The wave front set of oscillatory integrals with inhomogeneous phase function

Jochen Zahn

© The Author(s) 2011. This article is published with open access at Springerlink.com

Abstract   A generalized notion of oscillatory integrals that allows for inhomogeneous phase functions of arbitrary positive order is introduced. The wave front set of the resulting distributions is characterized in a way that generalizes the well-known result for phase functions that are homogeneous of order one.

1 Introduction

Oscillatory integrals play an important role in the theory of pseudodifferential operators. They are also a useful tool in Mathematical Physics, in particular in quantum field theory, where they are used to give meaning to formal Fourier integrals in the sense of distributions. For phase functions which are homogeneous of order one, this also leads to a characterization of the wave front set of the resulting distribution, as it is known to be contained in the manifold of stationary phase of the phase function. In these applications, the restriction to phase functions that are homogeneous of order one is often obstructive. In many cases, this restriction can be overcome by shifting a part of the would-be phase function to the symbol, cf. Example 1 below. However, such a shift is not always possible, for instance if the would-be phase function contains terms of order greater than one. Such phase functions are present in the twisted

---

J. Zahn (✉)
Courant Research Centre "Higher Order Structures", University of Göttingen,
Bunsenstraße 3–5, 37073 Göttingen, Germany
e-mail: jzahn@uni-math.gwdg.de

Ⓑ Birkhäuser

convolutions that occur in quantum field theory on Moyal space, cf. Examples 2 and 3 below. Up to now, a rigorous definition of these twisted convolution integrals could be given only in special cases and in such a way that the information on the wave front set is lost. Thus, it is highly desirable to generalize the notion of oscillatory integrals to encompass also phase functions that are not homogeneous of order one. Such generalizations were proposed by several authors. However, to the best of our knowledge, the wave front sets of the resulting distributions were not considered, except for one very special case. We comment on these settings below, cf. Remark 3.

It is shown here that the restriction to phase functions that are homogeneous of order one can indeed be weakened, without losing information on the wave front set. The generalization introduced here not only allows for inhomogeneous phase functions, but also for phase functions that are symbols of any positive order. However, one has to impose a condition that generalizes the usual nondegeneracy requirement. It is also shown that the wave front sets of the distributions thus obtained are contained in a set that generalizes the notion of the manifold of stationary phase. We conclude with a discussion of some applications.

Throughout, we use the following notation: For an open set Ω, $K \Subset \Omega$ means that K is a compact subset of Ω. $\dot{\mathbb{R}}^n$ stands for $\mathbb{R}^n \setminus \{0\}$. For a subset $M \subset \Omega \times \dot{\mathbb{R}}^m$, $\Pi_1 M$ stands for the projection on the first component. $C^k(\Omega)$ denotes the k times continuously differentiable functions supported on Ω and $C_0^k(\Omega)$ the set of elements of $C^k(\Omega)$ with compact support in Ω. The dual space of $C_0^k(\Omega)$ is denoted by $\mathcal{D}'^{(k)}(\Omega)$ and $\mathcal{D}'(\Omega) = \mathcal{D}'^{(\infty)}(\Omega)$. The pairing of a distribution F and a test function f is denoted by $\langle F, f \rangle$. The dot $\cdot$ stands for the scalar product on $\mathbb{R}^n$. $\sphericalangle(x, y)$ denotes the angle between two vectors $x, y \in \mathbb{R}^n$.

2 Generalized oscillatory integrals

As usual, cf. [1], we define a symbol as follows:

Definition 1 Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $a : \Omega \times \mathbb{R}^s \to \mathbb{C}$ is called a *symbol of order m* if for each $K \Subset \Omega$ and multiindices α, β, we have

$$\|a\|_{\alpha\beta K} = \sup_{x \in K, \theta} \left| \left(D_x^\alpha D_\theta^\beta a \right)(x, \theta) \right| (1 + |\theta|)^{|\beta| - m} < \infty.$$

The set of all such functions, equipped with these seminorms will be denoted by $S^m(\Omega, s)$. Furthermore, we denote $S^{-\infty}(\Omega, s) = \cap_m S^m(\Omega, s)$ and $S^\infty(\Omega, s) = \cup_m S^m(\Omega, s)$.

Remark 1 For simplicity, we restrict ourselves to these symbols. The generalization to the symbols $S_{\rho\delta}^m$ is straightforward. One then has to restrict to $1 - \mu < \rho \leq 1$, $0 \leq \delta < \mu$, where μ is the order of the phase function introduced below. Also the generalization to asymptotic symbols as discussed in [2] is straightforward.

