

Quantum feedback networks and control: A brief survey

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The purpose of this paper is to provide a brief review of some recent developments in quantum feedback networks and control. A quantum feedback network (QFN) is an interconnected system consisting of open quantum systems linked by free fields and/or direct physical couplings. Basic network constructs, including series connections as well as feedback loops, are discussed. The quantum feedback network theory provides a natural framework for analysis and design. Basic properties such as dissipation, stability, passivity and gain of open quantum systems are discussed. Control system design is also discussed, primarily in the context of open linear quantum stochastic systems. The issue of physical realizability is discussed, and explicit criteria for stability, positive real lemma, and bounded real lemma are presented. Finally for linear quantum systems, coherent H^∞ and LQG control are described.

open quantum systems, quantum feedback networks, physical realizability, H^∞ control, LQG control

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1 Introduction

Quantum technology is an interdisciplinary field that studies how to engineer devices by exploiting their quantum features. Regarded as the second quantum revolution, quantum technology has many potential far-reaching applications [1]. For example, Shor [2] presented a quantum algorithm which can offer exponential speedup over classical algorithms for factoring large integers into prime numbers. Bennett et al. [3] proposed a quantum teleportation protocol where an unknown quantum state can be disembodiedly transported to a desired receiver. Atomic lasers hold promising applications in nanotechnology such as atom lithography, atom optics and precision measurement [4]. Quantum technology (including quantum information technology) has more powerful capability than traditional technology and is one of the main focuses of scientists. Nevertheless, many challenging problems require to be systematically presented and successfully addressed in order to foster wider real-world applications of quantum technology in our life [1].

Recent years have seen a rapid growth of quantum feedback control theory [5–10]. If measurement is involved in the feedback loop, the feedback mechanism is conventionally called measurement-based feedback, e.g., [10–17]. Measurement-based feedback control of quantum systems is important in a number of areas of quantum technology, including quantum optical systems, nano-mechanical systems, and circuit QED systems. In measurement-based feedback control, the plant is a quantum system, while the controller is a classical (namely non-quantum) system. The classical controller processes the outcomes of measurement of an observable of the quantum system (e.g. the number of photons of an optical field) to determine the classical control actions (e.g. magnetic field) that are applied to control the behavior of the quantum system. Classical controllers are typically implemented using standard analog or digital electronics. However, for quantum systems that have bandwidth much higher than that of conventional electronics, an important practical issue for the implementation of measurement-based feedback control systems is the relatively slow speed of standard classical electronics, since the feedback system will not work properly unless the controller is fast enough.

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Alternatively, quantum components may be connected to each other without any measurement devices in the interconnections. For example, two optical cavities can be connected via electromagnetic fields (light beams). Such feedback mechanism is referred to as coherent feedback as originally proposed in [18–23]. The interconnection of a quantum plant and a quantum controller produces a fully quantum system; quantum information flows in this coherent feedback network, thus coherence is preserved in the whole quantum network. Moreover, a coherent feedback controller may have the similar time scale as the plant, and likely would be much faster than classical signal processing. Finally, it is becoming feasible to implement quantum networks in semiconductor materials, for example, photonic crystals are periodic optical nanostructures that are designed to affect the motion of photons in a similar way that periodicity of a semiconductor crystal affects the motion of electrons, and it may be desirable to implement control networks on the same chip (rather than interfacing to a separate system) [24–28]. For several reasons, then, it is desirable to implement controllers using the same or similar (e.g. in time scales) hardware as plants. Therefore, it might be advantageous to design coherent feedback networks.

Recently there is a growing interest in the study of coherent quantum feedback networks and control. For example, quantum feedback network structure has been studied [19, 22, 29–33]. The problem of H^∞ control has been discussed [33–36]. The problem of coherent LQG control has been investigated [33, 37]. The issue of physical realizability has been analyzed [33, 34, 38, 39]. The problem of network synthesis of quantum systems via optical devices have been studied [40–43]. There are also many papers investigating the applications of coherent feedback control, such as intracavity squeezing [19], optical field squeezing [23, 44, 45], H^∞ control [35], entanglement enhancement [46], polarization squeezing [47, 48], error correction in quantum memories [49, 50], optical switches [51], nonlinear coherent feedback control on chip [52], and photon pulse shaping [53].

The paper is organized as follows. Section 2 discusses closed quantum systems, in particular, closed quantum harmonic oscillators. Section 3 introduces Boson field and a basic model structure of open quantum systems. Section 4 discusses mechanisms by which open quantum systems interconnect. Section 5 presents results for quantum dissipative systems. Section 6 focuses on linear quantum systems. Section 7 discusses fundamental characteristics of linear quantum systems. Section 8 presents H^∞ and LQG controller synthesis of linear quantum systems. Section 9 touches on how linear quantum systems can be realized by means of optical devices. Section 10 concludes the paper.

Notation. i is the imaginary unit. δ_{jk} is Kronecker delta, and $\delta(t)$ is Dirac delta. Given a column vector of operators or

complex numbers $x = [x_1 \ \cdots \ x_m]^T$ where m is a positive integer, define $x^\# = [x_1^* \ \cdots \ x_m^*]^T$, where the asterisk $*$ indicates Hilbert space adjoint or complex conjugation. Denote $x^\dagger = (x^\#)^T = [x_1^* \ \cdots \ x_m^*]$. Furthermore, define the doubled-up column vector to be $\check{x} = [x^T \ (x^\#)^T]^T$. The matrix case can be defined analogously. Given two matrices $U, V \in \mathbb{C}^{r \times k}$, a doubled-up matrix $\Delta(U, V)$ is defined as $\Delta(U, V) := [U \ V; V^\# \ U^\#]$. Let I_n be an identity matrix. Define $J_n = \text{diag}(I_n, -I_n)$ and $\Theta_n = [0 \ I_n; -I_n \ 0]$ (The subscript “ n ” is always omitted). Then for a matrix $X \in \mathbb{C}^{2n \times 2m}$, define $X^b := J_m X^\dagger J_n$. Finally we also use I to denote identity operators.

2 Closed systems

In this paper, closed systems means systems that have no interactions with other systems and/or environment. In this section starting from the fundamental Schrodinger’s equation for closed quantum systems, we introduce closed quantum harmonic oscillators which in later sections will be allowed to interact with other systems or electromagnetic fields to produce open quantum systems.

Given a closed quantum system with Hamiltonian H , we have the following Schrodinger’s equation¹⁾

$$\frac{d}{dt}U(t) = -iHU(t), \quad U(0) = I. \quad (1)$$

Clearly, $U(t)$ is a unitary operator. The system variables $X(t)$ evolve according to $\dot{X}(t) = U^*(t)XU(t)$ with initial point $X(0) = X$, which satisfy, the Heisenberg picture,

$$\frac{d}{dt}X(t) = -i[X(t), H(t)]. \quad (2)$$

Note that for closed systems $H(t) \equiv H$ for all t due to preservation of energy.

Alternatively, the system density operator $\rho(t) = U(t)\rho U^*(t)$ with $\rho(0) = \rho$ satisfies, the Schrodinger picture,

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)]. \quad (3)$$

2.1 Closed quantum harmonic oscillators

An example of closed quantum harmonic oscillators is an optical cavity with $H = \omega a^*a$ (upon scaling), shown as Figure 1, where ω is the resonant frequency, and the annihilation operator a is the cavity mode (an operator on a Hilbert space). The adjoint operator a^* of a is called the creation operator. a and a^* satisfy the canonical commutation relation $[a(t), a^*(t)] = 1$ for all $t \geq 0$. Finally by eq. (2),

1) The reduced Planck constant \hbar is omitted throughout the paper.

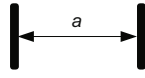


Figure 1 Closed optical cavity. Black rectangles denote fully reflecting mirrors at cavity resonant frequency.

$$\frac{d}{dt}a(t) = -i\omega a(t), \quad a(0) = a. \tag{4}$$

Therefore $a(t) = e^{-i\omega t}a$ — an oscillator.

In general, let G be a closed quantum system of interconnection of n quantum harmonic oscillators. The behavior of G is determined by the Hamiltonian

$$H_0 = \frac{1}{2}\check{a}^\dagger \begin{bmatrix} \Omega_- & \Omega_+ \\ \Omega_+^\# & \Omega_-^\# \end{bmatrix} \check{a}, \tag{5}$$

where Ω_- and Ω_+ are respectively $\mathbb{C}^{m \times n}$ matrices satisfying $\Omega_- = \Omega_-^\dagger$ and $\Omega_+ = \Omega_+^\dagger$. By eq. (2),

$$\dot{a}_j(t) = -i[a_j(t), H_0(t)], \quad a_j(0) = a_j, \quad (j = 1, \dots, n).$$

In a compact form we have the following linear differential equations:

$$\dot{\check{a}}(t) = A_0\check{a}(t) \tag{6}$$

with initial condition $\check{a}(0) = \check{a}$, where

$$A_0 = -\Delta(i\Omega_-, i\Omega_+). \tag{7}$$

3 Quantum fields and open quantum systems

3.1 Boson fields

The m -channel Boson field $b(t) = [b_1(t), \dots, b_m(t)]^T$ are operators on a Fock space \mathcal{F} [54], whose components satisfy the singular commutation relations

$$\begin{aligned} [b_j(t), b_k^*(t')] &= \delta_{jk}\delta(t-t'), \quad [b_j(t), b_k(t')] = 0, \\ [b_j^*(t), b_k^*(t')] &= 0, \quad (j, k = 1, \dots, m). \end{aligned} \tag{8}$$

The operators $b_j(t)$ may be regarded as quantum stochastic processes, see, eg. [55, Chapter 5]; when the field is in the vacuum state, namely absolutely zero temperature and completely dark, they are called standard quantum white noise (that is, $M = N = 0$ in [55, eq. (10.2.38)]). The integrated processes $B_j(t) = \int_0^t b_j(\tau)d\tau$ are quantum Wiener processes with Ito increments $dB_j(t) = B_j(t+dt) - B_j(t)$, ($j = 1, \dots, m$).

