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Lipschitz perturbations of a class of approximately controllable linear systems

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Abstract

This paper is devoted to the analysis of an approximately controllable system perturbed by a certain Lipschitz-continuous nonlinearity. By assuming that the intensity of the nonlinear influence is 'weak', we prove that the perturbed system is still approximately controllable. Using this perturbation theory, we prove that a third-order semilinear dispersion equation on a finite interval is approximately controllable whenever the nonlinearity effect is 'weak'. The result in the paper is a complement of the results obtained in (Zhou in SIAM J. Control Optim. 21:551-565, 1983), in which the control operator (resp. infinitesimal generator of the dynamics) of the unperturbed linear system is assumed to be bounded (resp. generated a differentiable semigroup on the state space).

Keywords: approximate controllability; Lipschitz perturbation; third-order semilinear dispersion equation

1 Introduction

Fix $T \in (0, \infty)$ and two real Hilbert spaces H and U. Let us consider the evolution equation/control system

$$\mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{f}\big(\mathbf{y}(t)\big) + B\mathbf{u}(t) \quad \text{for } t \in (0, T], \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}\}_{t\in[0,\infty)}$ on H, $B \in \mathscr{L}(U; [\mathcal{D}(A^*)]')$,[a] $\mathbf{u} \in L^2(0, T; U)$, and $\mathbf{f} \in \mathscr{C}(H; H)$ describes the nonlinearity effect. In the terminology from control community, B is a control operator, $\mathbf{u}$ is a control.

Several remarks are in order.

Remark 1.1 The fact that the control operator B can be unbounded, *i.e.*, $B \in \mathscr{L}(U; [\mathcal{D}(A^*)]')$, implies that our results for abstract systems may be applied to the study of partial differential equations subject to boundary controls and/or point controls.

Remark 1.2 There exists a pair $(M, \omega) \in [1, \infty) \times \mathbb{R}$ such that $\|e^{tA}\|_{\mathscr{L}(H)} \leq Me^{\omega t}$, $\forall t \in [0, \infty)$.

Remark 1.3 By [1], Corollary 10.6, p.41, A^* is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA^*}\}_{t\in[0,\infty)}$ on H, and $e^{tA^*} = (e^{tA})^*$, $\forall t \in [0, \infty)$.

In the paper, the following two assumptions always hold true.

Assumption 1 *There exists a* $K_1 \in (0,\infty)$ *such that*

$$\|\mathbf{f}(h_1)-\mathbf{f}(h_2)\|_H \le K_1\|h_1-h_2\|_H, \quad \forall (h_1,h_2)\in H^2.$$

Assumption 2 *There exists a* $K_2 \in (0,\infty)$ *depending only on* T *such that*

$$\begin{aligned}(K_2)^{-2}\int_0^T \|e^{(T-t)A^*}\eta\|_H^2\,dt &\le \int_0^T \|B^*e^{(T-t)A^*}\eta\|_H^2\,dt \\ &\le (K_2)^2\|\eta\|_H^2, \quad \forall \eta\in\mathcal{D}(A^*).\end{aligned} \tag{1.2}$$

Some further remarks need to be added.

Remark 1.4 The first '$\le$' in (1.2) describes a certain quantitative unique continuation property of solutions to the evolution equation

$$\mathbf{z}'(t) = -A^*\mathbf{z}(t) \quad \text{for } t\in[0,T).$$

This property implies, in particular, that the unperturbed linear system ((1.1) with $\mathbf{f}\equiv 0$) is approximately controllable in time T; see Lemma 3.2 and also [2], Theorem 11.2.1.

For every $\mathbf{y}^0\in H$, the control system (1.1) satisfying Assumptions 1 and 2 admits a unique trajectory $\mathbf{y}\in\mathscr{C}([0,T];H)$ such that $\mathbf{y}(0)=\mathbf{y}^0$; see Theorem 2.1 for details. The main focus of this paper is placed on the approximate controllability of the system (1.1); see Definition 2 for the notion of approximate controllability.

Controllability is a fundamentally important property of a control system. Therefore it comes as no surprise that the approximate controllability for various control systems has been extensively established in the literature; see [3–8] and the references cited therein. Seidman [4] considered the influence of nonlinearity on reachable sets; the results are closely related to exact/approximate controllability. References [3, 6, 7] investigated abstract systems with Lipschitz perturbations for their approximate controllability; see (1.4) for the precise problem considered in [3, 6, 7]. References [8–11] proved approximate controllability for systems described by semilinear heat equations. It is well known that the time-to-guarantee-approximate controllability for systems governed by linear heat equations can be short as desired; see [2, 9]. Zuazua [8] treated in a systematic way systems described by semilinear heat equations whose nonlinearities depend on the gradient of the state. The more recent approximate/exact controllability results can be found in [12–20], whose results are worth a brief introduction. Let us write still H and U for the state space and control-parameter space, respectively. The system considered in [12–20] is given formally as

$$\mathbf{y}'(t) = A(t)\mathbf{y}(t) + \mathbf{f}(\mathbf{y}(t),\mathbf{u}(t),t) + B(t)\mathbf{u}(t) \quad \text{for } t\in(0,T], \tag{1.3}$$

where the assumptions on $A(t)$, $B(t)$ and $\mathbf{f}$ are yet to be given explicitly. Bashirov, and Jneid [13] proved by Banach's fixed point theorem that the system (1.3) is partially exact-controllable in a fixed finite time T under the following assumptions: (★) $A = A(t)$ is the infinitesimal generator of a strongly continuous semigroup on H (■) $B = B(t)$ is a bounded linear operator of U into H, the linear system ((1.3) with $\mathbf{f} \equiv 0$) is partially exact-controllable in time T, and (▲) the nonlinearity $\mathbf{f}$ is Lipschitz-continuous with the first two arguments (uniformly in the third argument) and is not too intense. The notions of controllability of different types were introduced by Bashirov, Mahmudov, Semi, and Etikan [14]. Bashirov, and Ghahramanlou [12] proved that the system (1.3) is partially approximate controllable in a fixed finite time T under the following assumptions: (★), (■), (▲), and the linear system ((1.3) with $\mathbf{f} \equiv 0$) is partially approximate controllable in any time $t_0 \in (0, T]$. Leiva [17] proved later a total version of the results obtained in [12]. Leiva [15] proved by Rothe's fixed point theorem that the system (1.3) is exactly controllable in a fixed finite time T under the following assumptions: H and U are finite dimensional, the linear system ((1.3) with $\mathbf{f} \equiv 0$) is exactly controllable in time T, and the nonlinearity $\mathbf{f}$ is smooth enough and satisfies a sublinear growth condition. Leiva [16] proved (still by Rothe's fixed point theorem) similar results for systems studied in [15] and subject to impulses. Leiva, Merentes, and Sanchez [20] proved via Rothe's fixed point theorem that the system (1.3) is approximately controllable in a fixed finite time T under the following assumptions: $H = U = L^2(\Omega)$ where Ω is a bounded open nonempty subset of $\mathbb{R}^N$ with N a positive integer, $A = A(t)$ is the realization in H of Laplacian with the Dirichlet homogeneous boundary condition, $B = B(t)$ the characteristic function of ω (a nonempty subset of Ω), and the nonlinearity $\mathbf{f}$ is given by $\mathbf{f}(\phi, \psi, t)(x) = f(\phi(x), \psi(x), t)$ where $f: \mathbb{R}^2 \times [0, T] \to \mathbb{R}$ is smooth enough and satisfies $|f(y, u, t) - ay| \leq m_1|y|^{1-} + m_2$ for a certain triple $(a, m_1, m_2) \in \mathbb{R}^3$; Leiva and Merentes [18, 19] proved similar results for heat equations without appealing to Rothe's fixed point theorem. Reference [18] generalized [20] in several aspects: more functions f are considered in the former reference, the system studied in the former reference may incorporate delay in the nonlinearity f, the system studied in the former reference incorporates impulses. The key property used in [20] is that $\{e^{t\Delta}\}_{t\in[0,\infty)}$ is a compact semigroup; the key property used in [18] is that the linear heat equation is approximately controllable (by the internal control) in any finite time.

The most inspiring one of the various motivations to study the system (1.1) comes from [3, 6, 7] in which a similar control system was investigated for its approximate controllability. In particular, the author of [6] considered the system

$$\mathbf{y}'(t) = A\mathbf{y}(t) + \mathbf{f}\big(\mathbf{y}(t)\big) + \mathcal{B}\mathbf{v}(t) \quad \text{for } t \in (0, T]. \tag{1.4}$$

In [6], the semigroup $\{e^{tA}\}_{t\in[0,\infty)}$ generated by A was assumed to be differentiable, $\mathbf{f}$ was given in the same way as in Assumption 1, and the control operator $\mathcal{B}$ is 'bounded' in the sense that $\mathcal{B} \in \mathscr{L}(\mathscr{V}; L^2(0, T; H))$ where $\mathscr{V}$ is a real Hilbert space. Zhou [6] proved that the system (1.4) is approximately controllable in time T under the following assumptions:

Assumptions *There exists a* $t_0 \in [0, T)$ *such that for every pair* $(\varepsilon, \mathbf{p}) \in (0, \infty) \times L^2(t_0, T; H)$, *there exists a* $\mathbf{v} \in \mathscr{V}$ *such that*

$$\begin{cases} \| \int_{t_0}^T e^{(T-\tau)A}\mathcal{B}\mathbf{v}(\tau)\,d\tau - \int_{t_0}^T e^{(T-\tau)A}\mathbf{p}(\tau)\,d\tau \|_H < \varepsilon, \\ \int_{t_0}^T \|\mathcal{B}\mathbf{v}(\tau)\|_H^2\,d\tau \leq C^2 \int_{t_0}^T \|\mathbf{p}(\tau)\|_H^2\,d\tau, \end{cases}$$

where $C \in (0, \infty)$ *is a constant independent of* $\mathbf{p}$ *and* ε, *and it renders* $C(T - t_0)K_1$ *sufficiently small.*

Our principal purpose in this paper is to judge whether, and understand better how, an approximately controllable linear system preserves its approximate controllability when it is subject to Lipschitz perturbations. For this purpose, we first of all develop an approximate controllability theory for the unperturbed linear system ((1.1) with $\mathbf{f} \equiv 0$), and then attempt to analyze the perturbed system (1.1) for its approximate controllability by applying the afore-developed linear theory. Under Assumptions 1-2 and by assuming that K_1 is sufficiently small, we prove the approximate controllability is 'inherited' by perturbed nonlinear the system (1.1) from the unperturbed linear system. The main novelty in our proof is that a candidate approximation sequence for every approximate control is carefully constructed and is proved to converge exactly to a desired approximate control; the afore-developed linear theory takes crucial role throughout the procedure.

