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Relaxation of Three Solenoidal Wells and Characterization of Extremal Three-phase H-measures

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Abstract

We fully characterize quasiconvex hulls for three arbitrary solenoidal (divergence free) wells in dimension three. With this aim we establish weak lower semicontinuity of certain functionals with integrands restricted to generic two-dimensional planes and convex in (up to three) rank-2 directions within the planes. Within the framework of the theory of compensated compactness, the latter represents an example when the differential constraints fail the constant rank condition but nevertheless the so-called Λ-convexity still implies lower semicontinuity and $\mathscr{A}$-quasiconvexity (which essentially means that rank-2 convexity implies S-quasiconvexity—that is quasiconvexity in the sense of the divergence-free differential constraints—on the planes). The proof employs a version of Müller's estimates of Haar wavelet projections in terms of the Riesz transform. The above semicontinuity result is then applied to the three solenoidal wells problem via analogs of Šverák's "nontrivial" quasiconvex functions and connectedness properties of the rank-2 envelopes. As another application of the semicontinuity result, we obtain a "geometric" result of a more general nature: characterization of certain extremal three-point H-measures for three-phase mixtures (of three characteristic functions) in dimension three. We also discuss the applicability of the results to problems with other differential constrains, in particular to three linear elastic wells, and further generalizations.

1. Introduction

The problem of characterizing microstructures which may result from mixing a given set of component "phases" emerges for example in variational modeling of martensitic phase transformation, see, for example [1,26] and further references therein, as well as in bounding effective properties of composites, see for example [4,23] also containing numerous further references. The mathematical approaches use the notion of relaxation for the underlying (non-convex) energy minimization problem subjected to appropriate differential constraints and lead to the deployment of fundamental mathematical concepts of quasiconvexity, Young measures and associated mathematical theory of compensated compactness [28,45].

Within the above general framework, this paper addresses two distinct specific problems, which appear related at the fundamental level. The first problem is the characterization of the S-quasiconvex hull K_S^{qc}, of a set K of three 3×3 matrices subjected to divergence-free differential constraints. The second one is the characterization of extremal three-point H-measures for mixtures of three characteristic functions in dimension three. Both problems are resolved via, in particular, an application of a non-classical version of a semicontinuity result of the theory of compensated compactness, developed in this paper following the ideas in [24] and most recently in [19].

The first problem, although of an intrinsic mathematical interest, is also directly relevant to bounding effective properties of composites, cf., for example [11,23]. To be specific, it is stated as follows: given a set $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{3\times3}$ of three real 3×3 matrices, characterize the set K_S^{qc} of all matrices B_0 such that there exists a sequence $\{B_h\} \subset L^2_{\rm loc}(\mathbb{R}^3, \mathbb{M}^{3\times3})$, $L^2_{\rm loc}$-equi-integrable, Q-periodic, with $Q = (0, 2\pi)^3$, and such that

$$\begin{cases} \mathrm{Div} B_h = 0 & \text{in } \mathscr{D}'(\mathbb{R}^3, \mathbb{R}^3)\,, \\ \mathrm{dist}(B_h, K) \to 0 & \text{locally in measure as } h \to \infty, \\ ⨍_Q B_h = B_0 & \forall\, h. \end{cases} \tag{1}$$

When the fields B_h are curl-free rather then divergence-free, and, thus, are gradients of suitable vector-fields, the analogous problem was solved by Šverák [40]. More generally, problem (1) falls into the framework of $\mathscr{A}$-quasiconvexity where the differential constraint on the fields B_h is replaced by more general ones (see for example [9]). Notice that the problem is stated in (1) within the L^2-theory, although the results could be extended to the L^p-theory context, $1 < p < \infty$, see [19] containing all the relevant additional technical ingredients.

As recently shown by Palombaro and Ponsiglione [33], non-constant solutions to the "exact" version of problem (1), that is non-constant divergence-free fields taking values in K almost everywhere, may only exist if K contains rank-2 connections, more precisely if $\mathrm{rank}(A_i - A_j) \leqq 2$ for some $i \neq j$ (the rank-2 connections correspond to pairwise compatibility under the divergence-free differential constraints). In contrast, as shown by Garroni and Nesi [11], there exist sets K with no rank-2 connections for which the problem of "approximate rigidity" (1) admits solutions for some $B_0 \notin K$. Such examples in [11] correspond to the cases when the two-dimensional plane through K contains three distinct "rank-2 directions" and the mutual position of A_1, A_2 and A_3 is such that the rank-2 lines through them form an "inner triangle" inside the convex hull of K, like in Fig. 1(1) below. This is analogous to similar constructions, also involving an inner triangle, employed for example in [30] in the context of optimal microstructures for conducting polycrystals, in [2] in the context of mutual compatibility of three pairwise incompatible linearly elastic wells, and in [39] in the context of extremal three-phase

H-measures (see also below). The inner triangle construction is, in turn, reminiscent of the so-called T_4 configuration for four mutually compatible although pairwise incompatible gradient fields. The latter was used in different contexts by various authors, for example in [36] in the context of counterexamples to regularity of elliptic systems (see also [27] for recent advances) and by Tartar [48] as an example of a mutual compatibility of pairwise incompatible matrices with resulting failure of compactness for associated sequences of gradient fields, followed by numerous publications. The construction involving the inner triangle will play an important role in this paper too, and we refer to all three point sets that enjoy this property as sets of *Type 1*. One of the main results of this paper (Theorem 5) asserts that if K contains no rank-2 connections then the set K_S^{qc} is non-trivial, that is $K_S^{qc} \supsetneq K$, if and only if K is of *Type 1*. This is somewhat analogous to a recent result in [6] on the triviality, in the context of gradient fields, of the quasiconvex hulls for sets of 2×2 matrices containing neither rank-1 connections nor T_4 configurations.

The characterization of K_S^{qc} when K is of *Type 1*, is performed in two steps. First one seeks an inner bound for K_S^{qc}, and then one proves the optimality of such bound. An explicit construction for the inner bound is provided by an "infinite-rank" sequential lamination, the idea successfully exploited earlier in a number of different settings, see for instance [2,30,36,39,48]. All the essential details specifically for the divergence free (Div-free) context are found in [11].

Establishing the optimality of the inner bound requires an additional analysis. For the T_4 configuration of the approximate non-rigidity for four pairwise incompatible gradient wells, one way to prove the sharpness of the inner bound is by employing the Šverák's [41] "nontrivial" quasiconvex (but not polyconvex) function $\det^+$. Our motivation is somewhat similar in spirit. For this we construct a suitable modification of a function (originally introduced in the study of composites in homogenization, for example [46]), which is rank-2 convex and quadratic and, therefore, quasiconvex in the space of Div-free fields. This modification is a function defined only on a "model" two-dimensional plane determined by the three-wells and partially resembles the Šverák's function since it behaves like $\det^+$ function and is rank-2 convex on the plane (see Lemma 4).

A crucial accompanying ingredient is in establishing weak lower semicontinuity of functionals with rank-2 convex integrands on the above generic two-dimensional plane (the central Theorem 1), which may be of an independent interest. Within the framework of the theory of compensated compactness, the latter represents an example when the differential constraints fail the constant rank condition, thereby invalidating the classical proofs, for example [9,29]. Nevertheless, akin to examples in [19,24] for diagonal gradient fields, the weak lower semicontinuity is still equivalent to $\mathscr{A}$-quasiconvexity—in this particular context, to S-quasiconvexity, that is quasiconvexity in the sense of the divergence-free differential constraints—on the planes. Moreover, the so-called Λ-convexity (specifically, the rank-2 convexity on a generic plane) implies the S-quasiconvexity.

For proving the latter semicontinuity result, we closely follow Müller [24] and utilize in an appropriate way three-dimensional modifications of Haar wavelet projection estimates in terms of the Riesz transform. In [24] Müller, in the context of the L^2-theory and in dimension two, employs techniques of harmonic analysis such as Paley-Littlewood dyadic decomposition and almost orthogonality, see for example [38], while the most recent work [19] extends the results not only to arbitrary dimensions but also for an L^p-theory, $1 < p < \infty$, by additionally employing the advanced tools of the Calderon-Zygmund theory. Both [24] and [19] subsequently apply the Haar projectors—Riesz transforms estimates for deriving the semicontinuity for separately convex integrands on diagonal gradient fields. Nevertheless, we believe that our generic semicontinuity result, Theorem 1, is essentially new, as well as allows interesting applications and further generalizations as presented in this paper and beyond, see for example [32].

As a first application, Theorem 1 allows us to fully characterize the S-quasiconvex hull for all sets of *Type 1*. In other cases, when K does not contain any rank-2 connection but is not a set of *Type 1*, we first employ the above mentioned analog of Šverák's function to "disconnect" the set K_S^{qc} (which coincides with the rank-2 convex hull K^{r2}) and then employ connectedness properties of rank-2 envelopes, following from results of Kirchheim [16] and Matoušek [20] (see Lemma 6). This allows us to prove that in such a case necessarily $K_S^{qc} = K$ (Theorem 3). All the remaining cases (see Definitions 3 and 4 and the subsequent Remarks) can be treated without any special difficulty (see Theorem 4 and Proposition 4, and Theorem 5 for a full catalog).

The second, related, problem addressed in this paper is the characterization of the H-measures associated with three-phase mixtures in dimension three. Problem (1) is equivalently reformulated in terms of a relaxation of a three-well energy ($N = 3$) as follows. Given the function $F(\eta) = \frac{1}{2}\min\{|\eta - A_i|^2, i = 1, \ldots, N\}$, $\eta \in \mathbb{M}^{3\times 3}$, $N \in \mathbb{N}$, $N \geqq 2$, and $\theta \in (0,1)^N$ with $\sum_i \theta_i = 1$, characterize the S-quasiconvexification of F, $Q_S^\theta F$, at fixed volume fractions θ:

$$\forall \eta \in \mathbb{M}^{3\times 3} \quad Q_S^\theta F(\eta) := \inf_{⨍ \chi_i = \theta_i} \inf_{B\in V} \frac{1}{2} ⨍_Q \left| \eta + B(x) - \sum_{i=1}^{N} \chi_i A_i \right|^2 \mathrm{d}x. \qquad (2)$$

Here V is the space of $L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{M}^{3\times 3})$ functions $B(x)$ which are Q-periodic, $\mathrm{Div} B = 0$ in $\mathscr{D}'(\mathbb{R}^3, \mathbb{R}^3)$ and $\int_Q B_h = 0$, and χ_i's, $i = 1, \ldots, N$, are characteristic functions of disjoint measurable subsets of Q. Zero sets over all θ, that is such η that $Q_S^\theta F(\eta) = 0$ for some θ, equivalently re-define the S-quasiconvex hull K_S^{qc}. Problems similar to (2) emerge in the framework of linearized elasticity, with the function F being the minimum of N quadratic functions of the linearized strain with same elastic moduli but different stress-free strains (see, for example [2,17,39]). More generally, one could consider a problem analogous to (2) for rather arbitrary differential constraints (in the above-mentioned framework of $\mathscr{A}$-quasiconvexity).

One approach to these problems is based on the idea of using Fourier analysis, following earlier precedents in the metallurgical literature, for example [15]. As a result (2) is equivalently reformulated into a problem of minimization with respect to special objects characterizing the "intensity" of the oscillations (of the characteristic functions) in various directions, the H-measures, introduced by Tartar [47], and independently by Gerard [12], the idea proposed and advanced in this context by Kohn [17]. This approach was developed further by Smyshlyaev and

Willis [39], who first argued that for a rather generic nonconvex function with a quadratic growth (namely the difference of a quadratic function and of another convex function) the quasiconvexification problem can be reduced to ("kinematically unconstrained") non-local minimum energy principle. This followed the general methodology employed before for nonlinear composites by Talbot and Willis, for example [43], consistently with the thesis on the "non-locality" of quasiconvexity, [18]. Further use of the Fourier transform naturally leads to the reformulation in the language of H-measures, in particular for multi-well energies with quadratic wells as in (2). This reduces the problem of relaxation to that of characterizing the extremal points of the (weak* compact, convex) set of H-measures.

