

Wilson loops and Riemann theta functions II

Martin Kruczenski^a and Sannah Ziam^b

^a*Department of Physics, Purdue University,
525 Northwestern Avenue, W. Lafayette, IN 47907-2036, U.S.A.*

^b*Department of Physics and Astronomy, University of Kentucky,
505 Rose Street, Lexington, KY 40506-0055, U.S.A.*

E-mail: markru@purdue.edu, spziama@pa.uky.edu

ABSTRACT: In this paper we extend and simplify previous results regarding the computation of Euclidean Wilson loops in the context of the AdS/CFT correspondence, or, equivalently, the problem of finding minimal area surfaces in hyperbolic space (Euclidean AdS₃). If the Wilson loop is given by a boundary curve $\vec{X}(s)$ we define, using the integrable properties of the system, a family of curves $\vec{X}(\lambda, s)$ depending on a complex parameter λ known as the spectral parameter. This family has remarkable properties. As a function of λ , $\vec{X}(\lambda, s)$ has cuts and therefore is appropriately defined on a hyperelliptic Riemann surface, namely it determines the spectral curve of the problem. Moreover, $\vec{X}(\lambda, s)$ has an essential singularity at the origin $\lambda = 0$. The coefficients of the expansion of $\vec{X}(\lambda, s)$ around $\lambda = 0$, when appropriately integrated along the curve give the area of the corresponding minimal area surface.

Furthermore we show that the same construction allows the computation of certain surfaces with one or more boundaries corresponding to Wilson loop correlators. We extend the area formula for that case and give some concrete examples. As the main example we consider a surface ending on two concentric circles and show how the boundary circles can be deformed by introducing extra cuts in the spectral curve.

KEYWORDS: AdS-CFT Correspondence, Long strings

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1 Introduction

The AdS/CFT correspondence [1–3] set on firm footing the conjectured relation between string theory and gauge theories in the planar limit. It opened the possibility of computing gauge theory quantities in the strong coupling limit by using the dual string theory. Perhaps the most important observable in gauge theories is the Wilson loop. In AdS/CFT it is related to surfaces ending on the curve associated to the Wilson loop [4–6]. In the large 't Hooft coupling limit, the surface that needs to be considered is the one of minimal area. Among all Wilson loops the simplest ones are those determined by a two dimensional, closed, smooth curve. Until recently, the only such example whose dual minimal area surface could be computed was the circle [7–11]. The minimal area surface was simply given by a half sphere. This was somewhat surprising given the known integrability properties of the system [12, 13].

Recent progress in [14] provided a new, infinite parameter family of minimal area surfaces that can be used to further explore the AdS/CFT duality. In that work the minimal area surfaces were constructed analytically in terms of Riemann theta functions associated to hyperelliptic Riemann surfaces closely following previous work by M. Babich and A. Bobenko [15, 16]. It also follows related work where Wilson loops were studied or theta functions were used in similar problems, for example in [17–29, 34–38].¹ Much of that work was motivated by the relation to scattering amplitudes [28, 29] which we do not pursue here. A closely related approach can also be found in [39] where integrability properties are applied to the computation of Wilson loops as minimal area surfaces. More recent developments can be found in [40–50].

Here we continue the study of such solutions and show that, in particular, they also provide an infinite parameter family of examples of surfaces ending on multiple curves. The case we concentrate on is the one of concentric curves. The main technical difference with the previous case is that a certain periodicity condition is required, which implies a restriction on the position of the cuts and on the values of the spectral parameter. Since it is only one condition, it still implies an infinite parameter family of solutions.

The paper is organized as follows: in the next section we review briefly the previous results. Since the paper is essentially a continuation of the previous one we do not attempt to make this paper self-contained and should be read in conjunction with [14]. On the other hand we give new expressions for the boundary curve and the area that were derived by simplifying the ones in [14]. After that, we present the method to find minimal area surfaces ending on multiple curves and give a few examples with plots of the corresponding surfaces and boundaries. In the following section we compute the area for the case of multiple curves in complete parallel with the result for a single curve. Finally, as illustration, we give an example of a surface ending in eight different curves and give our conclusion in the last section.

2 Single curve case revisited

In [14] it was shown, based on [15, 16] how to find an infinite parameter family of minimal area surfaces ending on a curve at the boundary. The method uses integrability properties of the equations to solve them in terms of Riemann theta functions. The technique is common to many integrable non-linear differential equations and is well described in several references, for example, in the book [16].

In this section we briefly review those results and derive a simplified expression for the boundary curve which allows the explicit computation of other important quantities. For example, a simple expression is found for the Schwarzian derivative of the boundary curve. Finally, a remarkable formula for the area of the curve is found in terms of the behavior of the boundary curve near singular values of the spectral parameter.

To be concrete, we look for minimal area surfaces in hyperbolic space (EAdS₃) which can be parameterized by coordinates $(X, Y, Z > 0)$ with a metric

$$ds^2 = \frac{dX^2 + dY^2 + dZ^2}{Z^2} = \frac{dZ^2 + d\bar{X}dX}{Z^2}, \quad X = X + iY \tag{2.1}$$

¹See e.g. [30–33].

Given a contour $(X(s), Y(s), Z = 0) \equiv (X(s) = X(s) + iY(s), Z = 0)$ at the boundary $Z = 0$ we seek a surface of minimal area ending on it and parameterized as

$$Z(z, \bar{z}), \quad X + iY = X(z, \bar{z}) \tag{2.2}$$

where the coordinates (z, \bar{z}) parameterize the complex plane. In that plane we find a closed curve $z(s)$ such that $Z(z(s), \bar{z}(s)) = 0$. Such curve maps to the contour in the boundary of (EAdS₃) space, and the interior maps to a minimal area surface. At this time there is no generic solution to such problem of finding the minimal area surface for an arbitrary boundary. However we can give an infinite parameter family of minimal area surfaces and the corresponding contours where they end. It is an open problem if such solutions approximate all contours (and if so, in which sense).

The construction starts by choosing an auxiliary hyperelliptic Riemann surface \mathcal{M} , known as the spectral curve and given, as a subspace of \mathbb{C}^2 by

$$\mu^2 = \lambda \prod_{j=1}^{2g} (\lambda - \lambda_j) \tag{2.3}$$

where (μ, λ) parameterize \mathbb{C}^2 and $\lambda_i \neq \lambda_j$ if $i \neq j$. There is a reality condition that requires the set of branch points $\{0, \infty, \lambda_{j=1 \dots 2g}\}$ to be symmetric under the involution $T : \lambda \leftrightarrow -1/\bar{\lambda}$. A basis of one-cycles a_i, b_i is chosen such that the non-trivial intersections are $a_i \circ b_j = \delta_{ij}$ and that under the involution behave as

$$T a_i = -T_{ij} a_j, \quad T b_i = T_{ij} b_j \tag{2.4}$$

where T_{ij} is a $g \times g$ symmetric matrix such that $T^2 = 1$.

As a matter of notation, a generic point in the Riemann surface \mathcal{M} is denoted as $p_i = (\mu_{p_i}, \lambda_{p_i}) \in \mathcal{M}$. The coordinate λ_{p_i} is called the projection of p_i on the complex plane. The origin and infinity play an important role and are denoted as $p_1 = 0, p_3 = \infty$. Except for the branch points, given a point p_4 in one sheet there is another point denoted as $p_{\bar{4}}$ on the other sheet of the Riemann surface such that $\lambda_{p_4} = \lambda_{p_{\bar{4}}}$ and $\mu_{p_4} = -\mu_{p_{\bar{4}}}$. The relation between p_4 and $p_{\bar{4}}$ is the hyperelliptic involution associated with \mathcal{M} .

Now we consider the basis of holomorphic differentials

$$\nu_j = \frac{\lambda^{j-1}}{\mu(\lambda)} d\lambda, \quad i, j = 1 \dots g \tag{2.5}$$

and define the matrices

$$C_{ij} = \oint_{a_i} \nu_j, \quad \tilde{C}_{ij} = \oint_{b_i} \nu_j \tag{2.6}$$

and the basis of normalized holomorphic differentials $\omega_{i=1 \dots g}$

$$\omega_i = \nu_j (C^{-1})_{ji} = \sum_{j=1}^g C_{ji}^{-1} \lambda^{j-1} \frac{d\lambda}{\mu(\lambda)} \tag{2.7}$$

such that

$$\oint_{a_i} \omega_j = \delta_{ij}, \tag{2.8}$$

which defines the periodicity matrix

$$\oint_{b_i} \omega_j = \Pi_{ij}, \tag{2.9}$$

and the theta function

$$\theta(\zeta) = \sum_{n \in \mathbb{Z}^g} e^{i\pi n^t \Pi n + 2\pi i n^t \zeta}, \quad \zeta \in \mathbb{C}^g \tag{2.10}$$

It also defines the Jacobi map $\phi : \mathcal{M} \rightarrow \mathbb{C}^g$ for which the standard notation is

$$\phi(p_4) = \int_{p_1}^{p_4} \omega = \int_1^4 \tag{2.11}$$

where the point $p_1 = 0$ is chosen as a distinguished point, p_4 is an arbitrary point and \int_1^4 is just notation for the same function. There is an ambiguity in choosing the path of integration. Since two different paths from p_1 to p_4 can only differ by a closed cycle, $\phi(p_4)$ changes by

$$\phi(p_4) = \int_1^4 \rightarrow \int_1^4 + \epsilon_2 + \Pi \epsilon_1 \tag{2.12}$$

where ϵ_1 and ϵ_2 are vectors with integer components. Further, if p_4 is another branch point, the path $1 \rightarrow 4$ can be traced back on the lower sheet defining a generically non-trivial cycle. By choosing a certain path from $p_1 = 0$ to $p_3 = \infty$ we define the vectors $\Delta_{1,2} \in \mathbb{Z}^g$ through

$$\int_1^3 = \frac{1}{2} \Delta_2 + \frac{1}{2} \Pi \Delta_1 \tag{2.13}$$

The path should be chosen such that $\Delta_1^t \cdot \Delta_2$ is an odd integer. With this vector we define another function

$$\hat{\theta}(\zeta) = \exp \left\{ 2\pi i \left[\frac{1}{8} \Delta_1^t \Pi \Delta_1 + \frac{1}{2} \Delta_1^t \zeta + \frac{1}{4} \Delta_1^t \Delta_2 \right] \right\} \theta \left(\zeta + \int_1^3 \right) \tag{2.14}$$

Because $\Delta_1^t \cdot \Delta_2$ is odd it follows that $\hat{\theta}(-\zeta) = -\hat{\theta}(\zeta)$ and in particular $\hat{\theta}(0) = 0$.

Further, consider the vector $\omega = (\omega_1, \dots, \omega_g)$ and expand it around $p_1 = 0$. Since $p_1 = 0$ is a branch point an appropriate coordinate can be chosen as $y = -2i\sqrt{\lambda}$, namely $\lambda = -y^2/4$. It follows that

$$\omega = (\omega_1 + y^2 \omega_{12} + y^4 \omega_{14} + \dots) dy \tag{2.15}$$

for some constant vectors ω_1, ω_{12} , etc. that can be computed from eq. (2.7) by expanding $\mu(\lambda)$. Near infinity, on the other hand, an appropriate coordinate is $\tilde{y} = \frac{2}{\sqrt{\lambda}}$ and the expansion is

$$\omega = (\omega_3 + \tilde{y}^2 \omega_{32} + \tilde{y}^4 \omega_{34} + \dots) d\tilde{y} \tag{2.16}$$

The vectors $\omega_1, \omega_3, \omega_{12}, \omega_{32}, \dots \in \mathbb{C}^g$ play an important role and it is convenient to introduce a notation for the gradient of a function along those directions

$$D_1 F(\zeta) = (\omega_1 \cdot \nabla) F(\zeta) \tag{2.17}$$

$$D_1'' F(\zeta) = (\omega_{12} \cdot \nabla) F(\zeta) \tag{2.18}$$

$$D_3 F(\zeta) = (\omega_3 \cdot \nabla) F(\zeta) \tag{2.19}$$

$$D_3'' F(\zeta) = (\omega_{32} \cdot \nabla) F(\zeta) \tag{2.20}$$

Notice that D_1'' is a first derivative. Second derivatives are denoted e.g. as $D_1^2 F(\zeta) = (\omega_1 \cdot \nabla)(\omega_1 \cdot \nabla) F(\zeta)$. With all these ingredients in place, the solutions are given by

$$Z = \left| \frac{\hat{\theta}(2f_1^4)}{\hat{\theta}(f_1^4)\theta(f_1^4)} \right| \frac{|\theta(0)\theta(\zeta)\hat{\theta}(\zeta)|e^{\mu z + \nu \bar{z}}|^2}{|\hat{\theta}(\zeta - f_1^4)|^2 + |\theta(\zeta - f_1^4)|^2} \tag{2.21}$$

$$X + iY = X = e^{2\mu z + 2\nu \bar{z}} \frac{\theta(\zeta + f_1^4)\overline{\theta(\zeta - f_1^4)} - \hat{\theta}(\zeta + f_1^4)\overline{\hat{\theta}(\zeta - f_1^4)}}{|\hat{\theta}(\zeta - f_1^4)|^2 + |\theta(\zeta - f_1^4)|^2} \tag{2.22}$$

where the vector $\zeta(z, \bar{z}) \in \mathbb{C}^g$ is given by

$$\zeta = 2\omega_1 \bar{z} + 2\omega_3 z \tag{2.23}$$

and the constants μ, ν are given by

$$\mu = -2D_3 \ln \theta \left(\int_1^4 \right), \quad \nu = -2D_1 \ln \hat{\theta} \left(\int_1^4 \right) \tag{2.24}$$

which uses the directional derivatives $D_{1,3}$ defined in eq. (2.20). Notice that eq. (2.23) implies $\partial_z F(\zeta(z, \bar{z})) = 2D_1 F(\zeta(z, \bar{z}))$ and $\partial_{\bar{z}} F(\zeta(z, \bar{z})) = 2D_3 F(\zeta(z, \bar{z}))$. The solution contains an arbitrary parameter $p_4 = (\mu_{p_4}, \lambda_{p_4}) \in \mathcal{M}$. For eq. (2.22) to define a solution it is necessary that $|\lambda_{p_4}| = 1$. Therefore, each spectral curve \mathcal{M} actually defines a one real parameter family of surfaces. It was shown in [14] that all those surfaces have the same area.