Our results for the control system (1.1) can be viewed as a complement of the results obtained in [6]: The infinitesimal generator of dynamics of the linear system ((1.1) with $\mathbf{f} \equiv 0$) is only assumed to generate a strongly continuous semigroup, while the one in (1.4) was assumed to generate a differentiable semigroup; the control operator in (1.1) is local in the sense that the state at time t, $t \in [0, T]$, depends on the value of $\mathbf{u}$ in an arbitrarily 'small' neighborhood of t, while the one in (1.4) is 'non-local'; the control operator in (1.1) can be unbounded, while the one in (1.4) is bounded; our result can easily be applied to analyze the approximate controllability for partial differential equations, while the result in (1.4) seems to be an indirect criterion for approximate controllability but is indeed very general in the category of systems having control operators similar to the one in (1.4). In comparison with [3, 7, 12, 17–20], we do *not* assume any compactness (either assume that $\mathcal{D}(A)$ is compactly imbedded in H or else assume that the semigroup $\{e^{tA}\}_{t\in[0,\infty)}$ is compact), nor do we need assume that the linear system ((1.3) with $\mathbf{f} \equiv 0$) is approximately controllable in every time $t \in (0, T]$.

Motivated by both the mathematical interest and the desire to present an illustrative example for our result for (1.1), we study also a semilinear third-order dispersion equation on a finite interval. By assuming that the nonlinearity in the equation under consideration is 'weak', we prove the system in question is approximately controllable by applying our result for (1.1). We would like to mention that semilinear third-order dispersion equations on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, namely

$$\begin{cases} \partial_t y + \partial_x^3 y + \partial_x y + f(y,t) = u & \text{in } (0, 2\pi) \times (0, T), \\ \partial_x^\ell y(2\pi, \cdot) - \partial_x^\ell y(0, \cdot) = 0 & \text{in } (0, T), \ell = 0, 1, 2, \end{cases} \tag{1.5}$$

with u as control, have already been investigated in several references for their exact/approximate controllability. Russell and Zhang [21] proved exact controllability for (1.5) with $f \equiv 0$. As in the work by George, Chalishajar, and Nandakumaran [22], Tomar and Sukavanam [23] proved exact controllability for (1.5) with f given in a wide class. Chalishajar [24] proved exact controllability for (1.5) with memory in which $f(y(x,t),t)$ is replaced by $f(y(x,t), t, \int_0^t M(t,s,y(\cdot,s))\,ds)$ with M understood as the memory mechanism. Sakthivel, Mahmudov, and Ren [25] studied the system for approximate controllability. And more recently, Muthukumar and Rajivganthi [26] studied the semilinear stochastic dispersion equation on the torus for its approximate controllability. The aforementioned six references share the 'shape' of the control, that is, they chose their control as

$$u(x,t) = g(x)\left[v(x,t) - \int_0^{2\pi} g(\sigma)v(\sigma,t)\,d\sigma\right], \quad \forall (x,t) \in \mathbb{T} \times [0,\infty),$$

where $g: \mathbb{T} \to [0,\infty)$ is a piece-wise continuous function so that $\int_0^{2\pi} g(\sigma)\,d\sigma = 1$. The reader can consult [27] and the references cited therein for more studies on the controllability of third-order dispersion equations on a finite interval.

Usually, approximate controllability is an effective substitute for exact controllability when the latter is unavailable in a given control system (especially the one arising in engineering). Therefore, our results may have some significance in engineering.

The rest of the paper is organized as follows: The Cauchy problem for the evolution equation (1.1) is investigated in Section 2; the approximate controllability of the system (1.1) is established in Section 3; an illustrative example is provided in Section 4 for the main results of this paper; in Section 5 several concluding remarks are presented.

2 The Cauchy problem for the evolution equation (1.1)

In this section, we study the Cauchy problem for the evolution equation (1.1). Let us start by introducing the sense in which a solution to (1.1) can be given.

Definition 1 Let $u \in L^2(0,T;U)$. $\mathbf{y} \in \mathscr{C}([0,T];H)$ is called a mild solution to equation (1.1) if

$$\mathbf{y}(t) = e^{tA}\mathbf{y}(0) + \int_0^t e^{(t-\tau)A}\left[\mathbf{f}\big(\mathbf{y}(\tau)\big) + B\mathbf{u}(\tau)\right]d\tau, \quad \forall t \in [0,T]. \tag{2.1}$$

In the terminology from the control community, $\mathbf{y}$ is said to be a trajectory of the control system (1.1).

Now, we are going to give meaning to the integral $\int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau)\,d\tau$. Before that, we introduce some conventions and auxiliary tools.

We identify H with its dual space H' throughout this paper; by this identification, we have

$$\begin{cases} \mathcal{D}(A^*) \subset H \subset [\mathcal{D}(A^*)]', \\ \langle \psi, \eta\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} = \langle \psi, \eta\rangle_H, \quad \forall (\eta,\psi) \in H \times \mathcal{D}(A^*). \end{cases} \tag{2.2}$$

Define $S(t)$, for every $t \in [0,\infty)$, by giving its value $S(t)\eta$ at $\eta \in [\mathcal{D}(A^*)]'$ in this way:

$$\big\langle \psi, S(t)\eta\big\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} = \big\langle e^{tA^*}\psi, \eta\big\rangle_{H}, \quad \forall \psi \in \mathcal{D}\big(A^*\big). \tag{2.3}$$

Lemma 2.1 $\{S(t)\}_{t\in[0,\infty)}$ *is a strongly continuous semigroup on* $[\mathcal{D}(A^*)]'$, *and* $S(t)$ *extends* e^{tA} *for every* $t \in [0,\infty)$.

Thanks to this observation, we understand $\{e^{tA}\}_{t\in[0,\infty)}$ as $\{S(t)\}_{t\in[0,\infty)}$ when necessary. In particular, we understand the integral $\int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau)\,d\tau$ in the sense of $\int_0^t S(t-\tau)B\mathbf{u}(\tau)\,d\tau$; and indeed, we see that $[0,T] \ni \mathbf{u} \mapsto \int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau)\,d\tau \in [\mathcal{D}(A^*)]'$ belongs to $\mathscr{C}([0,T];[\mathcal{D}(A^*)]')$.

We relegate the proof of Lemma 2.1 to the Appendix.

If $\mathbf{y} \in \mathscr{C}([0,T];H)$ is a solution to equation (1.1), then the mapping

$$[0,T] \ni t \longmapsto \int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\,d\tau \in H,$$

defined initially as an element belonging to $\mathscr{C}([0,T];[\mathcal{D}(A^*)]')$, necessarily belongs to $\mathscr{C}([0,T];H)$. And this property can be obtained *a priori* from Assumption 2. More precisely, we have the following.

Lemma 2.2 *Suppose that Assumption* 2 *holds true. For every* $\mathbf{u} \in L^2(0,T;U)$, *the map* $t \mapsto \int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\,d\tau$ *belongs to* $\mathscr{C}([0,T];H)$ *and satisfies*

$$\max_{t\in[0,T]} \left\| \int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\,d\tau \right\|_H \le K_2 \|\mathbf{u}\|_{L^2(0,T;U)}. \tag{2.4}$$

The proof of Lemma 2.2 can be adapted directly from that of [2], Proposition 4.2.6, p.117, and, as with that of Lemma 2.1, is given in the Appendix.

Remark 2.1 The 'norm' of the inequality (2.4) comes from the 'norm' of the second '$\le$' in the string of inequalities (1.2).

We close this section by stating and proving that the Cauchy problem for the evolution equation (1.1) is well posed.

Theorem 2.1 *Suppose that Assumptions* 1-2 *hold true.*

(i) *Let* $\mathbf{u} \in L^2(0,T;U)$. *If* $\mathbf{y} \in \mathscr{C}([0,T];H)$ *is a solution to equation* (1.1), *then*

$$\begin{aligned}\|\mathbf{y}\|_{\mathscr{C}([0,T];H)} \le{}& \exp\bigl(K_1 M T e^{T|\omega|}\bigr) \\ &\times \bigl(M e^{T|\omega|} \|\mathbf{y}(0)\|_H + K_2 \|\mathbf{u}\|_{L^2(0,T;U)} + M T e^{T|\omega|} \|\mathbf{f}(0)\|_H\bigr).\end{aligned}$$

(ii) *Let* $(\mathbf{u}_1,\mathbf{u}_2) \in (L^2(0,T;U))^2$. *If* $\mathbf{y}_1$ *and* $\mathbf{y}_2$ *belong to* $\mathscr{C}([0,T];H)$ *and are solutions to equation* (1.1) *with* $\mathbf{u} = \mathbf{u}_1$ *and* $\mathbf{u} = \mathbf{u}_2$, *respectively, then*

$$\begin{aligned}\|\mathbf{y}_1 - \mathbf{y}_1\|_{\mathscr{C}([0,T];H)} \le{}& \exp\bigl(K_1 M T e^{T|\omega|}\bigr) \\ &\times \bigl(M e^{T|\omega|} \|\mathbf{y}_1(0) - \mathbf{y}_2(0)\|_H + K_2 \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(0,T;U)}\bigr).\end{aligned}$$

(iii) *For every* $(\mathbf{y}^0,\mathbf{u}) \in H \times L^2(0,T;U)$, *equation* (1.1) *admits a unique solution* $\mathbf{y} \in \mathscr{C}([0,T];H)$ *such that* $\mathbf{y}(0) = \mathbf{y}^0$.