There might exist scattering between channels, which is modeled by the gauge process:

$$\Lambda(t) = \int_0^t b^\#(\tau)b^T(\tau)d\tau = \begin{bmatrix} \Lambda_{11}(t) & \cdots & \Lambda_{1m}(t) \\ \vdots & \ddots & \vdots \\ \Lambda_{m1}(t) & \cdots & \Lambda_{mm}(t) \end{bmatrix}, \tag{9}$$

with operator entries Λ_{jk} on the Fock space \mathcal{F} . Finally in this paper it is assumed that these quantum stochastic processes

are canonical, that is, they have the following non-zero Ito products:

$$\begin{aligned} dB_j(t)dB_k^*(t) &= \delta_{jk}dt, \quad d\Lambda_{jk}dB_l^*(t) = \delta_{kl}dB_j^*(t), \\ dB_j(t)d\Lambda_{kl}(t) &= \delta_{jk}dB_l(t), \\ d\Lambda_{jk}(t)d\Lambda_{lm}(t) &= \delta_{kl}d\Lambda_{jm}(t), \quad (j, k, l = 1, \dots, m). \end{aligned} \tag{10}$$

3.2 Open quantum systems in the (S, L, H) parametrization

When a quantum system G is driven by a Boson field \mathcal{F} , we have an open quantum system. For example, if we allow the closed optical cavity in Figure 1 to interact with a Boson field, we end up with an open optical cavity (Figure 2). While the mutual influence between the system and field may be described rigorously from first principles in terms of an interaction Hamiltonian, it is much more convenient to use an idealized quantum noise model which is valid under suitable rotating wave and Markovian assumptions, as in many situations in quantum optics, e.g. cascaded open systems, see [18, 22, 56, 57] for detail. Let \mathcal{A}_G and A_F be physical variable spaces of the system G and the field \mathcal{F} respectively, then the physical variable space for the composite system is the tensor product space $\mathcal{A}_G \otimes A_F$.

Open quantum systems G studied in this paper can be parameterized by a triple (S, L, H) [30, 58]. Here, S is a scattering matrix with entries in the system space \mathcal{A}_G , $L \in \mathcal{A}_G$ is an coupling operator that provides interface between systems and fields, $H \in \mathcal{A}_G$ is the internal Hamiltonian of quantum system G .

With these parameters, and assuming that the input field is canonical, that is, eq. (10) holds, we have the following Schrodinger's equation for open quantum systems (in Ito form):

$$\begin{aligned} dU(t) &= \left\{ \text{tr}[(S - I_m)d\Lambda^T] + dB^\dagger(t)L - L^\dagger S dB(t) \right. \\ &\quad \left. - \left(iH + \frac{1}{2}L^\dagger L \right) dt \right\} U(t), \quad U(0) = I. \end{aligned} \tag{11}$$

Note that for a closed system, eq. (11) becomes the familiar Schrodinger's equation (1). This, together with the evolution $X(t) = U(t)^* X U(t)$, yields the following quantum stochastic differential equations (QSDEs), in Ito form:

$$\begin{aligned} dX(t) &= (-i[X(t), H(t)] + \mathcal{L}_{L(t)}(X(t)))dt \\ &\quad + dB^\dagger(t)S^\dagger(t)[X(t), L(t)] + [L^\dagger(t), X(t)]S(t)dB(t) \\ &\quad + \text{tr}[(S^\dagger(t)X(t)S(t) - X(t))d\Lambda^T(t)], \quad X(0) = X, \end{aligned} \tag{12}$$

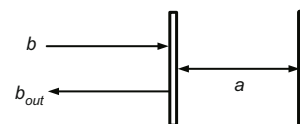


Figure 2 Open optical cavity. White rectangle denotes a partially transmitting mirror at cavity resonant frequency.

where the Lindblad operator \mathcal{L}_L is

$$\mathcal{L}_L(X) := \frac{1}{2}L^\dagger[X, L] + \frac{1}{2}[L^\dagger, X]L. \quad (13)$$

For later use, we define a generator operator

$$\mathcal{G}_G(X) := -i[X, H] + \mathcal{L}_L(X). \quad (14)$$

The output field $B_{out}(t) = U^*(t)B(t)U(t)$ satisfies

$$dB_{out}(t) = L(t)dt + S(t)dB(t). \quad (15)$$

The gauge process of the output field $\Lambda_{out}(t) := \int_0^t b_{out}^\#(s) b_{out}^T(s)ds = U^*(t)\Lambda(t)U(t)$ satisfies

$$d\Lambda_{out}(t) = S^\#(t)d\Lambda(t)S^T(t) + S^\#(t)dB^\#(t)L^T(t) + L^\#(t)dB^T(t)S^T(t) + L^\#(t)L^T(t)dt. \quad (16)$$

Finally, in the Schrodinger picture, the reduced system density operator $\hat{\rho}$ satisfies the master equation, c.f. [55, Sec. 11.2.5],

$$\frac{d}{dt}\hat{\rho}(t) = -i[H, \hat{\rho}(t)] + \mathcal{L}'_L(\hat{\rho}(t)), \quad (17)$$

where the operator \mathcal{L}'_L is defined to be

$$\mathcal{L}'_L(\hat{\rho}) := L^T\hat{\rho}L^\# - \frac{1}{2}L^\dagger L\hat{\rho} - \frac{1}{2}\hat{\rho}L^\dagger L. \quad (18)$$

Remark 1. Clearly, open quantum systems presented in this section are quantum Markov processes.

3.3 Examples

(i) Optical cavity. The one degree of freedom closed quantum harmonic oscillator in Figure 1 can be described by $(-, -, \omega a^*a)$, where the symbol “-” means that there is neither scattering nor coupling. The open optical cavity in Figure 2 may be described by $(1, \sqrt{\kappa}a, \omega a^*a)$, where κ is coupling coefficient and ω is the resonant frequency. According to eqs. (12)–(15),

$$da(t) = -(i\omega + \frac{\kappa}{2})a(t)dt - \sqrt{\kappa}dB(t), \quad a(0) = a, \quad (19)$$

$$dB_{out}(t) = \sqrt{\kappa}a(t)dt + dB(t). \quad (20)$$

(ii) Two-level systems. Given a two-level system parameterized by

$$S_- = 1, \quad L = \sqrt{\kappa}\sigma_-, \quad H = \frac{\omega}{2}\sigma_z,$$

with Pauli matrices

$$\begin{aligned} \sigma_z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\ \sigma_x &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \end{aligned} \quad (21)$$

by eq. (11) the unitary operator $U(t)$ evolves according to

$$dU(t) = \left\{ \sqrt{\kappa}dB^*(t)\sigma_- - \sqrt{\kappa}\sigma_+dB(t) - \frac{\kappa}{4}(\sigma_z + 1)dt - i\frac{\omega}{2}\sigma_z dt \right\} U(t), \quad U(0) = I. \quad (22)$$

By eqs. (12) and (13)

$$d\sigma_x(t) = -\left(\frac{\kappa}{2}\sigma_x(t) + \omega\sigma_y(t)\right)dt + \sqrt{\kappa}dB^*(t)\sigma_z(t) + \sqrt{\kappa}\sigma_z(t)dB(t), \quad (23)$$

$$d\sigma_y(t) = -\left(\frac{\kappa}{2}\sigma_y(t) - \omega\sigma_x(t)\right)dt - i\sqrt{\kappa}dB^*(t)\sigma_z(t) + i\sqrt{\kappa}\sigma_z(t)dB(t), \quad (24)$$

$$d\sigma_z(t) = -\kappa(I + \sigma_z(t))dt - \frac{\sqrt{\kappa}}{2}(\sigma_x(t) - i\sigma_y(t))dB^*(t) - \frac{\sqrt{\kappa}}{2}(\sigma_x(t) + i\sigma_y(t))dB(t). \quad (25)$$

On the other hand, the output field is

$$dB_{out}(t) = \sqrt{\kappa}\sigma_-(t)dt + dB(t), \quad (26)$$

$$d\Lambda_{out}(t) = d\Lambda(t) + \sqrt{\kappa}dB^*(t)\sigma_-(t) + \sqrt{\kappa}\sigma_+(t)dB(t) + \frac{\kappa}{2}(\sigma_z + 1)dt. \quad (27)$$

Remark 2. It can be seen from eqs. (23)–(25) that two-level systems are nonlinear quantum systems.

4 Interconnection

In this section we discuss how two quantum systems can be connected to each other. More specifically we discuss concatenation product, series product, direct coupling, and linear fractional transform. Several examples from the literature are used to illustrate these interconnections. Propagation delays are ignored in interconnections. Discussions of influence of propagation delays on system performance can be found in, e.g. [45].

4.1 Concatenation product

Given two open quantum systems $G_1 = (S_1, L_1, H_1)$ and $G_2 = (S_2, L_2, H_2)$, their concatenation product (Figure 3), is defined to be

$$G_1 \boxplus G_2 := \left(\begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, H_1 + H_2 \right). \quad (28)$$

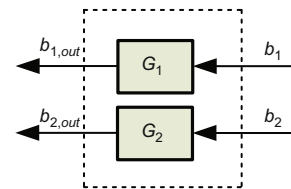


Figure 3 Concatenation product $G_1 \boxplus G_2$.