Since the boundary is at $Z = 0$, to find the curve where the surface ends we need to find the zeros of Z or, equivalently, the zeros of $\hat{\theta}$. Typically the zeros are given by isolated curves $z(s)$ on the world-sheet from which one can be chosen. The solution eq. (2.22) maps such curve $z(s)$ to a curve $X(s)$ at the boundary $Z = 0$ defining the shape of the Wilson loop. The region of the world-sheet inside the curve $z(s)$ is mapped, by eq. (2.22) to the minimal area surface ending at the curve $X(s)$.

2.1 Boundary curve

As just described the shape of the boundary curve, or equivalently of the dual Wilson loop, follows from finding a curve $z(s)$ on the world sheet such that

$$\hat{\theta}(\zeta_s) = 0 \tag{2.25}$$

where

$$\zeta_s = 2\omega_1 \bar{z}(s) + 2\omega_3 z(s) \tag{2.26}$$

Once this function ζ_s is found, the shape of the curve follows from replacing $\zeta \rightarrow \zeta_s$ in the expression for X , namely eq. (2.22) to get $X(s) = X(s) + iY(s)$ [14]:

$$X(s) = e^{2\mu z(s) + 2\nu \bar{z}(s)} \frac{\theta(\zeta_s + f_1^4)\overline{\theta(\zeta_s - f_1^4)} - \hat{\theta}(\zeta_s + f_1^4)\overline{\hat{\theta}(\zeta_s - f_1^4)}}{|\hat{\theta}(\zeta_s - f_1^4)|^2 + |\theta(\zeta_s - f_1^4)|^2} \tag{2.27}$$

However, it turns out that the result can be simplified by using eq. (A.9) derived in the appendix. With this identity, eq. (2.27) reduces to

$$X(s) + iY(s) = X(s) = -e^{2\mu z(s) + 2\nu \bar{z}(s)} \frac{\hat{\theta}(\zeta_s + \int_1^4)}{\hat{\theta}(\zeta_s - \int_1^4)} \tag{2.28}$$

which is a much simpler expression for the boundary curve. In fact the function $X(s)$ describes a family of curves parameterized by the point p_4 in the Riemann surface. Now we study the analytic properties of $X(s, p_4)$ as a function of p_4 .

2.2 Analytic properties of $X(s, p_4)$, analogy with the monodromy matrix

As discussed in the previous section, the function

$$X(s, p_4) = -e^{2\mu z(s) + 2\nu \bar{z}(s)} \frac{\hat{\theta}(\zeta_s + \int_1^4)}{\hat{\theta}(\zeta_s - \int_1^4)} \tag{2.29}$$

can be thought of as a family of curves parameterized by a point p_4 on the Riemann surface \mathcal{M} . When $|\lambda_{p_4}| = 1$ the curve is such that the minimal area surface can be found from eqs. (2.21) and (2.22). In this section we study the analytic properties of $X(s, p_4)$ as a function of p_4 and show that it is well defined on the Riemann surface but not on the complex plane parameterized by λ_{p_4} where it has cuts. In that manner, $X(s, p_4)$ defines the spectral curve in the same way as the monodromy matrix does in many integrable systems. Furthermore, it turns out that $X(s, p_4)$ has essential singularities at $p_4 = 0, \infty$, the precise behavior around the singularities determines the area of the corresponding surface. Finally, the Schwarzian derivative $\{X, s\}$ is computed, and shown to have a remarkable simple dependence on λ_{p_4} .

To understand the properties of $X(s, p_4)$ as a function of p_4 first it has to be proven that it is a well defined function on the Riemann surface. The reason it might not be is that there is an ambiguity in choosing the path of integration defining \int_1^4 . Choosing another path changes the integral by a period of the theta function. Using the quasi periodicity properties of $\hat{\theta}$ i.e. eq. (A.2), it follows that

$$\frac{\hat{\theta}(\zeta_s + \int_1^4 + \epsilon_2 + \Pi\epsilon_1)}{\hat{\theta}(\zeta_s - \int_1^4 + \epsilon_2 + \Pi\epsilon_1)} = e^{-4\pi i \epsilon_1^t \zeta e^{2\pi i [\Delta_1^t \epsilon_2 - \epsilon_1^t \Delta_2]}} \frac{\hat{\theta}(\zeta_s + \int_1^4)}{\hat{\theta}(\zeta_s - \int_1^4)} \tag{2.30}$$

$$= e^{-4\pi i \epsilon_1^t \zeta} \frac{\hat{\theta}(\zeta_s + \int_1^4)}{\hat{\theta}(\zeta_s - \int_1^4)} \tag{2.31}$$

where we used that $\Delta_1^t \epsilon_2 - \epsilon_1^t \Delta_2$ is an integer since $\epsilon_{1,2}$ and $\Delta_{1,2}$ are integer vectors. Therefore the ratio of theta functions is *not* a well defined function on the Riemann surface. However, the exponential factor also depends on p_4 through the constants μ, ν defined in eq. (2.24):

$$\mu = -2D_3 \ln \theta \left(\int_1^4 \right), \quad \nu = -2D_1 \ln \hat{\theta} \left(\int_1^4 \right) \tag{2.32}$$

To study their periodicity properties recall the definition

$$D_{p_i} F(\zeta) = \omega(p_i)_k \nabla_k F(\zeta) \tag{2.33}$$

and introduce a variable ζ to compute the derivatives

$$\mu \rightarrow \tilde{\mu} = -2 D_{p_3} \ln \theta \left(\zeta + \int_1^4 + \epsilon_2 + \Pi \epsilon_1 \right) \Big|_{\zeta=0} \quad (2.34)$$

$$= -2 D_{p_3} \left[\ln \theta \left(\zeta + \int_1^4 \right) - 2\pi i \epsilon_1^t \zeta \right] \Big|_{\zeta=0} \quad (2.35)$$

$$= \mu + 4\pi i \epsilon_1^t \omega_3 \quad (2.36)$$

Similarly

$$\nu \rightarrow \tilde{\nu} = \nu + 4\pi i \epsilon_1^t \omega_1 \quad (2.37)$$

Therefore the exponential factor transforms as

$$e^{2\mu z(s) + 2\nu \bar{z}(s)} \rightarrow e^{2\mu z(s) + 2\nu \bar{z}(s)} e^{4\pi i \epsilon_1^t (2\omega_3 z(s) + 2\omega_1 \bar{z}(s))} = e^{2\mu z(s) + 2\nu \bar{z}(s)} e^{4\pi i \epsilon_1^t \zeta_s} \quad (2.38)$$

The factor precisely cancels the factor in eq. (2.31) coming from the ratio of theta functions showing that $X(s, p_4)$ is the correct combination that is well defined on the Riemann surface.

Having proven that $X(s, p_4)$ is well defined on the Riemann surface, it is interesting to study if it takes different values on the two sheets of the Riemann surface. This is indeed so. Changing sheets on the Riemann surface is equivalent to change $4 \rightarrow \bar{4}$ or equivalently

$$\int_1^4 \rightarrow -\int_1^4 \quad (2.39)$$

Since θ is even and $\hat{\theta}$ is odd, the functions μ and ν change sign under such change. The ratio of theta functions is obviously inverted resulting in:

$$X(s, p_{\bar{4}}) = \frac{1}{X(s, p_4)} \quad (2.40)$$

Therefore the function $X(s, \lambda_{p_4})$ defined on the complex plane has cuts. Thus, in some sense, it is analogous to the monodromy matrix. Namely, the cuts in $X(s, \lambda_{p_4})$ can be removed by extending the function to $X(s, p_4)$ defined on a Riemann surface, the spectral curve of the problem.

Let us now study the analytical properties of $X(s, p_4)$. Riemann's theorem [54] ensures that theta functions defined on a Riemann surface have g zeros and no poles. The ratio that appears in the definition of $X(s, \lambda_{p_4})$ therefore can have at most g zeros and g poles. One however is at $p_4 = p_1$ and therefore cancels. In fact for the others it is clear that if p_4 is a zero then $p_{\bar{4}}$ is a pole. So $X(s, p_4)$ has $(g - 1)$ zeros and $(g - 1)$ poles conjugate from each other on the Riemann surface (that means on opposite sheets). On the other hand the exponential function is quite interesting. The function

$$\mu = -2 D_3 \ln \theta \left(\int_1^4 \right) = -2 \frac{D_3 \theta(\int_1^4)}{\theta(\int_1^4)} \quad (2.41)$$

has potential singularities on the g zeros of $\theta(\int_1^4)$. However, the derivative also vanishes on those zeros except at $p_4 = p_1$. Since μ has a pole at $p_4 = p_1 = 0$ then $X(s, p_4)$ has an essential singularity. The same happens at $p_4 = p_3$ in view of the function ν .

To summarize: $X(s, p_4)$ as a function of p_4 is a well defined function on the Riemann surface with $(g - 1)$ zeros and $(g - 1)$ poles and essential singularities at $p_4 = p_1 = 0$ and $p_4 = p_3 = \infty$.

Given the symmetry $\lambda \leftrightarrow -1/\bar{\lambda}$ it suffices to study the function near $\lambda = 0$. Since $\lambda = 0$ is a branch point a more appropriate coordinate y can be chosen as

$$y = -2i\sqrt{\lambda} \quad (2.42)$$

Using the expansion from eq. (2.15)

$$\omega(y) = (\omega_1 + \omega_{12}y^2 + \dots)dy \quad (2.43)$$

it follows that

$$\int_1^4 \omega = \omega_1 y + \frac{1}{3}\omega_{12}y^3 + \dots \quad (2.44)$$

A simple Taylor expansion in y gives

$$\ln \left(-\frac{\hat{\theta}(\zeta_s + \int_1^4)}{\hat{\theta}(\zeta_s - \int_1^4)} \right) = 1 + y \frac{D_1^2 \hat{\theta}(\zeta_s)}{D_1 \hat{\theta}(\zeta_s)} + \dots \quad (2.45)$$

Finally the expansion

$$\mu = -2D_{13} \ln \theta(0)y + \dots \quad (2.46)$$

$$\nu = -3\frac{2}{y} - \frac{2}{3}y \left(\frac{D_1^3 \hat{\theta}(0)}{D_1 \hat{\theta}(0)} - \frac{D_1'' \hat{\theta}(0)}{D_1 \hat{\theta}(0)} \right) + \dots \quad (2.47)$$

and eq. (A.37) in the appendix, result in

$$\begin{aligned} \ln X &= -\frac{4}{y}\bar{z}(s) \\ &+ y \left[-\frac{4}{3} \left(\frac{D_1^3 \hat{\theta}(0)}{D_1 \hat{\theta}(0)} - \frac{D_1'' \hat{\theta}(0)}{D_1 \hat{\theta}(0)} \right) \bar{z}(s) - 4D_{13} \ln \theta(0)z(s) + 2D_1 \ln \theta(0) \right] \\ &+ \mathcal{O}(y^3) \end{aligned} \quad (2.48)$$

This expansion is remarkable in two respects. First, the residue of the pole $\frac{1}{y}$ contains $z(s)$, namely the parameterization of the conformal world-sheet coordinate in terms of the boundary parameter. Second, as we will see later the constant term determines the area of the minimal area surface with this boundary. Actually the area of the whole family of minimal area surfaces parameterized by p_4 such that $|\lambda_{p_4}| = 1$. It is also instructive to compute the expansion of the Schwarzian derivative $\{X, p_4\}$ based on the expansion of $\ln X$. If, as shown, X has an expansion

$$X = \frac{a_1}{y} + a_2 y + \mathcal{O}(y^3) \quad (2.49)$$

and since

$$\{X, y\} = \{\ln X, y\} - \frac{1}{2}(\partial_y \ln X)^2 \quad (2.50)$$

it follows that

$$\{X, y\} = -\frac{a_1^2}{2y^4} + \frac{a_1 a_2}{y^2} + \mathcal{O}(1) \quad (2.51)$$

where the two leading terms come from the last term in eq. (2.50). It follows that

$$\begin{aligned} \{X, y\} = & -\frac{8}{y^4}(\bar{z}(s))^2 \\ & -\frac{4\bar{z}(s)}{y^2} \left[-\frac{4}{3} \left(\frac{D_1^3 \hat{\theta}(0)}{D_1 \hat{\theta}(0)} - \frac{D_1'' \hat{\theta}(0)}{D_1 \hat{\theta}(0)} \right) \bar{z}(s) - 4D_{13} \ln \theta(0) z(s) + 2D_1 \ln \theta(0) \right] \\ & + \mathcal{O}(1) \end{aligned} \quad (2.52)$$

2.3 Conformal invariant ratios and Schwarzian derivative

In the case of the monodromy matrix one can consider its trace which is a well defined function on the complex plane since the cuts are removed. In the present case we can consider functions of $X(s, p_4)$ which are invariant under inversion. Those are well defined in the complex plane. For example the functions

$$\begin{aligned} Y^{[4]}(s_1, s_2, s_3, s_4, \lambda_{p_4}) &= \frac{(X(s_1, p_4) - X(s_2, p_4))(X(s_3, p_4) - X(s_4, p_4))}{(X(s_1, p_4) - X(s_3, p_4))(X(s_4, p_4) - X(s_2, p_4))} \\ Y^{[2]}(s_1, s_2, \lambda_{p_4}) &= \frac{\partial_{s_1} X(s_1, p_4) \partial_{s_2} X(s_2, p_4)}{(X(s_1, p_4) - X(s_2, p_4))^2} \end{aligned} \quad (2.53)$$

$$Y^{[1]}(s, \lambda_{p_4}) = \{X(s), s\} = \frac{\partial_s^3 X(s, p_4)}{\partial_s X(s, p_4)} - \frac{3}{2} \left(\frac{\partial_s^2 X(s, p_4)}{\partial_s X(s, p_4)} \right)^2 \quad (2.54)$$

are invariants depending on four, two and one point of the curve respectively. In fact they are invariant under the full Möbius group. The last one introduces the Schwarzian derivative which is conformally invariant. On the left hand side we emphasized that these are functions of λ_{p_4} namely the projection of p_4 on the complex plane. As functions of λ_{p_4} they have no cuts. The invariant $Y^{[4]}$ can be simply written by using eq. (2.29). On the other hand $Y^{[2]}$ requires evaluating $\partial_s X$. From eqs. (2.29), (2.26) it follows that

$$\partial_s \ln X = 2\mu \partial_s z + 2\nu \partial_s \bar{z} + 2\partial_s z D_1 \ln \frac{\hat{\theta}(\zeta_s + f_1^4)}{\hat{\theta}(\zeta_s - f_1^4)} + 2\partial_s \bar{z} D_3 \ln \frac{\hat{\theta}(\zeta_s + f_1^4)}{\hat{\theta}(\zeta_s - f_1^4)} \quad (2.55)$$

Using the definition of μ, ν i.e. eq. (2.24) and eqs. (A.23), (A.29) for the derivatives of the theta function we immediately find that

$$\partial_s \ln X = -2\partial_s z \frac{\hat{\theta}(2f_1^4) D_3 \hat{\theta}(0)}{\theta^2(f_1^4)} \frac{\theta^2(\zeta)}{\hat{\theta}(\zeta_s + f_1^4) \hat{\theta}(\zeta_s - f_1^4)} \quad (2.56)$$

Equivalently

$$\partial_s X = 2\partial_s z e^{2\mu z(s) + 2\nu \bar{z}(s)} \frac{\hat{\theta}(2f_1^4) D_3 \hat{\theta}(0)}{\theta^2(f_1^4)} \frac{\theta^2(\zeta)}{\hat{\theta}^2(\zeta_s - f_1^4)} \quad (2.57)$$

Thus

$$\begin{aligned}
 Y^{[2]}(s_1, s_2, \lambda_{p_4}) &= \left(\frac{\hat{\theta}(2f_1^4) D_3 \hat{\theta}(0)}{\theta^2(f_1^4)} \right)^2 \times \\
 &\times \frac{4 \partial_s z_1 \partial_s z_2 \theta^2(\zeta_{s_1}) \theta^2(\zeta_{s_2})}{[e^{\mu z_{12} + \nu \bar{z}_{12}} \hat{\theta}(\zeta_{s_1} + f_1^4) \hat{\theta}(\zeta_{s_2} - f_1^4) - e^{-\mu z_{12} - \nu \bar{z}_{12}} \hat{\theta}(\zeta_{s_1} - f_1^4) \hat{\theta}(\zeta_{s_2} + f_1^4)]^2}
 \end{aligned} \tag{2.58}$$

with the notation $z_{12} = z_1 - z_2$, $\bar{z}_{12} = \bar{z}_1 - \bar{z}_2$ and $z_1 = z(s_1)$, $z_2 = z(s_2)$.