The following proposition is a straightforward consequence of the definition of $S^m(\Omega, s)$:

Proposition 1 *The maps* $D_x^\alpha : S^m(\Omega, s) \to S^m(\Omega, s)$, $D_\theta^\beta : S^m(\Omega, s) \to S^{m-|\beta|}(\Omega, s)$ *and the multiplication* $S^m(\Omega, s) \times S^{m'}(\Omega, s) \to S^{m+m'}(\Omega, s)$ *are continuous.*

The following proposition is proven in [1, Prop. 1.7]:

Proposition 2 *If* $m' > m$, *then* $S^{-\infty}(\Omega, s)$ *is dense in* $S^m(\Omega, s)$ *for the topology of* $S^{m'}(\Omega, s)$.

Now we introduce our new definition of a phase function.

Definition 2 A *phase function* of order μ on $\Omega \times \mathbb{R}^s$ is a function $\phi : \Omega \times \mathbb{R}^s \to \mathbb{R}$ such that

1. ϕ is a symbol of order $\mu > 0$.
2. For each $K \Subset \Omega$ there are positive C, D such that

$$\eta(x, \theta) := |\nabla_x \phi|^2 + |\theta|^2 |\nabla_\theta \phi|^2 \geq C |\theta|^{2\mu} \; \forall |\theta| \geq D, \quad x \in K.$$

Remark 2 Condition 2 generalizes the usual nondegeneracy requirement and ensures that ϕ oscillates rapidly enough for large θ. In particular it means that ϕ is not a symbol of order less than μ. It also means that one can choose $\chi \in C^\infty(\Omega \times \mathbb{R}^s)$ such that $\eta^{-1}(1 - \chi)$ is well-defined and a symbol of order -2μ. Here χ can be chosen such that $\chi(x, \cdot)$ is compact for each x.

Remark 3 Our definition of a phase function is a generalization of a definition introduced by Hörmander [3, Def. 2.3] in the context of pseudodifferential operators. He considered phase functions of order 1 (in the nomenclature introduced above) and characterized the singular support of the resulting distribution, but not its wave front set. Our characterization of the singular support (cf. Corollary 1) coincides with the one given by Hörmander [3, Thm. 2.6].

Inhomogeneous phase functions were also considered by Asada and Fujiwara [4] in the context of pseudodifferential operators on $L^2(\mathbb{R}^n)$. In their setting, $\Omega = \mathbb{R}^{2n}$, $s = n$ and there must be a positive constant C such that

$$|\det \begin{pmatrix} \partial_x \partial_y \phi & \partial_x \partial_\theta \phi \\ \partial_\theta \partial_y \phi & \partial_\theta \partial_\theta \phi \end{pmatrix}| \geq C. \tag{1}$$

Furthermore, all the entries of this matrix (and their derivatives) are required to be bounded. Thus, the phase function is asymptotically at least of order 1 and at most of order 2. The admissible amplitudes are at most of order 0. The wave front set of such operators on $L^2(\mathbb{R}^n)$ is not considered by Asada and Fujiwara. The same applies to the works of Boulkhemair [5] and Ruzhansky and Sugimoto [6], who work in a similar context.

Coriasco [7] considered a special case of Hörmander's framework, where again $\Omega = \mathbb{R}^{2n}$, $s = n$ and $\phi(x, \theta, y) = \varphi(x, \theta) - y \cdot \theta$ with $\varphi \in \mathbf{SG}^{(1,1)}$, a subset of the symbols of order 1. Furthermore, he imposed growth conditions on φ that are more restrictive than Condition 2. The resulting operators on $\mathcal{S}(\mathbb{R}^n)$ can then be extended to operators on $\mathcal{S}'(\mathbb{R}^n)$. If a further condition analogous to (1) is imposed, then also

the wave front set, which is there defined via **SG**-microregularity, can be characterized (at least implicitly, by the change of the wave front set under the action of the operator).

Proposition 3 *If ϕ is a phase function of order μ and $a \in S^m(\Omega, s)$ and there is a $p \in \mathbb{N}_0$ such that $m + p\mu < -s$, then*

$$D_\phi(a)(x) = \int a(x, \theta) e^{i\phi(x,\theta)} \mathrm{d}^s\theta \in C^p(\Omega) \tag{2}$$

and the map $D_\phi : S^m(\Omega, s) \to C^p(\Omega)$ is continuous.

Proof For $m < -s$ we have

$$\sup_{x \in K} \left| D_\phi(a)(x) \right| \leq \|a\|_{00K} \int (1 + |\theta|)^m \, \mathrm{d}^s\theta \leq C \|a\|_{00K},$$

so that $S^m(\Omega, s) \to C^0(\Omega)$ is continuous. Differentiation gives

$$\nabla_x D_\phi(a)(x) = \int \{i\nabla_x\phi(x, \theta)a(x, \theta) + \nabla_x a(x, \theta)\} \, e^{i\phi(x,\theta)} \mathrm{d}^s\theta.$$

The expression in curly brackets is a symbol of order $m + \mu$. With the same argument as before one can thus differentiate p times. □

We formulate the main theorem of this section analogously to [1, Thm. 1.11]. The proof is a straightforward generalization of the proof given there.