Proof Let $\mathbf{y} \in \mathscr{C}([0,T];H)$ be a solution to equation (1.1), *i.e.*, u satisfies (2.1). Therefore, we have

$$\begin{aligned}\|\mathbf{y}(t)\|_H &\le \left\| e^{tA}\mathbf{y}(0) + \int_0^t e^{(t-\tau)A}\bigl[\mathbf{f}\bigl(\mathbf{y}(\tau)\bigr) + B\mathbf{u}(\tau)\bigr]\,d\tau \right\|_H \\ &\le e^{tA}\mathbf{y}(0) + \int_0^t e^{(t-\tau)A}\mathbf{f}\bigl(\mathbf{y}(\tau)\bigr)\,d\tau + \int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\,d\tau\end{aligned}$$

$$\times \exp\bigl(K_1 MTe^{T|\omega|}\bigr)$$
$$\times \bigl(Me^{T|\omega|}\|\mathbf{y}(0)\|_H + K_2\|\mathbf{u}\|_{L^2(0,T;U)} + MTe^{T|\omega|}\|\mathbf{f}(0)\|_H\bigr),$$
$$\|\mathbf{y}(t)\|_H \le \bigl(Me^{T|\omega|}\|\mathbf{y}(0)\| + K_2\|u\|_{L^2(0,T;U)} + MTe^{T|\omega|}\|\mathbf{f}(0)\|_H\bigr) + MK_1e^{T|\omega|}\int_0^t \|\mathbf{y}(\tau)\|_H\,d\tau, \quad \forall t \in [0,T],$$

which, together with Gronwall's lemma, implies that the proof of (i) is complete.

Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two solutions to equation (1.1) with $\mathbf{u} = \mathbf{u}_1$ and $\mathbf{u} = \mathbf{u}_2$, respectively. Write $\mathbf{y} = \mathbf{y}_2 - \mathbf{y}_1$. Then $\mathbf{y}$ satisfies

$$\mathbf{y}(t) = e^{tA}\mathbf{y}(0) + \int_0^t e^{(t-\tau)A}\bigl[\mathbf{f}\bigl(\mathbf{y}_2(t)\bigr) - \mathbf{f}\bigl(\mathbf{y}_1(t)\bigr)\bigr]\,d\tau + \int_0^t e^{(t-\tau)A}B\bigl[\mathbf{u}_2(\tau) - \mathbf{u}_1(\tau)\bigr]\,d\tau, \quad \forall t \in [0,T].$$

This, together with Remark 1.2, implies

$$\|\mathbf{y}(t)\|_H \le Me^{\omega t}\|\mathbf{y}(0)\|_H + MK_1\int_0^t e^{\omega(t-\tau)}\|\mathbf{y}(\tau)\|_H\,d\tau + K_2\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(0,T;U)}, \quad \forall t \in [0,T].$$

By an argument used in the previous paragraph, we can deduce (ii) from the above observation.

The proof of (iii) follows from a standard contraction mapping fixed point argument, details are omitted. □

3 Approximate controllability of the system (1.1)

In this section we discuss the approximate controllability of the system (1.1). We prove first of all that the unperturbed linear system is approximately controllable, and then we apply this linear theory to construct approximation sequence for approximate control and to prove the afore-constructed approximation sequence converges to a desired approximate control. Let us start this section with the following remark.

Remark 3.1 It follows from Assumption 2 and the fact that $\mathcal{D}(A^*)$ is dense in H that

$$\mathcal{D}\bigl(A^*\bigr) \ni \eta \longmapsto \bigl[t \mapsto B^*e^{(T-t)A^*}\eta\bigr] \in L^2(0,T;H)$$

can be extended to a bounded linear operator of H into $L^2(0,T;H)$. From now on, we understand $t \mapsto B^*e^{(T-t)A^*}\eta$ in the above sense whenever it occurs that $\eta \in H$.

Two lemmata should be recorded as tools for the later development. The first one is the celebrated Douglas lemma, which relates the factorization, range inclusion, and majorization of operators defined between Hilbert spaces.

Lemma 3.1 *Let H_j, $j = 1,2,3$, be Hilbert spaces, $F \in \mathscr{L}(H_1;H_3)$ and $G \in \mathscr{L}(H_2;H_3)$. Then the following three assertions are equivalent*:

- *$\exists K \in \mathscr{L}(H_1;H_2)$ such that $F = GK$.*
- *The image FH_1 of F is contained in the image GH_2 of G.*
- *$\exists C \in (0,\infty)$ such that $\|F^*h_3\| \le C\|G^*h_3\|$, $\forall h_3 \in H_3$.*

If the last assertion holds true, then there exists a $\widetilde{K} \in \mathscr{L}(H_1;H_2)$ such that $F = G\widetilde{K}$ and $\|\widetilde{K}\|_{\mathscr{L}(H_1;H_2)} \le C$.

Lemma 3.1 was first obtained by Douglas in [28] in the special case $H_1 = H_2 = H_3$; the general cases can be proved in a way used in the previous reference. Therefore, we choose to omit the details of the proof.

Lemma 3.2 *Suppose that Assumption* 2 *holds true. The system*

$$\mathbf{y}'(t) = A\mathbf{y}(t) + B\mathbf{u}(t) \quad \text{for } t \in (0,T] \tag{3.1}$$

is approximately controllable in time T.

Before the proof of Lemma 3.2, we explain in detail the notion of approximate controllability. To each $\mathbf{y}^0 \in H$ we associate the set

$$\begin{aligned}\mathcal{R}^{(1.1)\ (\text{resp. } (3.1))}_{\mathbf{y}^0,T} := \bigl\{&\mathbf{y}^T \in H; \exists \mathbf{u}_0 \in L^2(0,T;U) \text{ s.t. } \mathbf{y}^T = \mathbf{y}(T) \text{ and} \\ &\mathbf{y}(0) = \mathbf{y}^0, \text{ where } \mathbf{y} \text{ is the solution} \\ &\text{to equation (1.1) (resp. (3.1)) with } \mathbf{u} = \mathbf{u}_0\bigr\}.\end{aligned} \tag{3.2}$$

Definition 2 The control system (1.1) (resp. (3.1)) is said to be approximately controllable in time T if for every $\mathbf{y}^0 \in H$, $\mathcal{R}^{(1.1)}_{\mathbf{y}^0,T}$ (resp. $\mathcal{R}^{(3.1)}_{\mathbf{y}^0,T}$) is dense in H.

Proof of Lemma 3.2 In principle, the second '$\le$' in (1.2), together with the qualitative unique continuation property that $B^*e^{(T-t)A^*}\eta = 0$, $t \in [0,T]$, implies that $\eta = 0$, which directly implies Lemma 3.2. Here we would like to give this proof in detail for two reasons: To get more (*i.e.*, the cost of the approximate control) than the approximate controllability from the quantitative unique continuation property (Assumption 2); and to get some clues to the proof of Theorem 3.1.

The following claim is necessary for later development.

Claim 3.1 *For every $\mathbf{x} \in L^2(0,T;H)$, there exists a $\mathbf{u} \in L^2(0,T;U)$ such that*

$$\begin{cases} \|\mathbf{u}\|_{L^2(0,T;U)} \le K_2\|\mathbf{x}\|_{L^2(0,T;H)}, \\ \int_0^T e^{(T-t)A}B\mathbf{u}(t)\,dt = \int_0^T e^{(T-t)A}\mathbf{x}(t)\,dt. \end{cases} \tag{3.3}$$

Proof of Claim 3.1 Define the following two bounded linear operators

$$\begin{aligned}\Phi: L^2(0,T;U) &\longrightarrow H, \\ \mathbf{u} &\longmapsto \int_0^T e^{(T-t)A}B\mathbf{u}(t)\,dt, \quad \text{and}\end{aligned} \tag{3.4}$$

$$\begin{aligned} \Psi : L^2(0,T;H) &\longrightarrow H, \\ \mathbf{x} &\longmapsto \int_0^T e^{(T-t)A}\mathbf{x}(t)\,dt. \end{aligned} \tag{3.5}$$

Some routine calculation implies that the adjoint operators of Φ and Ψ are

$$\begin{aligned} \Phi^* : H &\longrightarrow L^2(0,T;U), \\ \eta &\longmapsto \bigl[t \mapsto B^* e^{(T-t)A^*}\eta\bigr], \quad \text{and} \end{aligned} \tag{3.6}$$

$$\begin{aligned} \Psi^* : H &\longrightarrow L^2(0,T;H), \\ \eta &\longmapsto \bigl[t \mapsto e^{(T-t)A^*}\eta\bigr], \quad \text{respectively.} \end{aligned} \tag{3.7}$$

Thanks to Assumption 2, we have

$$\|\Psi^*\eta\|_{L^2(0,T;U)} \le K_2 \|\Phi^*\eta\|_{L^2(0,T;H)}, \quad \forall \eta \in H.$$

This, together with Lemma 3.1, implies that there exists a

$$\mathbb{K} \in \mathscr{L}\bigl(L^2(0,T;H); L^2(0,T;U)\bigr)$$

such that

$$\Psi = \Phi\mathbb{K} \quad \text{and} \quad \|\mathbb{K}\|_{\mathscr{L}(L^2(0,T;H);L^2(0,T;U))} \le K_2. \tag{3.8}$$

To every $\mathbf{x} \in L^2(0,T;H)$ we associate

$$\hat{\mathbf{u}} = \mathbb{K}\mathbf{x}. \tag{3.9}$$

Equation (3.8) implies that every $\mathbf{x} \in L^2(0,T;H)$, together with $\hat{\mathbf{u}}$ carefully specified as in (3.9) satisfies (3.3).