Finally, the Schwarzian derivative is harder to compute since it involves up to third order derivatives. However, it turns out that the result is remarkably simple. The second derivative can be computed in a similar way as the first one resulting in

$$\partial_\sigma \ln \partial_\sigma X = \frac{\partial_\sigma^2 X}{\partial_\sigma X} = \frac{\partial_\sigma^2 z}{\partial_\sigma z} + 2 \partial_\sigma z \left(\mu + 2 D_3 \ln \frac{\theta(\zeta)}{\hat{\theta}(\zeta - f_1^4)} \right) \tag{2.59}$$

In fact now, the Schwarzian derivative can be evaluated as

$$\begin{aligned}
 \{X, s\} &= \{z, s\} + 8(\partial_s z)^2 \left[D_3^2 \ln \frac{\theta(\zeta_s)}{\hat{\theta}(\zeta_s - f_1^4)} - \left(D_3 \ln \frac{\theta(\zeta_s)}{\theta(f_1^4) \hat{\theta}(\zeta_s - f_1^4)} \right)^2 \right] \\
 &\quad - 8(\partial_s \bar{z})^2 \left(\frac{D_1 \hat{\theta}(0) \theta(f_1^4)}{\theta(0) \hat{\theta}(f_1^4)} \right)^2
 \end{aligned} \tag{2.60}$$

The term in square brackets can be simplified by first using eq. (A.39) to replace $D_3^2 \ln \hat{\theta}(\zeta_s - f_1^4)$ and then eqs. (A.20), (A.44) to further reduce the result. The second line in eq. (2.60) is simplified by the identity (A.42). In that manner the result

$$\{X, s\} = \{z, s\} - 2\lambda_{p_4} (\partial_s z)^2 + \frac{2}{\lambda_{p_4}} (\partial_s \bar{z})^2 - 8(\partial_s z)^2 \left[\frac{D_3^3 \hat{\theta}(\zeta_s)}{D_3 \hat{\theta}(\zeta_s)} - 3 \frac{D_3^2 \theta(\zeta_s)}{\theta(\zeta_s)} \right] \tag{2.61}$$

is obtained. The last term can be rewritten using eq. (A.41)

$$\{X, s\} = \{z, s\} - 2\lambda_{p_4} (\partial_s z)^2 + \frac{2}{\lambda_{p_4}} (\partial_s \bar{z})^2 \tag{2.62}$$

$$+ 8(\partial_s z)^2 \left[2 D_3^2 \ln \theta(\zeta_s) - \frac{D_3^3 \hat{\theta}(0)}{D_3 \hat{\theta}(0)} + \frac{D_3^2 \theta(0)}{\theta(0)} \right] \tag{2.63}$$

A completely equivalent expression can be found in terms of derivatives D_1 :

$$\{X, s\} = \{\bar{z}, s\} - 2\lambda_{p_4} (\partial_s z)^2 + \frac{2}{\lambda_{p_4}} (\partial_s \bar{z})^2 \tag{2.64}$$

$$+ 8(\partial_s \bar{z})^2 \left[2 D_1^2 \ln \theta(\zeta_s) - \frac{D_1^3 \hat{\theta}(0)}{D_3 \hat{\theta}(0)} + \frac{D_1^2 \theta(0)}{\theta(0)} \right] \tag{2.65}$$

The most important point is that the dependence in λ is very explicit in eq. (2.61). We believe this to be one of the main results of this paper since $\{X, s\}$ gives a conformal invariant characterization of the Wilson loop.

2.4 Area

In [14] the following formula for the finite part of the area of the minimal area surface was derived:

$$A_f = 16D_{13} \ln \theta(0) \int d\sigma d\tau + \frac{1}{2} \oint \frac{\nabla^2 \hat{\theta}(\zeta_s)}{|\nabla \hat{\theta}(\zeta_s)} d\ell \quad (2.66)$$

where (σ, τ) are world-sheet coordinates such that $z = \sigma + i\tau$ and

$$d\ell = \sqrt{d\sigma^2 + d\tau^2} = \sqrt{(\partial_s \sigma)^2 + (\partial_s \tau)^2} ds \quad (2.67)$$

is the element of arc-length on the boundary curve $z(s) = \sigma(s) + i\tau(s)$. In this section we simplify the result and relate it to the expansion of $X(s, p_4)$ near the singularity at $p_4 = 0$. In fact, the second term in eq. (2.66) can be rewritten as

$$\frac{1}{2} \oint \frac{\nabla^2 \hat{\theta}(\zeta_s)}{|\nabla \hat{\theta}(\zeta_s)} d\ell = \oint \partial_z \partial_{\bar{z}} \hat{\theta}(\zeta_s) \sqrt{\partial_s z \partial_{\bar{s}} \bar{z}} \partial_z \hat{\theta}(\zeta_s) \partial_{\bar{z}} \hat{\theta}(\zeta_s) ds \quad (2.68)$$

Using that $\partial_z \hat{\theta} = 2D_3 \hat{\theta}$, $\partial_{\bar{z}} \hat{\theta} = 2D_1 \hat{\theta}$ and that the contour of integration is such that $\hat{\theta}(\zeta_s) = 0$, the formula for the area can be rewritten as

$$A_f = 16D_{13} \ln \theta(0) \int d\sigma d\theta - 2i \oint D_1 \ln D_3 \hat{\theta}(\zeta_s) ds \quad (2.69)$$

where the sign was chosen according to the conventions in figure 1.

We can further use that

$$\partial_\sigma \ln D_3 \hat{\theta}(\zeta_s) = 2\partial_s z D_3 \ln D_3 \hat{\theta}(\zeta_s) + 2\partial_s \bar{z} D_1 \ln D_3 \hat{\theta}(\zeta_s) \quad (2.70)$$

and eq. (A.35) to find

$$A_f = 16D_{13} \ln \theta(0) \int d\sigma d\tau + 4i \oint D_3 \ln \theta(\zeta_s) dz - i \ln D_3 \hat{\theta}(\zeta_s) \Big|_{s_i}^{s_f} \quad (2.71)$$

where $s_{i,f}$ are the initial and final values of s . Since the curve is periodic,

$$\ln D_3 \hat{\theta}(\zeta_s) \Big|_{s_i}^{s_f} = 2\pi i n \quad (2.72)$$

for an integer n known as the turning number of the curve. It is the number of times the unit normal to the curve goes around the unit circle when the point goes once around the curve. With the assumptions of figure 1 it is $n = 1$. The final result for the area is therefore

$$A_f = -2\pi n + 16D_{13} \ln \theta(0) \int d\sigma d\tau + 4i \oint D_3 \ln \theta(\zeta_s) dz \quad (2.73)$$

which is simpler than the one found in the previous paper [14]. Equivalently we can write

$$\begin{aligned} A_f &= -2\pi n - 8iD_{13} \ln \theta(0) \oint \bar{z}(s) \partial_s z(s) ds + 4i \oint D_3 \ln \theta(\zeta_s) \partial_s z(s) ds \\ &= -2\pi n + 8iD_{13} \ln \theta(0) \oint z(s) \partial_s \bar{z}(s) ds - 4i \oint D_1 \ln \theta(\zeta_s) \partial_s \bar{z}(s) ds \end{aligned}$$

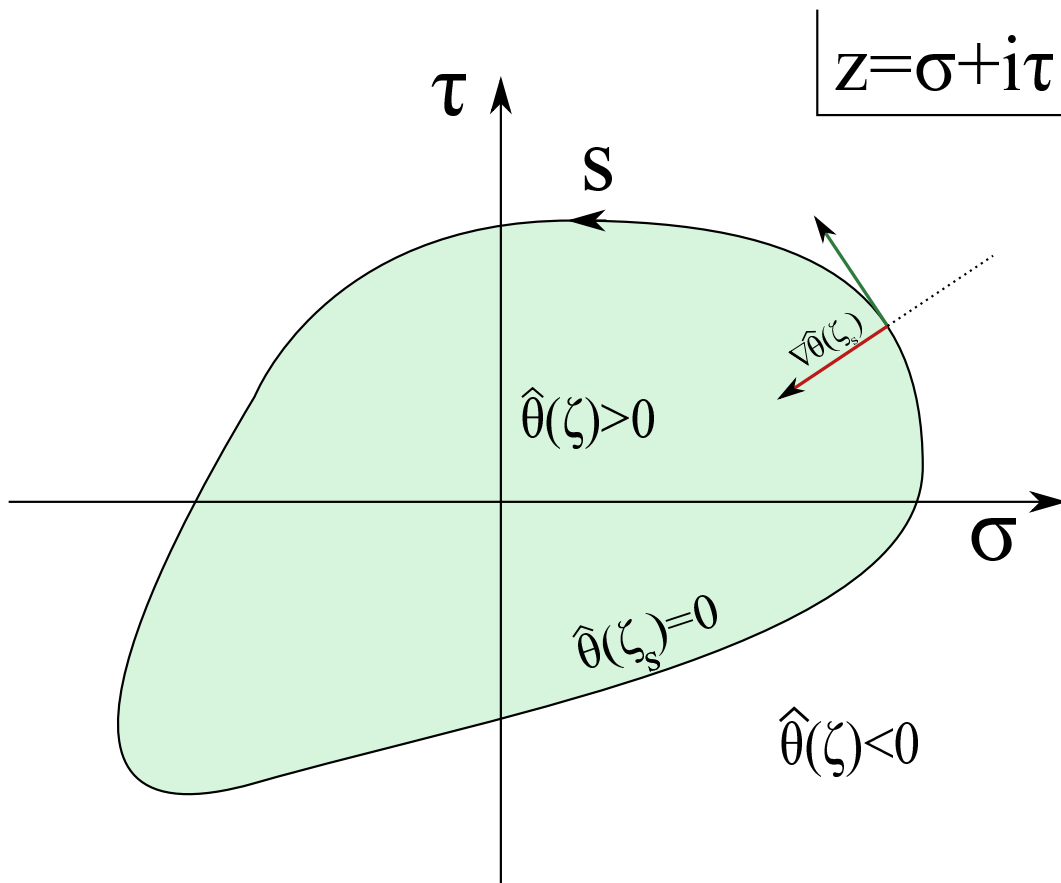


Figure 1. The world-sheet is parameterized by a complex coordinate z . The function $\hat{\theta}(\zeta)$ depends on z through the vector $\zeta = 2\omega(p_1)\bar{z} + 2\omega(p_3)z$ where $\omega_{p_{1,3}}$ are complex vectors. Our convention is that the shaded region where $\hat{\theta}(\zeta) > 0$ maps to the minimal area surface. The contour $\hat{\theta}(\zeta_s) = 0$ maps to the boundary of the surface, i.e. the Wilson loop. The contour is parameterized counterclockwise by s . The gradient $\nabla\hat{\theta}(\zeta_s)$ is normal to the boundary and point towards the interior. Finally, the region where $\hat{\theta}(\zeta_s) < 0$ also maps to a minimal area surface and the corresponding can be easily changed to cover that case. Slightly more involved is the case when $\hat{\theta}(\zeta)$ is purely imaginary but nothing fundamentally different changes.

$$(2.74)$$

Consider now the expansion in eq. (2.48). It immediately follows that

$$A_f = -2\pi n - \left(\frac{2i}{y} \oint \ln X \partial_s \bar{z} ds \right) \Big|_{y \rightarrow 0} \tag{2.75}$$

where we recall that $y = -2i\sqrt{\lambda_{p_4}}$ is the appropriate coordinate around the branch point $p_4 = 0$. Also the integral of a total derivative was discarded. Quite remarkably, the function $\bar{z}(s)$ is also contained in the expansion, in fact

$$-\frac{1}{4}y\partial_s \ln X \Big|_{y \rightarrow 0} = \partial_s \bar{z} \tag{2.76}$$

We can then write the slightly more cumbersome formula

$$A_f = -2\pi n - \frac{2i}{\pi^2} \left(\oint \left[\oint_{y=0} \frac{dy}{y^2} \ln X(s, y) \right] \left[\oint_{y'=0} dy' \partial_s \ln X(s, y') \right] ds \right) \quad (2.77)$$

The integrals on spectral parameters y, y' are complex contour integrals (given by the residues) around the points $y = 0, y' = 0$. The integral along the real parameter s is an integral around the boundary curve.