Theorem 1 *Let $\phi(x, \theta)$ be a phase function of order μ on $\Omega \times \mathbb{R}^s$. Then there is a unique way of defining $D_\phi(a) \in \mathcal{D}'(\Omega)$ for $a \in S^\infty(\Omega, s)$ such that $D_\phi(a)$ coincides with (2) when $a \in S^m(\Omega, s)$ for some $m < -s$ and such that, for all m, the map $D_\phi : S^m(\Omega, s) \to \mathcal{D}'(\Omega)$ is continuous. Moreover, if $p \in \mathbb{N}_0$ and $m - p\mu < -s$, then the map $D_\phi : S^m(\Omega, s) \to \mathcal{D}'^{(p)}(\Omega)$ is continuous.*

To prove this, we need the following Lemma:

Lemma 1 *Let ϕ be a phase function of order μ on $\Omega \times \mathbb{R}^s$. Then there exist $a_i \in S^{-\mu+1}(\Omega, s), b_j \in S^{-\mu}(\Omega, s)$ and $c \in S^{-\mu}(\Omega, s)$ such that for the differential operator*

$$Vf = a \cdot \nabla_\theta f + b \cdot \nabla_x f + cf$$

with adjoint

$$V^t f = -\nabla_\theta \cdot (af) - \nabla_x \cdot (bf) + cf$$

we have

$$V^t e^{i\phi(x,\theta)} = e^{i\phi(x,\theta)}. \tag{3}$$

Furthermore, V is a continuous map from $S^m(\Omega, s)$ to $S^{m-\mu}(\Omega, s)$.

Proof We choose η as in Definition 2 and χ as in Remark 2. We set

$$a_i = i\eta^{-1}(1 - \chi)|\theta|^2 \partial_{\theta_i}\phi,$$
$$b_j = i\eta^{-1}(1 - \chi)\partial_{x_j}\phi,$$
$$c = \nabla_\theta \cdot a + \nabla_x \cdot b + \chi.$$

Then we have

$$V^t e^{i\phi} = \left(-\nabla_\theta \cdot a - ia \cdot \nabla_\theta\phi - \nabla_x \cdot b - ib \cdot \nabla_x\phi + \nabla_\theta \cdot a + \nabla_x \cdot b_j + \chi\right) e^{i\phi}$$
$$= \left(\eta^{-1}(1 - \chi)\left(|\theta|^2 |\nabla_\theta\phi|^2 + |\nabla_x\phi|^2\right) + \chi\right) e^{i\phi}$$
$$= e^{i\phi}.$$

It is easy to see that a_i, b_j and c are symbols in the required way. The last statement follows from Proposition 1. □

Proof (*Theorem* 1) The uniqueness is a consequence of Proposition 2. For $a \in S^{-\infty}(\Omega, s)$ and $f \in C_0^\infty(\Omega)$, we have, with V as in Lemma 1,

$$\langle D_\phi(a), f \rangle = \int e^{i\phi(x,\theta)} a(x, \theta) f(x) \mathrm{d}^n x \mathrm{d}^s \theta$$
$$= \int (V^t)^p e^{i\phi(x,\theta)} a(x, \theta) f(x) \mathrm{d}^n x \mathrm{d}^s \theta$$
$$= \int e^{i\phi(x,\theta)} V^p[a(x, \theta) f(x)] \mathrm{d}^n x \mathrm{d}^s \theta,$$

for any $p \in \mathbb{N}_0$ and thus

$$\left|D_\phi(a)(f)\right| \le \int \left|V^p[a(x, \theta) f(x)]\right| \mathrm{d}^n x \mathrm{d}^s \theta.$$

Now the multiplication $S^m(\Omega, s) \times C_0^\infty(\Omega) \to S^m(\Omega, s)$ is continuous. Thus, if $a \in S^m(\Omega, s)$, then $V^p[af]$ is a symbol of order $m - p\mu$ and in particular we have, for each $K \Subset \Omega$,

$$\sup_{K \times \mathbb{R}^s} \left|V^p[a(x, \theta) f(x)]\right| (1 + |\theta|)^{p\mu - m} \le f_{p,K}(a) \sum_{|\alpha| \le p} \sup_{x \in K} \left|D^\alpha f\right|, \tag{4}$$

where $f_{p,K}$ is a seminorm on $S^m(\Omega, s)$. For $a \in S^m(\Omega, s)$, we may thus choose p such that $m - p\mu < -s$ and define

$$\langle D_\phi^p(a), f \rangle = \int e^{i\phi(x,\theta)} V^p[a(x, \theta) f(x)] \mathrm{d}^n x \mathrm{d}^s \theta.$$

As the sum on the r.h.s. of (4) is a seminorm on $\mathcal{D}'^{(p)}(\Omega)$, and due to the continuity properties discussed above, the map $D_\phi^p : S^m(\Omega, s) \to \mathcal{D}'^{(p)}(\Omega)$ is continuous. For $q > p$, we have $\langle D_\phi^q(a), f \rangle = \langle D_\phi^p(a), f \rangle$ by (3). Thus, we can unambiguously define $D_\phi(a) = D_\phi^p(a)$. $\square$

3 The wave front set

We may now further characterize the distributions that result from a generalized oscillatory integral.