The proof of Claim 3.1 is complete. □

Let us turn back to the proof of Lemma 3.2. Let $\eta \in \mathcal{D}(A)$. Write

$$\mathbf{x}_0(t) = \frac{1}{T}(\eta - tA\eta), \quad \forall t \in [0,T];$$

$\mathbf{x}_0$ belongs to $\in \mathscr{C}^\infty([0,T];H)$ (and hence $L^2(0,T;H)$) and satisfies $\eta = \int_0^T e^{(T-t)A}\mathbf{x}_0(t)\,dt$. This, together with Claim 3.1, implies that there exists a $\mathbf{u}_0 \in L^2(0,T;U)$ such that

$$\eta = \int_0^T e^{(T-t)A}B\mathbf{u}_0(t)\,dt.$$

To summarize, we have the following.

Claim 3.2 *For every $\eta \in \mathcal{D}(A)$, there exists a $\mathbf{u} \in L^2(0,T;H)$ such that*

$$\eta = \int_0^T e^{(T-t)A}B\mathbf{u}(t)\,dt.$$

Let us return to the proof after the above brief digression. By Claim 3.2, the fact that $\mathcal{D}(A)$ is dense in H implies that the range of the bounded linear operator given by (3.4) is dense in H. Noting that the system (3.1) is linear, we indeed complete the proof of Lemma 3.2. □

Let us state here the main result of this paper.

Theorem 3.1 *Suppose that Assumptions* 1-2 *hold true, and suppose*

$$K_1(K_2)^2\sqrt{T}\exp\bigl(K_1MTe^{T|\omega|}\bigr) < 1.$$

Then the system (1.1) *is approximately controllable in time* T.

Remark 3.2 The system (1.1) can be viewed as a nonlinear perturbation of the above approximately controllable linear system (3.1). The result stated in Theorem 3.1 means that the perturbed system is still approximately controllable when the perturbation is not severe.

Proof of Theorem 3.1 Let $(\mathbf{y}^0,\mathbf{y}^T) \in [\mathcal{D}(A)]^2$. Fix any $\mathbf{u}_0 \in L^2(0,T;U)$. By Theorem 2.1, equation (1.1) with $\mathbf{u} = \mathbf{u}_0$ admits a unique solution $\mathbf{y}_0 \in \mathscr{C}([0,T];H)$ such that $\mathbf{y}_0(0) = \mathbf{y}^0$. By Claims 3.1 and 3.2, the observation that $\mathbf{y}^T - e^{TA}\mathbf{y}^0 \in \mathcal{D}(A)$ implies that there exists a $\mathbf{u}_1 \in L^2(0,T;U)$ such that

$$\mathbf{y}^T - e^{TA}\mathbf{y}^0 - \int_0^T e^{(T-t)A}\mathbf{f}\bigl(\mathbf{y}_0(t)\bigr)\,dt = \int_0^T e^{(T-t)A}B\mathbf{u}_1(t)\,dt. \tag{3.10}$$

Again by Theorem 2.1, equation (1.1) with $\mathbf{u} = \mathbf{u}_1$ admits a unique solution $\mathbf{y}_1 \in \mathscr{C}([0,T];H)$ such that $\mathbf{y}_1(0) = \mathbf{y}^0$. By Claim 3.1, there exists a $\mathbf{v}_1 \in L^2(0,T;U)$ such that

$$\begin{cases} \|\mathbf{v}_1\|_{L^2(0,T;U)} \le K_2\{\int_0^T \|\mathbf{f}(\mathbf{y}_1(t)) - \mathbf{f}(\mathbf{y}_0(t))\|_H^2\,dt\}^{\frac{1}{2}}, \\ \int_0^T e^{(T-t)A}B\mathbf{v}_1(t)\,dt = \int_0^T e^{(T-t)A}\{\mathbf{f}(\mathbf{y}_1(t)) - \mathbf{f}(\mathbf{y}_0(t))\}\,dt. \end{cases} \tag{3.11}$$

From $(3.11)_1$ it follows that

$$\begin{aligned} \|\mathbf{v}_1\|_{L^2(0,T;U)} &\le \sqrt{T}K_1K_2\|\mathbf{y}_1 - \mathbf{y}_0\|_{\mathscr{C}([0,T];H)} \\ &\le K_1(K_2)^2\sqrt{T}\exp\bigl(K_1MTe^{T|\omega|}\bigr)\|\mathbf{u}_1 - \mathbf{u}_0\|_{L^2(0,T;U)}, \end{aligned} \tag{3.12}$$

where '$\le$' in the first line follows from Assumption 1, and '$\le$' in the second line follows from Theorem 2.1.

Write $\mathbf{u}_2 = \mathbf{u}_1 + \mathbf{v}_1$. It follows from (3.10) and $(3.11)_2$ that

$$\mathbf{y}^T - e^{TA}\mathbf{y}^0 - \int_0^T e^{(T-t)A}\mathbf{f}\bigl(\mathbf{y}_1(t)\bigr)\,dt = \int_0^T e^{(T-t)A}B\mathbf{u}_2(t)\,dt. \tag{3.13}$$

Let $k \in \mathbb{N}$ with $k \geq 2$, and let the sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}_0, j\leq k}$ be given in $L^2(0,T;U)$ such that

$$\begin{cases} \mathbf{y}^T - e^{TA}\mathbf{y}^0 - \int_0^T e^{(T-t)A}\mathbf{f}(\mathbf{y}_{j-1}(t))\,dt = \int_0^T e^{(T-t)A}B\mathbf{u}_j(t)\,dt, \\ \qquad \forall j \in \mathbb{N}_0 \text{ with } 1 \leq j \leq k, \\ \|\mathbf{u}_j - \mathbf{u}_{j-1}\|_{L^2(0,T;U)} \\ \quad \leq K_1(K_2)^2\sqrt{T}\exp(K_1MTe^{T|\omega|})\|\mathbf{u}_{j-1} - \mathbf{u}_{j-2}\|_{L^2(0,T;U)}, \\ \qquad \forall j \in \mathbb{N}_0 \text{ with } 2 \leq j \leq k, \end{cases} \tag{3.14}$$

where $\mathbf{y}_j$ is the unique solution to equation (1.1) with $\mathbf{u} = \mathbf{u}_j$ such that $\mathbf{y}_j(0) = \mathbf{y}^0$, $j \in \mathbb{N}_0$ with $j \leq k-1$. Employing an argument used to prove (3.12), we can deduce from Claim 3.1 that there exists a $\mathbf{v}_k \in L^2(0,T;U)$ such that

$$\begin{cases} \|\mathbf{v}_k\|_{L^2(0,T;U)} \leq K_1(K_2)^2\sqrt{T}\exp(K_1MTe^{T|\omega|})\|\mathbf{u}_k - \mathbf{u}_{k-1}\|_{L^2(0,T;U)}, \\ \int_0^T e^{(T-t)A}B\mathbf{v}_k(t)\,dt = \int_0^T e^{(T-t)A}\{\mathbf{f}(\mathbf{y}_k(t)) - \mathbf{f}(\mathbf{y}_{k-1}(t))\}\,dt. \end{cases} \tag{3.15}$$

Define $\mathbf{u}_{k+1} = \mathbf{u}_k + \mathbf{v}_k$. Collect $(3.14)_1$ and $(3.15)_2$ and conduct several routine calculations to obtain

$$\mathbf{y}^T - e^{TA}\mathbf{y}^0 - \int_0^T e^{(T-t)A}\mathbf{f}\big(\mathbf{y}_k(t)\big)\,dt = \int_0^T e^{(T-t)A}B\mathbf{u}_{k+1}(t)\,dt. \tag{3.16}$$