When the number of phases is two, $N = 2$, the set of the H-measures is known and the relaxation of a two-well energy may be explicitly computed [17] (see also [35]). In contrast, for $N > 2$, the H-measures are not fully characterized. It is known that they satisfy some necessary restrictions [17], but these are in general not sufficient. For $N = 3$, Smyshlyaev and Willis [39] explicitly characterized the bigger convex set (the "superset") described by the known restrictions for the H-measures (whose extremal points are matrix Borel measures supported in no more than three Dirac masses). They also provided a sufficient condition by showing that among these critical points there is a large class of actual H-measures, realized generally by an infinite-order sequential lamination, and considered applications to three-well problems with gradient and linear elastic constraints (see also [10, 13]).

In the present paper we prove that the above condition of realizability is essentially *necessary*, at least for all the measures supported on three linearly independent directions. As a result we are able to fully characterize certain extremal three-point H-measures (Theorems 7, 8 and 9). One strategy for achieving this is the following. We study problem (2) for $N = 3$, and, following the recipe of [17], rewrite it as a minimization over the H-measures. We use next an algorithm of [39] which allows one to compute a lower bound on $Q_S^\theta F$ by minimizing over all extremal points of the superset. We find that every three-point extremal measure of the superset supported on linearly independent directions is the unique minimizing measure delivering a zero lower bound on $Q_S^\theta F$ at $\eta = \sum_i \theta_i A_i$, for a suitable choice of A_1, A_2, A_3 and θ. Then we use the results of the first part of the paper to establish the attainability or otherwise of this lower bound. Namely, the measure in question is an H-measure if and only if the zero lower bound is attained, that is if and only if $Q_S^\theta F(\eta) = 0$, equivalently, $\eta \in K_S^{qc}$, with $K = \{A_1, A_2, A_3\}$. We briefly sketch also an argument for establishing these results *directly* for the H-measures (Remark 8), with both approaches equivalent at a fundamental level via crucially relying on the key Theorem 1.

An attractive feature of the H-measures is that those are purely "geometrical" objects, that is they do not depend on the differential constraint but only on the microgeometry of mixing the characteristic functions. They thereby separate the microgeometry of mixing from the differential constraints, which makes the new H-measures' results potentially applicable to other problems of relaxation, for example, just to mention one, to that of linearized elastic wells, as we also briefly explore in this paper.

The structure of the paper is as follows. In Section 2 we define the S-quasiconvex hull and discuss its relation to the S-relaxation with fixed volume fractions. Section 3 reviews the results from [11] and [33], provides the main tool for proving the (sharp) outer bound for the S-quasiconvex hull K_S^{qc} of an arbitrary three-point set (Lemma 4, Theorem 1 and Corollary 1) and finally gives the full characterization of K_S^{qc} (Theorem 5). Section 4 is devoted to the proof of Theorem 1 (with the key wavelet analysis and estimates in terms of the Riesz transform) and some other technical results. The reformulation of the relaxation problem in terms of minimization with respect to the H-measures is discussed in Section 5 and follows [17] and [39]. The main results on the H-measures are stated and proved in Section 6: Theorem 7 essentially establishes that the sufficient conditions [39, Proposition 6.1] for realizability of a class of extremal three-point measures of the superset by the H-measures are also necessary, while Theorems 8 and 9 rule out some extremal measures outside the above class (except for degenerate cases listed in accompanying Remarks). Section 7 completes the description of the remaining cases and summarizes the results. Section 8 discusses further applications of the results, in particular to the problem of three linear elastic wells, and Section 9 discusses some further generalizations and prospects. The Appendices prove a technical lemma, review the definition and some properties of the H-measures and specialize those to the three divergence-free wells problem.

2. Preliminaries S-quasiconvexification problem

In this section we set some notation, state the S-quasiconvexification problem and give its equivalent formulation in terms of minimization for a quadratic three-well problem.

Let $Q = (0, 2\pi)^d$ be the periodicity cell in $\mathbb{R}^d$, $d \geqq 2$ is the spatial dimension. Let K be a subset of real-valued $m \times d$ matrices, $K \subset \mathbb{M}^{m\times d}$, $m \geqq 1$.

Definition 1. The S-quasiconvex hull K_S^{qc} of K is the set of all $B_0 \in \mathbb{M}^{m\times d}$ such that there exists a sequence $\{B_h\} \subset L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{M}^{m\times d})$, L^2_{loc}-equi-integrable[1], Q-periodic and such that

$$\begin{cases} \mathrm{Div} B_h = 0 & \text{in } \mathscr{D}'(\mathbb{R}^d, \mathbb{R}^m)\,, \\ \mathrm{dist}(B_h, K) \to 0 & \text{locally in measure as } h \to \infty, \end{cases} \tag{3}$$

and $⨍_Q B_h = B_0, \ \forall\, h.$

Here $⨍_Q B_h$ stands for the volume average $\frac{1}{|Q|}\int_Q B_h$, $|Q| := (2\pi)^d$ is the volume of Q. In (3) $\mathrm{Div} B_h = 0$ means that each row in matrix $B_h(x)$ is divergence-free in

[1] A sequence $\{B_h\} \subset L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{M}^{m\times d})$ is called L^2_{loc}-equi-integrable if for any bounded $\Omega \subset \mathbb{R}^d$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all measurable $E \subset \Omega$ with $\mathrm{meas}(E) < \delta$, $\sup_h ||B_h||_{L^2(E)} < \varepsilon$. The equi-integrability prevents "concentration".

the sense of distributions. The convergence locally in measure means here that for all $\varepsilon > 0$, measure $\{x \in Q : \text{dist}(B_h, K) > \varepsilon\} \to 0$ as $h \to \infty$.

It is easily checked that $K \subseteq K_S^{qc} \subseteq K^c$, where K^c denotes the convex hull of K. In particular, let K be a finite set of N distinct matrices, $A_1, \ldots, A_N$ in $\mathbb{M}^{m\times d}$, where $N \geqq 2$ is the number of "phases". In this case the above problem is related to that of relaxation of a "multi-well energy" of the form

$$F(\eta) = \tfrac{1}{2}\min\{|\eta - A_i|^2, i = 1, \ldots, N\}, \quad \eta \in \mathbb{M}^{m\times d} \tag{4}$$

(in (4) for $A \in \mathbb{M}^{m\times d}$ we denote $|A| := \left(\text{Tr}(A^T A)\right)^{1/2}$). Here the "relaxation" is also to be understood in the context of solenoidal (divergence free) fields. We will, in fact, equivalently deal with the so-called "S-quasiconvexification at fixed volume fractions", defined as follows. Let $\theta = (\theta_1, \ldots, \theta_N) \in [0, 1]^N$, with $\sum_{i=1}^N \theta_i = 1$, where θ_i, $i = 1, \ldots, N$, are "the volume fractions". Denote $I(\theta)$ the set of all characteristic functions $\chi(x) = (\chi_1(x), \ldots, \chi_N(x))$ of non-intersecting measurable subsets comprising Q with fixed volume fractions θ, that is

$$I(\theta) = \Big\{\chi : \mathbb{R}^d \to \{0,1\}^N,\ Q-\text{periodic}$$
$$\text{and measurable} : \sum_{j=1}^N \chi_j = 1 \text{ almost everywhere}, \ \fint_Q \chi = \theta\Big\}. \tag{5}$$

Definition 2. For any $\theta \in [0, 1]^N$, with $\sum_{i=1}^N \theta_i = 1$, the S-quasiconvexification of F at fixed "volume fractions" θ, denoted $Q_S^\theta F$, is

$$\forall \eta \in \mathbb{M}^{d\times d} \quad Q_S^\theta F(\eta) := \inf_{\chi\in I(\theta)} \inf_{B\in V} \frac{1}{2}\fint_Q \left|\eta + B(x) - \sum_{i=1}^N \chi_i A_i\right|^2 \mathrm{d}x. \tag{6}$$

Here V is the space of Q-periodic divergence-free matrix fields with zero average on Q,

$$V := \Big\{B \in L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{M}^{m\times d}),\ Q-\text{periodic},$$
$$\fint_Q B(x)\mathrm{d}x = 0, \quad \text{Div} B = 0 \text{ in } \mathscr{D}'(\mathbb{R}^d, \mathbb{R}^m)\Big\}. \tag{7}$$

Definition 2 falls in the more general framework of $\mathscr{A}$-quasiconvexity (see, for example [9]). Indeed formula (6) involves matrix fields subject to differential constraints of "solenoidal" (that is divergence free) type; hence the label S, being a particular example of more general differential constraints.

It follows directly from the definitions that:

Proposition 1. *$B_0 \in K_S^{qc}$ if and only if there exists $\theta \in [0, 1]^N$ such that $Q_\theta^S F(B_0) = 0$.*

We consider in this paper three-point sets $K = \{A_1, A_2, A_3\} \in \mathbb{M}^{3\times 3}$, that is set $N = d = m = 3$, and give in the next section a full characterization of K_S^{qc}. In the second part of the paper, we exploit the above equivalence to use the obtained results for characterizing certain extremal H-measures.

3. Characterization of the S-quasiconvex hull of three-point sets

The main purpose of this section is to fully characterize the S-quasiconvex hull K_S^{qc} of any three-point set $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{3\times 3}$. The matrices $A_i, i = 1, 2, 3$ are assumed not to lie on a single straight line (which is a trivial case for purposes of finding K_S^{qc}: $K_S^{qc} = K$ if the direction of the line is not a rank-2 direction and $K_S^{qc} = K^c$ otherwise).[2] Remind that K^c denotes the convex hull of K.

A special role is played by the "rank-2 directions" in $\mathbb{M}^{3\times 3}$, along which any two matrices are "compatible" in the sense of divergence free fields: if $\operatorname{rank}(A - C) \leqq 2$ (equivalently, $\det(A - C) = 0$) then for any simple lamination in a direction $\xi \in S^2$ such that $(A - C)\xi = 0$ the matrix field $B(x)$ taking constant values A and C in the alternating layers is obviously divergence-free. We will conventionally refer to such matrices as rank-2 connected and the connecting straight line as a rank-2 direction. Further analysis will depend on the number of such rank-2 directions in the two-dimensional plane formed by A_1, A_2 and A_3. If A_i, $i = 1, 2, 3$ are pairwise connected, that is $\operatorname{rank}(A_i - A_j) \leqq 2$ for all $i \neq j$, trivially $K_S^{qc} = K^c$ via a two-stage sequential lamination. The aim is therefore to characterize K_S^{qc} when K contains at least one pair of rank-2 disconnected matrices, that is $\operatorname{rank}(A_i - A_j) = 3$ for some $i \neq j$. Then the plane cannot contain more than three rank-2 directions (since $\det(A_i - tA_j) = 0$ as cubic equation with respect to t cannot have more than three solutions), and the main effort will be towards the case when there are exactly three such directions (the cases of less than three directions will be treated thereafter by a direct adaptation). Then, depending on the position of the three rank-2 directions relative to the triangle formed by A_1, A_2 and A_3 on this plane, the set K may be of three different types, as follows. Each vertex A_j of the triangle (A_1, A_2, A_3) may contain zero to three rank-2 lines through it pointing strictly inside the triangle. Provided K is rank-2 disconnected, the total number of such lines over the three vertices is always three. This suggests the following classification.

Definition 3. We say that K is of Type 1 if there is precisely one rank-2 line through each vertex pointing inside the triangle, and those lines do not intersect in a single point (cf. Definition 4 below), that is form an "inner triangle" inside K^c [see Fig. 1(1)]. We say that K is of Type 2 if the mutual position of A_1, A_2 and A_3 is such that the three vertices have one, two and zero such rank-2 lines, respectively, see Fig. 1(2). We say that K is of Type 3 if one of the three vertices has three lines pointing inside the triangle (and hence the others have none), see Fig. 1(3).

Definition 4. We say that K is a set of degenerate Type 1 if the "inner triangle" degenerates into a single point that we denote by S_0 (see Fig. 4).

Remark 1. The plane through A_1, A_2, A_3 contains three distinct rank-2 directions if and only if, after reduction to $K = \{0, I, A\}$ (always possible by shifting by a constant matrix and left multiplying by an invertible matrix, cf. Lemma 1 below), A is diagonalizable with distinct real eigenvalues.

[2] This conclusion will itself follow from the results on this Section as a trivial limit case.