Definition 3 Let ϕ be a phase function of order μ on $\Omega \times \mathbb{R}^s$. We define

$$M(\phi) = \left\{ (x, \theta) \in \Omega \times \dot{\mathbb{R}}^s \, | \, \nexists C, D > 0 \text{ s.t. } |\nabla_\theta \phi(x, \lambda\theta)| \geq C\lambda^{\mu-1} \, \forall \lambda > D \right\},$$

$$SP(\phi) = \left\{ (x, k) \in \Omega \times \dot{\mathbb{R}}^n \, | \, \exists (x, \theta) \in M(\phi) \text{ s.t. } \nexists \alpha, D > 0 \text{ s.t.} \right.$$
$$\left. \sphericalangle(\nabla_x \phi(x, \lambda\theta), k) \geq \alpha \, \forall \lambda > D \right\}.$$

We call $SP(\phi)$ the *asymptotic manifold of stationary phase*. By definition, it is conic.

Lemma 2 $M(\phi)$ *is a closed conic subset of* $\Omega \times \dot{\mathbb{R}}^s$. $SP(\phi)$ *is a closed subset of* $\Omega \times \dot{\mathbb{R}}^n$.

Proof From the definition of $M(\phi)$ it follows that if $(x, \theta) \in M(\phi)$, then $(x, \lambda\theta) \in M(\phi)$ for all $\lambda > 0$, so $M(\phi)$ is conic. We now show that $(\Omega \times \dot{\mathbb{R}}^s) \setminus M(\phi)$ is open in $\Omega \times \dot{\mathbb{R}}^s$. Let (x_0, θ_0) be such that there are positive C, D such that

$$|\nabla_\theta \phi(x_0, \lambda\theta_0)| \geq C\lambda^{\mu-1} \quad \forall \lambda > D.$$

We set $F = \nabla_\theta \phi$. By Taylor's theorem we have

$$F(x, \lambda\theta) = F(x_0, \lambda\theta_0) + (x - x_0) \cdot G(x, \lambda\theta) + \lambda(\theta - \theta_0) \cdot H(x, \lambda\theta), \qquad (5)$$

where G and H fulfill the bounds

$$|G(x, \lambda\theta)| \leq \sup_{y \in B_\epsilon(x_0), \tau \in B_\epsilon(\theta_0)} |\nabla_x F(y, \lambda\tau)|,$$
$$|H(x, \lambda\theta)| \leq \sup_{y \in B_\epsilon(x_0), \tau \in B_\epsilon(\theta_0)} |\nabla_\theta F(y, \lambda\tau)|.$$

Here (x, θ) are chosen such that $|x - x_0| \leq \epsilon$, $|\theta - \theta_0| \leq \epsilon$. Furthermore, we restrict to $\epsilon < |\theta_0|$. Then we may use that F is a symbol of order $\mu - 1$ to conclude that there are positive constants C_ϵ, D_ϵ, which are bounded for $\epsilon \to 0$, for which

$$|G(x, \lambda\theta)| \leq C_\epsilon \lambda^{\mu-1} \, \forall \lambda > D_\epsilon,$$
$$|H(x, \lambda\theta)| \leq C_\epsilon \lambda^{\mu-2} \, \forall \lambda > D_\epsilon.$$

holds. As the zeroth order term in the Taylor expansion (5) grows faster than $C\lambda^{\mu-1}$ for a fixed positive constant C, we can make ϵ so small that $|F(x, \lambda\theta)|$ grows faster than $C'\lambda^{\mu-1}$ for some positive C'. Thus, $M(\phi)$ is closed in $\Omega \times \dot{\mathbb{R}}^s$.

In order to prove the closedness of $SP(\phi)$, we first note that if $x_0 \notin \Pi_1 M(\phi)$, then by the above there is a neighborhood of x_0 that does not intersect $\Pi_1 M(\phi)$. Thus, it suffices to show that for $x_0 \in M(\phi)$, $(x_0, k_0) \in (\Omega \times \dot{\mathbb{R}}^n) \setminus SP(\phi)$ there is a neighborhood that does not intersect $SP(\phi)$. By Condition 2 of Definition 2, there must be positive constants C, D such that

$$|\nabla_x \phi(x_0, \theta)| \geq C |\theta|^\mu \ \forall |\theta| > D, (x_0, \theta) \in M(\phi).$$