4.2 Series product

Given two open quantum systems $G_1 = (S_1, L_1, H_1)$ and $G_2 = (S_2, L_2, H_2)$ with the same number of input, their series product (Figure 4), is defined to be

$$G_2 \triangleleft G_1 := \left(S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \frac{1}{2i} (L_2^\dagger S_2 L_1 - L_1^\dagger S_2^\dagger L_2) \right). \tag{29}$$

Theorem 1 (Principle of Series Connections, [30, Theorem 5.5]). The parameters of the composite system $G_2 \triangleleft G_1$, obtained from $G_1 \boxplus G_2$ when the output of G_1 is used as input of G_2 , is given by the series product $G_2 \triangleleft G_1$.

4.3 Direct coupling

In quantum mechanics, two independent systems G_1 and G_2 may interact by exchanging energy. This energy exchange may be described by an interaction Hamiltonian H_{int} of the form $H_{int} = X_1^\dagger X_2 + X_2^\dagger X_1$, where $X_1 \in \mathcal{A}_{G_1}$ and $X_2 \in \mathcal{A}_{G_2}$; see, e.g. [19], [21], [33]. In this case, we say the two systems G_1 and G_2 are directly coupled, and the composite system is denoted $G_1 \bowtie G_2$ (Figure 5).

4.4 Linear fractional transform

Let G in Figure 6 be of the form $\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, H \right)$.

Assume that $(I - S_{22})^{-1}$ exists. Then the conventional linear fractional transform yields a feedback network

$$F(G) = (S_{11} + S_{12}(I - S_{22})^{-1}S_{21}, L_1 + S_{12}(I - S_{22})^{-1}L_2, H + \text{Im}\{L_1^\dagger S_{12}(I - S_{22})^{-1}L_2\} + \text{Im}\{L_2^\dagger S_{22}(I - S_{22})^{-1}L_2\})$$

from input b to output b_{out} .

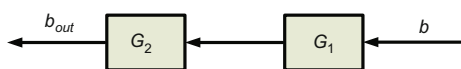


Figure 4 Series product $G_2 \triangleleft G_1$.

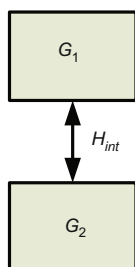


Figure 5 Directly coupled system $G_1 \bowtie G_2$.

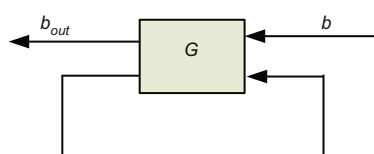


Figure 6 Linear fractional transformation $F(G)$.

4.5 Examples

In this section examples in the literature are used to demonstrate the usefulness of the parametrization (S, L, H) and interconnections. More examples can be found in [19, 30, 51, 59]. For the convenience of the readers to refer to the original papers we use symbols in those original papers.

Example 1 ([57]). In [57] quantum trajectory theory is formulated for interaction of open quantum systems via series product (Figure 4). Given two open systems $G_1 = (1, L_A, H_A), G_2 = (1, L_B, H_B)$ with $L_A = \sqrt{2\kappa_A}a_A, L_B = \sqrt{2\kappa_B}a_B$. Here κ_A and κ_B are coupling constants and a_A and a_B are annihilation operators for systems G_1 and G_2 respectively. The series product yields

$$G_2 \triangleleft G_1 = (1, \sqrt{2\kappa_A}a_A + \sqrt{2\kappa_B}a_B, H_A + H_B + i\sqrt{\kappa_A\kappa_B}(a_A^*a_B - a_B^*a_A)). \tag{30}$$

Identifying $H_A + H_B + i\sqrt{\kappa_A\kappa_B}(a_A^*a_B - a_B^*a_A)$ with \hat{H}_S in [57, eq. (7)] and $\sqrt{2\kappa_A}a_A + \sqrt{2\kappa_B}a_B$ with \hat{C} in [57, eq. (9)] respectively, eq. (17) re-produces the master equation [57, eq. (8)].

Example 2 ([56]). In [56] quantum Langevin equations and a quantum master equation were derived for a cascade of two two-level systems (Figure 7). The two two-level systems G_1 and G_2 in Figure 7 are respectively

$$G_1 = (1, \sqrt{\gamma_1}\sigma_1^-, 0) \boxplus ((1, \sqrt{\eta_1}\sigma_1^-, 0) \triangleleft (1, E, 0)),$$

$$G_2 = (1, \sqrt{\eta_2}\sigma_2^-, 0) \boxplus (1, \sqrt{\gamma_2}\sigma_2^-, 0).$$

Here, γ_1 and η_1 are coupling constants for system G_1 , E is an incident coherent electric drive (not an operator) of G_1 . γ_1 and η_1 are coupling constants for system G_2 . σ^- is defined in eq. (21):

$$\sigma_1^- = \sigma_2^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let G be a series product $G = (G_2 \boxplus (1, 0, 0)) \triangleleft ((1, 0, 0) \boxplus G_1)$. With these, according to eq. (17), [56, eq. (14)] can be re-produced (Notice the fact that in the interaction picture $H_{sys} = 0$ is used).

Example 3 ([47, 48]). A scheme is proposed in [47] to produce continuous-wave fields or pulses of polarization-squeezed light by passing classical, linearly polarized laser light through an atomic sample twice; that is, the field output of the first pass is fed back to the atomic sample again

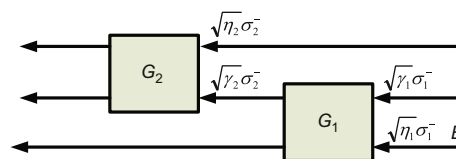


Figure 7 A re-draw of [56, Figure 1]. For clarity, coupling operators $\sqrt{\gamma_1}\sigma_1^-, \sqrt{\eta_1}\sigma_1^-, \sqrt{\gamma_2}\sigma_2^-, \sqrt{\eta_2}\sigma_2^-$ are shown explicitly. E is a coherent electric drive of G_1 .

so as to generate polarization-squeezed light. This scheme is confirmed and extended in [48]. The atomic sample can be modeled as an open quantum system $G = G_1 \boxplus G_2$ with

$$G_1 = \left(1, \frac{1}{\sqrt{2}}\alpha p, 0\right), \quad G_2 = \left(1, -\frac{i}{\sqrt{2}}\alpha x, 0\right).$$

Here α is a real constant, and x, p are position and momentum operators respectively, c.f. [48, eqs. (A1)–(A2)]. Double-pass of an electromagnetic field through the atomic sample introduces a series product, that is the overall quantum system is

$$G_2 \triangleleft G_1 = \left(1, \frac{\alpha}{2}(p - ix), \frac{\alpha^2}{4}(xp + px)\right),$$

whose Schrodinger equation is [48, eq. (1)].

Example 4 ([33]). Given two closed quantum Harmonic oscillators G_1 and G_2 as studied in Section 2.1, we take the interaction Hamiltonian H_{int} in Figure 5 to be

$$H_{int} = \frac{1}{2} \left(\check{a}^{(1)\dagger} \Xi^\dagger \check{a}^{(2)} + \check{a}^{(2)\dagger} \Xi \check{a}^{(1)} \right), \quad (31)$$

where $\Xi = \Delta(iK_-, iK_+)$ for matrices $K_-, K_+ \in \mathbb{C}^{n_2 \times n_1}$. The Hamiltonian for the directly coupled system $G_1 \bowtie G_2$ is

$$H = H_{0,1} + H_{int} + H_{0,2}, \quad (32)$$

where $H_{0,k} = \frac{1}{2} \check{a}^{(k)\dagger} \Delta(\Omega_-^{(k)}, \Omega_+^{(k)}) \check{a}^{(k)}$ is the self-Hamiltonian for G_k , and H_{int} is given by eq. (31). It is easy to show that system operators $\check{a}^{(j)}(t) = U^*(t) \check{a}^{(j)} U(t)$ ($j = 1, 2$) satisfy the following linear differential equations, in Stratonovich form:

$$\begin{aligned} \dot{\check{a}}^{(1)}(t) &= A_{0,1} \check{a}^{(1)}(t) + B_{12} \check{a}^{(2)}(t), \quad \check{a}^{(1)}(0) = \check{a}^{(1)}, \\ \dot{\check{a}}^{(2)}(t) &= A_{0,2} \check{a}^{(2)}(t) + B_{21} \check{a}^{(1)}(t), \quad \check{a}^{(2)}(0) = \check{a}^{(2)}, \end{aligned}$$

where

$$\begin{aligned} A_{0,j} &= -\Delta(i\Omega_-^{(j)}, i\Omega_+^{(j)}), \\ B_{12} &= -\Delta(K_-, K_+)^b, \\ B_{21} &= -B_{12}^b, \quad (j = 1, 2). \end{aligned}$$

5 Quantum dissipative systems

Open systems are systems that interact with other systems and/or their environment. In classical control theory, a general framework for the stability of open systems has been developed [60–63]. This classical theory abstracts energy concepts and provides fundamental relations for stability in terms of generalized energy inequalities. In this section we briefly review dissipation theory for open quantum systems [31]. In Figure 8 the open quantum system $P = (S, L, H)$ is the plant of interest whose space of variables is denoted \mathcal{A}_P . The other open quantum system $W = (R, w, D)$ is an external system or the environment, whose space of variables is denoted \mathcal{A}_{ex} . W is called an exosystem. Moreover, we allow W to vary in a class of such exosystems \mathcal{W} .

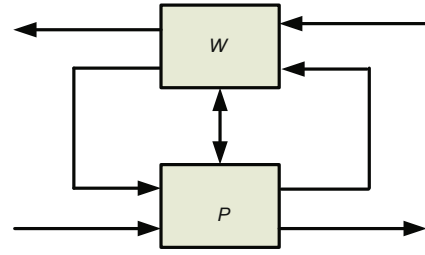


Figure 8 Plant-exosystem network $P \wedge W$.

5.1 Dissipativity, stability, passivity, gain

In this section we present concepts of dissipativity, stability, passivity and gain. As with the classical case, criteria of stability, passivity and gain follow those of dissipativity.

The following assumption is used in the sequel.

