0$, by (20) and (29) we get $\bar{y}\in N_{D_i}(\bar{z}_i)$, $i=1,\ldots,k$. This implies

$$\bar{y}^T\bar{z}_i=s(\bar{y}|D_i),\quad i=1,\ldots,k. \tag{34}$$

Assumption (b) implies $\bar{S}>0$. By (21), we similarly have $\bar{y}\in N_{F_i}(\bar{r}_i)$, $i=1,\ldots,k$. This implies

$$\bar{y}^T\bar{r}_i=s(\bar{y}|F_i),\quad i=1,\ldots,k. \tag{35}$$

Combining (25), (33), (34) and (35), we get

$$(f_i(\bar{x},\bar{y})+\bar{x}^T\bar{w}_i-s(\bar{y}|D_i))-\bar{S}_i\left(g_i(\bar{x},\bar{y})-\bar{x}^T\bar{t}_i+s(\bar{y}|F_i)\right)=0,\quad i=1,\ \ldots,\ k,$$

combining this with (31) and (32), by assumption (a), $(\bar{x},\bar{y},\bar{S},\bar{w}_1,\ldots,\bar{w}_k,\bar{t}_1,\ldots,\bar{t}_k,\bar{\lambda},\bar{q}=0)$ is a feasible solution of $(\mathrm{MFD})_W$.

Under the assumptions of Theorem 3.1, if $(\bar{x},\bar{y},\bar{S},\bar{w}_1,\ldots,\bar{w}_k,\bar{t}_1,\ldots,\bar{t}_k,\bar{\lambda},\bar{q}=0)$ is not an efficient solution of $(\mathrm{MFD})_W$, then there exists other feasible solution $(u,v,W,w_1,\ldots,w_k,t_1,\ldots,\ t_k,\bar{\lambda},q)$, of $(\mathrm{MFD})_W$ such that $\bar{S}\leq W$. Since $(\bar{x},\bar{y},\bar{S},\bar{z}_1,\ \ldots,\bar{z}_k,\bar{r}_1,\ \ldots,\bar{r}_k,\bar{\lambda},\bar{p})$ is a feasible solution of $(\mathrm{MFP})_S$, by Theorem 3.1, we have $\bar{S}\not\leq W$, hence the contradiction implies $(\bar{x},\bar{y},\bar{S},\bar{w}_1,\ldots,\bar{w}_k,\bar{t}_1,\ldots,\bar{t}_k,\bar{\lambda},\bar{q}=0)$ is an efficient solution of $(\mathrm{MFD})_W$.

If $(\bar{x},\bar{y},\bar{S},\bar{w}_1,\ldots,\bar{w}_k,\bar{t}_1,\ldots,\bar{t}_k,\bar{\lambda},\bar{q}=0)$ is not a properly efficient solution of $(\mathrm{MFD})_W$, then there exists other feasible solution $(u,v,W,w_1,\ldots,w_k,t_1,\ldots,\ t_k,\bar{\lambda},q)$ of $(\mathrm{MFD})_W$ such that for an index $i\in\{1,\ldots,k\}$ and any real number $M>0$, $W_i-\bar{S}_i>M(\bar{S}_j-W_j)$ for j satisfying $\bar{S}_j>W_j$ whenever $W_i>\bar{S}_i$ This implies $W_i>\bar{S}_i$ can be made arbitrarily large and this contradicts with Theorem 3.1. And it is easy to find that the two objective values are equal. □

Theorem 3.3 (*Strict converse duality*). Let $(\bar{u},\bar{v},\bar{W},\bar{w}_1,\ \ldots,\bar{w}_k,\bar{t}_1,\ \ldots,\bar{t}_k,\bar{\lambda},\bar{q})$ be a properly efficient solution of $(\mathrm{MFD})_W$, and fix $\lambda=\bar{\lambda}$ in $(\mathrm{MFP})_S$. Suppose that

(a) $\nabla_x\Phi_i(\bar{u},\bar{v},\ 0)=\nabla_x\Psi_i(\bar{u},\bar{v},\ 0)=0,\ \nabla_{q_i}\Phi_i(\bar{u},\bar{v},\ 0)=\nabla_{q_i}\Psi_i(\bar{u},\bar{v},\ 0)=0,$
$H_i(\bar{u},\bar{v},\ 0)=G_i(\bar{u},\bar{v},0)=0,\ \Phi_i(\bar{u},\bar{v},\ 0)=\Psi_i(\bar{u},\bar{v},\ 0)=0,\ \nabla_y\Phi_i(\bar{u},\bar{v},\ 0)=\nabla_y\Psi_i(\bar{u},\bar{v},\ 0)=0,$
$\nabla_{p_i}H_i(\bar{u},\bar{v},\ 0)=\nabla_{p_i}G_i(\bar{u},\bar{v},\ 0)=0,\quad i=1,\ldots,k.$

(b) For all $i\in\{1,\ldots,k\}$,

$$f_i(\bar{u},\bar{v})-s(\bar{v}|D_i)+\bar{u}^T\bar{w}_i+\Phi_i(\bar{u},\bar{v},\bar{q}_i)-\bar{q}_i^T\nabla_{q_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i)>0.$$

(c) (i) $\nabla_{q_iq_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_{q_iq_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i) \neq 0$, for $\bar{q}_i = 0$, $i = 1, ..., k$, and $\nabla_{q_iq_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_{q_iq_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i)$ is nonsingular for all $i = 1, ..., k$, and