By the induction principle, there exists a sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}_0}$ in $L^2(0,T;U)$ such that

$$\left.\begin{aligned} &\mathbf{y}^T - e^{TA}\mathbf{y}^0 - \int_0^T e^{(T-t)A}\mathbf{f}\big(\mathbf{y}_j(t)\big)\,dt \\ &\quad = \int_0^T e^{(T-t)A}B\mathbf{u}_{j+1}(t)\,dt, \\ &\|\mathbf{u}_{j+2} - \mathbf{u}_{j+1}\|_{L^2(0,T;U)} \\ &\quad \leq K_1(K_2)^2\sqrt{T}\exp\big(K_1MTe^{T|\omega|}\big)\|\mathbf{u}_{j+1} - \mathbf{u}_j\|_{L^2(0,T;U)}, \end{aligned}\right\} \quad \forall j \in \mathbb{N}_0, \tag{3.17}$$

where $\mathbf{y}_j$ is the unique solution to equation (1.1) with $\mathbf{u} = \mathbf{u}_j$ such that $\mathbf{y}_j(0) = \mathbf{y}^0$, $j \in \mathbb{N}_0$. Since $K_1(K_2)^2\sqrt{T}\exp(K_1MTe^{T|\omega|}) < 1$, $(3.17)_2$ implies that $\{\mathbf{u}_j\}_{j\in\mathbb{N}_0}$ is a Cauchy sequence in $L^2(0,T;U)$. Write $\bar{\mathbf{u}}$ for the limit of the sequence $\{\mathbf{u}_j\}_{j\in\mathbb{N}_0}$ in $L^2(0,T;U)$, and $\bar{\mathbf{y}}$ for the unique solution to equation (1.1) with $\mathbf{u} = \bar{\mathbf{u}}$ such that $\bar{\mathbf{y}}(0) = \mathbf{y}^0$. By the Lipschitz-continuity of the data-to-solution map (see (ii) of Theorem 2.1), we have $\mathbf{y}_j \to \bar{\mathbf{y}}$ in $\mathscr{C}([0,T];H)$ as $j \nearrow \infty$. On the other hand, we have

$$\begin{aligned} &\left\|\int_0^T e^{(T-t)A}\mathbf{f}\big(\mathbf{y}_j(t)\big)\,dt - \int_0^T e^{(T-t)A}\mathbf{f}\big[\bar{\mathbf{y}}(t)\big]\,dt\right\|_H \\ &\quad = \left\|\int_0^T e^{(T-t)A}\big\{\mathbf{f}\big(\mathbf{y}_j(t)\big)\,dt - \mathbf{f}\big[\bar{\mathbf{y}}(t)\big]\big\}\,dt\right\|_H \\ &\quad \leq K_1TMe^{T|\omega|}\|\mathbf{y}_j - \bar{\mathbf{y}}\|_{\mathscr{C}([0,T];H)}, \quad \forall j \in \mathbb{N}_0, \end{aligned} \tag{3.18}$$

which implies that

$$\lim_{j\nearrow}\left\| \int_0^T e^{(T-t)A}\mathbf{f}\big(\mathbf{y}_j(t)\big)\,dt - \int_0^T e^{(T-t)A}\mathbf{f}\big[\bar{\mathbf{y}}(t)\big]\,dt \right\|_H = 0.$$

This, together with $(3.17)_1$, implies

$$\mathbf{y}^T = e^{TA}\mathbf{y}^0 + \int_0^T e^{(T-t)A}\mathbf{f}\big[\bar{\mathbf{y}}(t)\big]\,dt + \int_0^T e^{(T-t)A}B\bar{\mathbf{u}}(t)\,dt, \tag{3.19}$$

where $\bar{\mathbf{y}} \in \mathscr{C}([0,T];H)$ is the unique solution to equation (1.1) with $\mathbf{u} = \bar{\mathbf{u}}$ such that $\bar{\mathbf{y}}(0) = \mathbf{y}^0$. Since $(\mathbf{y}^0, \mathbf{y}^T) \in [\mathcal{D}(A)]^2$ is arbitrary, we thus proved: For every $\mathbf{y}^0 \in \mathcal{D}(A)$, $\mathcal{D}(A) \subset \mathcal{R}^{(1.1)}_{\mathbf{y}^0,T}$. Thanks to the fact that $\mathcal{D}(A)$ is dense in H, we see that $\mathcal{R}^{(1.1)}_{\mathbf{y}^0,T}$ is dense in H for every $\mathbf{y}^0 \in \mathcal{D}(A)$.

In this paragraph, we prove that for every $\mathbf{y}^0 \in H \setminus \mathcal{D}(A)$, $\mathcal{D}(A)$ is contained in $\overline{\mathcal{R}^{(1.1)}_{\mathbf{y}^0,T}}$, the closure of $\mathcal{R}^{(1.1)}_{\mathbf{y}^0,T}$ in H. But before the proof we should first note that this implies directly $H = \overline{\mathcal{R}^{(1.1)}_{\mathbf{y}^0,T}}$. Let $(\mathbf{y}^0, \mathbf{y}^T) \in [H \setminus \mathcal{D}(A)] \times \mathcal{D}(A)$ and let $\varepsilon \in (0,1)$. Choose $\overline{\mathbf{y}^0} \in \mathcal{D}(A)$ such that

$$\big\|\overline{\mathbf{y}^0} - \mathbf{y}^0\big\|_H < \frac{\varepsilon}{M\exp(T|\omega| + K_1 MTe^{T|\omega|})}.$$

From the results in the previous paragraph we can deduce that there exists a $\bar{\mathbf{u}} \in L^2(0,T;U)$ such that

$$\mathbf{y}^T = e^{TA}\overline{\mathbf{y}^0} + \int_0^T e^{(T-t)A}\mathbf{f}\big[\bar{\mathbf{y}}(t)\big]\,dt + \int_0^T e^{(T-t)A}B\bar{\mathbf{u}}(t)\,dt, \tag{3.20}$$

where $\bar{\mathbf{y}} \in \mathscr{C}([0,T];H)$ is the unique solution to equation (1.1) with $\mathbf{u} = \bar{\mathbf{u}}$ such that $\bar{\mathbf{y}}(0) = \overline{\mathbf{y}^0}$. Besides, by Theorem 2.1, equation (1.1) with $\mathbf{u} = \bar{\mathbf{u}}$ admits a unique solution $\mathbf{y} \in \mathscr{C}([0,T];H)$ such that $\mathbf{y}(0) = \mathbf{y}^0$. Apply again Theorem 2.1 to obtain

$$\big\|\mathbf{y}(T) - \mathbf{y}^T\big\|_H \le \|\mathbf{y} - \bar{\mathbf{y}}\|_{\mathscr{C}([0,T];H)} \le M\exp\big(T|\omega| + K_1 MTe^{T|\omega|}\big)\big\|\mathbf{y}^0 - \overline{\mathbf{y}^0}\big\|_H < \varepsilon.$$

Since ε is arbitrary, $\mathbf{y}^T \in \overline{\mathcal{R}^{(1.1)}_{\mathbf{y}^0,T}}$. Since $\mathbf{y}^T \in \mathcal{D}(A)$ is arbitrary, $\mathcal{D}(A) \subset \overline{\mathcal{R}^{(1.1)}_{\mathbf{y}^0,T}}$. Thanks to the fact that $\mathbf{y}^0 \in H \setminus \mathcal{D}(A)$ is arbitrary, the aim of this paragraph is achieved. The proof is complete. □

4 An illustrative example. Third-order semilinear dispersion equation on a finite interval

Let $\mathrm{L} \in (0,\infty)$. Fix arbitrarily $T \in (0,\infty)$. In this section we are concerned with the approximate-control problem for a semilinear third-order dispersion equation posed on the interval $(0,\mathrm{L})$. More precisely, we consider the control system/boundary value problem

$$\begin{cases} \partial_t y + \partial_x^3 y + \partial_x y + f \circ y = 0 & \text{in } (0,\mathrm{L}) \times (0,T), \\ y(0,\cdot) = y(\mathrm{L},\cdot) = 0,\ \partial_x y(\mathrm{L},\cdot) = b & \text{in } (0,T). \end{cases} \tag{4.1}$$

Here and henceforth, $f \circ y$ is given in the usual sense of function composition, *i.e.*, $f \circ y(x,t) = f[y(x,t)]$, $\forall (x,t) \in [0,\mathrm{L}] \times [0,T]$; f is given in the following way:

$$\begin{aligned} &f \in \mathscr{C}(\mathbb{R};\mathbb{R}) \quad \text{verifies: } \exists K \in (0,\infty) \text{ such that} \\ &\quad \bigl|f(y_1) - f(y_2)\bigr| \le K|y_1 - y_2|, \quad \forall (y_1,y_2) \in \mathbb{R}^2. \end{aligned} \tag{4.2}$$

Let us define an unbounded linear operator A in $L^2(0,\mathrm{L})$ by

$$\begin{cases} \mathcal{D}(A) := \{\psi \in H^3(0,\mathrm{L}); \psi(0) = \psi(\mathrm{L}) = \psi'(\mathrm{L}) = 0\}, \\ A\psi = -\psi''' - \psi', \quad \forall \psi \in \mathcal{D}(A); \end{cases} \tag{4.3}$$

we can prove that its adjoint is given by

$$\begin{cases} \mathcal{D}(A^*) := \{\phi \in H^3(0,\mathrm{L}); \phi(0) = \phi(\mathrm{L}) = \phi'(0) = 0\}, \\ A^*\phi = \phi''' + \phi', \quad \forall \phi \in \mathcal{D}(A^*). \end{cases} \tag{4.4}$$

A can be extended into an unbounded linear operator in $[\mathcal{D}(A^*)]'$ in the way

$$\begin{cases} \mathcal{D}(A) := H, \\ \langle \phi, A\psi\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} \\ \quad = \int_0^{\mathrm{L}} [\phi'''(x) + \phi'(x)]\psi(x)\,dx, \quad \forall (\psi,\phi) \in H \times \mathcal{D}(A^*). \end{cases} \tag{4.5}$$

We can prove easily by [1], Corollary 4.4, p.15, that A is the infinitesimal generator of a strongly continuous semigroup $\{e^{tA}\}_{t\in[0,\infty)}$ of contractions on $L^2(0,\mathrm{L})$.