2.5 Genus $g = 1$ case

In the case of genus one the theta functions reduce to Jacobi theta functions and the solutions can be written in terms of elliptic functions or, equivalently, elliptic integrals. In the previous work [14] this case was not described since it had already been studied in [25] and [26]. More recently, these solutions were analyzed including the evaluation of the first quantum correction [38]. In this subsection however, revisiting this case is useful as a starting point for the next section. For genus one there is only two cuts, namely four branch points. They are taken to be $p_1 = 0, p_3 = \infty, \lambda_1 = a \in \mathbb{R}, \lambda_2 = -\frac{1}{a}$ in view of the $\lambda \leftrightarrow -1/\bar{\lambda}$ symmetry. The spectral curve is described by the equation

$$\mu^2 = \lambda(\lambda - a) \left(\lambda + \frac{1}{a} \right) \quad (2.78)$$

an is illustrated in figure 2 where it is also shown a choice of cycles a_1, b_1 . The only holomorphic differential is

$$\nu_1 = \frac{d\lambda}{\mu(\lambda)} \quad (2.79)$$

which needs to be properly normalized. Computing

$$C_1 = \oint_{a_1} \frac{d\lambda}{\mu(\lambda)} = 2i \int_0^a \frac{d\lambda}{\sqrt{(a-\lambda)\lambda(\lambda + \frac{1}{a})}} = 4i\sqrt{kk'} \mathbf{K} \quad (2.80)$$

$$\tilde{C}_1 = \oint_{b_1} \frac{d\lambda}{\mu(\lambda)} = -2 \int_{-\frac{1}{a}}^0 \frac{d\lambda}{\sqrt{(a-\lambda)(-\lambda)(\lambda + \frac{1}{a})}} = -4\sqrt{kk'} \mathbf{K}' \quad (2.81)$$

where the standard notation for elliptic functions was used (see [55])

$$k = \frac{a}{\sqrt{1+a^2}}, \quad k' = \sqrt{1-k^2} = \frac{1}{\sqrt{1+a^2}}, \quad \mathbf{K} = K(k), \quad \mathbf{K}' = K(k') \quad (2.82)$$

The formulas from [14] reduce to

$$\Pi = \tilde{C}_1 C_1^{-1} = i \frac{\mathbf{K}'}{\mathbf{K}} \quad (2.83)$$

$$\omega = -\frac{i}{4\sqrt{kk'} \mathbf{K}} \frac{d\lambda}{\mu(\lambda)} \quad (2.84)$$

$$\omega_1 = -\omega_3 = \frac{i}{4\sqrt{kk'} \mathbf{K}} \quad (2.85)$$

We also have $\Delta_1 = 1, \Delta_2 = 1$. Taking the involution $T = -1$ we find

$$\bar{\Pi} = -T\Pi T, \quad T\Delta_{1,2} = -\Delta_{1,2} \quad T\omega_3 = \bar{\omega}_1 \quad (2.86)$$

as it should be for the reality condition to be satisfied. Let us introduce also the half-period \hat{a} that defines $\hat{\theta}$ and which is a zero of θ :

$$\hat{a} = \frac{1 + \Pi}{2} \quad (2.87)$$

Now it is possible to define the variable ζ

$$\zeta = 2\omega_1\bar{z} + 2\omega_3z = \frac{\tau}{\sqrt{kk'}\mathbf{K}} \quad (2.88)$$

where $z = \sigma + i\tau$. Now we can relate the Riemann theta functions to Jacobi Theta functions (we use the convention in [55]) by:

$$\theta(\zeta) = \theta_3(\pi\zeta, q), \quad \hat{\theta}(\zeta) = \theta \left[\begin{matrix} \Delta_1 \\ \Delta_2 \end{matrix} \right] (\zeta) = -\theta_1(\pi\zeta, q), \quad \text{with} \quad q = e^{i\pi\Pi}, \quad (2.89)$$

Some convenient formulas are [55]

$$D_1\theta(\hat{a}) = \omega_1\theta'_3\left(\frac{1 + \Pi}{2}\right) = i\pi\omega_1q^{-\frac{1}{4}}\theta'(0) \quad (2.90)$$

and

$$\theta_3(0) = \sqrt{\frac{2\mathbf{K}}{\pi}}, \quad \theta'_1(0) = \theta_2(0)\theta_3(0)\theta_4(0) \quad \theta_2(0)\theta_4(0) = \sqrt{kk'}\theta_3^2(0) \quad (2.91)$$

The world-sheet metric (or Lagrange multiplier Λ) is determined by

$$e^{2\alpha} = \frac{\theta_3^2(\pi\zeta)}{\theta_1^2(\pi\zeta)} = \frac{1}{kk'} \left(\frac{\mathbf{dn}\left(\frac{2\tau}{\sqrt{kk'}}, k\right)}{\mathbf{sn}\left(\frac{2\tau}{\sqrt{kk'}}, k\right)} \right)^2 \quad (2.92)$$

In the last equation α was written in terms of Jacobi elliptic functions, perhaps the easiest expression to check the validity of the cosh-Gordon equation

$$\frac{1}{4}\partial_\tau^2\alpha = e^{2\alpha} + e^{-2\alpha} \quad (2.93)$$

for such function. The last step is to choose the spectral parameter λ such that $|\lambda| = 1$. In terms of λ we define

$$v = \int_1^4 = -\frac{i}{4\sqrt{kk'}\mathbf{K}} \int_0^\lambda \frac{d\lambda}{\mu(\lambda)} \quad (2.94)$$

The solution can then be written in terms of Jacobi theta functions as

$$\mu = 2\pi\omega_1\partial_v \ln \theta_3(v) \quad (2.95)$$

$$\nu = \mp 2\pi\omega_1\partial_v \ln \theta_1(v) \quad (2.96)$$

$$\psi_1 = -2\pi\omega_1 \frac{\theta'_1(0)\theta_1(v)\theta_1(x+v)}{\theta_3(0)\theta_3(v)\theta_1(x)} e^{-\frac{\alpha}{2}} e^{\mu z + \nu \bar{z}} \quad (2.97)$$

$$\psi_2 = \pm \frac{\theta_3(x+v)}{\theta_3(x)} e^{\frac{\alpha}{2}} e^{\mu z + \nu \bar{z}} \quad (2.98)$$

$$\tilde{\psi}_1 = 2\pi\omega_1 \frac{\theta_1'(0)\theta_1(v)\theta_1(x-v)}{\theta_3(0)\theta_3(v)\theta_1(x)} e^{-\frac{\alpha}{2}} e^{-\mu z - \nu \bar{z}} \quad (2.99)$$

$$\tilde{\psi}_2 = \pm \frac{\theta_3(x-v)}{\theta_3(x)} e^{\frac{\alpha}{2}} e^{-\mu z - \nu \bar{z}} \quad (2.100)$$

$$\psi_1 \tilde{\psi}_2 - \psi_2 \tilde{\psi}_1 = \left| \frac{\theta_1(2v)\theta_3(0)}{\theta_3^2(v)} \right| \quad (2.101)$$

$$\zeta = 2\omega_1(\pm \bar{z} - z) \quad (2.102)$$

$$\lambda = \left(\frac{\theta_1(v)}{\theta_3(v)} \right)^2 \quad (2.103)$$

where v is an additional parameter which corresponds to \int_1^4 in eq. (2.22). The formulas for $\psi_{1,2}$ and $\tilde{\psi}_{1,2}$ refer to the functions defined in [14] and are included for completeness. They are not used in this paper. Finally, the solution is written as

$$Z = \left| \frac{\theta_1(2v)\theta_3(0)}{\theta_3^2(v)} e^{-2\mu z - 2\nu \bar{z}} \right| \frac{\theta_1(x)\theta_3(x)}{|\theta_1(x+v)|^2 + |\theta_3(x+v)|^2} \quad (2.104)$$

$$X + iY = e^{-2\bar{\mu}\bar{z} - 2\bar{\nu}z} \frac{\theta_3(x+v)\bar{\theta}_3(x-v) - \theta_1(x+v)\bar{\theta}_1(x-v)}{|\theta_1(x+v)|^2 + |\theta_3(x+v)|^2} \quad (2.105)$$

As before the boundary curves are found by finding the zeros of $\hat{\theta} = -\theta_1$. Since the variable ζ_+ depends only on τ (or ζ_- on σ), the world-sheet region is a strip between two zeros of θ_1 . Generically the edges of the strip map to two infinite boundary curves with the shape of a spiral. However if we take $\lambda = 1$ for $a > 1$ and $\lambda = -1$ for $a < 1$ we have the following interesting result

- $a > 1$, $\lambda = 1$, the Wilson loop is given by two concentric circles of radii $r_{1,2}$, satisfying

$$\ln \frac{r_2}{r_1} = \pi \operatorname{Re} \left(\frac{\theta_3'(v)}{\theta_3(v)} + \frac{\theta_1'(v)}{\theta_1(v)} \right) \quad (2.106)$$

The ratio, by definition, is smaller than one, but, analyzing the previous result, it shows that it has a minimum value larger than zero. For each value of the ratio $\frac{r_2}{r_1}$ in that range there are two possible values of v . However they do not give the same solution, namely, for each allowed ratio $\frac{r_2}{r_1}$ there are two solutions, one world-sheet is close to the boundary and the other extends towards the interior.

- $a < 1$, $\lambda = -1$ the Wilson loop is given by a cusp.

The following limits are of interest

- $a \rightarrow \infty$ gives a circle.
- $a \rightarrow 1^+$ gives two parallel lines.
- $a \rightarrow 1^-$ gives two parallel lines.

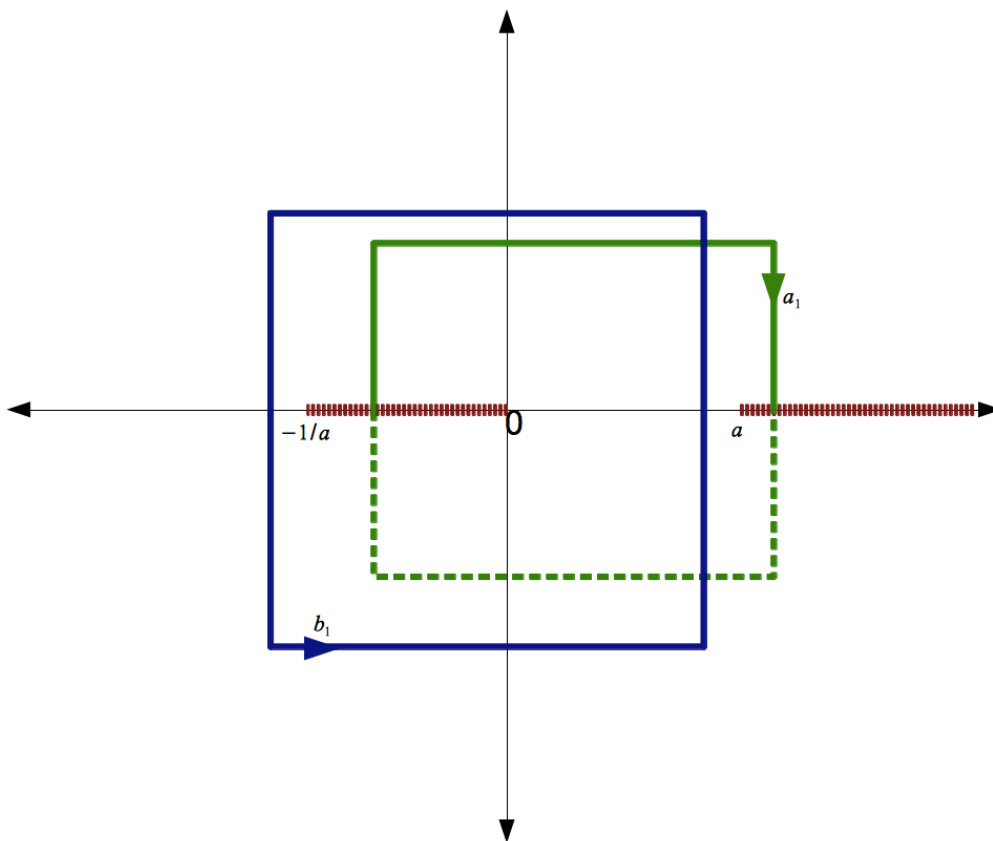


Figure 2. $g=1$ Riemann surface.

- $a \rightarrow 0$ gives a straight line.

For the case $a = 1$ corresponding to two parallel line the spectral curve coincides with the one given in [39]:

$$\mu^2 = \lambda(\lambda^2 - 1) \tag{2.107}$$

suggesting that both approaches are equivalent.

3 Surfaces ending on two contours

In the previous section we discussed, following [26] a solution ending on two concentric circles and corresponding to two cuts. It seems evident that adding two small cuts to the spectral curve will just slightly deform the solution. This is correct except that generically the periodicity is broken and therefore the curves become infinite spirals instead of two closed curves. To be precise, the theta functions are periodic but there is also the exponential factor in eq. (2.28) which is periodic when the exponent is purely imaginary, a condition that needs to be satisfied. Both periods have to match. It is clear that for particular positions of the cuts the periodicity would be restored but it requires some numerical effort to find them. In this section we present several examples. The number of

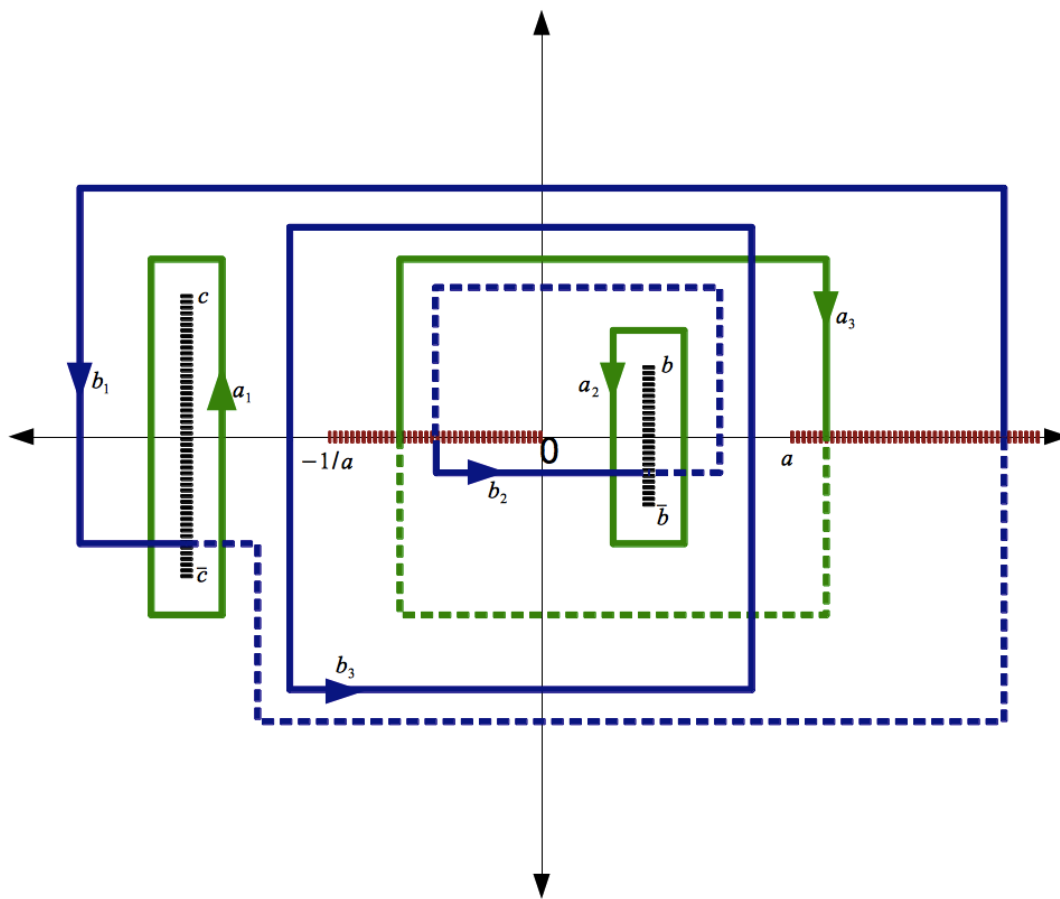


Figure 3. $g=3$ hyperelliptic Riemann surface.

branch cuts is four, the situation is depicted in figure 3 where also the choice of cycles is made evident. Due to the involution $\lambda \rightarrow -1/\bar{\lambda}$, the branch point $c = -1/b$. The points \bar{b} and \bar{c} are the complex conjugates of b and c , respectively.