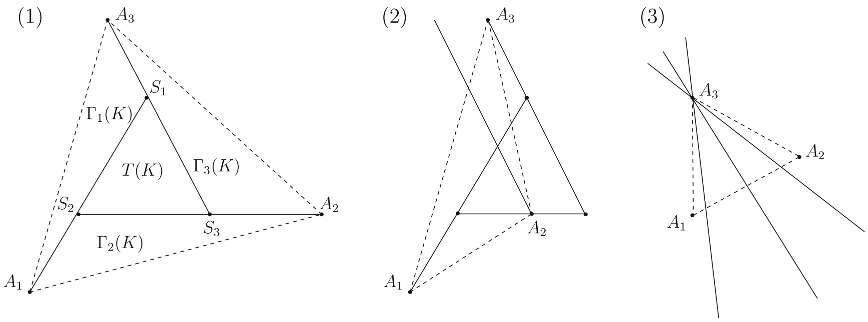

Fig. 1. (*1*) The "inner triangle" $S_1S_2S_3$ formed by the rank-2 lines for the sets of *Type 1*. (*2*) A set of *Type 2*. (*3*) A set of *Type 3*. In (1–3) the *dashed lines* delimit the convex hull while the *solid lines* are rank-2 lines

Remark 2. Cases when some of the matrices in K are rank-2 connected are "borderlines" between the above three types, some of which could be included as limit cases, while others have to be treated separately. For example, if on Fig. 1(1) the point S_1 merges with A_3 and/or S_2 with A_1, S_3 with A_2, those cases can, for purposes of the forthcoming analysis, be included as limit cases into Type 1. However similar limit cases for Types 2 and 3 will have to be considered separately.

We first study the sets of *Type 1* and then separately the sets of *degenerate Type 1*, see Proposition 3. We will show that if (not fully rank-2 connected) K is of *Type 1*, then $K \subsetneq K_S^{qc} \subsetneq K^c$ (Corollary 2), while for sets of *Type 2* the S-quasiconvex hull is trivial, that is $K_S^{qc} = K$ (Theorem 3), except for the limit cases containing rank-2 connections. The sets of *Type 3* with no rank-2 connections have trivial S-quasiconvex hulls, too. Their study does not present any special difficulty and is postponed to Section 7. The case of A having multiple eigenvalues follows essentially the same approach as for the sets of *Type 1* and *Type 2* and the related results are stated in Theorem 4. Finally, the case of non-diagonalizable A is treated in Section 7. We give in Theorem 5 a complete account of all the cases.

Remark 3. If $K = \{A_1, A_2, A_3\}$ does not contain any rank-2 connection, then there exists no "exact" divergence free matrix field B such that $B \in K$ almost everywhere, and $⨍ B = \sum_{i=1}^{3} \theta_i A_i$ with $\theta_i \in (0, 1)\ \forall\, i = 1, 2, 3$ (see [33]).

The next lemma shows that for the purpose of characterizing the S-quasiconvex hull of a set, one can make a convenient change of variables. In particular it allows us to reduce the problem to the diagonal case when dealing with sets of *Types 1, 2* and *3*.

Lemma 1. *Suppose* $K = \{A_1, A_2, A_3\} \subset \mathbb{M}^{3\times 3}$ *and* $\bar{K} = \{NA_1G+M,\ NA_2G+M,\ NA_3G+M\}$ *with* $G, N \in GL(3,\mathbb{R})$, $M \in \mathbb{M}^{3\times 3}$. *Then* $B_0 \in K_S^{qc}$ *if and only if* $NB_0G + M \in \bar{K}_S^{qc}$.

The proof of Lemma 1 is given in Appendix A.

We will now focus on the sets of *Type 1*. Before stating the main results of this section, we briefly explain how the sets of *Type 1* look (see [11,31,33] for further details). A key "geometric" property of every set $K = \{A_1, A_2, A_3\}$ of *Type 1* which does not contain rank-2 connections, see Fig. 1, is that one can find three matrices $S_1, S_2, S_3 \in \mathbb{M}^{3\times 3}$ such that

$$\begin{aligned} S_2 &= q_1 A_1 + (1-q_1) S_1, \\ S_3 &= q_2 A_2 + (1-q_2) S_2, \\ S_1 &= q_3 A_3 + (1-q_3) S_3, \end{aligned} \tag{8}$$

where $q_1, q_2, q_3 \in (0,1)$, and $\det(A_i - S_i) = 0$, $i = 1, 2, 3$, that is A_i and S_i are rank-2 connected. The rank-2 lines $A_1 S_1$, $A_2 S_2$, $A_3 S_3$ intersect forming the triangle $S_1 S_2 S_3$ as in Fig. 1. The following lemma describes algebraically all the sets of *Type 1* which do not contain rank-2 connections (in fact, it follows by direct calculation from (8) via Lemma 1 with appropriate diagonalization of matrices).

Lemma 2. *[33] Suppose $K \subset \mathbb{M}^{3\times 3}$ does not contain any rank-2 connection. Then K is of Type 1 if and only if there exist $q_1, q_2, q_3 \in (0,1)$, $G, N \in GL(3,\mathbb{R})$, $M \in \mathbb{M}^{3\times 3}$ such that*

$$K = \{M, N + M, NA + M\}, \tag{9}$$

where

$$A = \frac{1}{q_3}\left[\left(1 - \prod_{i=1}^{3}(1-q_i)\right) G^{-1} \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)\, G - q_2(1-q_3) I\right], \tag{10}$$

with

$$\lambda_1 = 0, \quad \lambda_2 = 1/(1-q_1), \quad \lambda_3 = q_2/(q_1 + q_2 - q_1 q_2). \tag{11}$$

Notation. For every $K = \{A_1, A_2, A_3\}$ of *Type 1*, we set (see Fig. 1):

$$\begin{aligned} \Gamma_1(K) = \Bigg\{ \xi \in \mathbb{M}^{3\times 3} : \xi &= t_1 A_1 + t_2 S_1 + t_3 A_3, \\ t_i &\in [0,1), t_1, t_3 \neq 0, \sum_{i=1}^{3} t_i = 1 \Bigg\}, \\ \Gamma_2(K) = \Bigg\{ \xi \in \mathbb{M}^{3\times 3} : \xi &= t_1 A_2 + t_2 S_2 + t_3 A_1, \\ t_i &\in [0,1), t_1, t_3 \neq 0, \sum_{i=1}^{3} t_i = 1 \Bigg\}, \\ \Gamma_3(K) = \Bigg\{ \xi \in \mathbb{M}^{3\times 3} : \xi &= t_1 A_3 + t_2 S_3 + t_3 A_2, \\ t_i &\in [0,1), t_1, t_3 \neq 0, \sum_{i=1}^{3} t_i = 1 \Bigg\}, \end{aligned}$$

$$T(K) := K^c - \bigcup_{i=1}^{3} \Gamma_i(K) \tag{12}$$

(hence $T(K)$ is the union of the closed triangle $S_1S_2S_3$ and the three "arms" $[A_1S_2)$, $[A_2S_3)$ and $[A_3S_1)$). The above includes the limit cases when K contains one or two rank-2 connections. For example, if $\mathrm{rank}(A_1 - A_2) \leqq 2$, then $A_1 = S_2$ and $T(K)$ is given by the union of the closed triangle $S_1S_2S_3$ and the two "arms" $[A_2S_3)$ and $[A_3S_1)$.

The next result provides an inner bound for the S-quasiconvex hulls of the sets of *Type 1*.

Lemma 3. *If K is of Type 1, then $T(K) \subseteq K_S^{qc}$.*

Proof. This follows from [11] (see in particular Lemmas 4.1 and 4.2 therein). An explicit construction realizing a point in $T(K)$ is that of infinite-rank sequential lamination, cf. [2,30,36,48]. □

In the sequel we will show that in fact $K_S^{qc} = T(K)$. By Lemma 3, we only need to prove $K_S^{qc} \subseteq T(K)$. By Lemma 1 it suffices to prove the latter in the diagonal case, that is when K is of the form $K = \{0, I, D(q)\}$, where $D(q)$ is given by (10) with $G = I$ and (q_1, q_2, q_3) an arbitrary point in $(0, 1)^3$.

To proceed, we first recall the notion of S-quasiconvexity (see [9] for the general setting).

Definition 5. A continuous function $f : \mathbb{M}^{3\times3} \to \mathbb{R}$ with quadratic growth is said to be S-quasiconvex if for every Q-periodic divergence free matrix field $B \in L^2_{\mathrm{loc}}(\mathbb{R}^3, \mathbb{M}^{3\times3})$

$$⨍_Q f(B)\,\mathrm{d}x \geqq f\left(⨍_Q B\,\mathrm{d}x\right). \tag{13}$$

If f is S-quasiconvex, $B_0 \in K_S^{qc}$ and $\{B_h\}$ satisfies (3) for $K = \{A_1, A_2, A_3\}$ and $⨍_Q B_h = B_0$, then necessarily $B_0 = \sum_{i=1}^3 \theta_i A_i$ for some $\theta \in [0, 1]^3$, with $\sum_{i=1}^3 \theta_i = 1$, and

$$f(B_0) \leqq \sum_{i=1}^{3} \theta_i f(A_i). \tag{14}$$

Hence if for a given $B_0 = \sum_{i=1}^3 \theta_i A_i$, for some S-quasiconvex f holds $f(B_0) > \sum_{i=1}^3 \theta_i f(A_i)$, then $B_0 \notin K_S^{qc}$. Unfortunately, we do not know any explicit S-quasiconvex function which can provide the optimal bound on K_S^{qc} when the set K is of the type (9). Therefore the characterization $K_S^{qc} = T(K)$ will be performed in several steps. The plan is briefly as follows:

Step 1. We consider a "model" plane π generated by two rank-2 matrices

$$V_1 = \mathrm{diag}(1, 1, 0), \quad V_2 = \mathrm{diag}(-1, 0, -1)\,. \tag{15}$$

Hence $\pi := \{M \in \mathbb{M}^{3\times 3} : M = uV_1 + vV_2 \text{ for some } u, v \in \mathbb{R}\}$. We construct a particular function $\mathcal{T}^+$ on π which is rank-2 convex, that is convex along all the (three) rank-2 directions contained in π (Lemma 4).

Step 2. In this central step we prove that inequality (14) holds true whenever $K \subset \pi$, $B_0 \in K_S^{qc}$ and f is a rank-2 convex function on π. This will follow from the key Theorem 1, establishing appropriate weak lower semicontinuity and Corollary 1.

Step 3. We show that, up to a transformation, the considered sets K are subsets of the plane π. Namely, for every $i = 1, 2, 3$, there exists a transformation of the type described in Lemma 1 that maps the rank-2 lines $A_i S_i$ and $A_{i+1} S_{i+1}$ into V_1 and V_2, respectively ($A_4 S_4 := A_1 S_1$). This will allow us to use the function $\mathcal{T}^+$ to check that for $B_0 \notin T(K)$ the inequality (14) fails and hence $B_0 \notin K_S^{qc}$, establishing $K_S^{qc} = T(K)$ (Theorem 2).

Step 1. Denote by π^+ the subset of π defined as follows

$$\pi^+ := \{M \in \pi : M = uV_1 + vV_2 \text{ for some } u > 0, v > 0\}.$$

Recall that a function $f : \mathbb{M}^{3\times 3} \to \mathbb{R}$ is said to be rank-2 convex if f is convex along all the rank-2 lines, that is if $t \to f(M + tV)$ is convex in t for every $M, V \in \mathbb{M}^{3\times 3}$ with $\mathrm{rank}(V) \leqq 2$.