Assumption A1. The inputs to the composite system $P \wedge W$ are all canonical vacuum fields, c.f. Section 3.1.

Let $r_P(W)$ be a self-adjoint operator in the composite plant-exosystem space $\mathcal{A}_P \otimes \mathcal{A}_{ex}$. $r_P(W)$ is usually called supply rate. We have the following definition of dissipativity for open quantum systems P .

Definition 1 (Dissipation, [31, Sec. III-A]). The plant P is said to be dissipative with supply rate $r_P(W)$ with respect to a class of exosystems \mathcal{W} if there exists a non-negative plant observable $V \in \mathcal{A}_P$, called storage function, such that the dissipation inequality

$$\underbrace{\mathbb{E}_0 [V(t) - V(0)]}_{\text{stored energy}} \leq \underbrace{\int_0^t \mathbb{E}_0 [r_P(W)(s)] ds}_{\text{supplied energy}} \quad (33)$$

holds for all $W \in \mathcal{W}$ and all $t \geq 0$, where \mathbb{E}_0 is vacuum expectation [54, Chapter 26]. In particular, when “=” in (33) holds for all $W \in \mathcal{W}$ and all $t \geq 0$, P is called lossless.

The combination of Definition 5.1 and the following property of vacuum expectation [54, Proposition 26.6]

$$\mathbb{E}_\tau [V(t)] = V(\tau) + \int_\tau^t \mathbb{E}_0 [\mathcal{G}_{P \wedge W}(V(r))] dr, \quad (34)$$

yields an infinitesimal version (namely independent of the time variable) of the dissipation inequality (33).

Theorem 2 (Dissipation, [31, Theorem 3.1]). Pertaining to Figure 8, the plant P is dissipative with a supply rate $r_P(W)$ with respect to a class of exosystems \mathcal{W} if and only if there exists a non-negative plant observable $V \in \mathcal{A}_P$ such that

$$\mathcal{G}_{P \wedge W}(V) - r_P(W) \leq 0 \quad (35)$$

holds for all $W \in \mathcal{W}$.

Given a quantum system P , assume it has indirect and/or direct connections to external systems and/or its environment. With slight abuse of notation we still call P an open quantum system. Let \mathcal{W}_u denote the class of all the quantum systems

that can be connected to P , directly or indirectly. The following result shows that P is lossless with respect to \mathcal{W}_u .

Theorem 3 ([31, Theorem 3.3]). Pertaining to Figure 8, for any given storage function V_0 , which is a non-negative observable in \mathcal{A}_P , the open quantum system P is lossless with a supply rate

$$r_0(W) = \mathcal{G}_{P \wedge W}(V_0) \tag{36}$$

with respect to \mathcal{W}_u .

Theorem 3 shows that any open quantum system is dissipative in some sense.

Next we study stability of open quantum systems which is characterized in terms of the evolution of mean values.

Definition 2 (Exponential stability, [31, Sec. III-B]). An open quantum system P is said to be exponentially stable if there exists a non-negative observable $V \in \mathcal{A}_P$, scalars $c > 0$ and $\lambda \geq 0$ such that

$$\langle V(t) \rangle \leq e^{-ct} \langle V \rangle + \frac{\lambda}{c} \tag{37}$$

holds for any plant state and all time $t \geq 0$. Moreover, if $\lambda = 0$, then $\lim_{t \rightarrow \infty} \langle V(t) \rangle = 0$.

The combination of eq. (34) and Definition 5.1 gives the following stability result for open quantum systems P .

Theorem 4 (Stability, [31, Lemma 3.4]). If there exists a nonnegative observable $V \in \mathcal{A}_P$, scalars $c > 0$ and $\lambda \geq 0$ such that

$$\mathcal{G}_P(V) + cV \leq \lambda,$$

then the open quantum system P is exponentially stable. Moreover, if $\lambda = 0$, then $\lim_{t \rightarrow \infty} \langle V(t) \rangle = 0$.

In what follows we focus on the series product of P and W with additional direct coupling (Figure 9). That is, the composite system is

$$P \wedge W = (P \triangleleft W) \boxplus (-, -, H_{int}),$$

where $P = (I, L, H)$, $W = (I, w, 0)$, $H_{int} = -i(M^\dagger v - v^\dagger M)$ with $w, v \in \mathcal{A}_{ex}$ and $M \in \mathcal{A}_P$. With slight abuse of notation, we write

$$P \wedge W = P \triangleleft W,$$

where $W = (I, w, -i(M^\dagger v - v^\dagger M))$. That is, direct coupling is absorbed into the exosystem W .

Let $V \in \mathcal{A}_P$ be a non-negative observable. Assume that $S = I$. By eq. (14)

$$\begin{aligned} \mathcal{G}_{P \wedge W}(V) &= \mathcal{G}_P(V) + \mathcal{L}_w(V) + [w^\dagger \ v^\dagger]Z + Z^\dagger \begin{bmatrix} w \\ v \end{bmatrix} \\ &\quad + [V, v^\dagger]M - M^\dagger[V, v], \end{aligned} \tag{38}$$

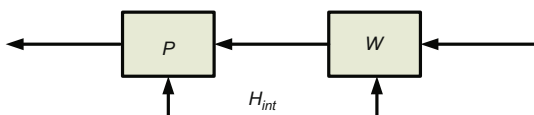


Figure 9 Series product plus direct coupling.

where $Z = \begin{bmatrix} V, \begin{bmatrix} L \\ M \end{bmatrix} \end{bmatrix}$.

For fixed $M \in \mathcal{A}_P$, define a class of exosystems

$$\mathcal{W}_1 = \{W = (I, w, -i(M^\dagger v - v^\dagger M)) : w, v \text{ commute with } \mathcal{A}_P\}. \tag{39}$$

Then we have the following definition of passivity for the system P in Figure 9.

Definition 3 (Passivity, [31, Sec. III-C]). Given $M \in \mathcal{A}_P$, the plant $P = (I, L, H)$ is said to be passive with respect to the class of exosystems \mathcal{W}_1 in eq. (39) if it is dissipative with the supply rate

$$r_P(W) = -N^\dagger N + [w^\dagger \ v^\dagger]Z + Z^\dagger \begin{bmatrix} w \\ v \end{bmatrix} + \lambda \tag{40}$$

for some non-negative real number λ , and $N, Z \in \mathcal{A}_P$. P is said to be strictly passive if $N^\dagger N$ is strictly positive.

The combination of Definition 5.1, Theorem 5.1 and eq. (38) gives the following passivity result.

Theorem 5 (Positive Real Lemma, [31, Theorem 3.6]). A plant $P = (I, L, H)$ is passive with respect to the class of exosystems \mathcal{W}_1 in eq. (39) if and only if there exists a non-negative observable $V \in \mathcal{A}_P$, an operator $N \in \mathcal{A}_P$, and a non-negative real number λ such that

$$\mathcal{G}_P(V) + N^\dagger N - \lambda \leq 0, \tag{41}$$

$$Z = \begin{bmatrix} V, \begin{bmatrix} L \\ M \end{bmatrix} \end{bmatrix}. \tag{42}$$

As with the classical case, strict passivity implies stability.

Theorem 6. The open quantum system P is exponentially stable if it is strictly passive with respect to the exosystem $W = (I, 0, 0)$.

The bounded real lemma is used to determine L^2 gain of an open system, and in conjunction with the small gain theorem, can be used for robust stability analysis and design. In what follows we discuss L^2 gain of open quantum systems.

Define a class of exosystems

$$\mathcal{W}_2 = \{W = (I, w, 0) : w \text{ commutes with } \mathcal{A}_P\}. \tag{43}$$

Note that in this case there is no direct coupling.

Definition 4 (L^2 gain, [31, Sec. III-C]). The plant $P = (I, L, H)$ is said to have L^2 gain $g > 0$ with respect to the class of exosystems \mathcal{W}_2 in eq. (43) if it is dissipative with the supply rate

$$r_P(W) = g^2 w^\dagger w - (N + ZW)^\dagger (N + ZW) + \lambda \tag{44}$$

for some non-negative real number λ , and $N, Z \in \mathcal{A}_P$.

The combination of Definition 5.1 and Theorem 5.1 gives the following result.

Theorem 7 (Bounded Real Lemma, [31, Theorem 3.7]). A plant $P = (I, L, H)$ has L^2 gain $g > 0$ with respect to \mathcal{W}_2 if and only if there exists a non-negative plant variable $V \in \mathcal{A}_P$,

an operator $N \in \mathcal{A}_P$, and a non-negative real number λ such that

$$\Gamma = g^2 - Z^\dagger Z \geq 0 \tag{45}$$

and

$$\mathcal{G}_P(V) + N^\dagger N - w^\dagger \Gamma w + w^\dagger ([V, L] + Z^\dagger N) + ([V, L] + Z^\dagger N)^\dagger w - \lambda \leq 0 \tag{46}$$

for all $w \in \mathcal{A}_{ex}$. If Γ^{-1} exists, then plant $P = (I, L, H)$ has gain $g > 0$ with respect to \mathcal{W}_2 if and only if

$$\mathcal{G}_P(V) + N^\dagger N + ([V, L] + Z^\dagger N)^\dagger \Gamma^{-1} ([V, L] + Z^\dagger N) - \lambda \leq 0 \tag{47}$$

for all $w \in \mathcal{A}_{ex}$. In the latter case, the plant P is strictly bounded real.