As we shrink the $b\bar{b}$ and $c\bar{c}$ branch cuts we should get results close to the concentric circles for $a > 1$. Unlike the $g = 1$ case, even if we take $\lambda = 1$, the solutions are not automatically periodic; we must pick a suitable value of $a > 1$ as we move b to get periodic concentric curves.

The ratio of Riemann theta functions in eq. (2.28) is periodic if the argument shifts by an integer vector, also, the exponential function is periodic under a 2π shift in the imaginary direction. Both periods have to match. This can be achieved in practice by moving the branch points around. We found values for a and b such that we can match 2,3,4 and 5 periods of the theta function into a single period of the exponential. In each case we obtained concentric Wilson loops. The curves deviate from a circle in a way such that the periodicity is manifest as can be seen in figures 4 and 5. The corresponding positions of the branch points is shown in table 1 below. It is also possible to fit one period of the theta function into e.g. two periods of the exponential in (2.28). It turns out that we get a boundary made of self-intersecting concentric curves. For appropriate choices of a and b

n=2	n=3	n=4	n=5
$a = 1.28088$	$a = 1.102149$	$a = 1.035312$	$a = 1.0304752$
$b = 0.5 + 0.01i$	$b = 0.5 + 0.1i$	$b = 0.7 + 0.2i$	$b = 0.7 + 0.3i$

Table 1. Positions of the branch points; n corresponds to the number of periods of the exponential prefactor in each period of the ratio of theta function.

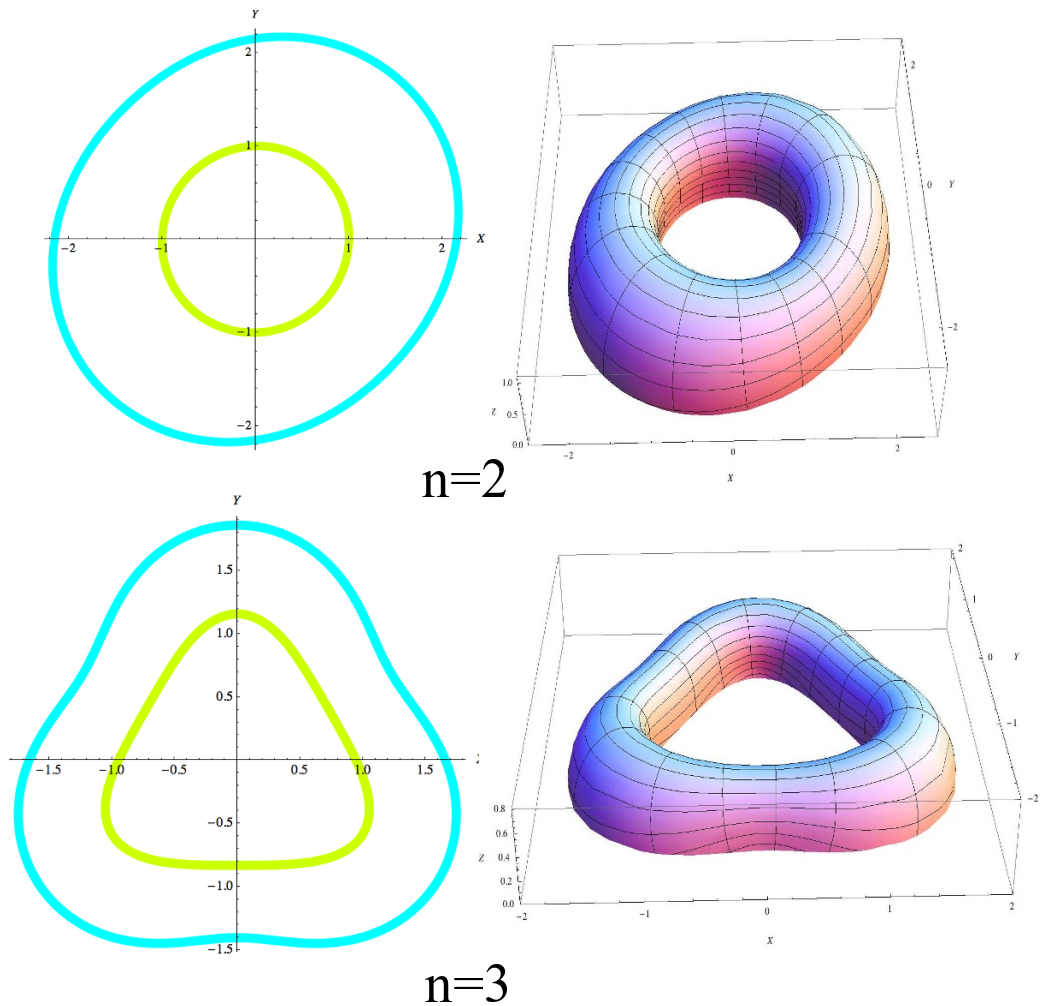


Figure 4. Wilson loops and corresponding minimal area surfaces for two concentric curves. In this case, two and three periods of the theta functions are fit into one period of the exponential as is evident from the shape of the curves.

each individual curve resembles a circle which is described twice as shown in figure 6. It is also made clear the claim that as we shrink the $b\bar{b}$ branch cut the surface approaches the one corresponding to the concentric circular Wilson loops.

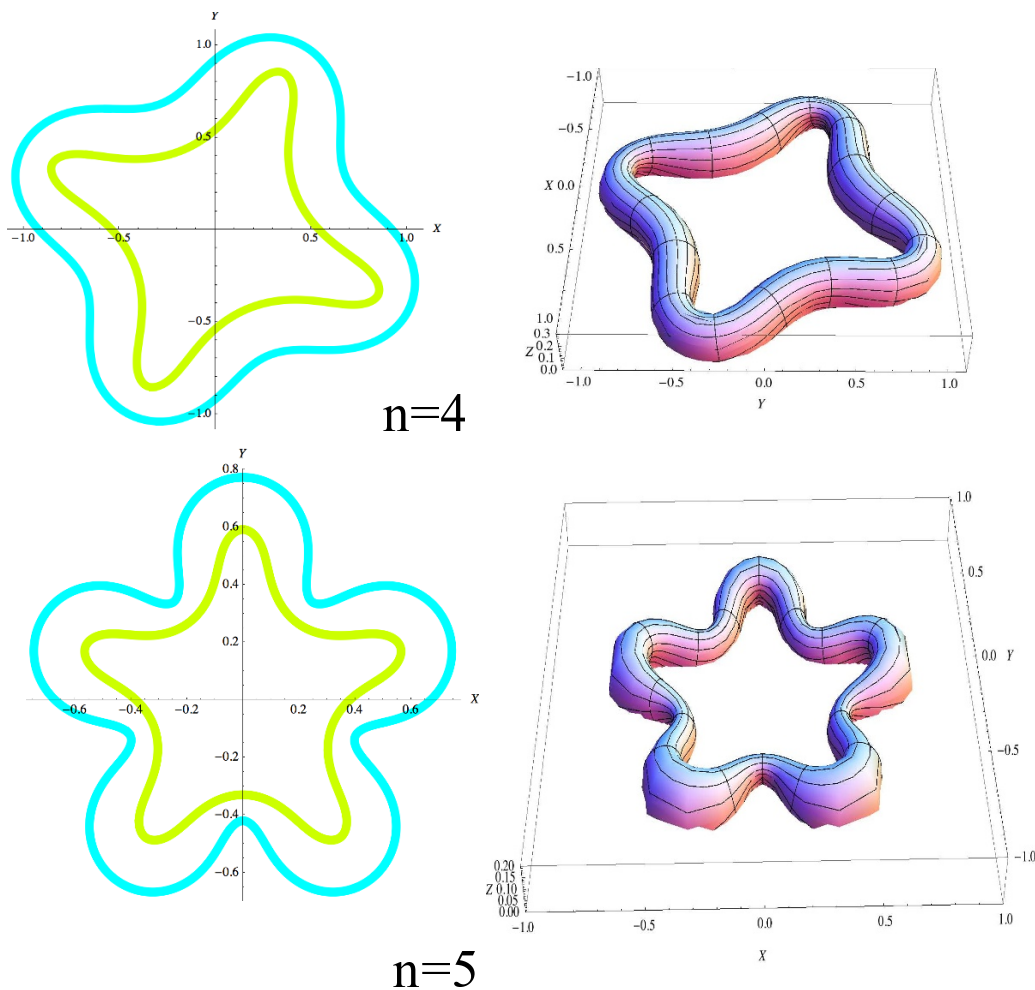


Figure 5. Wilson loops and corresponding minimal area surfaces for two concentric curves. Now, four and five periods of the theta functions are fit into one period of the exponential as is evident from the shape of the curves.

4 More than two boundary curves

As the size of the cuts is varied, closed curves appear in the region between the two periodic curves giving rise to surfaces with more boundaries. An example is shown in figure 7. The curves depicted are zeros of $\hat{\theta}$ in the world-sheet and correspondingly of the function $Z(z, \bar{z})$. That means that they map into multiple curves at the boundary. The region surrounded by one of the closed curves maps into a minimal area surface ending in the corresponding single curve. On the other hand the complement of those regions maps to a minimal area surface ending in multiple curves, those closed curves we described plus two infinite curves (of spiral shape) given by the upper and lower periodic curves. Again, for $\lambda = 1$ and particular positions of the cuts, the period of those curves match those of the exponential and the result is a surface ending in multiple closed curves. A particular example is illustrated in figures 8, 9 and 10. Admittedly, the surface is not simple to describe or plot since it ends in eight different and self-intersecting curves. The boundary curves are shown

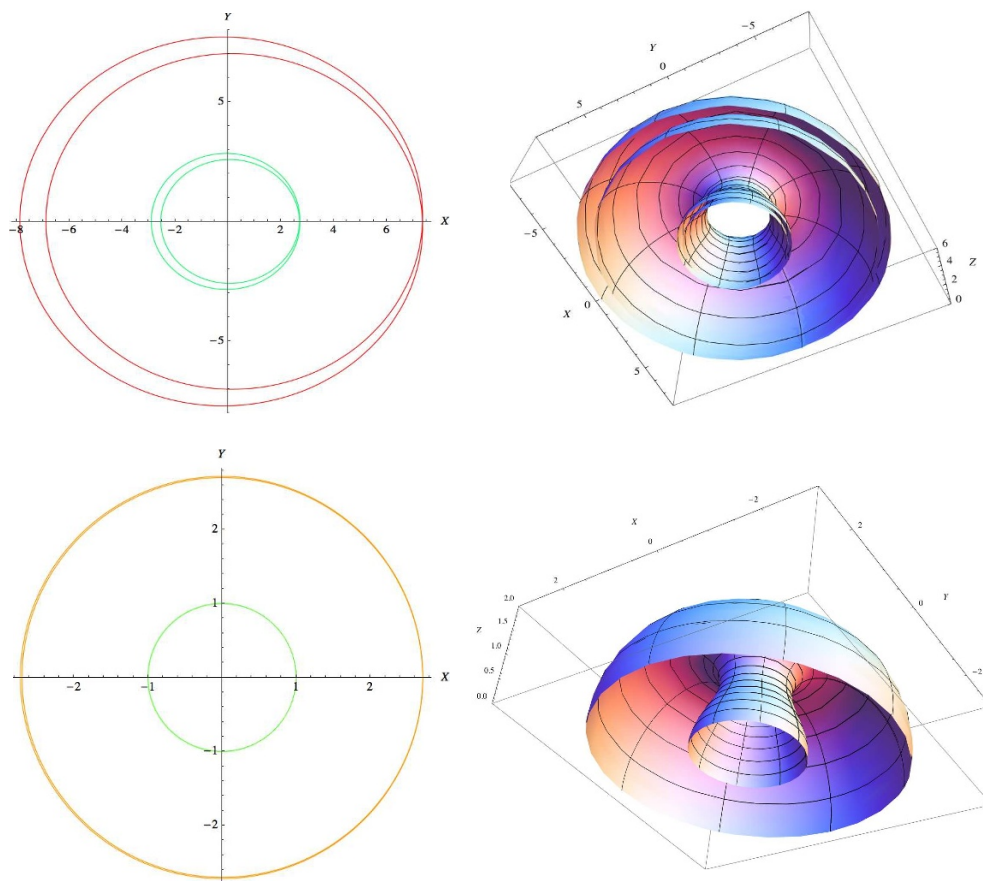


Figure 6. In this case, two periods of the exponential are fit in two periods of the theta function giving rise to concentric self-intersecting curves. In the first example the position of the cuts is given by $a = 2.412712, b = 2 + 0.05I$ and in the second $a = 2.414205, b = 2 + 0.003I$. The second one is shown as illustration of the fact that the surface can be made to approximate the (double cover) of the concentric circles.

in figure 9, notice however that different scales are used. As an intermediate representation, contours of equal Z are depicted in figure 8. Finally, the surface is plotted in figure 10.