Lemma 4. *(Construction of $\mathcal{T}^+$) There exists a continuous function $\mathcal{T}^+ : \pi \to \mathbb{R}$ with a quadratic growth, such that:*

$$\mathcal{T}^+ \text{ is rank-2 convex on } \pi, \text{ that is, } t \to f(M + tV) \text{ is convex for every } M, V \in \pi \tag{16}$$

with rank$(V) \leqq 2$;

$$\mathcal{T}^+(M) > 0 \quad \text{if } M \in \pi^+; \tag{17}$$
$$\mathcal{T}^+(M) = 0 \quad \text{if } M \in \pi \backslash \pi^+. \tag{18}$$

Proof. As a motivation, consider first the function $\mathcal{T} : \mathbb{M}^{3\times 3} \to \mathbb{R}$ given by

$$\mathcal{T}(M) = 2\mathrm{tr}\,(M^T M) - (\mathrm{tr}\, M)^2. \tag{19}$$

This is an S-quasiconvex function, see for example [47], satisfying (16) and (17), but not (18). The idea is to appropriately modify the restriction of $\mathcal{T}$ to the plane π to achieve (18). First, the restriction of $\mathcal{T}$ to π is, via (15):

$$\forall u, v \in \mathbb{R} \quad \mathcal{T}(uV_1 + vV_2) = 2[(u - v)^2 + u^2 + v^2] - 4(u - v)^2 = 4uv.$$

Define $\mathcal{T}^+ : \pi \to \mathbb{R}$ in the following way:

$$\forall u, v \in \mathbb{R} \qquad \mathcal{T}^+(uV_1 + vV_2) = u^+ v^+, \tag{20}$$

where u^+ and v^+ denote the positive parts of u and v : $u^+ := \max\{0, u\}$, $v^+ := \max\{0, v\}$ (the function $\mathcal{T}^+$ is loosely analogous to the function $\det^+$ of Šverák [41]). The function $\mathcal{T}^+$ satisfies (17) and (18) by construction. It is also rank-2 convex by direct inspection since the only rank-2 directions in π are V_1, V_2, $V_1 + V_2$, as follows from (15). □

Step 2. In this central step, we prove that inequality (14) holds for all rank-2 convex functions on π whenever $K \subset \pi$. For notational simplicity we regard any function f defined on π as a function on $\mathbb{R}^2$ via the identification:

$$(y_1, y_2) \in \mathbb{R}^2 \to y_1 V_1 + y_2 V_2 \in \pi,$$

and, when no ambiguity arises, we write $f(y_1, y_2)$ instead of $f(y_1 V_1 + y_2 V_2)$. Hence, if f is a rank-2 convex function on π, then as a function on $\mathbb{R}^2$ it is separately convex (that is convex in both y_1 and y_2), and is additionally *convex in the diagonal direction* $(1, 1)$, that is $t \in \mathbb{R} \to f(y_1 + t, y_2 + t)$ is convex for every $(y_1, y_2) \in \mathbb{R}^2$. This immediately follows from the fact that the only rank-2 directions in π are those along V_1, V_2 and $V_1 + V_2$.

Before introducing the main result, to clarify the motivation further consider Q-periodic divergence free matrix fields whose values are restricted to π, that is $B \in V_\pi$ where

$$V_\pi = \Big\{ B \in L^2_{\text{loc}}(\mathbb{R}^3, \mathbb{M}^{3\times 3}),\ Q-\text{periodic},\ \text{Div} B = 0 \text{ in } \mathscr{D}'(\mathbb{R}^3, \mathbb{R}^3), \\ B \in \pi \ \textit{almost everywhere} \Big\}.$$

Lemma 5. *Let $B \in V_\pi$. Then there exist $\eta_1, \eta_2, \eta_3 \in L^2_{\text{loc}}(\mathbb{R})$, $(0, 2\pi)$-periodic, such that*

$$B(x_1, x_2, x_3) = (\eta_3(x_3) - \eta_1(x_1))V_1 + (\eta_2(x_2) - \eta_1(x_1))V_2\,, \ \textit{almost everywhere}. \tag{21}$$

Moreover, for every rank-2 convex function f on π with quadratic growth

$$⨍_Q f(B) \geqq f\Big(⨍_Q B\Big). \tag{22}$$

Proof. By assumptions there exist $u, v \in L^2_{\text{loc}}(\mathbb{R}^3)$, $(0, 2\pi)^3$-periodic such that

$$B(x) = u(x)V_1 + v(x)V_2 \quad \textit{almost everywhere}.$$

The equation $\text{Div} B = 0$ then yields, see (15),

$$\partial_1(u - v) = 0, \quad \partial_2 u = 0, \quad \partial_3 v = 0, \quad \text{in } \mathscr{D}'(\mathbb{R}^3), \tag{23}$$

with shorthand notation $\partial_j := \frac{\partial}{\partial x_j}$, $j = 1, 2, 3$. Then (21) follows from (23) by explicit integration.

Further,

$$⨍_Q f(B)\,\mathrm{d}x = ⨍_Q f\,(\eta_3(x_3) - \eta_1(x_1), \eta_2(x_2) - \eta_1(x_1))\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 \\ \geqq f(\bar\eta_3 - \bar\eta_1, \bar\eta_2 - \bar\eta_1) = f\Big(⨍_Q B\,\mathrm{d}x\Big),$$

where $\bar\eta_1, \bar\eta_2, \bar\eta_3$ denote the averages of η_1, η_2, η_3 over $(0, 2\pi)$. In the last inequality we have, sequentially, used the convexity of the integrand in $\eta_2(x_2)$, $\eta_3(x_3)$ and in $\eta_1(x_1)$. $\square$

Notice in passing that relations (23) somewhat resemble those in the proof of [25, Theorem 1], although the precise detail is fundamentally different: the example in [25] is for gradient fields, which would essentially imply in the present notation, see (24) below, a stronger requirement of the H^{-1}_{loc} convergence of the *gradients* of $\partial_2 u_h$, $\partial_3 v_h$ and $\partial_1(u_h - v_h)$. As a result, less sophisticated methods suffice in [25].

The next key result builds on the above motivation and provides the central tool for proving our main claim. It is a modification of a result due to Müller [24, Theorem 1] (see also [19]), appropriately adjusted to the present setting of divergence-free differential constraints in three dimensions, although is of a more general interest, as the rest of this paper partly demonstrates, cf. for example Remark 8 (see also [32] for a further generalization).

Theorem 1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a separately convex function additionally convex in the direction $(1, 1)$, satisfying $0 \leqq f(y) \leqq C(1 + |y|^2)$. Let $U \subset \mathbb{R}^3$ be open and suppose that*

$$\begin{aligned} &u_h \rightharpoonup u_\infty, \quad v_h \rightharpoonup v_\infty \quad \text{in } L^2_{\text{loc}}(U) \text{ as } h \to \infty, \\ &\partial_2 u_h \to \partial_2 u_\infty, \quad \partial_3 v_h \to \partial_3 v_\infty, \\ &\partial_1(u_h - v_h) \to \partial_1(u_\infty - v_\infty) \quad \text{in } H^{-1}_{\text{loc}}(U) \text{ as } h \to \infty. \end{aligned} \tag{24}$$

Then for every open set $V \subset U$

$$\int_V f(u_\infty, v_\infty)\,\mathrm{d}x \leqq \liminf_{h\to\infty} \int_V f(u_h, v_h)\,\mathrm{d}x. \tag{25}$$

The proof of Theorem 1 is postponed to Section 4.

Remark 4. Theorem 1 can be interpreted as a statement that rank-2 convexity is equivalent to S-quasiconvexity on two-dimensional plane π, cf. [19,24] for analogous interpretation for gradient fields on diagonal matrices. This allows various further interpretations, for example, in terms of divergence free fields Young measures supported on π being laminates and of the existence of S-quasiconvex functions ε-close on any compact subsets of π to a given function rank-2 convex on π, cf. [24, Theorem 2 and Corollary 3], and $K_S^{qc} = K^{r2}$ (see Definition 6 below).

On the other hand notice that, alike in [19,24], within the framework of the theory of compensated compactness (24) represents an example of differential constraints failing the constant rank condition, thereby invalidating the classical proofs, for example [9,29].

Corollary 1. *Let $K = \{A_1, A_2, A_3\} \subset \pi$ and let $B_0 = \sum_{i=1}^{3} \theta_i A_i \in K_S^{qc}$. Then* (14) *holds true for every function f satisfying the assumptions of Theorem 1 (that is rank-2 convex on π and with quadratic growth).*

Proof. By definition of K_S^{qc} there exists a sequence $\{B_h\}$ satisfying (3) with $⨍_Q B_h = B_0 := u_\infty V_1 + v_\infty V_2$. Then it directly follows from (3), cf. also Proposition 1, that there exists a sequence $\{\chi^h\} \subset I(\theta)$ of Q-periodic characteristic functions such that:

$$A^h := \sum_{i=1}^{3} \chi_i^h A_i \in \pi \text{ almost everywhere}, \quad \fint_Q A^h = B_0,$$
$$\operatorname{Div} A^h \to 0 \quad \text{strongly in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^3), \qquad A^h \overset{*}{\rightharpoonup} B_0 \quad \text{in } L^\infty(\mathbb{R}^3)$$

(for example, we can take "periodic rescaling" $\chi^h(x) = \tilde{\chi}^h(n(h)x)$, where $\tilde{\chi}^h(x)$ is a minimizing sequence in (6) associated with B_h and $n(h) \in \mathbb{N}$, $n(h) \to \infty$).

Therefore $A^h = u_h V_1 + v_h V_2$ for some functions u_h, v_h which satisfy

$$u_h \rightharpoonup u_\infty,\ v_h \rightharpoonup v_\infty \quad \text{weakly in } L^2_{\mathrm{loc}}(\mathbb{R}^3),$$
$$\partial_2 u_h \to 0,\ \partial_3 v_h \to 0,\ \partial_1(u_h - v_h) \to 0 \quad \text{strongly in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^3).$$

We then apply Theorem 1 with $V = Q$ to get

$$\fint_Q f(u_\infty, v_\infty)\, \mathrm{d}x \leqq \liminf_{h\to\infty} \fint_Q f(u_h, v_h)\, \mathrm{d}x.$$

Since $\liminf_{h\to\infty} \fint_Q f(u_h, v_h)\, \mathrm{d}x = \sum_{i=1}^{3} \theta_i f(A_i)$, (14) follows from $f(u_\infty, v_\infty) \equiv f(B_0)$. □

Step 3. We are now ready to demonstrate that $K_S^{qc} = T(K)$ for all sets K of the form $K = \{0, I, D(q)\}$, where $D(q)$ is defined by (10), (11) with $G = I$ and $q = (q_1, q_2, q_3) \in (0, 1)^3$. It will be convenient to give the explicit expressions for the matrices $D(q)$, $S_1(q)$, $S_2(q)$, $S_3(q)$ in this case, which follows by straightforward calculation from (10), (11) and (8) with $A_1 = 0$, $A_2 = I$, $A_3 = D(q)$:

$$\begin{aligned}
D(q) &= \operatorname{diag}\left(-\frac{q_2}{q_3}(1-q_3),\ \frac{q_1+q_3-q_1q_3}{q_3(1-q_1)},\ \frac{q_2}{q_1+q_2-q_1q_2}\right),\\
S_1(q) &= \operatorname{diag}\left(0,\ \frac{1}{1-q_1},\ \frac{q_2}{q_1+q_2-q_1q_2}\right),\\
S_2(q) &= \operatorname{diag}\left(0,\ 1,\ \frac{q_2(1-q_1)}{q_1+q_2-q_1q_2}\right),\\
S_3(q) &= \operatorname{diag}\left(q_2,\ 1,\ \frac{q_2}{q_1+q_2-q_1q_2}\right).
\end{aligned} \tag{26}$$

Theorem 2. *Let $q \in (0, 1)^3$ and let $K = \{0, I, D(q)\}$. Then $K_S^{qc} = T(K)$.*

Proof. Let $B_0 \in K_S^{qc}$. By Lemma 3, it suffices to prove that $B_0 \notin \bigcup_{i=1}^3 \Gamma_i(K)$; see (12). This is achieved by a version of a "biting-out" argument as follows. For simplicity we omit displaying the dependence on q in the matrices (26). We first show that $B_0 \notin \Gamma_3(K)$. Recall that the lines S_3S_1 and S_2S_3 are rank-2 lines. Then, for $N = \operatorname{diag}\left(-\frac{1}{q_2},\ \frac{1-q_1}{q_1},\ -\frac{q_1+q_2-q_1q_2}{q_1q_2}\right)$ we find

$$N(S_1 - S_3) = V_1 \ \text{ and } \ N(S_3 - S_2) = V_2.$$

By assumption there exists a sequence $\{B_h\}$ satisfying (3) with $⨍_Q B_h = B_0$. We now define the new sequence $\{B'_h\}$ and the set K' in the following way:

$$\forall h \quad B'_h := N(B_h - S_3)\,, \quad K' := \{-NS_3, N(I - S_3), N(D - S_3)\}\,.$$

It is readily seen that $\{B'_h\}$ satisfies the following:

$$\begin{cases} \operatorname{Div} B'_h = 0 & \text{in } \mathscr{D}'(\mathbb{R}^3, \mathbb{R}^3), \\ \operatorname{dist}(B'_h, K') \to 0 & \text{in measure}, \\ ⨍_Q B'_h = N_1(B_0 - S_3). \end{cases} \tag{27}$$

Next notice that the condition $B_0 \notin \Gamma_3(K)$ is equivalent to $N(B_0 - S_3) \in \pi \backslash \pi^+$. In order to prove the latter inclusion we use the function $\mathscr{T}^+$ of Lemma 4. Since, by construction, $\mathscr{T}^+|_{K'} = 0$, using (27) and Corollary 1, one gets

$$\mathscr{T}^+(N(B_0 - S_3)) \leqq 0,$$

implying the desired inclusion via (17).