5.2 Example

The following example illustrates the above results for stability, passivity, and L^2 gain. Consider a two-level system P and an exosystem W of the form

$$P = (1, \sqrt{\gamma}\sigma_+, \frac{\omega}{2}\sigma_z), \quad W = (1, w, 0),$$

where w commutes with \mathcal{A}_P . Assume there is no direct coupling between P and W . Choose a storage function $V_0 = \frac{1}{2}(I - \sigma_z) = \sigma_- \sigma_+$ and a supply rate

$$r_P(W) = \mathcal{G}_{P \wedge W}(V_0) = \mathcal{G}_P(V_0) + w^* Z + Z^* w, \tag{48}$$

where $Z = [V_0, \sqrt{\gamma}\sigma_+] = -\sqrt{\gamma}\sigma_+$. Clearly, $\mathcal{G}_P(V_0) = -\gamma V_0$. As a result, eq. (48) becomes

$$\begin{aligned} r_P(W) &= -\gamma V_0 - \sqrt{\gamma}(w^* \sigma_+ + \sigma_- w) \\ &= -(\sqrt{\gamma}\sigma_+ + w)^* (\sqrt{\gamma}\sigma_+ + w) + w^* w. \end{aligned}$$

Choose $N = \sqrt{\gamma}\sigma_+$ and $Z = -N$, by Theorem 5 we see that the system is passive. Choose $N = \sqrt{\gamma}\sigma_+$ and $Z = 1$, by Theorem 7 we find that the system has L^2 gain 1. Finally, when $W = (1, 0, 0)$, $r_P(W) = r_P(I) = -\gamma V_0$, then by Theorem 4 the system P is exponentially stable.

6 Linear quantum systems

Linear quantum systems are those for which certain conjugate operators evolve linearly, the optical cavity being a basic example, c.f. Section 3.3(i). Linear systems have the advantage that they are much more computationally tractable than general nonlinear systems, and indeed, powerful methods from linear algebra may be exploited.

6.1 General model

Open linear quantum systems discussed in this paper are open quantum harmonic oscillators with direct and indirect couplings to other quantum systems and/or external fields. In this section we present a general model for an open linear quantum system G Figure 10, based on the ingredients discussed in the previous sections. Here, G is an open quantum system with parametrization (I, L, H_0) , where $L = C_- a + C_+ a^\#$ with C_-, C_+ being constant complex-valued matrices. The internal Hamiltonian H_0 is that given in eq. (5). Moreover, G is allowed to coupled directly to another (independent) quantum system W_d via an interaction Hamiltonian

$$H_{int} = \frac{1}{2} (\check{a}^\dagger \Xi^\dagger \check{v} + \check{v}^\dagger \Xi \check{a}), \tag{49}$$

where $\Xi = \Delta(iK_-, iK_+)$. Our interest is in the influence of external systems/fields on the given system G . The performance characteristics of interest are encoded in a performance variable²⁾ z .

Building upon the discussions in previous sections, the equations for G (including direct coupling, indirect coupling and performance variable) are

$$\dot{\check{a}}(t) = A\check{a}(t) + B_d\check{v}(t) + B_f\check{w}(t) + B_f\check{b}(t), \quad \check{a}(0) = \check{a}, \tag{50}$$

$$\check{b}_{out}(t) = C_f\check{a}(t) + \check{w}(t) + \check{b}(t), \tag{51}$$

$$\check{z}(t) = C_p\check{a}(t) + D_{pd}\check{v}(t) + D_{pf}\check{w}(t). \tag{52}$$

The complex matrices in eqs. (50) and (51) are given by

$$A = -\frac{1}{2}C_f^\dagger C_f - \Delta(i\Omega_-, i\Omega_+), \quad B_d = -\Delta(K_-, K_+)^b, \tag{53}$$

$$C_f = \Delta(C_-, C_+), \quad B_f = -C_f^b. \tag{54}$$

The matrices A and B_f are specified by the parameters Ω_\pm , and C_\pm . In eqs. (50) and (51), $b(t)$ and $b_{out}(t)$ are respectively, the input and output fields for G . The term v in eq. (50) is an exogenous quantity associated with W_d with which G is directly coupled via the interaction Hamiltonian H_{int} , c.f. Section 4.5.

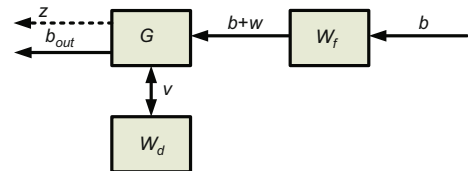


Figure 10 General model.

2) A performance variable is chosen to capture control performance, such as an error quantity, and so may involve external variables, like a reference signal. Performance variables need not have anything to do with the output quantities associated with direct or indirect couplings to other systems, c.f. Sections 7 and 8.

The term w in eq. (50) is another exogenous quantity associated with another (independent) system W_f with which G is indirectly coupled through a series product. W_f may be a quantum system of the form $(1, w, 0)$ where w is an operator on some Fock space, it can also denote modulation so that w coherent drive modulates the vacuum field b c.f. E in Example 2 of Section 4.5. Because of the assumed independence, w and v commute with the mode operators a_j, a_j^* for G . While v and w are arbitrary external variables, the time evolutions $v(t)$ and $w(t)$ (when it is an operator) are determined by the evolution of the overall composite system. The matrices C_p, D_{pd} and D_{pf} specify the performance variable z . In brief, system G is specified by the parameters $G = (\Omega_{\pm}, C_{\pm}, K_{\pm}, C_p, D_{pd}, D_{pf})$. Of these, Ω_{\pm}, C_{\pm} and K_{\pm} are physical parameters.

In particular, when all the plus terms are zero, namely $C_+ = 0, \Omega_+ = 0, K_+ = 0$, all matrices A, B_f, B_d, C_f are block diagonal, system (50)–(51) is equivalent to

$$\begin{aligned} \dot{a}(t) &= -(i\Omega_- + \frac{1}{2}C_-^\dagger C_-)a(t) - K_-^\dagger v(t) - C_-^\dagger w(t) \\ &\quad - C_-^\dagger b(t), \quad a(0) = a, \end{aligned} \tag{55}$$

$$b_{out}(t) = C_- a(t) + b(t), \tag{56}$$

c.f. optical cavity (19)–(20). It can be readily shown that system (55)–(56) is passive. Passive systems have been studied in, e.g. [33, 36, 39, 43, 53].

6.2 Physical realizability

It can be readily verified that the following relations for system matrices (53)–(54) hold

$$J_n A + A^\dagger J_n + C_f^\dagger J_m C_f = 0, \tag{57}$$

$$B_f = -C_f^b, \tag{58}$$

$$B_d = -\Delta(K_-, K_+)^b. \tag{59}$$

Eq. (57) characterizes preservation of the canonical commutation relations, namely

$$[\check{a}_j(t), \check{a}_k^*(t)] = [\check{a}_j, \check{a}_k^*] = (J_n)_{jk}, \quad \forall t \geq 0, \quad (j, k = 1, \dots, n). \tag{60}$$

Eq. (58) reflects the input and output relation, while eq. (59) is for direct coupling.

The relations (57)–(59) are called physical realizability relations, which generalize results in [34, Theorem 3.4], [38], [39, Theorem 5.1], [38, Theorem 3]. These conditions guarantee that the equations correspond to a physical system.

6.3 Quadrature representation

So far, annihilation-creation representation has been used to represent linear quantum systems in terms of the notation $\check{a} = [a^T \ a^\dagger]^T$, the resulting matrices are complex-valued matrices. In this section we introduce an alternative representation, the so-called quadrature representation, which leads to equations with real-valued matrices.

Define the unitary matrix

$$\Lambda = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -iI & iI \end{bmatrix} \tag{61}$$

and the vector of self-adjoint operators

$$\check{a} = \begin{bmatrix} q \\ p \end{bmatrix} \tag{62}$$

by the relation

$$\check{a} = \Lambda \check{a}. \tag{63}$$

The vector $q = \frac{1}{\sqrt{2}}[I \ I]\check{a}$ is known as the real quadrature, while $p = \frac{1}{\sqrt{2}}[-iI \ iI]\check{a}$ is called the imaginary or phase quadrature [64].

Similarly define unitary matrices Λ_f, Λ_d and Λ_p of suitable dimension, of the form (61), and define quadrature vectors

$$\check{b} = \Lambda_f \check{b}, \quad \check{b}_{out} = \Lambda_f \check{b}_{out}, \quad \check{w} = \Lambda_f \check{w}, \quad \check{v} = \Lambda_d \check{v}, \quad \check{z} = \Lambda_p \check{z}.$$

Then in quadrature form G is in the form

$$\begin{aligned} \dot{\check{a}}(t) &= \check{A}\check{a}(t) + \check{B}_d\check{v}(t) + \check{B}_f\check{w}(t) + \check{B}_f\check{b}(t), \quad \check{a}(0) = \check{a}, \tag{64} \\ \check{b}_{out}(t) &= \check{C}_f\check{a}(t) + \check{w}(t) + \check{b}(t), \\ \check{z}(t) &= \check{C}_p\check{a}(t) + \check{D}_{pd}\check{v}(t) + \check{D}_{pf}\check{w}(t), \end{aligned}$$

where $\check{A} = \Lambda A \Lambda^\dagger, \check{B}_d = \Lambda B_d \Lambda_d^\dagger, \check{B}_f = \Lambda B_f \Lambda_f^\dagger, \check{C}_f = \Lambda_f C_f \Lambda^\dagger, \check{C}_p = \Lambda_p C_p \Lambda^\dagger, \check{D}_{pd} = \Lambda_p D_{pd} \Lambda_d^\dagger, \check{D}_{pf} = \Lambda_p D_{pf} \Lambda_f^\dagger$. Note that all entries of the matrices in this representation are real.

6.4 Series products for linear quantum systems

Assume both G_1 and G_2 in Figure 4 are linear, in this section we present the explicit form of $G = G_2 \triangleleft G_1$.