In the mean time, we define the control operator $B: \mathbb{R} \to [\mathcal{D}(A^*)]'$ by

$$B\alpha = -\alpha[A - \mathrm{id}_{[\mathcal{D}(A^*)]'}]\xi = -\alpha\delta'(\cdot - \mathrm{L}), \quad \forall \alpha \in \mathbb{R}, \tag{4.6}$$

where δ is the Dirichlet distribution centered at 0 and A is given as in (4.5) (noting that the function ξ defined next belongs to $L^2(0,\mathrm{L})$ but *not* $\mathcal{D}(A)$), and $\xi \in \mathscr{C}^\infty([0,\mathrm{L}];\mathbb{R})$ is the unique solution to the boundary value problem

$$\begin{cases} \xi''' + \xi' + \xi = 0 \quad \text{in } (0,\mathrm{L}), \\ \xi(0) = \xi(\mathrm{L}) = 0, \quad \xi'(\mathrm{L}) = 1. \end{cases} \tag{4.7}$$

With the aid of the previous observations, we have

$$\begin{aligned} &\langle \phi, B\alpha\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} \\ &\quad = \bigl\langle \phi, -\alpha[A - \mathrm{id}_{[\mathcal{D}(A^*)]'}]\xi\bigr\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} \\ &\quad = -\alpha\langle \phi, A\xi\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} + \alpha\langle \phi, \xi\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} \\ &\quad = -\alpha\int_0^{\mathrm{L}} \bigl[\phi'''(x) + \phi'(x)\bigr]\xi(x)\,dx \\ &\qquad + \alpha\int_0^{\mathrm{L}} \phi(x)\xi(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&= \alpha\big[-\phi''(x)\xi(x) + \phi'(x)\xi'(x) - \phi(x)\xi''(x) - \phi(x)\xi(x)\big]_0^{\mathrm{L}} \\
&\quad + \alpha\int_0^{\mathrm{L}} \big[\xi'''(x) + \xi'(x) + \xi(x)\big]\phi(x)\,dx \\
&= \alpha\phi'(\mathrm{L}) = \langle \alpha, B^*\phi\rangle_{\mathbb{R}}, \quad \forall(\alpha,\phi) \in \mathbb{R}\times\mathcal{D}(A^*);
\end{aligned}
$$

that is, the adjoint $B^* \in \mathscr{L}(\mathcal{D}(A^*),\mathbb{R})$ to B is given by

$$
B^*\phi = \phi'(\mathrm{L}), \quad \phi \in \mathcal{D}(A^*). \tag{4.8}
$$

With A and B at hand, we have the following.

Lemma 4.1 *Let A be given by* (4.3), *and let $B:\mathbb{R} \to [\mathcal{D}(A^*)]'$ be given such that its dual B^* is defined by* (4.8). *Then for every $\eta \in \mathcal{D}(A^*)$, the function $t \mapsto B^*e^{(T-t)A^*}\eta$ belongs to $L^2(0,T)$ and satisfies*

$$
\int_0^T \big|B^*e^{(T-t)A^*}\eta\big|^2\,dt \le \|\eta\|_{L^2(0,\mathrm{L})}^2. \tag{4.9}
$$

Moreover, the assertion

$$
\exists C = C_{\mathrm{L},T} \quad \text{such that } \|\eta\|_{L^2(0,\mathrm{L})}^2 \le C^2\int_0^T \big|B^*e^{(T-t)A^*}\eta\big|^2\,dt, \quad \forall\eta \in \mathcal{D}(A^*), \tag{4.10}
$$

holds if and only if

$$
\mathrm{L} \notin \left\{2\pi\sqrt{\frac{k^2+k\ell+\ell^2}{3}}; (k,\ell) \in \mathbb{N}^2\right\}. \tag{4.11}
$$

Remark 4.1 Hereafter, we write $\widetilde{K} = \max(C\sqrt{T},1)$ whenever L satisfies (4.11).

Proof of Lemma 4.1 The first conclusion is just a re-statement of [29], Proposition 3.2, and the latter one is a re-statement of [29], Proposition 3.3 and Remark 3.6. □

We shall say that $y \in \mathscr{C}([0,T];L^2(0,\mathrm{L})) \cap L^2(0,T;L^2(0,\mathrm{L}))$ is a solution to the BVP (4.1) if $y(0,\cdot)$, $y(\mathrm{L},\cdot)$ and $\partial_x y(\mathrm{L},\cdot)$ belong to $L^2(0,T)$, respectively, and $y(0,\cdot) = y(\mathrm{L},\cdot) = 0$, $\partial_x y(\mathrm{L},\cdot) = b$ a.e. in $(0,T)$, and if y satisfies $(4.1)_1$ in the weak sense. By a standard contraction mapping argument, we can deduce the following lemma from the well-posedness results of the associated linearized problem.

Lemma 4.2 *Assume* (4.2). *For every pair $(y^0,h) \in L^2(0,\mathrm{L}) \times L^2(0,T)$, BVP* (4.1) *admits a unique solution $y \in \mathscr{C}([0,T];L^2(0,\mathrm{L})) \cap L^2(0,T;H_0^1(0,\mathrm{L}))$ such that $y(\cdot,0) = y^0$. Moreover, we have*

$$
\left.
\begin{aligned}
&\|y\|_{\mathscr{C}([0,T];L^2(0,\mathrm{L}))} \\
&\quad \le e^{KT}\big(\|y(\cdot,0)\|_{L^2(0,\mathrm{L})} + \|\partial_x y(\mathrm{L},\cdot)\|_{L^2(0,T)} + T\sqrt{\mathrm{L}}|f(0)|\big), \\
&\|y_1 - y_2\|_{\mathscr{C}([0,T];L^2(0,\mathrm{L}))} \\
&\quad \le e^{KT}\big[\|y_1(\cdot,0) - y_2(\cdot,0)\|_{L^2(0,\mathrm{L})} + \|\partial_x y_1(\mathrm{L},\cdot) - \partial_x y_2(\mathrm{L},\cdot)\|_{L^2(0,T)}\big],
\end{aligned}
\right\} \tag{4.12}
$$

whenever y, y_1 *and* y_2, *belonging to* $\mathscr{C}([0,T];L^2(0,\mathrm{L}))\cap L^2(0,T;H_0^1(0,\mathrm{L}))$, *are solutions to BVP* (4.1).

Remark 4.2 By a standard multiplier argument, we can prove that there exists a $C\in(0,\infty)$ depending only on L and T such that

$$\|y\|_{L^2(0,T;L^2(0,\mathrm{L}))}\le C\bigl(\bigl\|y(\cdot,0)\bigr\|_{L^2(0,\mathrm{L})}+\|b\|_{L^2(0,T)}+\bigl|f(0)\bigr|\bigr), \tag{4.13}$$

whenever $y\in\mathscr{C}([0,T];L^2(0,\mathrm{L}))\cap L^2(0,T;H_0^1(0,\mathrm{L}))$ is a solution to BVP (4.1).

Theorem 3.1-A *Assume* (4.2) *and* (4.11). *If* K *is sufficiently small, say* $Ke^{TK}<\widetilde{K}^{-2}T^{-\frac{1}{2}}$ (*where* $\widetilde{K}$ *is given as in Remark* 4.1), *then the system* (4.1) *is approximately controllable in time* T.

Proof In this proof, we write A and B for the operators given as in Lemma 4.1. Using these operators, we can rewrite the system (4.1) in the following abstract form:

$$\mathbf{y}'(t)=A\mathbf{y}+\mathbf{f}\bigl(\mathbf{y}(t)\bigr)+Bb(t)\quad\text{for } t\in(0,T], \tag{4.14}$$

where $\mathbf{y}(t)=y(\cdot,t)$, $H=L^2(0,\mathrm{L})$, $\mathbf{f}\in\mathscr{C}(H;H)$ is given by $\mathbf{f}(\psi)(x)=f[\psi(x)]$, $x\in[0,\mathrm{L}]$ and satisfies

$$\bigl\|\mathbf{f}(\psi)-\mathbf{f}(\phi)\bigr\|_H\le K\|\psi-\phi\|_H,\quad\forall(\psi,\phi)\in H^2.$$

Since L satisfies (4.11), it follows from Lemma 4.1 (in particular (4.10)) and Remark 4.1 that

$$\begin{aligned}\widetilde{K}^{-2}\int_0^T\bigl|e^{(T-t)A^*}\eta\bigr|_H^2\,dt&\le\frac{1}{TC^2}\bigl(T\|\eta\|_H^2\bigr)\\&\le\int_0^T\bigl|B^*e^{(T-t)A^*}\eta\bigr|^2\,dt\\&\le\widetilde{K}^2\|\eta\|_H^2,\quad\forall\eta\in\mathcal{D}\bigl(A^*\bigr).\end{aligned} \tag{4.15}$$

Combining all the information collected in this proof, we apply Theorem 3.1 and Lemma 4.2 to complete the proof. □

The above analysis, especially Theorem 3.1-A, shows that the nonlinear system (4.1) is approximately controllable in time T. The main idea used in the proof of the approximate controllability is to view the nonlinear system as a Lipschitz-perturbation of an approximately controllable linear system which is indicated by (4.15).

In the proof of Theorem 3.1-A, we applied (4.10) in an indirect way. Next, we would like to apply (4.10) more directly to analyze the system (4.1). Let us start by stressing that in the rest of the paper, H, A, B, and $\mathbf{f}$ are given as in the proof of Theorem 3.1-A. With the aid of these 'tools', we deduce easily from (4.10) that the system

$$\mathbf{y}'(t)=A\mathbf{y}(t)+Bb(t)\quad\text{for } t\in(0,T] \tag{4.16}$$

is exactly controllable in time T, *i.e.*, there exists a $b \in L^2(0,T)$ for every pair $(\mathbf{y}^0,\mathbf{y}^T) \in H^2$ such that the solutions $\mathbf{y} \in \mathscr{C}([0,T];H)$ to (4.16) verifying $\mathbf{y}(0) = \mathbf{y}^0$ satisfy also $\mathbf{y}(T) = \mathbf{y}^T$. To give a short proof of this property, we introduce first of all $\Lambda \in \mathscr{L}(H;H)$ by

$$\langle \Lambda\phi,\psi\rangle_H = \int_0^T \langle B^* e^{(T-t)A^*}\phi, B^* e^{(T-t)A^*}\psi\rangle_H \,dt, \quad \forall(\phi,\psi) \in H^2. \tag{4.17}$$

By the Lax-Milgram lemma, we can deduce from (4.10) that 0 belongs to the resolvent set $\rho(\Lambda)$ of Λ. For every pair $(\mathbf{y}^0,\mathbf{y}^T) \in H^2$, we see easily that the control $b(t) = B^* e^{(T-t)A^*}\Lambda^{-1}(\mathbf{y}^T - e^{TA}\mathbf{y}^0)$ can transfer the state $\mathbf{y}^0$ to the state $\mathbf{y}^T$ in time T. That is, the system (4.16) is exactly controllable. This property is more stringent than the one (*i.e.*, approximate controllability) used to prove Theorem 3.1-A. In view of the previous information, one natural question should be

Is the nonlinear system (4.14) *exactly controllable in time* T*?*

The next theorem is a partial answer to the above question. By 'partial' we mean that the theorem only concerns the case in which the nonlinearity $\mathbf{f}$ is 'weak'. More precisely, the answer states the following.