To prove that $B_0 \notin \Gamma_1(K)$, one first finds a diagonal matrix N' such that

$$N'(S_1 - S_3) = \operatorname{diag}(1, 1, 0)\,, \quad N'(S_2 - S_1) = \operatorname{diag}(0, -1, -1),$$

which is possible since S_1S_3 and S_2S_1 are rank-2 lines. Then one employs Lemma 1 to make a change of variable (via a simple permutation matrix in this case) and reduces to the previous case. In a fully analogous way, one shows that $B_0 \notin \Gamma_2(K)$. □

Corollary 2. *If K is a set of Type 1, then $K_S^{qc} = T(K)$.*

Proof. This follows from Lemmas 1, 2 and Theorem 2. □

We now turn to the characterization of the S-quasiconvex hull of the sets of *Type 2*. To proceed, we need the following standard definition.

Definition 6. We define the rank-2 convex hull K^{r2} of a set K as

$$K^{r2} = \{M \in \mathbb{M}^{3\times 3} : f(M) \leqq \sup_K f, \text{ for all rank-2 convex } f\}\,.$$

Trivially, the rank-2 convex hull provides an inner approximation of the S-quasiconvex hull of a set: $K^{r2} \subseteq K_S^{qc}$.

An immediate consequence of Corollary 1 (see also Remark 4) is the following

Corollary 3. *Let $K = \{A_1, A_2, A_3\} \subset \pi$. Then $K_S^{qc} = K^{r2}$.*

Hence, to characterize the S-quasiconvex hull of the sets of *Type 2*, we can equivalently deal with the rank-2 convex hull. We will employ the following lemma, which is a particular case of a more general result first claimed in [21,34] and later proved in [16,20].

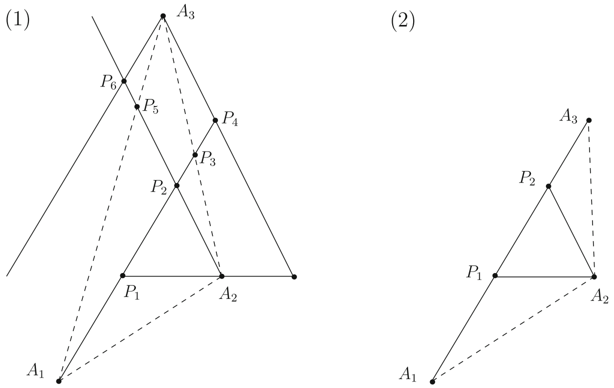

Fig. 2. *Type 2*: (*1*) generic case, (*2*) degenerate case

Lemma 6. *Let $C_1, \ldots, C_k$ be disjoint compact sets and let $K^{r2} \subset \cup_{i=1}^{k} C_i$. Then $K^{r2} = \cup_{i=1}^{k}(K \cap C_i)^{r2}$.*

Proof. Follows by direct adaptation of [20, Proposition 6.1] to present setting. □

Theorem 3. *If K is a set of Type 2 containing no rank-2 connection, then $K_S^{qc} = K$.*

Proof. Using Lemma 1 we can make a suitable change of variables and reduce as before to the case $K \subset \pi$. Then, employing the same "biting-out" arguments as in the study of *Type 1*, we can rule out from K_S^{qc} the three triangles [see Fig. 2(1)]: $A_1P_1A_2$, $A_1P_3A_3$ ($\subset A_1P_4A_3$), $A_2P_5A_3$ ($\subset A_2P_6A_3$). Therefore, it only remains to eliminate the triangle $P_1P_2A_2$ and the "arm" $(A_1, P_1]$. Let $C_1 = \{A_3\}$ and let C_2 be $P_1P_2A_2 \cup [A_1, P_1]$. Then, by Lemma 6, $K^{r2} = A_3 \cup (K \cap C_2)^{r2}$. Clearly $(K \cap C_2)^{r2} = \{A_1, A_2\}^{r2} = \{A_1, A_2\}$ since A_1 and A_2 are rank-two disconnected. Hence $K^{r2} = \{A_1, A_2, A_3\} = K$. Finally, by Corollary 3 $K_S^{qc} = K$ as required. □

Remark 5. The limit case of *Type 2* when K contains a rank-2 direction is, of course, special and results in a non-trivial K_S^{qc}. Assume A_1 and A_3 are (rank-2) connected. Let A_2 be connected to points P_1 and P_2, see Fig. 2(2), $P_j = t_jA_1 + (1 - t_j)A_3$, $j = 1, 2$, $0 \leqq t_2 < t_1 \leqq 1$. Then K_S^{qc} is the union of $[A_1, A_3]$ and the closed triangle $[A_2P_1P_2]$. The inner inclusion holds by a simple sequential lamination. The triangles $A_1P_1A_2$ and $A_3P_2A_2$ are eliminated by the same method as above. The other limit case corresponds to A_2 connected to no point in $[A_1, A_3]$, in which case K_S^{qc}=$[A_1, A_3] \cup A_2$, via the same argument as in Theorem 3.

We conclude this section with treating the case when $K = \{0, I, A\}$ and A is diagonalizable (on $\mathbb{R}$) with multiple eigenvalues.

Theorem 4. *Assume that the set $K = \{0, I, A\}$ does not contain any rank-2 connection and that the matrix A is diagonalizable with a real eigenvalue of multiplicity two or three. Then $K_S^{qc} = K$.*

Proof. If A has an eigenvalue of multiplicity three, then $A = aI$ for some $a \in \mathbb{R}$, $a \neq 1$, $a \neq 0$. Then the original Tartar's S-quasiconvex function $\mathscr{T}$, see (19), provides the desired bound. Namely, inequality (14) for $f = \mathscr{T}$ reads

$$-3(\theta_2 + a\theta_3)^2 \leqq -3\theta_2 - 3a^2\theta_3,$$

which is equivalent to $(a-1)^2\theta_2\theta_3 + \theta_1\theta_2 + a^2\theta_3\theta_1 \leqq 0$ and is, hence, never satisfied unless $\theta_i = 1$ for some $i = 1, 2, 3$.

Now assume that A has two distinct eigenvalues, one of multiplicity two. In this case the two-dimensional subspace generated by I and A contains only two rank-2 lines (one of which should in fact be rank-1). The case when the corresponding affine rank-2 lines through 0, I and A do not intersect inside K^c does not present any difficulty and is treated in Section 7 together with the sets of *Type 3*. Here we assume that there is one point of intersection inside K^c. Then the proof is similar to that of Theorems 2 and 3 and is, therefore, only sketched here. Namely, up to a transformation, we may regard A diagonal and $K \subset \tilde{\pi}$, where $\tilde{\pi}$ is the plane generated by the matrices $W_1 := \mathrm{diag}(0, 1, 1)$ and $W_2 := \mathrm{diag}(1, 0, 0)$:

$$\tilde{\pi} := \{M \in \mathbb{M}^{3\times 3} : M = uW_1 + vW_2 \text{ for some } u, v \in \mathbb{R}\}.$$

Then we follow the same approach as before. We define the function $\widetilde{\mathscr{T}}^+ : \tilde{\pi} \to \mathbb{R}$ via $\widetilde{\mathscr{T}}^+(uW_1 + vW_2) = u^+v^+$ and observe that $\widetilde{\mathscr{T}}^+$ is rank-2 convex on $\tilde{\pi}$ since the only rank-2 (or rank-1) directions contained in $\tilde{\pi}$ are those generated by W_1 and W_2. Next, up to a further modification of Theorem 1 as sketched below and Corollary 1, we show that rank-2 convexity on $\tilde{\pi}$ implies S-quasiconvexity on $\tilde{\pi}$ (see also [32, Theorem 1.3] for a further generalization unifying in a sense this case with that in Theorem 1). Namely, the assumptions (24) in Theorem 1 are replaced by

$$\partial_2 u_h \to \partial_2 u_\infty, \quad \partial_3 u_h \to \partial_3 u_\infty, \quad \partial_1 v_h \to \partial_1 v_\infty \quad \text{in } H^{-1}_{\mathrm{loc}}(\mathbb{R}^3) \text{ as } h \to \infty,$$

with the same conclusion held for any f separately convex with quadratic growth. The proof relies again on Theorem 6, which implies an appropriately modified version of Lemma 7: in (28) only the term containing $h^{(1,0,0)}$ is kept for u, and the summand for v contains three similar terms with $h^{(0,1,0)}$, $h^{(0,0,1)}$ and $h^{(0,1,1)}$ and arbitrary coefficients. Then, arguing as for the sets of *Type 2*, we first show that K^{r2} is the union of two disjoint sets and apply Lemma 6. □

The following theorem gives a full catalog of all the possible cases.

Theorem 5. *Let* $K = \{A_1, A_2, A_3\}$. *Then:*

(i) *if* $\mathrm{rank}(A_i - A_j) \leqq 2 \ \ \forall i, j = 1, 2, 3$, *then* $K^{qc}_S = K^c$*;*
(ii) *if* K *is a set of Type* 1*, including the limit cases (see Remark 2), then* $K^{qc}_S = T(K)$*, see Fig.* 1(1)*;*
(iii) *if* K *is a set of degenerate Type* 1*, then* $K^{qc}_S = [A_1, S_0] \cup [A_2, S_0] \cup [A_3, S_0]$ *[cf. Fig.* 4(1)*];*
(iv) *if* K *is Type* 2 *or Type* 3 *and contains no rank-2 connected matrices, then* $K^{qc} = K$ *(this includes the limit case of all the three matrices lying on a single non-rank-2 directed line);*

(v) *in the degenerate Type 2 case, for example* $\det(A_1 - A_3) = 0$*: either* $\det(A_2 - P_j) = 0$*,* $j = 1, 2$*, with* $P_j = t_j A_1 + (1 - t_j)A_3$*,* $0 \leqq t_2 < t_1 \leqq 1$*, with* $K_S^{qc} = [A_1 A_3] \cup [A_2 P_1 P_2]$ *[Fig. 2(2)]; or* $\det\big(A_2 - (tA_1 + (1-t)A_3)\big) \neq 0\ \forall t \in [0, 1]$ *in which case* $K^{qc} = [A_1, A_3] \cup A_2$*;*

(vi) *in the degenerate Type 3 case, for example* $\det(A_1 - A_3) = 0$*,* $K^{qc} = [A_1, A_3] \cup A_2$*;*

(vii) *if there are two rank-2 directions in the plane* $A_1 A_2 A_3$*,* $K_S^{qc} = K$*, unless at least one of those coincides with, for example the* $A_1 A_3$ *line, in which case*

(vii-1) *either there exists* $t \in [0, 1]$ *such that* $\det(A_2 - (tA_1 + (1-t)A_3)) = 0$ *and then* $K_S^{qc} = [A_1 A_3] \cup [A_2, tA_1 + (1-t)A_3]$*;*

(vii-2) *or* $K_S^{qc} = [A_1 A_3] \cup A_2$*;*

(viii) *in all other cases (of a single or no rank-2 direction in the plane)* $K_S^{qc} = K$*, unless the (single) direction coincides, for example with* $A_1 A_3$ *line, in which case* $K_S^{qc} = [A_1 A_3] \cup A_2$.

Proof. The cases *(i),(ii),(v)* and *(vi)* either are trivial or are covered by the arguments used in the previous part of this section. The case *(iii)* is treated in the proof of Proposition 3 in Section 6 below. In the case *(vii)* after reduction to $K = \{0, I, A\}$, A has a multiple eigenvalue, and we apply the arguments in the proof of Theorem 4.