By the same argument as above, such a bound holds true in a conic neighborhood Γ of $M(\phi) \cap \{x_0\} \times \dot{\mathbb{R}}^s$. By the definition of $SP(\phi)$, there are posititve α, D such that

$$\sphericalangle(\nabla_x \phi(x_0, \lambda\theta_0), k_0) \geq \alpha \ \forall \lambda > D. \tag{6}$$

We now want to show that one can choose a conic neighborhood Γ_0 of (x_0, θ_0), contained in Γ, such that an analogous bound holds, i.e., there are positive α, D so that

$$\sphericalangle(\nabla_x \phi(x, \theta), k_0) \geq \alpha \ \forall |\theta| > D, (x, \theta) \in \Gamma_0. \tag{7}$$

By the above construction, $|\nabla_x \phi(x, \lambda\theta)|$ grows as λ^μ in Γ. The deviations that occur by varying x and θ also scale as λ^μ, as ϕ is a symbol of order μ. Recalling

$$\cos \sphericalangle(\nabla_x \phi(x, \theta), k_0) = \frac{\nabla_x \phi(x, \theta) \cdot k_0}{|\nabla_x \phi(x, \theta)| \, |k_0|}$$

and again using Taylor's theorem, one shows that by making Γ_0 small enough, one still retains an inequality of the form (6). By suitably restricting Γ_0 in Ω, we can ensure that for $x \in \Pi_1\Gamma_0 \cap \Pi_1 M(\phi)$ we have $(x, \theta) \in \Gamma_0$ whenever $(x, \theta) \in M(\phi)$. Then no new direction θ for which we would have to check the bound (7) can appear while varying x in $\Pi_1\Gamma_0$. Given (7), it is clear that we can also take a conic neighborhood Γ_1 of k_0, by tilting it by angles less than α. Choosing $U = \Pi_1\Gamma_0 \times \Gamma_1$ gives a neighborhood of (x_0, k_0) that does not intersect $SP(\phi)$. □

Proposition 4 *If the support of the symbol a does not intersect $M(\phi)$, then $D_\phi(a)$ is smooth.*

Proof We choose a neighborhood W of supp a whose closure does not intersect $M(\phi)$. We choose a smooth function χ that is equal to one in a neighborhood of $M(\phi)$ and vanishes on $\overline{W}$. We choose another smooth function ψ on $\Omega \times \mathbb{R}^s$ with support in W which is identical to one whenever $|\nabla_\theta \phi(x, \theta)| = 0$ for $(x, \theta) \in$ supp a. By definition of $M(\phi)$, the set

$$N_x = \left\{\theta \in \mathbb{R}^s | (x, \theta) \in \text{supp } a, |\nabla_\theta \phi(x, \theta)| = 0\right\}$$

is bounded, so we can choose ψ such that $\psi(x, \cdot)$ is compact for each x. Then we define

$$
\begin{aligned}
\eta &= |\nabla_\theta \phi|^2 \,, \\
a_i &= i\eta^{-1}(1 - \chi)(1 - \psi)\partial_{\theta_i}\phi, \\
c &= \nabla_\theta \cdot a + \psi, \\
V &= a \cdot \nabla_\theta + c.
\end{aligned}
$$

By the definition of $M(\phi)$ and χ, we have $a_i \in S^{-\mu+1}(\Omega, s)$ and $c \in S^{-\mu}(\Omega, s)$. With these definitions, we have

$$
V^t e^{i\phi} = \left(-\nabla_\theta \cdot a - ia \cdot \nabla_\theta \phi + \nabla_\theta \cdot a + \psi\right) e^{i\phi} = (1 - \chi)e^{i\phi}.
$$

Here we used that χ and ψ have nonoverlapping supports. As $\chi a = 0$ and V differentiates only w.r.t. θ, we thus have $D_\phi(a) = D_\phi(V^p a)$ for arbitrary integer p. As V maps symbols of order m to symbols of order $m - \mu$, $D_\phi(a)$ is smooth by Proposition 3. $\square$

Corollary 1 *The singular support of $D_\phi(a)$ is contained in $\Pi_1 M(\phi)$.*

Theorem 2 *The wave front set of $D_\phi(a)$ is contained in $SP(\phi)$.*

Lemma 3 *Let $(x_0, \theta_0) \in M(\phi)$, $(x_0, k_0) \in (\Omega \times \dot{\mathbb{R}}^n) \setminus SP(\phi)$. Then there is a conic neighborhood Γ of (x_0, θ_0), a conic neighborhood V of k_0 and positive constants C, D, α such that*

$$
|\nabla_x \phi(x, \theta)| \geq C |\theta|^\mu \ \forall |\theta| > D, (x, \theta) \in \Gamma, \tag{8}
$$

$$
\sphericalangle(\nabla_x \phi(x, \theta), k) \geq \alpha \ \forall |\theta| > D, (x, \theta) \in \Gamma, k \in V. \tag{9}
$$