For ease of presentation we assume both G_1 and G_2 are passive with parametrization $G_j = (I, (C_-^{(j)} a^{(j)}, 0), (j = 1, 2)$. Therefore

$$\dot{a}^{(j)}(t) = -\frac{1}{2}(C_-^{(j)})^\dagger C_-^{(j)} a^{(j)}(t) - (C_-^{(j)})^\dagger b^{(j)}(t), \quad a^{(j)}(0) = a^{(j)}, \tag{65}$$

$$b_{out}^{(j)}(t) = C_-^{(j)} a^{(j)}(t) + b^{(j)}(t), \quad (j = 1, 2). \tag{66}$$

According to eq. (29), the composite linear quantum system G is

$$\begin{aligned} G &= \left(I, [C_-^{(1)} \ C_-^{(2)}] \begin{bmatrix} a^{(1)} \\ a^{(2)} \end{bmatrix}, \frac{1}{2i}[(a^{(1)})^\dagger \ (a^{(2)})^\dagger] \right. \\ &\quad \left. \times \begin{bmatrix} 0 & -(C_-^{(1)})^\dagger C_-^{(2)} \\ (C_-^{(2)})^\dagger C_-^{(1)} & 0 \end{bmatrix} \begin{bmatrix} a^{(1)} \\ a^{(2)} \end{bmatrix} \right). \end{aligned} \tag{67}$$

By eqs. (55)–(56), G is in the form of

$$\begin{bmatrix} \dot{a}^{(1)}(t) \\ \dot{a}^{(2)}(t) \end{bmatrix} = - \begin{bmatrix} \frac{1}{2}(C_-^{(1)})^\dagger C_-^{(1)} & 0 \\ (C_-^{(2)})^\dagger C_-^{(1)} & (C_-^{(2)})^\dagger C_-^{(2)} \end{bmatrix} \begin{bmatrix} a^{(1)}(t) \\ a^{(2)}(t) \end{bmatrix}$$

$$-\begin{bmatrix} (C_-^{(1)})^\dagger \\ -(C_-^{(2)})^\dagger \end{bmatrix} b^{(1)}(t), \tag{68}$$

$$b_{out}^{(2)}(t) = [C_-^{(1)} \ C_-^{(2)}] \begin{bmatrix} a^{(1)}(t) \\ a^{(2)}(t) \end{bmatrix} + b^{(1)}(t). \tag{69}$$

If we identify $b^{(2)}(t)$ with $b_{out}^{(1)}(t)$ in eqs. (65)–(66), then eqs. (65)–(66) give rise to system (68)–(69) too. This fact is useful in forming closed-loop coherent feedback control systems, c.f., Section 8.1.

7 Performance specifications for linear quantum systems

In Section 5 we have established criteria for stability, passivity, and L^2 gain for general quantum dissipative systems studied in Section 3.2. These criteria are expressed in terms of operators. In this section we specialize those results to linear quantum systems introduced in Section 6. It can be seen that for linear quantum systems such criteria can be expressed in terms of constant matrices.

7.1 Stability, passivity, gain

Perhaps the most basic performance characteristic is stability. For system G of open quantum harmonic oscillators presented in Section 6.1, stability may be evaluated in terms of the behavior of the number of quanta (e.g. photons) stored in the system, $N = a^\dagger a = \sum_{j=1}^n a_j^* a_j$. We introduce the following definition of stability.

Definition 5 (Stability, [33, Sec. III-A]). Let $w = 0$ and $v = 0$ in eq. (50), that is there is no energy input to system G . We say that G is (i) exponentially stable if there exist scalars $c_0 > 0, c_1 > 0$, and $c_2 \geq 0$ such that $\langle N(t) \rangle \leq c_0 e^{-c_1 t} \langle N \rangle + c_2$; (ii) marginally stable if there exist scalars $c_1 > 0$ and $c_2 \geq 0$ such that $\langle N(t) \rangle \leq c_1 \langle N \rangle + c_2 t$; and (iii) exponentially unstable if there exists an initial system state and real numbers $c_0 > 0, c_1 > 0$ and c_2 such that $\langle N(t) \rangle \geq c_0 e^{c_1 t} \langle N \rangle + c_2$.

For example, for the closed optical cavity in Section 3.3(i) (Figure 1), $a(t) = \exp(-i\omega t)a$, and $a^*(t)a(t) = a^*a$ for all t , which means that G is marginally stable but not exponentially stable—it oscillates—hence the name “oscillator”. However, an open cavity ($1, \sqrt{\kappa}a, \omega a^*a$) (Figure 2) is exponentially stable, a damped oscillator.

The number operator $N = a^\dagger a$, whose mean value is the total number of quanta, is a natural Lyapunov function for G , and is directly related to the energy of the system. However we find it more convenient to use storage functions of the form $V = \frac{1}{2} \check{a}^\dagger P \check{a}$ for non-negative Hermitian matrices P . For such storage functions, the generator function (14) becomes

$$\mathcal{G}_G(V) = \frac{1}{2} \check{a}^\dagger (A^\dagger P + PA) \check{a}. \tag{70}$$

With this simple yet important observation, the results in Section 5 can be specialized to linear quantum systems.

Define the matrix F by

$$F dt = (d\check{B}^\#(t) d\check{B}^T(t))^T = \begin{bmatrix} 0_m & 0 \\ 0 & I_m \end{bmatrix} dt. \tag{71}$$

The following result is a simple criterion for stability of linear quantum system G , which is a linear version of Theorems 4 and 6.

Theorem 8 (Stability, [33, Theorem 1]). If there exist constant matrices $P \geq 0$ and $Q \geq cP$ for a scalar $c > 0$ such that

$$A^\dagger P + PA + Q \leq 0, \tag{72}$$

then inequality

$$\langle \check{a}^\dagger(t) P \check{a}(t) \rangle \leq e^{-ct} \langle \check{a}^\dagger P \check{a} \rangle + \frac{\lambda}{2c} \tag{73}$$

holds, where $\lambda = \text{tr}[B_f^\dagger P B_f F]$ with F given by eq. (71). If also $P \geq \alpha I$ ($\alpha > 0$), then $\langle \check{a}^\dagger(t) a(t) \rangle \leq \frac{1}{\alpha} e^{-ct} \langle \check{a}^\dagger P \check{a} \rangle + \frac{\lambda}{2c\alpha}$. In this case, G is exponentially stable.

In a similar way, by choosing linear versions of supply rate functions positive real lemma and bounded real lemma can be established for linear quantum systems.

In order to simplify the notation we write $u = [w^T \ v^T]^T$ for the doubled-up vector of external variables, and define accordingly

$$B := [B_f \ B_d] \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{74}$$

where dimensions of identity matrices are implicitly assumed to be conformal to those of v and w .

Define a supply rate

$$r(\check{a}, \check{u}) = \frac{1}{2} (-\check{a}^\dagger Q \check{a} + \check{u}^\dagger \check{z} + \check{z}^\dagger \check{u}). \tag{75}$$

Then we have the following positive real lemma.

Theorem 9 (Positive Real Lemma, [33, Theorem 3]). The system G with performance variable $\check{z} = C_p \check{a}$ is passive if and only if there exist non-negative definite Hermitian matrices P and Q such that

$$\begin{bmatrix} PA + A^\dagger P + Q & PB - C_p^\dagger \\ B^\dagger P - C_p & 0 \end{bmatrix} \leq 0. \tag{76}$$

Moreover, $\lambda = \text{tr}[B_f^\dagger P B_f F]$.

Remark 3. When

$$P = H_0 = \begin{bmatrix} \Omega_- & \Omega_+ \\ \Omega_+^\# & \Omega_-^\# \end{bmatrix}, V = \frac{1}{2} \check{a}^\dagger P \check{a},$$

$$L = C_- a + C_+ a^\#, M = K_- a + K_+ a^\#,$$

the operator Z in eq. (42) satisfies $\check{Z} = \check{z} = C_p \check{a}$. That is, Theorem 9 is a special case of Theorem 5.

In what follows we discuss L^2 gain of linear quantum systems. Denote $D_p = [D_{pf} \ D_{pd}]$, the performance variable can be rewritten as $\check{z} = C_p \check{a} + D_p \check{u}$. Define a supply rate

$$r(\check{a}, \check{u}) = -\frac{1}{2}(\check{z}^\dagger \check{z} - g^2 \check{u}^\dagger \check{u}), \tag{77}$$

where $g \geq 0$ is a real gain parameter. Then we have the following results.

Theorem 10 (Bounded Real Lemma, [33, Theorem 4]). The system G with performance variable $\check{z} = C_p \check{a} + D_p \check{u}$ is bounded real with finite L^2 gain less than g if and only if there exists a non-negative Hermitian matrix P such that

$$\begin{bmatrix} PA + A^\dagger P + C_p^\dagger C_p & PB + C_p^\dagger D_p \\ B^\dagger P + D_p^\dagger C_p & D_p^\dagger D_p - g^2 I \end{bmatrix} \leq 0. \tag{78}$$

Moreover, $\lambda = \text{tr}\{B_f^\dagger P B_f F\}$.

Theorem 11 (Strict Bounded Real Lemma, [33, Theorem 5]). The following statements are equivalent.

- (i) The quantum system G defined in eqs. (50)–(52) is strictly bounded real with disturbance attenuation g .
- (ii) A is stable and $\|C_p (sI - A)^{-1} B + D_p\|_\infty < g$.
- (iii) $g^2 I - D_p^\dagger D_p > 0$ and there exists a Hermitian matrix $P_1 > 0$ satisfying inequality

$$\begin{bmatrix} A^\dagger P_1 + P_1 A & P_1 B & C_p^\dagger \\ B^\dagger P_1 & -gI & D_p^\dagger \\ C_p & D_p & -gI \end{bmatrix} < 0. \tag{79}$$

- (iv) $g^2 I - D_p^\dagger D_p > 0$ and there exists a Hermitian matrix $P_2 > 0$ satisfying the algebraic Riccati equation

$$\begin{aligned} & A^\dagger P_2 + P_2 A + (P_2 B + C_p^\dagger D_p) \\ & \times (g^2 I - D_p^\dagger D_p)^{-1} (B^\dagger P_2 + D_p^\dagger C_p) \\ & = 0 \end{aligned}$$

with $A + B B^\dagger P_2$ being Hurwitz.