The cases which are left to complete the proof of Theorem 5 are first when K is a set of *Type 3* and second when A is not diagonalizable. These cases are treated in Proposition 4 when K does not contain any rank-2 connection. However the proof extends as well to the case of rank-2 connected A_1 and A_3. □

4. Proof of Theorem 1

The proof of Theorem 1 follows by an adaptation of the approach of [24], see also [19]. Let $h : \mathbb{R} \to \mathbb{R}$ be defined as $h = 1$ on $(0, 1/2]$, $h = -1$ on $(1/2, 1]$ and $h = 0$ elsewhere. For $j \in \mathbb{Z}$, $k \in \mathbb{Z}^3$, $\varepsilon \in \{0, 1\}^3 \setminus (0, 0, 0)$ we define the three-dimensional Haar wavelet basis, cf. for example [22,38], $\{h_{j,k}^{(\varepsilon)}(x)\}_{k,j,\varepsilon}$ as

$$h_{j,k}^{(\varepsilon)}(x) = h^{(\varepsilon)}(2^j x - k)$$

where $h^{(\varepsilon_1,\varepsilon_2,\varepsilon_3)}(x) := (h(x_1))^{\varepsilon_1}(h(x_2))^{\varepsilon_2}(h(x_3))^{\varepsilon_3}$ (with the adopted convention $(-1)^0 = 1, 0^0 = 0$). For every $u \in L^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} u\,\mathrm{d}x = 0$ we consider the expansion of u into the Haar wavelets

$$u = \sum_{j,k,\varepsilon} a_{j,k}^{(\varepsilon)} h_{j,k}^{(\varepsilon)}$$

and define the projection operator $P^{(\varepsilon)}$ by

$$P^{(\varepsilon)} u := \sum_{j,k} a_{j,k}^{(\varepsilon)} h_{j,k}^{(\varepsilon)}.$$

The following theorem is an adaptation of [24, Theorem 5] to the three-dimensional case, see also [19, Theorem 1.1], and plays a central role providing key estimates of the wavelet projectors in terms of the Riesz transforms $R_k := -i\partial_k(-\Delta)^{-1/2}$, $k = 1, 2, 3$.

Theorem 6. *The operator $P^{(\varepsilon)}$ can be extended to a bounded operator on L^2 and $\forall\, k = 1, 2, 3$, $\forall\, \varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ with $\varepsilon_k = 1$ one has*

$$\|P^{(\varepsilon)}u\|_2 \leqq C\|u\|_2^{1/2}\|R_k u\|_2^{1/2},$$

where $\|\cdot\|_2$ denotes the standard norm in $L^2(\mathbb{R}^3)$.

The proof of Theorem 6 is a direct line-by-line adaptation of the Müller's proof [24, Theorem 5] to the three-dimensional case, employing the deep ideas from harmonic analysis (see, for example, [38]) of Littlewood-Paley dyadic decomposition and of "almost orthogonality", and is not reproduced here. [The Müller's proof survives the change of dimensionality, with in particular the assumption $\varepsilon_k = 1$ still implying that $h^{(\varepsilon)}$ has a compactly supported primitive in x_k which eventually ensures the estimate of the Haar wavelet projector $P^{(\varepsilon)}$ in terms of the Riesz transform R_k. See also [19] where further generalizations have been obtained most recently. In particular, [19, Theorem 1.1] implies Theorem 6, although the proof is technically more advanced since it is developed for an arbitrary p-growth, $1 < p < \infty$, by additionally invoking advanced tools of the Calderon-Zygmund theory.]

We will next need the following lemma, which is a modification of Lemma 6 in [24].

Lemma 7. *Let f satisfy the assumptions of Theorem 1. Assume that $u, v \in L^2(\mathbb{R}^3)$ have the finite expansions in the Haar basis*

$$\begin{aligned} u &= \sum_{j=J}^{K}\sum_{k\in\mathbb{Z}^3}\left[a_{j,k}^{(0,0,1)}h_{j,k}^{(0,0,1)} + c_{j,k}^{(1,0,0)}h_{j,k}^{(1,0,0)}\right], \\ v &= \sum_{j=J}^{K}\sum_{k\in\mathbb{Z}^3}\left[b_{j,k}^{(0,1,0)}h_{j,k}^{(0,1,0)} + c_{j,k}^{(1,0,0)}h_{j,k}^{(1,0,0)}\right]. \end{aligned} \tag{28}$$

Then

$$\int_{\mathbb{R}^3}(f(u,v) - f(0,0))\,\mathrm{d}x \geqq 0.$$

Proof. The proof essentially follows [24, Lemma 6] by induction in K and employs Jensen's inequality to the integrations in x_3, x_2 and x_1 via the convexity of $f(u, v)$ in the directions $(1, 0)$, $(0, 1)$ and $(1, 1)$, respectively (cf. also Lemma 5 above). □

We are now in position to prove Theorem 1.

Proof of Theorem 1. As in the proof of [24, Theorem 1], we can assume without loss of generality $V = Q$ and $u_\infty = v_\infty = 0$. The assumptions (24) then imply

$$\|R_2 u_h\|_2 \to 0, \quad \|R_3 v_h\|_2 \to 0, \quad \|R_1(u_h - v_h)\|_2 \to 0. \tag{29}$$

Hence, by Theorem 6, it follows that

$$P^{(\varepsilon)} u_h \to 0 \quad \text{in } L^2 \quad \forall \varepsilon \text{ such that } \varepsilon_2 = 1, \tag{30}$$

$$P^{(\varepsilon)} v_h \to 0 \quad \text{in } L^2 \quad \forall \varepsilon \text{ such that } \varepsilon_3 = 1, \tag{31}$$

$$P^{(\varepsilon)} (u_h - v_h) \to 0 \quad \text{in } L^2 \quad \forall \varepsilon \text{ such that } \varepsilon_1 = 1. \tag{32}$$

Additionally, we obtain the following:

$$(31) + (32) \Rightarrow P^{(1,0,1)} u_h \to 0 \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^3)\,, \tag{33}$$

$$(30) - (32) \Rightarrow P^{(1,1,0)} v_h \to 0 \quad \text{in} \quad L^2_{\text{loc}}(\mathbb{R}^3)\,. \tag{34}$$

Thus (30)–(34) yield

$$\|P^{(0,0,1)} u_h + P^{(1,0,0)} v_h - u_h\|_2 \;\to\; 0$$
$$\|P^{(0,1,0)} v_h + P^{(1,0,0)} v_h - v_h\|_2 \;\to\; 0.$$

This allows us to reduce the rest of the proof essentially to Lemma 7 adapting the Müller's techniques in a straightforward way. □

5. Relaxation and H-measures

We return now to the original problem and develop an alternative way for its solution, exploiting the equivalence Proposition 1. We use Fourier analysis to execute the "internal" minimization in (6) for an arbitrary number N of "wells", essentially following the same method as used by Kohn [17], with appropriate modifications for the solenoidal fields.

Let us fix $\chi \in I(\theta)$ and compute the inner infimum (in fact, the minimum) over B in (6). Elementary manipulation transforms the integral in (6) into

$$\frac{1}{2} \fint_Q \left| B(x) + \eta - \sum_{i=1}^N \chi_i A_i \right|^2 \mathrm{d}x$$
$$= \frac{1}{2} \left\{ \left| \eta - \sum_{i=1}^N \theta_i A_i \right|^2 + \fint_Q \left| B(x) \;-\; \sum_{i=1}^N (\chi_i - \theta_i) A_i \right|^2 \mathrm{d}x \right\}. \tag{35}$$

The last term can be rewritten in the Fourier space using the Plancherel's formula in the form

$$\frac{1}{2} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left| \hat{B}(k) - \sum_{i=1}^N \hat{\chi}_i(k) A_i \right|^2, \tag{36}$$

where $\hat{B}(k)$ and $\hat{\chi}_i(k)$ are Fourier coefficients for the Q-periodic functions B and χ_i, respectively. Notice that the frequency $k = 0$ does not contribute to (36), since $\hat{B}(0) = 0$ and $\hat{\chi}_i(0) = \theta_i$.

Minimization of (36) can be done separately for each k, with respect to all $\hat{B}(k)$ consistent with the divergence-free constraint (7). For any $k \neq 0$, the minimizing value of $\hat{B}(k)$ as a result is

$$\hat{B}(k) = \sum_{i=1}^{N} \hat{\chi}_i(k) \Pi_{V(k)} A_i. \tag{37}$$

Here $\Pi_{V(k)} A_i$ denotes the orthogonal projection, in the sense of the inner product $\langle A, B\rangle := \mathrm{Tr}(A^T \bar{B})$ of (possibly complex) matrices $A, B \in \mathbb{M}^{m\times d}$, onto the space

$$V(k) = \{\zeta \in \mathbb{M}^{m\times d} :\ \zeta k = 0\}.$$

Here $V(k)$ describes the space of Fourier transforms of divergence free fields "of frequency k" and depends actually only on the "direction of oscillation" $k/|k|$. The orthogonal space to $V(k)$ is given by the space $V(k)^{\perp}$ of Fourier transforms of gradient fields:

$$V(k)^{\perp} = \{\zeta \in \mathbb{M}^{m\times d} : \zeta = v \otimes k \ \text{ for some } v \in \mathbb{R}^m\}.$$

Therefore, for every $\zeta \in \mathbb{M}^{m\times d}$, we have

$$\Pi_{V(k)}\zeta = \zeta - (\zeta k) \otimes k/|k|^2, \quad \Pi_{V(k)^{\perp}}\zeta = (\zeta k) \otimes k/|k|^2. \tag{38}$$

Plugging (37) into (36), we find that the minimum value of (36) is given by

$$\tfrac{1}{2}\sum_{k\neq 0}\left|\sum_{i=1}^{N} \hat{\chi}_i(k)\Pi_{V(k)^{\perp}} A_i\right|^2. \tag{39}$$

The latter can be conveniently re-written as follows:

$$\tfrac{1}{2}\sum_{k\neq 0}\left|\sum_{i=1}^{N} \hat{\chi}_i(k)\Pi_{V(k)^{\perp}} A_i\right|^2 = \tfrac{1}{2}\sum_{i,j=1}^{N}\int_{S^{d-1}}\left\langle \Pi_{V(k)^{\perp}} A_i, \Pi_{V(k)^{\perp}} A_j\right\rangle \mathrm{d}\mu_{ij},$$

where $\mu = (\mu_{ij})_{i,j}$ is the H-measure generated by χ, see Appendix B, (B.1). Next we set

$$\forall \xi \in S^{d-1} \quad f^{ij}(\xi) := \tfrac{1}{2}\left\langle \Pi_{V(\xi)^{\perp}} A_i, \Pi_{V(\xi)^{\perp}} A_j\right\rangle, \tag{40}$$

and by (38), we find that

$$f^{ij}(\xi) := \tfrac{1}{2}\langle A_i\xi, A_j\xi\rangle, \tag{41}$$

(with $\langle\cdot,\cdot\rangle$ denoting here the conventional inner product of vectors in $\mathbb{R}^m$).

Then, taking in (6) into account (35)–(40) and (B.1), the minimization problem for $Q_S^\theta F$ becomes

$$Q_\theta^S F(\eta) = \frac{1}{2}\left|\eta - \sum_{i=1}^N \theta_i A_i\right|^2 + \inf_{\mu\in Y^H(\theta)} \sum_{i,j=1}^N \int_{S^{d-1}} f^{ij}(\xi)\,\mathrm{d}\mu_{ij}(\xi). \quad (42)$$

The minimization is with respect to all H-measures associated with N characteristic functions. Notice that the weak* limits (B.2) have been included into the minimization [which is allowed since $f^{ij} \in C(S^{d-1})$]. The Krein–Milman theorem, for example [5], assures further that the infimum in (42) is in fact the minimum and is achieved at the set $Y_e^H(\theta)$ of *extremal points* of (weak* compact and convex) $Y^H(\theta)$:

$$Q_\theta^S F(\eta) = \frac{1}{2}\left|\eta - \sum_{i=1}^N \theta_i A_i\right|^2 + \min_{\mu\in Y_e^H(\theta)} \sum_{i,j=1}^N \int_{S^{d-1}} f^{ij}(\xi)\,\mathrm{d}\mu_{ij}(\xi). \quad (43)$$

The latter in combination with Proposition 1 implies the following important equivalence:

Proposition 2. *For $N \geqq 2$ and $K = \{A_1, \ldots, A_N\}$ and any $B_0 \in \mathbb{M}^{m\times d}$, $B_0 \in K_S^{qc}$ if and only if there exists $\theta \in [0,1]^N$ and an extremal H-measure $\mu \in Y_e^H(\theta)$, such that*

$$B_0 = \sum_{i=1}^N \theta_i A_i \quad \textit{and} \quad \sum_{i,j=1}^N \int_{S^{d-1}} f^{ij}(\xi)\,\mathrm{d}\mu_{ij}(\xi) = 0. \quad (44)$$

The above equivalence implies that, as long as we are able to characterize K_S^{qc}, we are potentially able to clarify which candidate matrix S^{d-1}-Borel measures are and which are not extremal points of the H-measures, thereby clarifying further the structure of the set $Y^H(\theta)$ of the H-measures themselves. For the two-well case ($N = 2$) this approach leads to the exact computation of $Q_S^\theta F$, see [31], and the above equivalence re-establishes the full characterization of the two-phase H-measures [17]. We further specialize to the case of three-wells in the dimension three ($N = d = m = 3$), to exploit in this context the results of Section 3.