Furthermore, there is a positive C' such that

$$
|\nabla_x \phi(x, \theta) - k| \geq C'(|\theta|^\mu + |k|) \ \forall |\theta| > D, (x, \theta) \in \Gamma, k \in V.
$$

Proof Condition (8) is fulfilled for (x_0, θ_0) by Condition 2 of Definition 2. That it is also fulfilled in a neighborhood of (x_0, θ_0) can be shown analogously to the proof of the closedness of $M(\phi)$ in Lemma 2. Condition (9) is fulfilled for (x_0, θ_0, k_0). That it is also fulfilled in a neighborhood of (x_0, θ_0, k_0) can again be shown as in Lemma 2. In order to prove the last statement, we note that by (9) we have
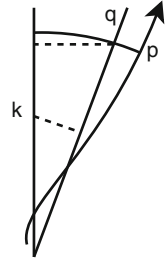
$$
|\nabla_x \phi(x, \theta) - k| \geq |q(\theta) - k| \ \forall |\theta| > D,
$$

where $q(\theta) \in \mathbb{R}^n$ has length $|\nabla_x \phi(x, \theta)|$ and lies on the cone with angle α around k (see Fig. 1, where $\nabla_x \phi(x, \theta)$ is denoted by p). For the distance of k and $q(\theta)$ we have the bound (see the dashed lines in Fig. 1)

$$
|q(\theta) - k| \geq \frac{1}{2} \sin \alpha \left(|q(\theta)| + |k|\right).
$$

Using (8), we then obtain the above statement. $\square$

Fig. 1 Sketch indicating the construction of $q(\theta)$ and the bound of its distance to k.

Proof (*Theorem* 2) By Corollary 1, it suffices to consider points $x_0 \in \Pi_1 M(\phi)$. Let $(x_0, k_0) \in (\Omega \times \dot{\mathbb{R}}^n) \setminus SP(\phi)$. Due to Proposition 4, we may assume that a is supported in an arbitrarily small closed conic neighborhood Γ of $M(\phi)$. We thus need to show that there is a $\psi \in C_0^\infty(\Omega)$, identically one near x_0 and a conic neighborhood V of k_0 such that for each N there is a seminorm $C_{m,N}$ on $S^m(\Omega, s)$ such that

$$\left| \widehat{\psi D_\phi(a)}(k) \right| \leq C_{m,N}(a)(1 + |k|)^{-N} \ \forall N, k \in V, \tag{10}$$

for all a supported in Γ. As in the proof of Theorem 1, it suffices to construct such a bound for $a \in S^{-\infty}(\Omega, s)$ and then make use of Proposition 2.

Let Γ, V, D be as in Lemma 3. Choose $\zeta \in C_0^\infty(\Omega)$ that is identically one near x_0 and whose support is contained in $\Pi_1\Gamma$. Choose a $\chi \in C_0^\infty(\mathbb{R}^s)$ that is identical to one on $|\theta| \leq D$. Now we set

$$\begin{aligned}
\eta &= |\nabla_x \phi - k|^2, \\
b_j &= i\eta^{-1}\zeta(1 - \chi)(\partial_{x_j}\phi - k_j), \\
c &= \nabla_x \cdot b, \\
V_k &= b \cdot \nabla_x + c.
\end{aligned}$$

Then for $k \in V$ we have

$$\begin{aligned}
V_k^t e^{i(\phi(x,\theta)-ik\cdot x)} &= (-\nabla_x \cdot b - ib \cdot (\nabla_x\phi - k) + \nabla_x \cdot b) \, e^{i(\phi(x,\theta)-ik\cdot x)} \\
&= \zeta(1 - \chi)e^{i(\phi(x,\theta)-ik\cdot x)}.
\end{aligned}$$

Now we choose $\psi \in C_0^\infty(\Omega)$, identical to one near x_0 and with support in $\zeta^{-1}(1)$. We also choose $\xi \in C_0^\infty(\mathbb{R}^s)$ which is identically one on supp χ. We consider $a = (1 - \xi)a + \xi a$. Then by Proposition 3, the second term yields a smooth function, so that the above bound is fulfilled. It remains to consider

$$\left|\psi\,\widehat{D_\phi((1-\xi)a)}(k)\right| = \left|\iint e^{i\phi(x,\theta)-ik\cdot x}\psi(x)(1-\xi(\theta))a(x,\theta)\mathrm{d}^n x\mathrm{d}^s\theta\right|$$

$$= \left|\iint e^{i\phi(x,\theta)-ik\cdot x}V_k^p[\psi(x)(1-\xi(\theta))a(x,\theta)]\mathrm{d}^n x\mathrm{d}^s\theta\right|$$

$$\leq \int \left|V_k^p[\psi(x)(1-\xi(\theta))a(x,\theta)]\right|\mathrm{d}^n x\mathrm{d}^s\theta.$$