Furthermore, if these statements hold, then $P_1 < P_2$.

7.2 LQG performance

In this section a quantum LQG cost function is first defined in the annihilation-creation form, and whose evaluation is connected to a Lyapunov equation in the complex domain. After that the real domain case is presented. More discussions can be found in, e.g. [33, 37].

Consider the following stable linear quantum system

$$d\check{a}(t) = A\check{a}(t)dt + B_f d\check{B}(t), \tag{80}$$

where $B(t)$ is a quantum Wiener process introduced in Section 3.1. Given a performance variable $\check{z}(t) = C_p \check{a}(t)$, along the line of [37], the infinite-horizon LQG cost is

$$\begin{aligned} \mathfrak{J}_\infty & := \lim_{t_f \rightarrow \infty} \frac{1}{t_f} \int_0^{t_f} \frac{1}{2} \langle \check{z}^\dagger(t) \check{z}(t) + \check{z}^T(t) \check{z}^\#(t) \rangle dt \\ & = \lim_{t_f \rightarrow \infty} \frac{1}{t_f} \int_0^{t_f} \text{Tr} \{ C_p P_{LQG}(t) C_p^\dagger \} dt \\ & = \text{Tr} \{ C_p P_{LQG} C_p^\dagger \}, \end{aligned} \tag{81}$$

where the constant Hermitian matrix $P_{LQG} \geq 0$ satisfies the following Lyapunov equation

$$A P_{LQG} + P_{LQG} A^\dagger + \frac{1}{2} B_f B_f^\dagger = 0. \tag{82}$$

In quadrature form, given a stable linear quantum system

$$d\check{a}(t) = \check{A}\check{a}(t)dt + \check{B}_f d\check{B}(t) \tag{83}$$

with performance variable $\check{z}(t) = \check{C}_p \check{a}(t)$. Assume that the constant real matrix $\check{P}_{LQG} \geq 0$ is the (unique) solution to the following Lyapunov equation in the real domain

$$\check{A} \check{P}_{LQG} + \check{P}_{LQG} \check{A}^T + \check{B}_f \check{B}_f^T = 0. \tag{84}$$

Then

$$\mathfrak{J}_\infty = \text{Tr} \{ \check{C}_p \check{P}_{LQG} \check{C}_p^\dagger \}. \tag{85}$$

8 Coherent feedback control

We have discussed interconnections of quantum systems (Section 4), open linear quantum systems (Section 6), and their performance specifications (Section 7). We are now in a position to study synthesis of open linear quantum systems; that is, how to connect a plant of interest to another system (namely controller) so as to achieve pre-specified control performance.

8.1 Closed-loop plant-controller system

In Figure 11, P is the plant to be controlled, and K is the controller to be designed. Clearly, this feedback system involves both direct and indirect couplings between P and K .

The plant P is described by a system of quantum stochastic differential equations (QSDEs)

$$\begin{aligned} \dot{\check{a}}(t) & = A\check{a}(t) + B_{12}\check{a}_K(t) + B_v \check{b}_v(t) + B_f \check{w}(t) \\ & \quad + B_f \check{b}(t) + B_u \check{u}(t), \quad \check{a}(0) = \check{a}, \\ \check{y}(t) & = C\check{a}(t) + D_v \check{b}_v(t) + D_f \check{w}(t) + D_f \check{b}(t). \end{aligned} \tag{86}$$

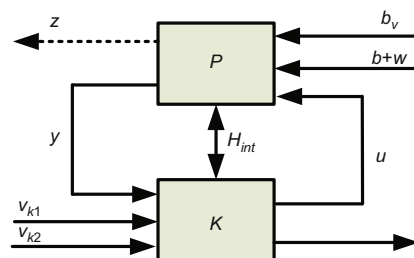


Figure 11 Coherent feedback control arrangement.

The inputs $\check{w}(t)$ and $\check{b}(t)$ are defined in Section 6. $\check{y}(t)$ is a selection of output field channels from the plant. $\check{b}_v(t)$ is a vector of additional quantum white noises; $\check{u}(t)$ is a quantum field signal from the to-be-designed controller K , hence it is a vector of physical variables. The term $B_{12}\check{a}_K(t)$ is due to direct coupling between P and K .

The fully quantum controller K is a linear quantum system of the form³⁾

$$\begin{aligned} \check{a}_K(t) &= A_K\check{a}_K(t) + B_{21}\check{a}(t) + B_K\check{y}(t) + B_{K1}\check{b}_{v_{K1}}(t) \\ &\quad + B_{K2}\check{b}_{v_{K2}}(t), \quad \check{a}_K(0) = \check{a}_K, \\ \check{u}(t) &= C_K\check{a}_K(t) + \check{b}_{v_{K1}}(t). \end{aligned} \quad (87)$$

This structure allows for direct coupling and indirect coupling between the plant P and the controller K . Here, $\check{b}_{v_{K1}}(t)$ and $\check{b}_{v_{K2}}(t)$ are independent quantum white noises, and $\check{u}(t)$ is the field output of the controller corresponding to $\check{b}_{v_{K1}}(t)$. Finally the terms $B_{12}\check{a}_K(t)$ and $B_{21}\check{a}(t)$ are due to the direct coupling between the plant and controller in terms of an interaction Hamiltonian

$$H_{int} = \frac{1}{2} (\check{a}^\dagger \Xi^\dagger \check{a}_K + \check{a}_K^\dagger \Xi \check{a}), \quad (88)$$

where $\Xi = \Delta(iK_-, iK_+)$ for complex matrices K_- and K_+ of suitable dimensions, c.f. Section 4.3.

The controller matrices K_-, K_+ , (or B_{12}, B_{21}) for direct coupling, and $A_K, B_K, C_K, B_{K1}, B_{K2}$ for indirect coupling are to be found to optimize performance criteria defined in terms of the closed-loop performance variable

$$\check{z}(t) = [C_p \quad D_u C_K] \begin{bmatrix} \check{a}(t) \\ \check{a}_K(t) \end{bmatrix} + \check{D}_{pf} \check{w}(t). \quad (89)$$

Because standard matrix algorithms will be used in H^∞ synthesis and LQG synthesis in later sections, we resort to quadrature representation discussed in Section 6.3. Let $\tilde{a}, \tilde{a}_K, \tilde{w}, \tilde{b}, \tilde{b}_v, \tilde{u}, \tilde{z}, \tilde{y}, \tilde{b}_{v_{K1}}, \tilde{b}_{v_{K2}}$ be the quadrature counterparts of $\check{a}, \check{a}_K, \check{w}, \check{b}, \check{b}_v, \check{u}, \check{z}, \check{y}, \check{b}_{v_{K1}}, \check{b}_{v_{K2}}$ respectively. Define

$$\tilde{A}_{cl} = \begin{bmatrix} \tilde{A} & \tilde{B}_u \tilde{C}_K \\ \tilde{B}_K \tilde{C} & \tilde{A}_K \end{bmatrix} + \tilde{\Xi}, \quad \tilde{B}_{cl} = \begin{bmatrix} \tilde{B}_f \\ \tilde{B}_K \tilde{D}_f \end{bmatrix},$$

$$\tilde{G}_{cl} = \begin{bmatrix} \tilde{B}_f & \tilde{B}_v & \tilde{B}_u & 0 \\ \tilde{B}_K \tilde{D}_f & \tilde{B}_K \tilde{D}_v & \tilde{B}_{K1} & \tilde{B}_{K2} \end{bmatrix},$$

$$\tilde{C}_{cl} = \begin{bmatrix} \tilde{C}_p & \tilde{D}_u \tilde{C}_K \end{bmatrix}, \quad \tilde{D}_{cl} = \tilde{D}_{pf},$$

where $\tilde{\Xi} = [0 \quad \tilde{B}_{12}; \tilde{B}_{21} \quad 0]$ satisfies $\tilde{B}_{21} = \Theta \tilde{B}_{12}^T \Theta$. Then the closed-loop system in the quadrature representation is given

by

$$\begin{bmatrix} \dot{\tilde{a}}(t) \\ \dot{\tilde{a}}_K(t) \end{bmatrix} = \tilde{A}_{cl} \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}_K(t) \end{bmatrix} + \tilde{B}_{cl} \tilde{w}(t) + \tilde{G}_{cl} \begin{bmatrix} \tilde{b}(t) \\ \tilde{b}_v(t) \\ \tilde{b}_{v_{K1}}(t) \\ \tilde{b}_{v_{K2}}(t) \end{bmatrix}, \quad (90)$$

$$\tilde{z}(t) = \tilde{C}_{cl} \begin{bmatrix} \tilde{a}(t) \\ \tilde{a}_K(t) \end{bmatrix} + \tilde{D}_{cl} \tilde{w}(t). \quad (91)$$

8.2 H^∞ control

As in the classical case, the bounded real lemmas stated in Section 6.4 can be used for H^∞ controller synthesis of open linear quantum systems. It is shown in [34] that for open linear quantum systems H^∞ control performance and physical realizability condition of controllers can be treated separately. Adding direct coupling between plants and controllers complicates H^∞ controller synthesis. Nonetheless, the separation of H^∞ control performance and physical realizability condition still holds. This is a unique feature of quantum H^∞ controller synthesis: To guarantee physical realizability, vacuum noise is added, while such noise does not affect H^∞ control performance [34].

(i) LMI formulation. In this section we present a general formulation using LMIs for H^∞ synthesis of open linear quantum stochastic systems.