6. Extremal three-point H-measures

In the present section we turn to the problem of characterizing the H-measures $Y^H(\theta)$ for $N = d = m = 3$. Those arise in (43) but are intrinsically more general geometric objects describing possible mixtures of characteristic functions. For a full characterization, it would be sufficient describing the set $Y_e^H(\theta)$ of the extremal point of the H-measures. This is not known explicitly but it was in effect shown in [17] that $Y^H(\theta)$ is contained in an explicitly described "superset" $Y(\theta)$, see Appendix B.2 below and specifically (B.8). It was shown in [39] and then further generalized in [13] that the extremal points $Y_e(\theta)$ of $Y(\theta)$ are in turn explicitly characterized, being certain matrix Borel measures supported in no more than three Dirac masses, and that a substantial sub-class of those are in fact the H-measures. This is reviewed and clarified further in Appendix B.2. In particular, $Y_e(\theta)$ is explicitly described in Proposition 5 and the associated geometric construction, see in particular Fig. 6.

The aim of the present section is to characterize, as far as possible, $Y_e(\theta) \cap Y^H(\theta)$. We show that the solution to the problem discussed in Section 3, that is finding K_S^{qc}, also provides a key to the solution of this problem. More precisely, referring to Appendix B.2 for further details, by Proposition 5 all the extremal points $Y_e(\theta)$ are three-point measures of the form $\mu(\xi) = \sum_{r=1}^{3} \mu^r \, \delta_{\xi_r}$, with $\mu^r = m^r \otimes m^r$. We consider all those for which ξ_1, ξ_2, ξ_3 are *arbitrary linearly independent vectors*. The answer depends on the position of the associated normalized masses, that is the points $\mu_{cs}^r = m_{cs}^r \otimes m_{cs}^r$ on the boundary circle C on the (c, b)-plane, see Figs 6, 3, relative to the "basic" points ν_1, ν_2 and ν_3 [the latter correspond to the three pairwise mixings of the three phases, cf. (B.12)]. Namely, we will distinguish two cases. The first case is when μ_{cs}^r, $r = 1, 2, 3$, lie on the circular segments $\nu_2\nu_1$, $\nu_1\nu_3$, $\nu_3\nu_2$, *one on each segment*, see Figs 3 and 4. Lemma 8 states that characterizing the H-measures in this case is *equivalent* to characterizing the S-quasiconvex hull of the sets of *Type 1*. As a consequence we obtain criteria (Theorem 7) which allows us to identify all the H-measures among the three-point measures having normalized masses on different arches.

The second case is when *two* of the normalized masses lie on the *same* circular segment, see Fig 5. Theorem 8 asserts that such measures are *not* H-measures. This result is, in turn, equivalent to the characterization of the S-quasiconvex hull of the sets of *Type 2* (as shown in the proof of Theorem 8). Finally, the case when two of the normalized masses merge is ruled out by Theorem 9. Appropriate limit cases are covered in the Remarks 7, 11 and 12. Remark 8 briefly sketches how all these results could be derived directly for the H-measures, that is without explicitly exploiting the above equivalence (with both approaches seemingly equivalent at a fundamental level by crucially relying on the compensated compactness property provided by Theorem 1).

The precise plan is as follows: to each measure $\mu \in Y_e(\theta)$ with linearly independent ξ_r, $r = 1, 2, 3$, we associate a set $K = \{A_1, A_2, A_3\}$ for which μ is *the only* minimizing measure in the lower bound $L(\theta)$ on $Q_S^\theta F$, see (B.14), and $L(\theta) = 0$. Then we use the knowledge of the S-quasiconvex hull of K (Section 3) in combination with the equivalence in Proposition 1, to establish the attainability or otherwise of the lower bound. Namely, if $B_0(\theta) := \sum_{i=1}^{3} \theta_i A_i \in K_S^{qc}$ then the lower bound $L(\theta)$ *is* attained, and hence $\mu \in Y^H(\theta)$, otherwise $\mu \notin Y^H(\theta)$.

For a three-point measure $\mu = \sum_{r=1}^{3} \mu^r \delta_{\xi_r} \in Y_e(\theta)$, let ϕ_r denote the angle associated with the mass $\mu_{cs}^r \in C$ via (B.13), and let t_r be defined as follows (assuming $\phi_r \neq \pi$):

$$t_r := \tan \frac{\phi_r}{2}\,, \quad r = 1, 2, 3\,. \tag{45}$$

Lemma 8. *Let $d = 3$, and let, for a range of $\theta \in (0, 1)^3$, $\bar{\mu} \in Y_e(\theta)$ be supported on three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in S^2$:*

$$\bar{\mu}(\xi) = \sum_{r=1}^{3} \bar{\mu}^r \delta_{\xi_r} .$$

Suppose that for the $\bar{\phi}_r$ associated with $\bar{\mu}^r$ and for t_r related to $\bar{\phi}_r$ via (45)

$$\bar{\phi}_1 \in (0, \pi), \quad \bar{\phi}_2 \in (\pi, \tfrac{3}{2}\pi), \quad \bar{\phi}_3 \in (\tfrac{3}{2}\pi, 2\pi), \quad \textit{and} \quad t_1(1+t_3) \neq t_3(1+t_2). \tag{46}$$

Then there exists a set $K = \{A_1, A_2, A_3\}$, depending on $\bar{\phi}_r$, $r = 1, 2, 3$ but independent of θ, of the form (9), (10) *such that*

$$\bar{\mu} \in Y^H(\theta) \Longleftrightarrow \sum_{i=1}^{3} \theta_i A_i \in K_S^{qc} . \tag{47}$$

Proof. We first assume that $\bar{\mu}$ is supported on the canonical basis of $\mathbb{R}^3$, (e_1, e_2, e_3), that is $\xi_r = e_r$, $r = 1, 2, 3$. By assumption, either $t_1(1+t_3) < t_3(1+t_2)$ or $t_1(1+t_3) > t_3(1+t_2)$. We consider these two cases separately.
Case (i). Assume $t_1(1+t_3) < t_3(1+t_2)$. Define $q = (q_1, q_2, q_3)$ as follows:

$$\begin{aligned} q_1 &= \frac{t_1(1+t_3) - t_3(1+t_2)}{t_3(t_1 - t_2)}, \quad q_2 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_2 - t_3}, \\ q_3 &= \frac{t_1(1+t_3) - t_3(1+t_2)}{(1+t_2)(t_1 - t_3)}. \end{aligned} \tag{48}$$

Since by assumption the numerators are negative and by (46) $t_1 \in (0, +\infty)$, $t_2 \in (-\infty, -1)$, $t_3 \in (-1, 0)$, we find that $q \in (0, 1)^3$. It is further directly checked that for $D(q)$ defined by (26) and q as in (48),

$$D(q) = \mathrm{diag}(-t_1, -t_2, -t_3). \tag{49}$$

[Hence (48) may be viewed as the inversion of (26)]. Set

$$\begin{aligned} &A_1 = 0, \quad A_2 = I, \quad A_3 = D(q), \quad K = \{A_1, A_2, A_3\}, \\ &B_0 = \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3 . \end{aligned} \tag{50}$$

By construction K is of the form (9), (10), that is by Lemma 2 is of *Type 1*.

Consider next $Q_\theta^S F(B_0)$ and the lower bound $L(\theta)$ associated with it, see (B.14). We will show that $L(\theta) = 0$ with $\bar{\mu}$ the *only* minimizing measure. Then, by Proposition 6, $Q_\theta^S F(B_0) = 0$ if and only if $\bar{\mu} \in Y^H(\theta)$ and the assertion of the lemma follows via the equivalence Proposition 1.

To evaluate $L(\theta)$, use the algorithm stated in Lemma 9. To this end, evaluate the function $\psi(\mu) = \psi(a, b, c)$ defined by (B.16) and note that by (B.7) and (41)

$$a(\xi) f^{22}(\xi) + 2b(\xi) f^{23}(\xi) + c(\xi) f^{33}(\xi) \geqq 0$$

for any $\xi \in S^2$. Therefore it is enough to prove that the function $\psi(\mu)$ vanishes only at the points $\mu = t\bar{\mu}^r$, $t \geqq 0$, $r = 1, 2, 3$, equivalently *only* at the three cross-sectional points $\bar{\mu}_{cs}^r$, see Figs. 3(i), 6. Parametrize the cross-sectional boundary circle C by the angle ϕ as in (B.13) and set $e(\phi) = \sin(\phi/2)A_2 + \cos(\phi/2)A_3$. Evaluation of $\psi(\mu)$ for μ belonging to C yields, cf. (B.16), (41) and (B.13):

$$\psi(a,b,c) = \tfrac{1}{2}\min_{\xi\in S^2}\left\{a\,|A_2\xi|^2 + 2b\,\langle A_2\xi, A_3\xi\rangle + c\,|A_3\xi|^2\right\}$$
$$= \tfrac{1}{2}\min_{|\xi|=1}|e(\phi)\xi|^2 = \tfrac{1}{2}\min_{|\xi|=1}\left\langle e(\phi)^T e(\phi)\xi, \xi\right\rangle. \quad (51)$$

Therefore $\psi(a,b,c)$ is the smallest eigenvalue of the symmetric non-negative matrix

$$\frac{1}{2}e(\phi)^T e(\phi) = \frac{1}{2}\left(\sin\frac{\phi}{2}A_2 + \cos\frac{\phi}{2}A_3\right)^T\left(\sin\frac{\phi}{2}A_2 + \cos\frac{\phi}{2}A_3\right).$$

Recalling (50),

$$e(\phi)^T e(\phi) = \left(\sin\frac{\phi}{2}I + \cos\frac{\phi}{2}D(q)\right)^2. \quad (52)$$

Hence, via (49), the eigenvalues are $\lambda_r = \frac{1}{2}\left(\sin(\phi/2) - t_r\cos(\phi/2)\right)$, $r = 1, 2, 3$, and

$$\psi(a,b,c) = \tfrac{1}{2}\min_{r=1,2,3}\left(\sin\frac{\phi}{2} - t_r\cos\frac{\phi}{2}\right)^2.$$

Hence $\psi(a,b,c) = 0$ if and only if $\tan(\phi/2) = t_r$, $r = 1, 2, 3$, that is, via (45), ϕ takes one of three possible values, $\phi = \bar{\phi}_r$, $r = 1, 2, 3$. Moreover, for every $r = 1, 2, 3$, the minimizing point in (51) for $\phi = \bar{\phi}_r$ is the eigenvector of $e(\bar{\phi}_r)^T e(\bar{\phi}_r)$ corresponding to the zero eigenvalue, that is e_r. Since the decomposition (B.17) of the cross-sectional total mass M_{cs} into the convex combination of $\bar{\mu}^r_{cs}$ is unique, $\mu = \bar{\mu}$ is the *only* minimizing measure in (B.15) and hence $Q^S_\theta F(B_0) = 0$ if and only if $\bar{\mu} \in Y^H(\theta)$, and (47) follows.
Case (ii). Now let $t_1(1+t_3) > t_3(1+t_2)$. Then choose $q = (q_1, q_2, q_3) \in (0,1)^3$ in the following way:

$$q_1 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_1 - t_3}, \quad q_2 = \frac{t_1(1+t_3) - t_3(1+t_2)}{t_1(t_3 - t_2)},$$
$$q_3 = \frac{t_1(1+t_3) - t_3(1+t_2)}{(1+t_3)(t_1 - t_2)}, \quad (53)$$

and set

$$A_1 = 0, \quad A_2 = PD(q)P, \quad A_3 = I, \quad K = \{A_1, A_2, A_3\},$$
$$B_0 = \theta_1 A_1 + \theta_2 A_2 + \theta_3 A_3, \quad (54)$$

where P is the permutation matrix $\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$ and $D(q)$ is given by (26) with q as in (53). By Lemma 2, K is still of *Type 1*. In particular $PD(q)P = \mathrm{diag}(-t_1^{-1}, -t_2^{-1}, -t_3^{-1})$. Then one considers $Q^S_\theta(B_0)$ corresponding to (54) and proceeds as in the previous case.