Here we used that $\zeta(1-\chi)$ is identically one on the support of $\psi(1-\xi)$. By Lemma 3, one now has

$$\left|V_k^p[\psi(x)(1-\xi(\theta))a(x,\theta)]\right| \leq C_{m,p}(a)(1+|\theta|)^m(|\theta|^\mu+|k|)^{-p},$$

where $C_{m,p}$ is a seminorm on $S^m(\Omega,s)$. By using

$$(|\theta|^\mu+|k|)^{-1} \leq (D^\mu+|k|)^{-1},$$

$$(|\theta|^\mu+|k|)^{-1} \leq |\theta|^{-\mu} \leq \left(\frac{D}{2}+\frac{D}{2}|\theta|\right)^{-\mu},$$

one can make the integral convergent and assure (10) by choosing p large enough. $\quad\square$

4 Applications

Example 1 We consider the two-point function Δ_+ of a free massive scalar relativistic field, where one has $\Omega = \mathbb{R}^4$, $s = 3$ and

$$\phi(x,\theta) = -x_0\omega(\theta) + \mathbf{x}\cdot\theta, \tag{11}$$

with

$$\omega(\theta) = \sqrt{|\theta|^2 + m^2}. \tag{12}$$

Here, we use the notation $x = (x_0, \mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^3$. Note that ϕ is not homogeneous. In [2] this problem is circumvented by using $-x_0|\theta| + \mathbf{x}\cdot\theta$ as phase function and multiplying the symbol with the function $e^{ix_0(|\theta|-\omega(\theta))}$. It is then no longer a symbol (as it is not smooth in $\theta = 0$), so one has to allow for so-called asymptotic symbols. Furthermore, one has to show that the multiplication with such a term does not spoil the fall-off properties, in particular that differentiation w.r.t. θ lowers the order.

In the present approach, this is not necessary. ϕ is a phase function in the sense of Definition 2 and therefore, by Theorem 1, it defines an oscillatory integral for every symbol a. In order to find the wave front set, we compute

$$\nabla_\theta\phi(x,\theta) = -\frac{\theta}{\omega(\theta)}x_0 + \mathbf{x}.$$

It is easy to see that its modulus is bounded from below by a positive constant unless $x = 0$ or $|x_0| = |x|$ and $\mathbf{x} = \text{sign}(x_0)\theta$. Thus, we have

$$M(\phi) = \{(x, \theta) | x = 0 \text{ or } |x_0| = |x| \neq 0, \theta = \lambda \mathbf{x}/x_0, \lambda > 0\}.$$

Furthermore,

$$\nabla_x \phi(x, \theta) = (-\omega(\theta), \theta).$$

For large θ, this behaves as

$$\nabla_x \phi(x, \theta) \sim (-|\theta|, \theta) + R(\theta),$$

where the remainder term R scales as $|\theta|^{-1}$. Thus, only in the directions $(-|\theta|, \theta)$ the bound on the angle of k and $\nabla_x \phi$ can not be fulfilled. Hence, we obtain the well-known result[1]

$$\text{WF}(\Delta_+) \subset \text{SP}(\phi) = \{(0, (-|\mathbf{k}|, \mathbf{k}))\} \cup \{((\pm|\mathbf{x}|, \mathbf{x}), (-\lambda|\mathbf{x}|, \pm\lambda\mathbf{x}))|\lambda > 0\}.$$

A variant of this example is obtained by considering phase functions of the form

$$\phi(x, \theta) = -x_0\sqrt{|\theta|^2 + f(\theta)} + \mathbf{x} \cdot \theta, \tag{13}$$

with f a positive function that is a symbol of order $\nu < 2$. Such expressions occur for example in quantum field theory on the Moyal plane with hyperbolic signature and signify a distortion of the dispersion relations, cf. [8,9]. The above trick to put $e^{-ix_0(\sqrt{|\theta|^2 + f(\theta)} - |\theta|)}$ into the symbol still works, but then the symbol will be of type $\rho' = \min(\rho, 2 - \nu)$, where ρ is original type of the symbol. It is straightforward to check that (13) still defines a phase function of order 1 in the sense defined here, and that its stationary phase is as above.

If the function f in (13) is a symbol of order $\nu > 2$,[2] then the shift of $e^{-ix_0(\sqrt{|\theta|^2 + f(\theta)} - |\theta|)}$ to the symbol is not possible, as this would no longer give a symbol of type $\rho > 0$. Thus, a treatment of (13) in the context of phase functions that are homogeneous of order 1 is not possible. However, one can still interpret (13) as a phase function of order $\nu/2$ and easily computes

$$\text{SP}(\phi) = \{(x, k) | x_0 = 0, \mathbf{k} = 0, k_0 \neq 0\}.$$

---

[1] In the physical literature, the Fourier transform is computed by integration with $e^{i(k_0x_0 - \mathbf{k}\cdot\mathbf{x})}$. In that convention, the sign of the zeroth component in the cotangent bundle has to be reversed.