5 Area

Consider again the surfaces ending on two concentric curves, namely those in figure 4, 5 and compute their area. The formula in eq. (2.75) applies to this case if we take the turning number of the curve to be $n = 0$. Remember that this is the turning number of the boundary curve in the world-sheet. In figure 7 it is clear that, as one moves along the upper and lower curves, the tangent to the curves does not wrap around the unit circle. The area of the world-sheet can also be explicitly written term of a parameterization $\tau_{1,2}(\sigma)$ of the

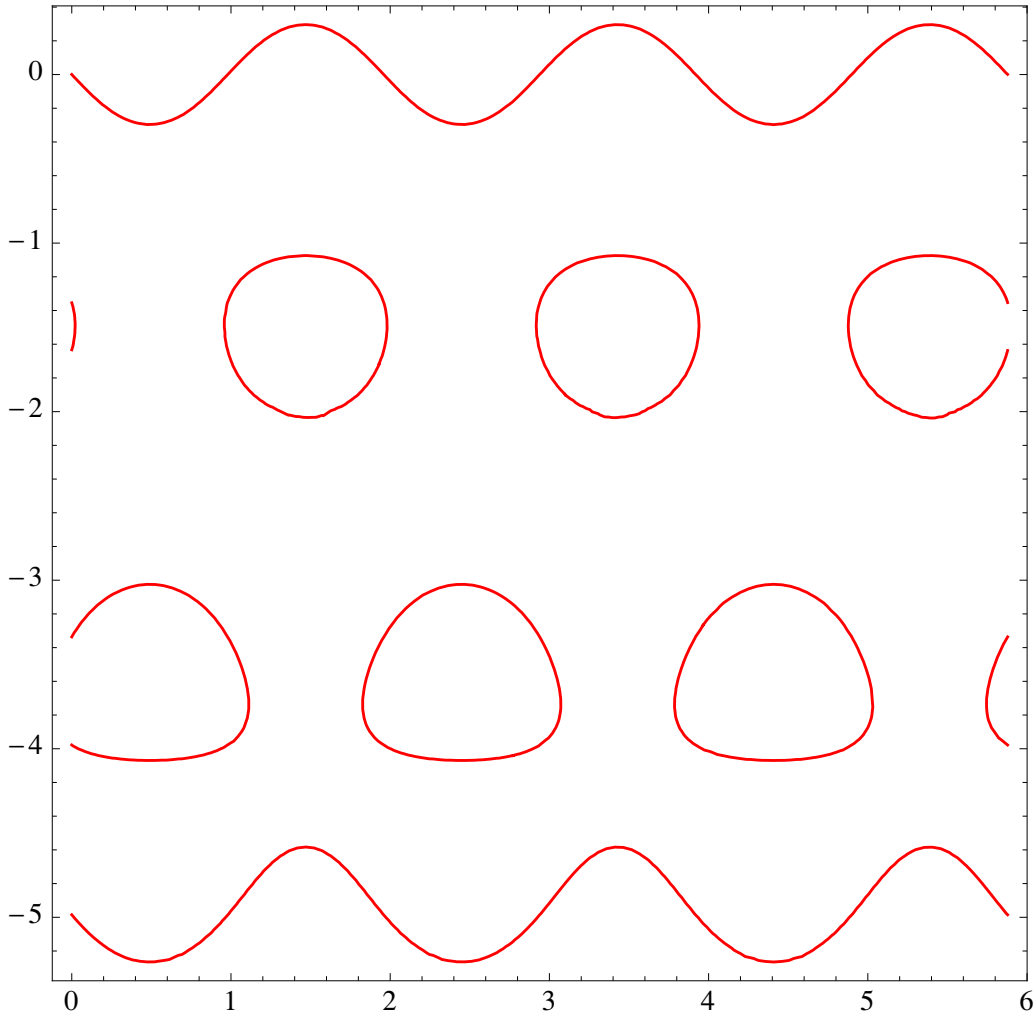


Figure 7. World-sheet of the string where the zeros of $\hat{\theta}$ are depicted. Those curves map to the boundary. For particular position of the cuts, the period of the figure in the σ direction matches the period of the exponential function and a surface ending in several closed curves is obtained.

upper and lower curves giving

$$\begin{aligned}
 A_f &= 4D_{p_1 p_3} \log \theta(0) \int (\tau_2(\sigma) - \tau_1(\sigma)) d\sigma + \frac{1}{2} \left(\int_{\tau_2(\sigma)} + \int_{\tau_1(\sigma)} \right) \frac{\nabla^2 \hat{\theta}}{|\nabla \hat{\theta}|} dl \\
 &= -2\Im \left\{ D_{13} \log \theta(0) \oint z d\bar{z} + \left(\int_{\tau_2(\sigma)} - \int_{\tau_1(\sigma)} \right) D_1 \log \theta(\zeta) d\bar{z} \right\} \quad (5.1)
 \end{aligned}$$

Recall also that the total area is given by

$$A_{\text{total}} = \frac{L}{\epsilon} + A_f \quad (5.2)$$

where L is the length of the Wilson loop and $\epsilon \rightarrow 0$ is the regulator defined by cutting the surface at $Z = \epsilon$. The analytical results thus obtained are in agreement with a direct numerical evaluation of the Area as shown in table 2.

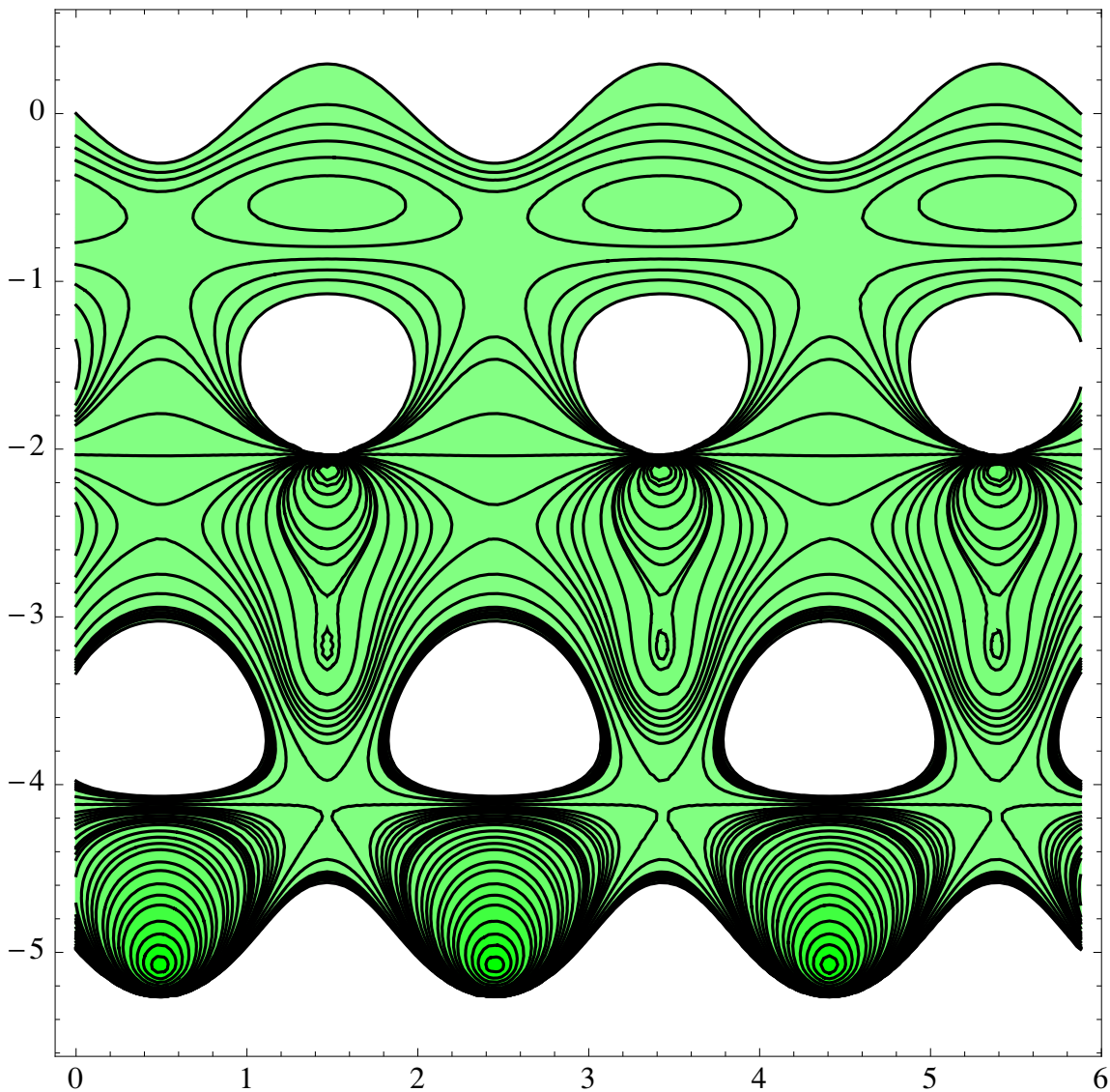


Figure 8. Contour lines indicating the surface heights above the boundary Wilson loop.

Areas	n=2	n=3	n=4	n=5
A_{total} by fit	$-13.80 + \frac{19.8644}{\epsilon}$	$-20.55 + \frac{17.25}{\epsilon}$	$-40.23 + \frac{12.23}{\epsilon}$	$-55.48 + \frac{9.64}{\epsilon}$
A_{total} by eq. (5.2)	$-13.80 + \frac{19.86}{\epsilon}$	$-20.55 + \frac{17.25}{\epsilon}$	$-40.27 + \frac{12.23}{\epsilon}$	$-55.66 + \frac{9.63658}{\epsilon}$
A_f by eq. (2.74)	-13.80	-20.55	-40.27	-55.66

Table 2. Numerical values of the Area of minimal surfaces in figures 4 and 5.

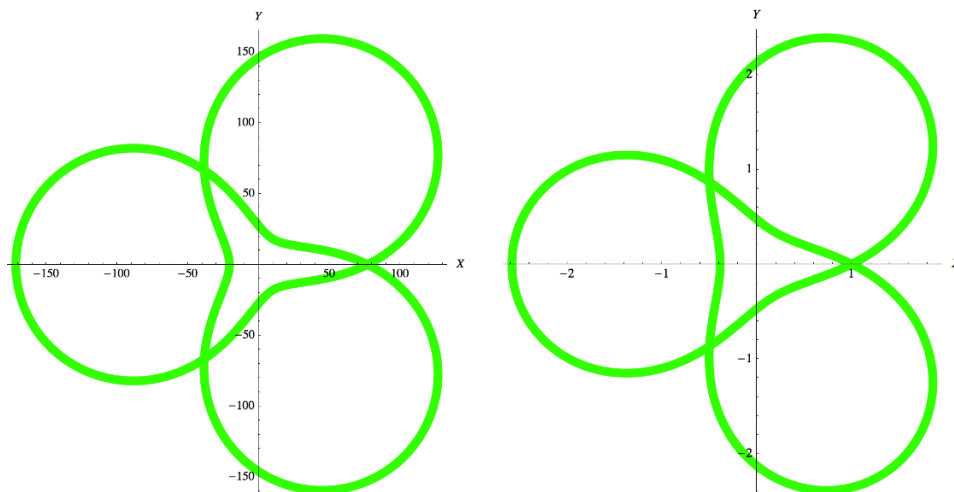


Figure 9. Image on the boundary of the curves depicted in figure 7.

6 Conclusions

In this paper we have simplified and extended the results in [14]. Given a boundary curve $X(s)$, a one complex parameter family of curves $X(s, \lambda)$ can be given by solving the linear problem associated with the flat connection. Studying the analytical properties of $X(s, \lambda)$ it turns out that the function is naturally defined on a hyperelliptic Riemann surface \mathcal{M} , the spectral curve. We then denote it as $X(s, p_4)$ with $p_4 \in \mathcal{M}$. In that way it plays a similar role to the monodromy matrix for standard integrable systems. The first part of the paper is devoted to the study of such function. In particular we find a simple expression for the Schwarzian derivative $\{X, s\}$. It is also shown that $X(s, p_4)$ has an essential singularity at $p_4 \rightarrow 0$ and the behavior near the singularity determines the area of the surface.

The second part of the paper is devoted to the study of surfaces ending in multiple contours. In particular examples for surfaces ending in two or more contours are given, generalizing previous results of Drukker and Fiol [26]. Again, a relatively simple expression is given for the area of such curves.