Theorem 3.1-B *Assume* (4.2) *and* (4.11). *If* K *is sufficiently small, say* $K < T^{-1}(C^2+1)^{-1}$ (*where* C *is given as in* (4.10)), *then the system* (4.1) *is exactly controllable in time* T.

Proof Let $(\mathbf{y}^0,\mathbf{y}^T) \in H^2$. Define a map $\Gamma : \mathscr{C}([0,T];H) \to \mathscr{C}([0,T];H)$ by

$$\begin{aligned}\Gamma\mathbf{y}(t) = {}& \int_0^t e^{(t-\tau)A}BB^* e^{(T-\tau)A^*}\Lambda^{-1}\left\{\mathbf{y}^T - e^{TA}\mathbf{y}^0 - \int_0^T e^{(T-\sigma)A}\mathbf{f}\big(\mathbf{y}(\sigma)\big)\,d\sigma\right\}d\tau \\ & + e^{tA}\mathbf{y}^0 + \int_0^t e^{(t-\tau)A}\mathbf{f}\big(\mathbf{y}(\tau)\big)\,d\tau, \quad \forall t \in [0,T].\end{aligned}$$

Let $(\mathbf{y}_1,\mathbf{y}_2) \in [\mathscr{C}([0,T];H)]^2$. By definition, we have immediately

$$\begin{aligned}&\|\Gamma\mathbf{y}_2 - \Gamma\mathbf{y}_1\|_{\mathscr{C}([0,T];H)} \\ &\quad\le \max_{t\in[0,T]}\left\|\int_0^t e^{(t-\tau)A}BB^* e^{(T-\tau)A^*}\Lambda^{-1}\left(\int_0^T e^{(T-\sigma)A}\big\{\mathbf{f}\big(\mathbf{y}_2(\tau)\big) - \mathbf{f}_1\big(\mathbf{y}(\sigma)\big)\big\}\,d\sigma\right)d\tau\right\|_H \\ &\qquad + \max_{t\in[0,T]}\left\|\int_0^t e^{(t-\tau)A}\big\{\mathbf{f}\big(\mathbf{y}_2(\tau)\big) - \mathbf{f}\big(\mathbf{y}_1(\tau)\big)\big\}\,d\tau\right\|_H.\end{aligned}$$

But

$$\max_{t\in[0,T]}\left\|\int_0^t e^{(t-\tau)A}\big\{\mathbf{f}\big(\mathbf{y}_2(\tau)\big) - \mathbf{f}\big(\mathbf{y}_1(\tau)\big)\big\}\,d\tau\right\|_H \le KT\|\mathbf{y}_2 - \mathbf{y}_1\|_{\mathscr{C}([0,T];H)}$$

and

$$\begin{aligned}&\max_{t\in[0,T]}\left\|\int_0^t e^{(t-\tau)A}BB^* e^{(T-\tau)A^*}\Lambda^{-1}\left(\int_0^T e^{(T-\sigma)A}\big\{\mathbf{f}\big(\mathbf{y}_2(\sigma)\big) - \mathbf{f}\big(\mathbf{y}_1(\sigma)\big)\big\}\,d\sigma\right)d\tau\right\|_H \\ &\quad\le \left(\int_0^T \left\|B^* e^{(T-\tau)A^*}\Lambda^{-1}\left(\int_0^T e^{(T-\sigma)A}\big\{\mathbf{f}\big(\mathbf{y}_2(\sigma)\big) - \mathbf{f}\big(\mathbf{y}_1(\sigma)\big)\big\}\,d\sigma\right)\right\|_H^2 d\tau\right)^{\frac{1}{2}}\end{aligned}$$

$$\leq \left\| \Lambda^{-1}\left(\int_0^T e^{(T-\sigma)A}\left\{ \mathbf{f}\big(\mathbf{y}_2(\sigma)\big) - \mathbf{f}\big(\mathbf{y}_1(\sigma)\big)\right\} d\sigma \right) \right\|_H$$
$$\leq C^2 \left\| \int_0^T e^{(T-\sigma)A}\left\{ \mathbf{f}\big(\mathbf{y}_2(\sigma)\big) - \mathbf{f}\big(\mathbf{y}_1(\sigma)\big)\right\} d\sigma \right\|_H$$
$$\leq C^2 KT \|\mathbf{y}_2 - \mathbf{y}_1\|_{\mathscr{C}([0,T];H)},$$

in which the '≤' in the second line follows from (4.9), Lemma 2.2, and Remark 2.1, the '≤' in the third line follows from (4.9), the '≤' in the fourth line follows from (4.10) and (4.17), and C is exactly the one in (4.10). Therefore,

$$\|\varGamma \mathbf{y}_2 - \varGamma \mathbf{y}_1\|_{\mathscr{C}([0,T];H)} \leq KT\big(C^2+1\big)\|\mathbf{y}_2 - \mathbf{y}_1\|_{\mathscr{C}([0,T];H)}.$$

This, together with the assumption $K < T^{-1}(C^2+1)^{-1}$ and the fact that $\mathbf{y}_1$ and $\mathbf{y}_2$ are chosen arbitrarily, implies $\varGamma$ is a (strict) contraction mapping on $\mathscr{C}([0,T];H)$. Therefore, there exists a unique $\bar{\mathbf{y}}$ such that $\bar{\mathbf{y}} = \varGamma\bar{\mathbf{y}}$.

On the other hand, it is obvious that $\varGamma\mathbf{y}(0) = \mathbf{y}^0$ and $\varGamma\mathbf{y}(T) = \mathbf{y}^T$ for every $\mathbf{y} \in \mathscr{C}([0,T];H)$. In particular,

$$\begin{cases} \bar{\mathbf{y}}(0) = \varGamma\bar{\mathbf{y}}(0) = \mathbf{y}^0, \\ \bar{\mathbf{y}}(T) = \varGamma\bar{\mathbf{y}}(T) = \mathbf{y}^T. \end{cases}$$

Since $(\mathbf{y}^0, \mathbf{y}^T) \in H^2$ is arbitrarily given, the proof is complete. □

5 Conclusion

In this paper we studied the influence of a Lipschitz perturbation on an approximately controllable linear system. The approximate controllability of the unperturbed linear system is described by a quantitative unique continuation property for trajectories of the system dual to the unperturbed one. We proved that when the perturbation influence is not intense, the perturbed nonlinear system is approximately controllable.

We also applied the afore-obtained 'perturbation theory' to check the approximate controllability of a semilinear third-order dispersion equation in which the unperturbed linear third-order dispersion equation is exactly controllable. By using the afore-obtained 'perturbation theory', we proved that the semilinear third-order dispersion equation is approximately controllable. By exploiting the exact controllability of the unperturbed linear third-order dispersion equation, and by applying the classical contraction mapping fixed point argument, we proved under the condition that the nonlinearity is weak that the semilinear dispersion equation in question is still exactly controllable.

Appendix: Proof of Lemmata 2.1 and 2.2

Proof of Lemma 2.1 For every $(t,\eta) \in [0,\infty) \times H$,

$$\begin{aligned} \big\langle \psi, S(t)\eta \big\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} &= \big\langle e^{tA^*}\psi, \eta \big\rangle_H \\ &= \big\langle \psi, e^{tA}\eta \big\rangle_H \\ &= \big\langle \psi, e^{tA}\eta \big\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'}, \quad \forall \psi \in \mathcal{D}\big(A^*\big), \end{aligned} \tag{A.1}$$

where the '=' in the third line follows from (2.2). In other words, we have proved just now that $S(t)$ extends e^{tA} for every $t \in [0, \infty)$.