(ii) $\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{xx}f_i(\bar{u},\bar{v}) - \bar{W}_i\nabla_{xx}g_i(\bar{u},\bar{v}))$ is positive definite and $\bar{q}_i^T((\nabla_x\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_x\Psi_i(\bar{u},\bar{v},\bar{q}_i)) - (\nabla_{q_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_{q_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i))) \geqq 0$ for all $i = 1, ..., k$, or $\sum_{i=1}^{k}\bar{\lambda}_i(\nabla_{xx}f_i(\bar{u},\bar{v}) - \bar{W}_i\nabla_{xx}g_i(\bar{u},\bar{v}))$ is negative definite and $\bar{q}_i^T((\nabla_x\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_x\Psi_i(\bar{u},\bar{v},\bar{q}_i)) - (\nabla_{q_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i\nabla_{q_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i))) \leqq 0$ for all $i = 1, ..., k$.

(iii) $\{\nabla_x f_i(\bar{u},\bar{v}) + \bar{w}_i + \nabla_{q_i}\Phi_i(\bar{u},\bar{v},\bar{q}_i) - \bar{W}_i(\nabla_x g_i(\bar{u},\bar{v}) - \bar{t}_i + \nabla_{q_i}\Psi_i(\bar{u},\bar{v},\bar{q}_i)) : i = 1, \ldots, k\}$ is linearly independent.

Then $\bar{q} = 0$, and there exist $\bar{z}_i \in D_i$ and $\bar{r}_i \in F_i$, $i = 1, ..., k$ such that $(\bar{u},\bar{v},\bar{W},\bar{z}_1,...,\bar{z}_k,\bar{r}_1,\ldots,\bar{r}_k,\bar{\lambda},\bar{p} = 0)$ is a feasible solution of (MFP)$_S$. Furthermore, if the hypotheses in Theorem 3.1 are satisfied, then $(\bar{u},\bar{v},\bar{W},\bar{z}_1,...,\bar{z}_k,\bar{r}_1,\ldots,\bar{r}_k,\bar{\lambda},\bar{p} = 0)$ is a properly efficient solution of (MFP)$_S$, and the two objective values are equal. □

Remark 3.1.(1) If $k = 1$, $H_1(x,y,p_1) = \frac{1}{2}p_1^T\nabla_{yy}f_1(x,y)p_1$, $\Phi_1(u, v, q_1) = \frac{1}{2}q_1^T\nabla_{xx}f_1(u, v)q_1$, $\Phi_1(u, v, q_1) = \frac{1}{2}q_1^T\nabla_{xx}f_1(u, v)q_1$, and $g_1(u,v) + s(v|F_1) - u^Tt_1 + \Psi_1(u,v,q_1) - q_1^T\nabla_{q1}\Psi_1(u,v,q_1) = 1$, then (MFP)$_S$ and (MFD)$_W$ becomes the problems considered by Hou and Yang [17].

(2) If $k = 1$, $g_1(x,y) - s(x|E_1) + y^Tr_1 + G_1(x,y,p_1) - p_1^T\nabla_{p_1}G_1(x,y,p_1) = 1$, and $g_1(u,v) + s(v|F_1) - u^Tt_1 + \Psi_1(u,v,q_1) - q_1^T\nabla_{q1}\Psi_1(u,v,q_1) = 1$, then (MFP)$_S$ and (MFD)$_W$ becomes the problems considered by Mishra [18].

(3) If $g_i(x, y) - s(x|E_i) + y^Tr_i + G_i(x, y,p_i) - p_i^T\nabla_{p_i}G_i(x, y,p_i) = 1$, and $g_i(u, v) + s(v|F_i) - u^Tt_i + \Psi_i(u, v, q_i) - q_i^T\nabla_{q_i}\Psi_i(u, v, q_i) = 1$ for all i {1, ..., k}, then (MFP)$_S$ and (MFD)$_W$ becomes the problems considered by Chen [14].

(4) If $g_i(x, y) - s(x|E_i) + y^Tr_i + G_i(x, y,p_i) - p_i^T\nabla_{p_i}G_i(x, y,p_i) = 1$, $H_i(x, y,p_i) = \frac{1}{2}p_i^T\nabla_{yy}f_i(x, y)p_i$, $\Phi_i(u, v, q_i) = \frac{1}{2}q_i^T\nabla_{xx}f_i(u, v)q_i$, $H_i(x, y,p_i) = \frac{1}{2}p_i^T\nabla_{yy}f_i(x, y)p_i$, $\Phi_i(u, v, q_i) = \frac{1}{2}q_i^T\nabla_{xx}f_i(u, v)q_i$, for all $i \in$ {1, ..., k}, and there is not the condition $\lambda^T e = 1$ in (MFP)$_S$ and (MFD)$_W$, then the two problems reduce to the problems considered by Yang et al. [19].

Acknowledgements
This work was partially supported by the National Science Foundation of China (11126348), the Education Committee Project Research Foundation of Chongqing (KJ110624, KJ120628), the special fund of Chongqing key laboratory (CSTC,2011KLORSE03) and the Doctoral Foundation of Chongqing Normal University(No.10XLB015).

Competing interests
The authors declare that they have no competing interests.

Received: 28 December 2011 Accepted: 20 June 2012 Published: 20 June 2012

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doi:10.1186/1029-242X-2012-142
Cite this article as: Ying: **Higher-order symmetric duality for a class of multiobjective fractional programming problems.** *Journal of Inequalities and Applications* 2012 **2012**:142.