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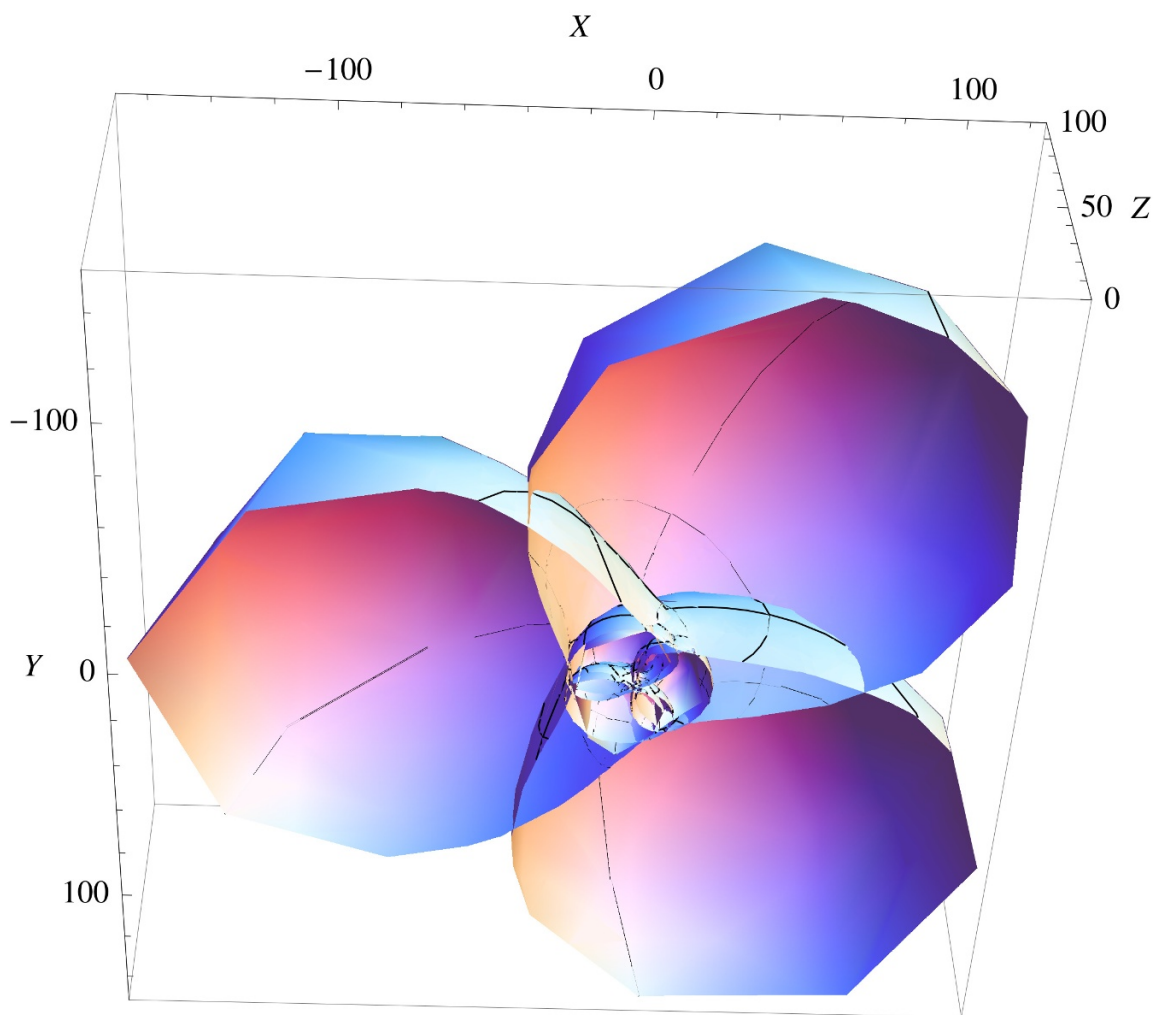


Figure 10. Example of minimal area surface embedded in Euclidean AdS_3 and ending on multiple contours.

A Theta functions identities

In this appendix we derive several identities for the Riemann theta functions that are needed in the main text. Some of the calculations are done in detail to illustrate the procedure. We use a basic knowledge of the theory of theta functions as can be found, for example, in the classic book on Riemann surfaces by Farkas and Kra [54]. In fact everything follows from the quasi periodicity property and the trisecant identity. Specialized books on theta functions such as [51–53] contain many of these results, we just briefly rederive them in this particular context and using the notation of this paper.

A.1 Quasi periodicity of θ and $\hat{\theta}$

Given two integer vectors $\epsilon_{1,2}$ it follows from the definition of $\theta(\zeta)$ and $\hat{\theta}(\zeta)$, i.e. eqs. (2.10), (2.14) that

$$\theta(\zeta + \epsilon_2 + \Pi\epsilon_1) = e^{-2\pi i[\epsilon_1^t \zeta + \frac{1}{2}\epsilon_1^t \Pi\epsilon_1]} \theta(\zeta) \tag{A.1}$$

and

$$\hat{\theta}(\zeta + \epsilon_2 + \Pi\epsilon_1) = e^{-2\pi i[\epsilon_1^t \zeta + \frac{1}{2}\epsilon_1^t \Pi\epsilon_1]} e^{i\pi[\Delta_1^t \epsilon_2 - \epsilon_1^t \Delta_2]} \hat{\theta}(\zeta) \tag{A.2}$$

As discussed in eq. (2.12), the function $\phi(p_4) = \int_1^4$ is holomorphic on \mathcal{M} but defined only up to integer periods. The quasi periodicity of the theta function allows to write holomorphic functions on \mathcal{M} as ratios

$$f(p_4) = \frac{\prod_{v=1}^n \theta(\zeta_v + \int_1^4)}{\prod_{\tilde{v}=1}^n \theta(\tilde{\zeta}_{\tilde{v}} + \int_1^4)}, \quad \text{with} \quad \sum_{v=1}^n \zeta_v = \sum_{\tilde{v}=1}^n \tilde{\zeta}_{\tilde{v}} \tag{A.3}$$

for arbitrary constant vectors $\zeta_{v=1\dots n}, \tilde{\zeta}_{\tilde{v}=1\dots n}$. Indeed, under such condition $f(p_4)$ is independent of the path chosen in \int_1^4 and therefore is a well defined function on the Riemann surface. A particular useful example is the generalized cross ratio defined in eq. (A.5) below.

A.2 Trisecant identity

By studying the analytic properties of ratios such as those in eq. (A.2) Fay derived the following fundamental identity known as trisecant identity [54]:

$$\theta(\zeta) \theta\left(\zeta + \int_j^i + \int_k^l\right) = \gamma_{ijkl} \theta\left(\zeta + \int_j^i\right) \theta\left(\zeta + \int_k^l\right) + \gamma_{ikjl} \theta\left(\zeta + \int_k^i\right) \theta\left(\zeta + \int_j^l\right) \tag{A.4}$$

valid for an arbitrary vector $\zeta \in \mathbb{C}$ and four arbitrary points $p_{i,j,k,l}$ on the Riemann surface. The coefficient γ_{ijkl} is called a generalized crossed ratio and defined as

$$\gamma_{ijkl} = \frac{\theta(a + \int_k^i) \theta(a + \int_l^j)}{\theta(a + \int_l^i) \theta(a + \int_k^j)} \tag{A.5}$$

where, a is a non-singular zero of the theta function i.e. $\theta(a) = 0$ but $|\nabla\theta| \neq 0$. Under such conditions it is an important property of the cross ratio that it does not depend on the zero a chosen to defined it. In this paper we defined $\hat{\theta}$, see eq. (2.14), through the odd period \int_1^3 . Since $\hat{\theta}(0) = 0$ it implies that $\theta(\int_1^3) = 0$. Setting $a = \int_1^3$ in the cross ratio gives

$$\gamma_{ijkl} = \frac{\hat{\theta}(\int_k^i) \hat{\theta}(\int_l^j)}{\hat{\theta}(\int_l^i) \hat{\theta}(\int_k^j)} \tag{A.6}$$

The trisecant identity plays a central role in using the theta function to solve non-linear differential equations in integrable systems. For us, it is the starting point for deriving all the identities below. It is interesting to note that in string theory it has an interpretation as a bosonization identity [56].

A.3 Generalized cross ratio

As it was just stated, γ_{ijkl} in eq. (A.5) is independent of a as long as a is a zero of the theta function ($\theta(a) = 0$). In the case at hand, the world-sheet boundary curve $z(s)$ defines a set of zeros of $\hat{\theta}$, namely ζ_s , for any s is a zero of $\hat{\theta}$ (see eqs. (2.25), (2.26)). However, from the definition of $\hat{\theta}$ i.e. eq. (2.14) it is clear that $\hat{\theta}(\zeta_s) = 0 \iff \theta(\zeta_s + \int_1^3) = 0$, it follows that we can take

$$a = \zeta_s + \int_1^3 \quad (\text{A.7})$$

Now choose the points $p_i = p_4$, $p_k = p_3$, $p_l = p_1$, $p_j = \bar{4}$. Remember that $p_1 = 0$ and $p_3 = \infty$. Also p_4 and $p_{\bar{4}}$ are the same (arbitrary) point in the complex plane but on opposite sheets of the Riemann surface, implying $\int_1^4 = -\int_1^{\bar{4}}$ since $p_1 = 0$ is a branch cut. Up to a constant independent of ζ_s , the cross ratio can be written as

$$\tilde{\gamma} = \frac{\theta(\zeta_s + \int_1^4) \hat{\theta}(\zeta_s + \int_1^{\bar{4}})}{\theta(\zeta_s + \int_1^{\bar{4}}) \hat{\theta}(\zeta_s + \int_1^4)} \quad (\text{A.8})$$

which is therefore independent of ζ_s as long as it is a zero of $\hat{\theta}$. However, it is also true that $\hat{\theta}(0) = 0$. Replacing $\zeta_s \rightarrow 0$ and using that θ is symmetric whereas $\hat{\theta}$ is antisymmetric it immediately follows that $\tilde{\gamma} = -1$. Therefore the simple but very useful identity

$$\frac{\theta(\zeta_s + \int_1^4)}{\theta(\zeta_s - \int_1^4)} = -\frac{\hat{\theta}(\zeta_s + \int_1^4)}{\hat{\theta}(\zeta_s - \int_1^4)} \quad \text{if } \hat{\theta}(\zeta_s) = 0 \quad (\text{A.9})$$

is derived.

A.4 First derivatives

Identities for the derivatives of the theta functions can be obtained from the trisecant identity by deriving with respect to the position of the points p_i , p_j , p_k , p_l . Consider deriving the trisecant identity with respect of the position of p_i and afterwards taking the limit $p_j \rightarrow p_i$. The result reads

$$D_i \ln \frac{\theta(\zeta)}{\theta(\zeta + \int_k^l)} = D_i \ln \frac{\hat{\theta}(\int_l^i)}{\hat{\theta}(\int_k^i)} - \frac{\hat{\theta}(\int_l^k) D_i \hat{\theta}(0)}{\hat{\theta}(\int_i^k) \hat{\theta}(\int_l^i)} \frac{\theta(\zeta + \int_i^l) \theta(\zeta + \int_k^i)}{\theta(\zeta + \int_k^l) \theta(\zeta)} \quad (\text{A.10})$$

valid for any vector $\zeta \in \mathbb{C}^g$ and any three points p_i , p_k , p_l in \mathcal{M} . In our case, such points are usually chosen among $p_1 = 0$, $p_3 = \infty$, and p_4 , $p_{\bar{4}}$ arbitrary and in opposite sheets of \mathcal{M} . As particular cases it is useful to derive

$$\begin{aligned} D_3 \ln \frac{\theta(\zeta)}{\theta(\zeta + \int_1^4)} &= -D_3 \ln \theta \left(\int_1^4 \right) - \frac{\hat{\theta}(\int_1^4) D_3 \hat{\theta}(0)}{\theta(\int_1^4) \theta(0)} \frac{\hat{\theta}(\zeta) \hat{\theta}(\zeta + \int_1^4)}{\theta(\zeta) \theta(\zeta + \int_1^4)} \\ D_3 \ln \frac{\hat{\theta}(\zeta)}{\hat{\theta}(\zeta + \int_1^4)} &= -D_3 \ln \theta \left(\int_1^4 \right) + \frac{\hat{\theta}(\int_1^4) D_3 \hat{\theta}(0)}{\theta(\int_1^4) \theta(0)} \frac{\theta(\zeta) \theta(\zeta + \int_1^4)}{\hat{\theta}(\zeta) \hat{\theta}(\zeta + \int_1^4)} \end{aligned}$$

$$\begin{aligned}
 D_1 \ln \frac{\theta(\zeta)}{\hat{\theta}(\zeta + f_1^4)} &= -D_1 \ln \hat{\theta} \left(\int_1^4 \right) + \frac{\theta(f_1^4) D_1 \hat{\theta}(0)}{\hat{\theta}(f_1^4) \theta(0)} \frac{\hat{\theta}(\zeta) \theta(\zeta + f_1^4)}{\theta(\zeta) \hat{\theta}(\zeta + f_1^4)} \\
 D_1 \ln \frac{\hat{\theta}(\zeta)}{\theta(\zeta + f_1^4)} &= -D_1 \ln \hat{\theta} \left(\int_1^4 \right) + \frac{\theta(f_1^4) D_1 \hat{\theta}(0)}{\hat{\theta}(f_1^4) \theta(0)} \frac{\theta(\zeta) \hat{\theta}(\zeta + f_1^4)}{\hat{\theta}(\zeta) \theta(\zeta + f_1^4)}
 \end{aligned} \tag{A.11}$$

valid for any vector ζ . The function $\hat{\theta}$ is defined in eq. (2.14) and is odd, a fact used in writing eq. (A.11). In the solutions eq. (2.22), the vector ζ is a function of the world-sheet coordinates (z, \bar{z}) :

$$\zeta(z, \bar{z}) = 2\omega_1 \bar{z} + 2\omega_3 z \tag{A.12}$$

and therefore

$$\partial_z F(\zeta(z, \bar{z})) = 2D_1 F(\zeta), \quad \partial_{\bar{z}} F(\zeta(z, \bar{z})) = 2D_3 F(\zeta) \tag{A.13}$$

In that way, the identities (A.11) can be used as a table of derivatives of the theta functions and allow the verification of the equations of motion.

A.5 Second derivatives

Second derivatives can be computed in the same way as first ones. In this subsection we just derive a special case that is useful for our purposes. Consider eq. (A.10) and take the points $p_i = p_3$, $p_l = p_4$ and use the definition of $\hat{\theta}$, i.e. eq. (2.14). After reordering the terms, the result is

$$D_3 \ln \frac{\theta(\zeta)}{\hat{\theta}(\zeta + f_1^4 - f_3^k)} = D_3 \ln \frac{\hat{\theta}(f_3^k)}{\theta(f_1^4)} - \frac{D_3 \hat{\theta}(0) \hat{\theta}(\zeta + f_1^4)}{\theta(f_1^4) \theta(\zeta)} \frac{\theta(\zeta - f_3^k) \theta(f_1^4 - f_3^k)}{\hat{\theta}(f_3^k) \hat{\theta}(\zeta + f_1^4 - f_3^k)} \tag{A.14}$$

The point p_k is now taken to be very close to p_3 so that the expansion in eq. (2.16) can be used, namely

$$\int_3^k = \omega_3 \tilde{y} + \frac{1}{3} \tilde{y}^3 \omega_{32} + \dots \tag{A.15}$$

with $\tilde{y} = \frac{2}{\sqrt{\lambda}}$. Multiplying eq. (A.14) by $\hat{\theta}(f_3^k)$ and expanding in powers of \tilde{y} gives, at order \tilde{y}^2 ,

$$D_3^2 \ln \left[\theta(\zeta) \theta \left(\int_1^4 \right) \hat{\theta} \left(\zeta + \int_1^4 \right) \right] + \left[D_3 \ln \frac{\theta(\zeta) \theta(f_1^4)}{\hat{\theta}(\zeta + f_1^4)} \right]^2 = \frac{D_3^3 \hat{\theta}(0)}{D_3 \hat{\theta}(0)} \tag{A.16}$$

Similarly, taking $p_i = p_1$ and expanding from $p_k \rightarrow p_1$ it follows that

$$D_1^2 \ln \left[\theta(\zeta) \hat{\theta} \left(\int_1^4 \right) \theta \left(\zeta + \int_1^4 \right) \right] + \left[D_1 \ln \frac{\theta(\zeta) \hat{\theta}(f_1^4)}{\theta(\zeta + f_1^4)} \right]^2 = \frac{D_1^3 \hat{\theta}(0)}{D_1 \hat{\theta}(0)} \tag{A.17}$$

A.6 First derivatives at $\zeta = \zeta_s$

A special role in the construction is played by the world-sheet boundary curve $z(s)$ defined such that

$$\hat{\theta}(\zeta_s) = 0 \tag{A.18}$$

where

$$\zeta_s = \zeta(z(s), \bar{z}(s)) = 2\omega_1 \bar{z}(s) + 2\omega_3 z(s) \quad (\text{A.19})$$

It therefore becomes necessary to evaluate various derivatives at such values of $\zeta = \zeta_s$.