We deduce from (A.1) that

$$S(0) = \mathrm{id}_{[\mathcal{D}(A^*)]'}. \tag{A.2}$$

For every $(\mu, \sigma, \eta) \in [0, \infty)^2 \times H$,

$$\begin{aligned} \langle \psi, S(\mu + \sigma)\eta \rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} &= \langle e^{(\mu+\sigma)A^*}\psi, \eta \rangle_H \\ &= \langle e^{\sigma A^*} e^{\mu A^*}\psi, \eta \rangle_H \\ &= \langle \psi, S(\mu)S(\sigma)\eta \rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'}, \quad \forall \psi \in \mathcal{D}(A^*); \end{aligned} \tag{A.3}$$

in other words,

$$S(\mu + \sigma) = S(\mu)S(\sigma), \quad \forall (\mu, \sigma) \in [0, \infty)^2. \tag{A.4}$$

For every $\eta \in [\mathcal{D}(A^*)]'$,

$$\begin{aligned} \|S(t)\eta - \eta\|_{[\mathcal{D}(A^*)]'} &= \sup_{\psi \in \mathcal{D}(A^*), \|\psi\|_{\mathcal{D}(A^*)} \le 1} \left| \langle \psi, S(t)\eta - \eta \rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} \right| \\ &= \sup_{\psi \in \mathcal{D}(A^*), \|\psi\|_{\mathcal{D}(A^*)} \le 1} \left| \langle e^{tA^*}\psi - \psi, \eta \rangle_H \right| \\ &\le \sup_{\psi \in \mathcal{D}(A^*), \|\psi\|_{\mathcal{D}(A^*)} \le 1} \left\| e^{tA^*}\psi - \psi \right\|_H \|\eta\|_H \\ &\le \sup_{\psi \in \mathcal{D}(A^*), \|\psi\|_{\mathcal{D}(A^*)} \le 1} \left\| \int_0^t e^{\tau A^*} A^*\psi \, d\tau \right\|_H \|\eta\|_H \\ &\le \sup_{\psi \in \mathcal{D}(A^*), \|\psi\|_{\mathcal{D}(A^*)} \le 1} \int_0^t \left\| e^{\tau A^*} A^*\psi \right\|_H d\tau \, \|\eta\|_H \\ &\le \sup_{\psi \in \mathcal{D}(A^*), \|\psi\|_{\mathcal{D}(A^*)} \le 1} M \int_0^t e^{\omega\tau} \, d\tau \, \|\psi\|_{\mathcal{D}(A^*)} \|\eta\|_H \\ &= M\|\eta\|_H \int_0^t e^{\omega\tau} \, d\tau \searrow 0, \quad \text{as } t \searrow 0, \end{aligned} \tag{A.5}$$

where the '=' in the second line follows from

$$\langle \psi, S(t)\eta - \eta \rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} = \langle e^{tA^*}\psi - \psi, \eta \rangle_H, \quad \forall (\psi, \eta) \in \mathcal{D}(A^*) \times [\mathcal{D}(A^*)]'. \tag{A.6}$$

The proof is complete. □

Proof of Lemma 2.2 Let $\mathbf{u} \in L^2(0, T; U)$, and $t \in [0, T]$. We have

$$\begin{aligned} &\left| \left\langle \psi, \int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau) \, d\tau \right\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} \right| \\ &\quad = \left| \int_0^t \langle B^* e^{(t-\tau)A^*}\psi, \mathbf{u}(\tau) \rangle_H \, d\tau \right| \end{aligned}$$

$$\leq \left(\int_0^t \left\| B^* e^{(t-\tau)A^*} \psi \right\|_H^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \left\| \mathbf{u}(\tau) \right\|_H^2 d\tau \right)^{\frac{1}{2}}$$
$$\leq \left(\int_0^T \left\| B^* e^{(T-\tau)A^*} \psi \right\|_H^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \left\| \mathbf{u}(\tau) \right\|_H^2 d\tau \right)^{\frac{1}{2}}$$
$$\leq K_2 \|\psi\|_H \left(\int_0^t \left\| \mathbf{u}(\tau) \right\|_H^2 d\tau \right)^{\frac{1}{2}}, \quad \forall \psi \in \mathcal{D}(A^*). \tag{A.7}$$

This, together with the Hahn-Banach extension theorem and with Riesz' representation theorem, implies that there exists an $\eta \in H$ such that

$$\left\langle \psi, \int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\, d\tau \right\rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'} = \langle \psi, \eta \rangle_H$$
$$= \langle \psi, \eta \rangle_{\mathcal{D}(A^*),[\mathcal{D}(A^*)]'}, \quad \forall \psi \in \mathcal{D}(A^*), \tag{A.8}$$

or

$$\int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\, d\tau = \eta (\in H).$$

To sum up, for every $\mathbf{u} \in L^2(0,T;U)$, the function $[0,T] \ni t \mapsto \int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\, d\tau \in H$ is well defined, and

$$\left\| \int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\, d\tau \right\|_H \leq K_2 \left(\int_0^t \left\| \mathbf{u}(\tau) \right\|_H^2 d\tau \right)^{\frac{1}{2}}, \quad \forall t \in [0,T]. \tag{A.9}$$

It remains to prove that the mapping $[0,T] \ni t \mapsto \int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\, d\tau \in H$ is continuous.

Let $\mathbf{u} \in L^2(0,T;U)$.

Fix $t_1 \in [0,T)$. For every $t \in [0,T]$ with $t > t_1$, we have

$$\int_0^t e^{(t-\tau)A} B\mathbf{u}(\tau)\, d\tau = \int_0^{t_1} e^{(t-\tau)A} B\mathbf{u}(\tau)\, d\tau + \int_{t_1}^t e^{(t-\tau)A} B\mathbf{u}(\tau)\, d\tau$$
$$= e^{(t-t_1)A} \int_0^{t_1} e^{(t_1-\tau)A} B\mathbf{u}(\tau)\, d\tau$$
$$+ \int_0^{t-t_1} e^{(t-t_1-\tau)A} B\mathbf{u}(\tau + t_1)\, d\tau. \tag{A.10}$$

But

$$\lim_{t \searrow t_1} e^{(t-t_1)A} \int_0^{t_1} e^{(t_1-\tau)A} B\mathbf{u}(\tau)\, d\tau = \int_0^{t_1} e^{(t_1-\tau)A} B\mathbf{u}(\tau)\, d\tau, \tag{A.11}$$

which follows from the fact that $\{e^{tA}\}_{t\in[0,\infty)}$ is a strongly continuous semigroup on H and $\int_0^{t_1} e^{(t_1-\tau)A} B\mathbf{u}(\tau)\, d\tau \in H$, and

$$\left\| \int_0^{t-t_1} e^{(t-t_1-\tau)A} B\mathbf{u}(\tau + t_1)\, d\tau \right\|_H$$
$$\leq K_2 \left(\int_0^{t-t_1} \left\| \mathbf{u}(\tau + t_1) \right\|_U^2 \right)^{\frac{1}{2}} \searrow 0, \quad \text{as } t \searrow t_1, \tag{A.12}$$

where the '$\leq$' in the second line follows from (A.9). Therefore,

$$\lim_{t\searrow t_1}\int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau)\,d\tau=\int_0^{t_1} e^{(t_1-\tau)A}B\mathbf{u}(\tau)\,d\tau. \tag{A.13}$$

To summarize, we have the following:

$$\text{The mapping } [0,T]\ni t\longmapsto \int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau)\,d\tau\in H \text{ is right-continuous.} \tag{A.14}$$

Fix $t_2\in(0,T]$. For every $t\in[0,T]$ with $t<t_2$, we have

$$\begin{aligned}\int_0^{t_2} e^{(t_2-\tau)A}B\mathbf{u}(\tau)\,d\tau&=\int_0^{t_2-t} e^{(t_2-\tau)A}B\mathbf{u}(\tau)\,d\tau+\int_{t_2-t}^{t_2} e^{(t_2-\tau)A}B\mathbf{u}(\tau)\,d\tau\\&=e^{tA}\int_0^{t_2-t} e^{(t_2-t-\tau)A}B\mathbf{u}(\tau)\,d\tau\\&\quad+\int_0^{t} e^{(t-\tau)A}B\mathbf{u}(\tau+t_2-t)\,d\tau.\end{aligned} \tag{A.15}$$

But

$$\begin{aligned}&\left\|\int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau+t_2-t)\,d\tau-\int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau)\,d\tau\right\|_H\\&\quad=\left\|\int_0^t e^{(t-\tau)A}B\big[\mathbf{u}(\tau+t_2-t)-\mathbf{u}(\tau)\big]\,d\tau\right\|_H\\&\quad\leq K_2\left(\int_0^t\left\|\mathbf{u}(\tau+t_2-t)-\mathbf{u}(\tau)\right\|_U^2\,d\tau\right)^{\frac{1}{2}}\\&\quad\leq K_2\sqrt{t_2}\left(\int_0^1\left\|\mathbf{u}(t\theta+t_2-t)-\mathbf{u}(t\theta)\right\|_U^2\,d\theta\right)^{\frac{1}{2}}\longrightarrow 0,\quad\text{as } t\nearrow t_2,\end{aligned} \tag{A.16}$$

and

$$\begin{aligned}&\left\|e^{tA}\int_0^{t_2-t} e^{(t_2-t-\tau)A}B\mathbf{u}(\tau)\,d\tau\right\|_H\\&\quad\leq MK_2e^{|\omega|t_2}\left(\int_0^{t_2-t}\left\|\mathbf{u}(\tau)\right\|_U^2\,d\tau\right)^{\frac{1}{2}}\searrow 0,\quad\text{as } t\nearrow t_2.\end{aligned} \tag{A.17}$$

Therefore,

$$\lim_{t\nearrow t_2}\int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau)\,d\tau=\int_0^{t_2} e^{(t_2-\tau)A}B\mathbf{u}(\tau)\,d\tau. \tag{A.18}$$

Thus, we proved that the mapping $[0,T]\ni t\mapsto\int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau)\,d\tau\in H$ is left-continuous. This, together with (A.14), implies that the mapping $[0,T]\ni t\mapsto\int_0^t e^{(t-\tau)A}B\mathbf{u}(\tau)\,d\tau$ is continuous.

The proof is complete. □

Competing interests
The author declares to have no competing interests.

Author's contributions
The author contributed to the work totally, and read and approved the final version of the manuscript.

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Endnote
[a] Throughout this paper, we denote by X' the topological dual to X whenever X is a Banach space, by A^* the adjoint to A whenever A is the infinitesimal generator of the dynamics of a linear control system with H as state space, by B^* the dual to B whenever B is a control operator, and by $\mathcal{D}(A)$ (resp. $\mathcal{D}(A^*)$) the domain of A (resp. A^*).

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