According to the strict bounded real lemma (Theorem 6.4), the closed-loop system (90)–(91) is internally stable and strictly bounded real (from \tilde{w} to \tilde{z}) with disturbance attenuation g if and only if there is a real symmetric matrix \mathcal{P} such that

$$\mathcal{P} > 0, \quad (92)$$

$$\begin{bmatrix} \tilde{A}_{cl}^T \mathcal{P} + \mathcal{P} \tilde{A}_{cl} & \mathcal{P} \tilde{B}_{cl} & \tilde{C}_{cl}^T \\ \tilde{B}_{cl}^T \mathcal{P} & -gI & \tilde{D}_{cl}^T \\ \tilde{C}_{cl} & \tilde{D}_{cl} & -gI \end{bmatrix} < 0. \quad (93)$$

The H^∞ controller synthesis is to find indirect coupling parameters $\tilde{A}_K, \tilde{B}_K, \tilde{C}_K$ and direct coupling parameters $\tilde{\Xi}$ such that eqs. (92)–(93) hold.

Partition \mathcal{P} and its inverse \mathcal{P}^{-1} to be

$$\mathcal{P} = \begin{bmatrix} \mathbf{Y} & \mathbf{N} \\ \mathbf{N}^T & * \end{bmatrix}, \quad \mathcal{P}^{-1} = \begin{bmatrix} \mathbf{X} & \mathbf{M} \\ \mathbf{M}^T & * \end{bmatrix}.$$

Define matrices

$$\Pi_1 = \begin{bmatrix} \mathbf{X} & \mathbf{I} \\ \mathbf{M}^T & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} \mathbf{I} & \mathbf{Y} \\ 0 & \mathbf{N}^T \end{bmatrix}.$$

3) We assume that all the variables and matrices of the plant and the controller have compatible dimension, but we do not bother to specify them explicitly.

And also define a change of variables

$$\begin{aligned} \hat{A} &= N(\tilde{A}_K M^T + \tilde{B}_K \tilde{C} X) + Y(\tilde{B}_u \tilde{C}_K M^T + \tilde{A} X), \\ \hat{B} &= N \tilde{B}_K, \\ \hat{C} &= \tilde{C}_K M^T, \\ \Omega &= \Pi_1^T \mathcal{P} \tilde{\Xi} \Pi_1. \end{aligned} \tag{94}$$

With these notations, eqs. (92) and (93) hold if and only if the following inequalities hold:

$$-\begin{bmatrix} X & I \\ I & Y \end{bmatrix} < 0, \tag{95}$$

$$\begin{bmatrix} \tilde{A} X + X \tilde{A}^T + \tilde{B}_u \hat{C} + (\tilde{B}_u \hat{C})^T \\ \hat{A} + \tilde{A}^T \\ \tilde{B}_f^T \\ \tilde{C}_p X + \tilde{D}_u \hat{C} \\ \tilde{A} + \hat{A}^T & * & * \\ \tilde{A}^T Y + Y \tilde{A} + \hat{B} \hat{C} + (\hat{B} \hat{C})^T & * & * \\ (Y \tilde{B}_f + \hat{B} \tilde{D}_f)^T & -gI & * \\ \tilde{C}_p & \tilde{D}_{cl} & -gI \end{bmatrix} + \begin{bmatrix} \tilde{B}_{12} M^T + (\tilde{B}_{12} M^T)^T & (N \tilde{B}_{21} X)^T + (Y \tilde{B}_{12} M^T)^T \\ N \tilde{B}_{21} X + Y \tilde{B}_{12} M^T & N \tilde{B}_{21} + (N \tilde{B}_{21})^T \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} < 0. \tag{96}$$

If eqs. (95) and (96) are simultaneously soluble, according to eq. (94), the following matrices can be obtained:

$$\begin{aligned} \tilde{B}_K &= N^{-1} \hat{B}, \\ \tilde{C}_K &= \hat{C} (M^T)^{-1}, \\ \tilde{A}_K &= N^{-1} (\hat{A} - N \tilde{B}_K \tilde{C} X - Y(\tilde{B}_u \tilde{C}_K M^T + \tilde{A} X)) M^{-T}, \\ \tilde{\Xi} &= \mathcal{P}^{-1} (\Pi_1^{-T}) \Omega \Pi_1^{-1}. \end{aligned} \tag{97}$$

Unfortunately, notice that there are such terms as $N \tilde{B}_{21} X$ and $Y \tilde{B}_{12} M^T$ in inequality (96), which induce nonlinearity. The above analysis shows it is hard to directly utilize LMI techniques to do controller design when direct coupling is involved.

(ii) Multi-step optimization. In this section, we attempt to circumvent the above difficulty by proposing a multi-step optimization procedure which is formulated as follows:

Initialization. Set $\tilde{B}_{12} = 0$ and $\tilde{B}_{21} = 0$.

Step 1. Solve linear matrix inequalities (95) and (96) for parameters \hat{A} , \hat{B} , \hat{C} , X , Y and disturbance gain g , then choose matrices M and N satisfying $MN^{-1} = I - XY$.

Step 2. Pertaining to Step 1. Solve inequality (96) for direct coupling parameters \tilde{B}_{12} , \tilde{B}_{21} and disturbance gain g .

Step 3. Fix \tilde{B}_{12} and \tilde{B}_{21} obtained in Step 2 and M and N in Step 1, go to Step 1.

After the above iterative procedure is complete, use the values \tilde{B}_{12} , \tilde{B}_{21} , \tilde{A}_K , \tilde{B}_K , \tilde{C}_K obtained to find \tilde{B}_{K1} , \tilde{B}_{K2} to ensure physical realizability of the controller. A complete procedure of finding matrices \tilde{B}_{K1} , \tilde{B}_{K2} is given in [34, Sec. V-D].

Remark 4. Steps 1 and 2 are standard LMI problems which can be solved efficiently using the Matlab LMI toolbox. However, there is some delicate issue in Step 3. Assume that \tilde{B}_{12} and \tilde{B}_{21} have been obtained in Step 2. According to the second item in (96), constant matrices M and N must be specified in order to render (96) linear in parameters \hat{A} , \hat{B} , \hat{C} , X , Y , and disturbance gain g . In Step 3, M and N obtained in Step 1 is used. Unfortunately, this choice of M and N sometimes may generate a controller whose parameters are ill-conditioned. Due to this reason, M and N in Step 3 might have to be chosen carefully to produce a physically meaningful controller. This fact is illuminated by an example in [33, Sec. IV-C6].

Finally we discuss robustness briefly. It is demonstrated in [33] that direct coupling may improve robustness of closed-loop quantum feedback systems. For instance, for the example studied in [34, Sec. VII.A], using coupling coefficients $\kappa_1 = 2.6$, $\kappa_2 = \kappa_3 = 0.2$, we implement Step 1 of the above multi-step optimization procedure to design an indirect coupling, and obtain closed-loop L^2 gain 0.0487. We implement Step 2 to design direct coupling and obtain an L^2 gain of 0.0498. This is a bit worse than the previous one, however the difference is quite small. Now we assume there is uncertainty in the coupling coefficient κ_1 , say the actual value of κ_1 is 1.3. In this case, the L^2 gain of the closed-loop with indirect coupling becomes 0.1702, which is a significant performance degradation. However, the L^2 gain of the closed-loop with both direct and indirect couplings is 0.0595, which is still close to the original 0.0498.

8.3 LQG control

In this section we study the problem of coherent quantum LQG control by means of both direct and indirect couplings. In contrast to the coherent quantum H^∞ controller synthesis presented in Section 6.4, the nice property of separation of control and physical realizability does not hold any more. This is evident as LQG control concerns the influence of quantum white noise on the plant, the addition of quantum noise that guarantees the physical realizability of the to-be-designed controller affects the overall LQG control performance.

In the following we just give a brief formulation of the coherent quantum LQG control problem. In-depth discussions can be found in [33, 37, 65].

We make the following assumption.

Assumption A2. There are no quantum signal $\check{w}(t)$ and noise input $\check{b}_v(t)$ in the quantum plant P in (86).

Following the development in Section 7.2, the LQG control objective is to design a controller (87) such that the performance index $\mathfrak{J}_\infty = \text{Tr}\{\tilde{C}_c \tilde{P}_{LQG} \tilde{C}_c^*\}$ is minimized, subject to equation (84) and the quadrature counterpart of the physical realizability condition (57)–(59).

As yet, quantum LQG coherent feedback is still an outstanding problem, there are no analytic solutions. In [37] an indirect coupling is designed to address the coherent quantum LQG control problem, where a numerical procedure based on semidefinite programming is proposed to design the indirect coupling. In order to design both direct and indirect couplings. In [33, Sec. IV-D] a multi-step optimization algorithm is developed to incorporate direct coupling into numerical design procedures.

9 Network synthesis

A linear quantum controller, obtained from either coherent H^∞ or LQG control synthesis, is in the form of a set of linear quantum stochastic differential equations. Network synthesis theory is concerned with how to physically implement such controllers by means of physical devices like optical instruments. This problem has been addressed in [40–43, 66]. The general result is: A general linear quantum dynamical system can be (approximately) physically implemented by linear and nonlinear quantum optical elements such as optical cavities, parametric oscillators, beam splitters, and phase shifters.

Lately, the Mabuchi group at Stanford [28] has developed a Quantum Hardware Description Language (QHDL) to facilitate the analysis and synthesis of quantum feedback networks described in this survey. As a subset of the standard Very High Speed Integrated Circuit (VHSIC) Hardware Description Language (VHDL), QHDL provides high-level modular representations of quantum feedback networks. This user-friendly interface will be helpful to the fabrication of complex photonic circuits.

10 Conclusions

In this survey we have presented a brief look at recent results concerning quantum feedback networks and control. On the basis of this model interconnection structures of quantum systems have been presented. Fundamental characteristics of quantum systems such as stability, passivity, and L^2 gain have been described. It turns out that for linear quantum systems these fundamental characteristics have very explicit forms. The problem of coherent H^∞ control and coherent LQG control have been discussed.

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