Consider the first and third equations in (A.11) and evaluate them at $\zeta = \zeta_s$. The result is

$$D_3 \ln \frac{\theta(\zeta_s)}{\theta(\zeta_s + f_1^4)} = -D_3 \ln \theta \left(\int_1^4 \right) \quad (\text{A.20})$$

$$D_1 \ln \frac{\theta(\zeta_s)}{\hat{\theta}(\zeta_s + f_1^4)} = -D_1 \ln \hat{\theta} \left(\int_1^4 \right) \quad (\text{A.21})$$

Furthermore, using that $\hat{\theta}$ is antisymmetric and θ symmetric it follows that

$$D_3 \ln \frac{\theta(\zeta_s + f_1^4)}{\theta(\zeta_s - f_1^4)} = 2D_3 \ln \theta \left(\int_1^4 \right) \quad (\text{A.22})$$

$$D_1 \ln \frac{\hat{\theta}(\zeta_s + f_1^4)}{\hat{\theta}(\zeta_s - f_1^4)} = 2D_1 \ln \hat{\theta} \left(\int_1^4 \right) \quad (\text{A.23})$$

Consider now the second equation in (A.11). Both sides have poles when $\hat{\theta}(\zeta) = 0$ but after multiplying it by $\hat{\theta}(\zeta)$, it reads

$$D_3 \hat{\theta}(\zeta) + \hat{\theta}(\zeta) D_3 \ln \frac{\theta(f_1^4)}{\hat{\theta}(\zeta + f_1^4)} = \frac{\hat{\theta}(f_1^4) D_3 \hat{\theta}(0)}{\theta(f_1^4) \theta(0)} \frac{\theta(\zeta) \theta(\zeta + f_1^4)}{\hat{\theta}(\zeta + f_1^4)} \quad (\text{A.24})$$

Evaluating at $\zeta = \zeta_s$ gives

$$D_3 \hat{\theta}(\zeta_s) = \frac{\hat{\theta}(f_1^4) D_3 \hat{\theta}(0)}{\theta(f_1^4) \theta(0)} \frac{\theta(\zeta_s) \theta(\zeta_s + f_1^4)}{\hat{\theta}(\zeta_s + f_1^4)} \quad (\text{A.25})$$

Similarly, from the last equation in (A.11) it follows that

$$D_1 \hat{\theta}(\zeta_s) = \frac{\theta(f_1^4) D_1 \hat{\theta}(0)}{\hat{\theta}(f_1^4) \theta(0)} \frac{\theta(\zeta_s) \hat{\theta}(\zeta_s + f_1^4)}{\theta(\zeta_s + f_1^4)} \quad (\text{A.26})$$

Another possibility to cancel the pole at $\hat{\theta}(\zeta) = 0$ is by subtracting the second equation in (A.11) from itself evaluated at p_4 :

$$D_3 \ln \frac{\hat{\theta}(\zeta + f_1^4)}{\hat{\theta}(\zeta - f_1^4)} = 2D_3 \ln \theta \left(\int_1^4 \right) - \frac{\hat{\theta}(f_1^4) D_3 \hat{\theta}(0)}{\theta(f_1^4) \theta(0)} \frac{\theta(\zeta)}{\hat{\theta}(\zeta)} \left[\frac{\theta(\zeta + f_1^4)}{\hat{\theta}(\zeta + f_1^4)} + \frac{\theta(\zeta - f_1^4)}{\hat{\theta}(\zeta - f_1^4)} \right] \quad (\text{A.27})$$

where again we used that $\hat{\theta}$ is odd: $\hat{\theta}(-f_1^4) = -\hat{\theta}(f_1^4)$. From the trisecant identity, taking the points $p_i = p_4, p_j = p_1, p_k = p_3, p_l = p_4$ and using that $f_1^{\bar{4}} = -f_1^4$ one obtains

$$\left[\frac{\theta(\zeta + f_1^4)}{\hat{\theta}(\zeta + f_1^4)} + \frac{\theta(\zeta - f_1^4)}{\hat{\theta}(\zeta - f_1^4)} \right] = \frac{\hat{\theta}(2f_1^4) \theta(0)}{\hat{\theta}(f_1^4) \theta(f_1^4)} \frac{\theta(\zeta) \hat{\theta}(\zeta)}{\hat{\theta}(\zeta + f_1^4) \hat{\theta}(\zeta - f_1^4)} \quad (\text{A.28})$$

Replacing in eq. (A.27), $\hat{\theta}(\zeta)$ indeed cancels resulting in

$$D_3 \ln \frac{\hat{\theta}(\zeta + f_1^4)}{\hat{\theta}(\zeta - f_1^4)} = 2D_3 \ln \theta \left(\int_1^4 \right) - \frac{\hat{\theta}(2f_1^4)D_3\hat{\theta}(0)}{\theta^2(f_1^4)} \frac{\theta^2(\zeta)}{\hat{\theta}(\zeta + f_1^4)\hat{\theta}(\zeta - f_1^4)} \quad (\text{A.29})$$

which is now safe to evaluate at $\zeta = \zeta_s$. Similarly, from the last equation in eq. (A.11) it follows that

$$D_1 \ln \frac{\theta(\zeta + f_1^4)}{\theta(\zeta - f_1^4)} = 2D_1 \ln \hat{\theta} \left(\int_1^4 \right) - \frac{\theta(f_1^4)D_1\hat{\theta}(0)}{\hat{\theta}(f_1^4)\theta(0)} \frac{\theta(\zeta)}{\hat{\theta}(\zeta)} \left[\frac{\hat{\theta}(\zeta + f_1^4)}{\theta(\zeta + f_1^4)} + \frac{\hat{\theta}(\zeta - f_1^4)}{\theta(\zeta - f_1^4)} \right] \quad (\text{A.30})$$

leading to

$$D_1 \ln \frac{\theta(\zeta + f_1^4)}{\theta(\zeta - f_1^4)} = 2D_1 \ln \hat{\theta} \left(\int_1^4 \right) - \frac{\hat{\theta}(2f_1^4)D_1\hat{\theta}(0)}{\theta^2(f_1^4)} \frac{\theta^2(\zeta)}{\theta(\zeta + f_1^4)\theta(\zeta - f_1^4)} \quad (\text{A.31})$$

A.7 Higher derivatives at $\zeta = \zeta_s$

Consider eq. (A.20).and take $p_4 \rightarrow p_3$. Both sides develop a pole so we should carefully expand using eq. (2.16)

$$\omega(p_4) = \omega_3 + \tilde{y}^2 \omega_{32} + \dots \quad (\text{A.32})$$

$$\int_1^4 \omega = \int_1^3 \omega + \omega_3 \tilde{y} + \frac{1}{3} \tilde{y}^3 \omega_{32} + \dots \quad (\text{A.33})$$

where $\tilde{y} = \frac{2}{\sqrt{\lambda}}$. It follows that

$$\frac{D_3 \hat{\theta}(\omega_3 \tilde{y} + \frac{1}{3} \tilde{y}^3 \omega_{32} + \dots)}{\hat{\theta}(\omega_3 \tilde{y} + \frac{1}{3} \tilde{y}^3 \omega_{32} + \dots)} = \frac{D_3 \hat{\theta}(\zeta_s + \omega_3 \tilde{y} + \frac{1}{3} \tilde{y}^3 \omega_{32} + \dots)}{\hat{\theta}(\zeta_s + \omega_3 \tilde{y} + \frac{1}{3} \tilde{y}^3 \omega_{32} + \dots)} - D_3 \ln \theta(\zeta_s) \quad (\text{A.34})$$

Expanding both sides in \tilde{y} two useful identities are derived:

$$\frac{D_3^2 \hat{\theta}(\zeta_s)}{D_3 \hat{\theta}(\zeta_s)} = 2D_3 \ln \theta(\zeta_s) \quad (\text{A.35})$$

$$\frac{D_3^3 \hat{\theta}(\zeta_s)}{D_3 \hat{\theta}(\zeta_s)} = \frac{D_3^3 \hat{\theta}(0)}{D_3 \hat{\theta}(0)} - \frac{D_3'' \hat{\theta}(0)}{D_3 \hat{\theta}(0)} + \frac{D_3'' \hat{\theta}(\zeta_s)}{D_3 \hat{\theta}(\zeta_s)} + 3 (D_3 \ln \theta(\zeta_s))^2 \quad (\text{A.36})$$

where D_3'' denotes the derivative along the direction ω_{32} and we also took into account that $\hat{\theta}$ is an odd function and therefore its even derivatives vanish at 0. Similarly, starting from eq. (A.21) one finds

$$\frac{D_1^2 \hat{\theta}(\zeta_s)}{D_1 \hat{\theta}(\zeta_s)} = 2D_1 \ln \theta(\zeta_s) \quad (\text{A.37})$$

$$\frac{D_1^3 \hat{\theta}(\zeta_s)}{D_1 \hat{\theta}(\zeta_s)} = \frac{D_1^3 \hat{\theta}(0)}{D_1 \hat{\theta}(0)} - \frac{D_1'' \hat{\theta}(0)}{D_1 \hat{\theta}(0)} + \frac{D_1'' \hat{\theta}(\zeta_s)}{D_1 \hat{\theta}(\zeta_s)} + 3 (D_1 \ln \theta(\zeta_s))^2 \quad (\text{A.38})$$

Finally, taking two derivatives in direction ω_3 i.e. applying D_3^2 to eq. (A.24) and evaluating at $\zeta = \zeta_s$ follows that

$$D_3^2 \ln \left[\theta(\zeta_s) \theta \left(\zeta_s + \int_1^4 \right) \hat{\theta} \left(\zeta_s + \int_1^4 \right) \right] + \left(D_3 \ln \frac{\theta(\zeta_s + \int_1^4)}{\hat{\theta}(\zeta_s + \int_1^4)} \right)^2 = \frac{D_3^3 \hat{\theta}(\zeta_s)}{D_3 \hat{\theta}(\zeta_s)} - 3 (D_3 \ln \theta(\zeta_s))^2 \quad (\text{A.39})$$

where eqs. (A.20), (A.25) and eq. (A.29) were used to simplify the result. Similarly

$$D_1^2 \ln \left[\theta(\zeta_s) \theta \left(\zeta_s + \int_1^4 \right) \hat{\theta} \left(\zeta_s + \int_1^4 \right) \right] + \left(D_1 \ln \frac{\theta(\zeta_s + \int_1^4)}{\hat{\theta}(\zeta_s + \int_1^4)} \right)^2 = \frac{D_1^3 \hat{\theta}(\zeta_s)}{D_1 \hat{\theta}(\zeta_s)} - 3 (D_1 \ln \theta(\zeta_s))^2 \quad (\text{A.40})$$

is obtained. Instead for this derivation one can simply replace $\zeta = \zeta_s$ in eqs. (A.16), (A.17). The result is not exactly the same, they only agree if

$$\frac{D_3^3 \hat{\theta}(\zeta_s)}{D_3 \hat{\theta}(\zeta_s)} - \frac{D_3^2 \theta(\zeta_s)}{\theta(\zeta_s)} - 2 (D_3 \ln \theta(\zeta_s))^2 = \frac{D_3^3 \hat{\theta}(0)}{D_3 \hat{\theta}(0)} - \frac{D_3^2 \theta(0)}{\theta(0)} \quad (\text{A.41})$$

which is another useful identity.

A.8 Other identities

In [14] the following result was obtained

$$\left(\frac{\theta(0) \hat{\theta}(\int_1^4)}{D_1 \hat{\theta}(0) \theta(\int_1^4)} \right)^2 = -4\lambda_{p_4} \quad (\text{A.42})$$

$$\left(\frac{D_3 \hat{\theta}(0) \hat{\theta}(\int_1^4)}{\theta(0) \theta(\int_1^4)} \right)^2 = -\frac{1}{4} \lambda_{p_4} \quad (\text{A.43})$$

Consider now the first equation in eq. (A.11), apply D_3 and evaluate at $\zeta = \zeta_s$:

$$D_3^2 \ln \frac{\theta(\zeta_s)}{\theta(\zeta_s + \int_1^4)} = - \left(\frac{\hat{\theta}(\int_1^4) D_3 \hat{\theta}(0)}{\theta(\int_1^4) \theta(0)} \right)^2 = \frac{1}{4} \lambda_{p_4} \quad (\text{A.44})$$

where eq. (A.25) and eq. (A.43) were used to simplify the result. Similarly, from the third equation in (A.11) and using eqs. (A.26), (A.42) it follows that

$$D_1^2 \ln \frac{\theta(\zeta_s)}{\hat{\theta}(\zeta_s + \int_1^4)} = \left(\frac{\theta(\int_1^4) D_1 \hat{\theta}(0)}{\hat{\theta}(\int_1^4) \theta(0)} \right)^2 = -\frac{1}{4\lambda_{p_4}} \quad (\text{A.45})$$

These identities are valid whenever ζ_s is a zero of $\hat{\theta}$. In particular we can set $\zeta_s = 0$ obtaining

$$D_1^2 \ln \hat{\theta} \left(\int_1^4 \right) = \frac{1}{4\lambda} + \frac{D_1^2 \theta(0)}{\theta(0)} \quad (\text{A.46})$$

$$D_3^2 \ln \theta \left(\int_1^4 \right) = -\frac{\lambda}{4} + \frac{D_3^2 \theta(0)}{\theta(0)} \quad (\text{A.47})$$

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