[2] The phase function ϕ of (11) corresponds to solutions of the hyperbolic wave equation $(\partial_t^2 - \Delta_x + m^2)\psi = 0$. The modification suggested here means that the underlying PDE is no longer hyperbolic.

Example 2 In quantum field theory on the Moyal plane of even dimension d with Euclidean signature[3], one frequently finds phase funtions of the form

$$\mathbb{R}^d \times \mathbb{R}^{2d} \ni (x, \theta_1, \theta_2) \mapsto \phi(x, \theta) = x \cdot (\theta_1 + \theta_2) + \theta_1^t \sigma \theta_2.$$

Here σ is some real antisymmetric $d \times d$ matrix of maximal rank d. The above is clearly a symbol of order 2, and we have

$$\nabla_{\theta_1} \phi(x, \theta) = x + \sigma \theta_2,$$
$$\nabla_{\theta_2} \phi(x, \theta) = x - \sigma \theta_1.$$

As σ has rank d, Condition 2 of Definition 2 is fulfilled. From the above it follows that $M(\phi) = \emptyset$ and thus also $SP(\phi) = \emptyset$, so that the resulting distributions are smooth.

We note that up to now such integrals could only be treated in the so-called adiabatic limit [11]. But then one loses the information about the singular behaviour in position space, contrary to the present case, where the wave front set is known completely.

Example 3 In quantum field theory on the Moyal plane with hyperbolic signature, one frequently finds phase functions of the form

$$\mathbb{R}^d \times \mathbb{R}^{2(d-1)} \ni (x, \theta_1, \theta_2) \mapsto \phi(x, \theta) = x \cdot (\tilde{\theta}_1 + \tilde{\theta}_2) + \tilde{\theta}_1^t \sigma \tilde{\theta}_2,$$

where $\tilde{\theta} = (\omega(\theta), \theta)$ with ω as in (12) and σ as in Example 2. The above is a symbol of order 2, but it is not a phase function as defined here, as can most easily be seen in the case $d = 2$. Then with $\sigma = \epsilon$ one obtains

$$\partial_{\theta_1} \phi(x, \theta) = x_0 \frac{\theta_1}{\omega(\theta_1)} + x_1 + \theta_2 \frac{\theta_1}{\omega(\theta_1)} - \omega(\theta_2),$$
$$\partial_{\theta_2} \phi(x, \theta) = x_0 \frac{\theta_2}{\omega(\theta_2)} + x_1 - \theta_1 \frac{\theta_2}{\omega(\theta_2)} + \omega(\theta_1).$$

If the signs of θ_1 and θ_2 coincide, then the above derivatives tend to a constant as a function of θ, so that Condition 2 of Definition 2 is not fulfilled. The rigourous treatment of such integrals is an open problem, which we plan to address in future work[4].

---

[3] For an overview on the subject, we refer to [10].

[4] Joint work with Dorothea Bahns.

References

1. Grigis, A., Sjöstrand, J.: Microlocal Analysis for Differential Operators. Cambridge University Press, Cambridge (1994)
2. Reed, M., Simon, B.: Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness. Academic Press, Orlando (1975)
3. Hörmander, L.: The spectral function of an elliptic operator. Acta Math. **12**(1), 193 (1968)
4. Asada, K., Fujiwara, D.: On some oscillatory integral transformations in $L^2(\mathbb{R}^n)$. Jpn. J. Math. **4**(2), 299 (1978)
5. Boulkhemair, A.: Estimations L^2 précisées pour des intégrales oscillantes. Commun. Partial Diff. Equ. **22**(1–2), 165 (1997)
6. Ruzhansky, M., Sugimoto, M.: Global L^2-boundedness theorems for a class of Fourier integral operators. Commun. Partial Diff. Equ. **31**(4–6), 547 (2006)
7. Coriasco, S.: Fourier integral operators in SG classes. I. Composition theorems and action on SG Sobolev spaces. Rend. Sem. Mat. Univ. Politec. Torino **57**(4), 249 (2002)
8. Bahns, D., Doplicher, S., Fredenhagen, K., Piacitelli, G.: Field theory on noncommutative spacetimes: quasiplanar Wick products. Phys. Rev. D **71**, 025022 (2005)
9. Döscher, C., Zahn, J.: Dispersion relations in the noncommutative ϕ^3 and Wess-Zumino model in the Yang-Feldman formalism. Annales Henri Poincare **10**, 35 (2009)
10. Wulkenhaar, R.: Field theories on deformed spaces. J. Geom. Phys. **56**, 108 (2006)
11. Döscher, C.: Yang-Feldman formalism on noncommutative Minkowski space. Ph.D. thesis, Hamburg (2006)