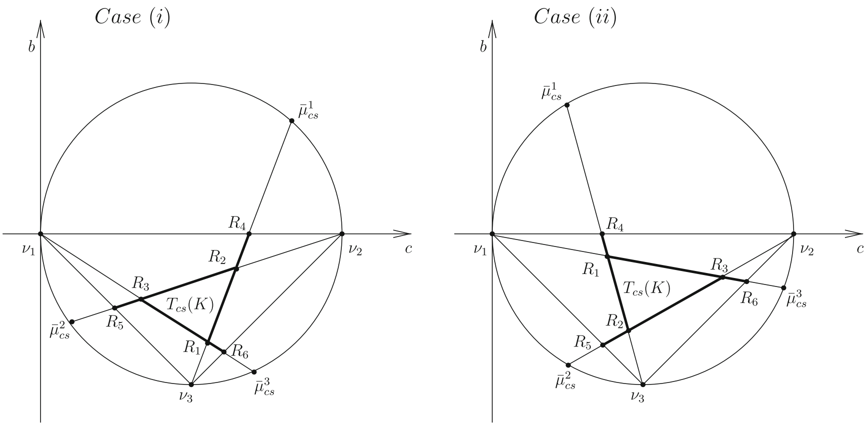

Fig. 3. The set $T_{cs}(K)$ in cases (i) and (ii) of the proof of Lemma 8

To conclude the proof we observe that if the measure $\bar{\mu}$ is supported on any three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in S^2$, then it is enough to replace the matrix $D(q)$ in (50) and (54) by $GD(q)G^{-1}$, where G is the matrix with columns ξ_1, ξ_2, ξ_3: $G := (\xi_1 \,|\xi_2|\, \xi_3)$. □

Remark 6. Lemma 8 establishes the equivalence between two problems: the one on whether a three-point measure in $Y_e(\theta)$ is an H-measure and the one on characterizing the S-quasiconvex hull of the set K given by (50) or (54). The nature of this equivalence can be visualized as follows, see Fig. 3. On the cross-section $\mathscr{K}_{cs}$ in the (c, b)-plane consider the triangle specified by the points $\nu_1 = (0, 0)$, $\nu_2 = (1, 0)$, $\nu_3 = (\frac{1}{2}, -\frac{1}{2})$. For given A_1, A_2, A_3, every point in the interior of K^c can be identified with a point inside the triangle $\nu_1\nu_2\nu_3$ via the mapping

$$\sum_{i=1}^{3} \theta_i A_i \in \mathrm{Int}(K^c)$$
$$\xrightarrow{\rho} \left(\frac{\theta_3(1-\theta_3)}{\theta_2(1-\theta_2)+\theta_3(1-\theta_3)}, \frac{-\theta_2\theta_3}{\theta_2(1-\theta_2)+\theta_3(1-\theta_3)} \right) \in \mathscr{K}_{cs}, \qquad (55)$$

describing the projection M_{cs} of the total mass $M(\theta)$, see (B.11), on the (c, b)-plane of unit trace matrices. Now let $\bar{\mu} \in Y_e(\theta)$ satisfy the assumptions of Lemma 8 and let K be the set associated with $\bar{\mu}$ via (50) or (54). Then set

$$T_{cs}(K) := \rho\left(T(K) \cap \mathrm{Int}(K^c)\right),$$

where $T(K)$ is the set defined by (12). Since by Corollary 2, $K^{qc}_S = T(K)$, Lemma 8 can be equivalently re-stated by saying that $\bar{\mu}$ is an H-measure *if and only if* M_{cs} on $\mathscr{K}_{cs}$ belongs to $T_{cs}(K)$. It can be further checked that $T_{cs}(K)$ is the region delimited by the lines $\nu_1\bar{\mu}^3_{cs}$, $\nu_2\bar{\mu}^2_{cs}$ and $\nu_3\bar{\mu}^1_{cs}$ on the (c, b)-plane (see Fig. 3). We briefly illustrate its construction. On the (c, b)-plane draw the three segments $\nu_3\bar{\mu}^1_{cs}$, $\nu_2\bar{\mu}^2_{cs}$,

$\nu_1\bar{\mu}^3_{cs}$. Consider the intersections of each of the segments with the two others and with the segments $\nu_1\nu_2$, $\nu_1\nu_3$, $\nu_2\nu_3$:

$$\{R_1\} = \nu_3\bar{\mu}^1_{cs} \cap \nu_1\bar{\mu}^3_{cs}\,, \quad \{R_2\} = \nu_3\bar{\mu}^1_{cs} \cap \nu_2\bar{\mu}^2_{cs}\,, \quad \{R_3\} = \nu_2\bar{\mu}^2_{cs} \cap \nu_1\bar{\mu}^3_{cs}\,,$$
$$\{R_4\} = \nu_3\bar{\mu}^1_{cs} \cap \nu_1\nu_2\,, \quad \{R_5\} = \nu_2\bar{\mu}^2_{cs} \cap \nu_1\nu_3\,, \quad \{R_6\} = \nu_1\bar{\mu}^3_{cs} \cap \nu_3\nu_2\,.$$

Then the set $T_{cs}(K)$ is given by the union of the closed triangle $R_1R_2R_3$ and the segments $[R_1, R_6)$, $[R_2, R_4)$, $[R_3, R_5)$.

Keeping the notation introduced in Remark 6, we can then state the following

Theorem 7. *Let $\bar{\mu} \in Y_e(\theta)$ satisfy the assumptions of Lemma 8. Then*

$$\bar{\mu} \in Y^H(\theta) \Longleftrightarrow M_{cs} \in T_{cs}(K) = R_1R_2R_3 \cup [R_1, R_6) \cup [R_2, R_4) \cup [R_3, R_5).$$

Remark 7. The results of Theorem 7 can be extended to the case when the measure $\bar{\mu}$ is such that one or more of the points $\bar{\mu}^r_{cs}$ coincide with some of the basic points ν_s, $s = 1, 2, 3$. In this case some of the points R_1, R_2, R_3 in Fig. 3 would merge with some of the points ν_s. The set associated with μ in the sense of Lemma 8 is still a set of *Type 1*, but with rank-2 connections, cf. Remark 2.

Conversely, if we study problem (6) when the set K contains one or more rank-2 connections, then the resulting extremizing measure will have one or more of the normalized masses coinciding with some of the basic points ν_s. In particular, if the matrices A_1, A_2, A_3 are pairwise rank-2 connected, then $\{A_1, A_2, A_3\}^{qc}_S = \{A_1, A_2, A_3\}^c$ and the minimizing measure μ will have normalized masses equal to ν_1, ν_2 and ν_3.

Remark 8. Theorem 7 essentially establishes that the sufficient conditions [39, Proposition 6.1] for realizability of some extremal three-point measures of $Y(\theta)$ by the H-measures are also necessary. This result crucially relies, via Lemma 8 and Theorem 5, on the key lower semicontinuity result of Theorem 1 which has been, in turn, proved using advanced tools of harmonic analysis, and, apparently, could not be derived from polyconvexity/ quadratic translation-type arguments only (cf. for example [7,8]). In principle, it could have been derived directly from Theorem 1, that is without explicitly appealing to the three divergence-free wells problem, namely *directly for the H-measures*, as we briefly sketch below. At a fundamental level the two approaches seem equivalent since in both cases crucially rely on Theorem 1.

Consider $\bar{\mu} \in Y_e(\theta)$, hence in the form (B.9), and assume (again without loss of generality) that $\xi_r = e_r$, $r = 1, 2, 3$. Suppose $\bar{\mu} \in Y^H(\theta)$. Then there exists a sequence of characteristic functions $\chi^h \in I(\theta)$ such that for $\bar{\mu}^h \in Y^H(\theta)$ associated to them via (B.1) $\bar{\mu}^h \overset{*}{\rightharpoonup} \bar{\mu}$, and (up to a periodic rescaling) $\chi^h \rightharpoonup \theta$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$. Next, for each $r = 1, 2, 3$, consider a non-zero $n^r \in \mathbb{R}^3$ orthogonal to m^r, see (B.9), ($\sum_{j=1}^3 n^r_j m^r_j = 0$), with $\sum_{j=1}^3 n^r_j = 0$. Consider next a sequence of periodic functions $\psi^h_r(x) := \sum_{j=1}^3 n^r_j \left(\chi^h_j(x) - \theta_j\right)$. We claim that, for each $r = 1, 2, 3$,

$\partial_r \psi_r^h \to 0$ in $H^{-1}_{\text{per}}(Q) \subset H^{-1}_{\text{loc}}(\mathbb{R}^3)$ with $\|F\|^2_{H^{-1}_{\text{per}}(Q)} = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} |k|^{-2} \left|\hat{F}(k)\right|^2$:

$$\left\|\partial_r \psi_r^h(x)\right\|^2_{H^{-1}_{\text{per}}(Q)} = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \left(\frac{k_r^2}{|k|^2} \sum_{i,j=1}^{3} n_i^r n_j^r \hat{\chi_i}^h(k) \overline{\hat{\chi_j}^h(k)} \right)$$
$$= \int_{S^2} \sum_{i,j=1}^{3} \xi_r^2 n_i^r n_j^r \mathrm{d}\mu_{ij}^h(\xi) \longrightarrow \int_{S^2} \sum_{i,j,s=1}^{3} \xi_r^2 n_i^r n_j^r m_i^s m_j^s \delta_{e_s} = 0.$$

(In the above we have used that if $\xi = e_s$ then the r-th component ξ_r of ξ is zero unless $s = r$ whereas in the latter case the summand is zero by orthogonality of n^r and m^r.) Next, since n^r, $r = 1, 2, 3$, are co-planar, $\psi_r^h(x)$, $r = 1, 2, 3$, are linearly dependent. Hence there exist constants c_2 and c_3 (independent on h) such that $\psi_1^h(x) \equiv c_2\psi_2^h(x) - c_3\psi_3^h(x)$. Let $u_h(x) := c_2\psi_2^h(x)$ and $v_h(x) := c_3\psi_3^h(x)$. Then (29) holds as stated, equivalently implying that the assumptions of Theorem 1 are satisfied for $U = \mathbb{R}^3$ and u_h, v_h as above, up to a subsequence. Hence the conclusion (25) of Theorem 1 can be directly applied to the above sequence of characteristic functions. In particular, producing a function f satisfying the assumptions of the Theorem but violating (25) [for example the one akin to (20)] establishes the contradiction and the fact that $\bar{\mu} \notin Y^H(\theta)$. Alternatively, $\bar{\mu} \in Y^H(\theta)$ by an explicit infinite rank lamination construction, cf [39, Proposition 6.1].

Remark 9. We emphasize that Theorem 7 holds under the restriction of ξ_1, ξ_2 and ξ_3 being linearly independent (that is not co-planar vectors). If ξ_r, $r = 1, 2, 3$, are *linearly dependent* then the result still holds one way: if $M_{cs} \in T_{cs}(K)$ then still $\bar{\mu} \in Y^H(\theta)$, by continuity and the closedness of the H-measures. However we have been unable to prove, by the present methods, the converse statement. Resolving this may require further modifications of the methods of harmonic analysis cf. [24]. On the other hand, it is curious to notice in this context that the Šverák's counterexample of a rank-one convex function which is not quasiconvex [26,42] employs a gradient field that oscillates in three co-planar directions, that is relates to an H-measure supported exactly in three linearly dependent directions. This may indicate at insufficiency of the arguments based on separate convexity in this case, as well as keeps the possibility of a similar in spirit counterexample realizing an H-measure outside $T_{cs}(K)$ (the latter would imply existence of sequences of mixtures of characteristic functions which could not be mimicked by sequential lamination, in the sense of H-measures).

We discuss next the case when the triangle $R_1R_2R_3$ on the (c, b)-plane degenerates into one single point, which we denote by R_0 [see Fig. 4(2)]. In this case the associated measure μ satisfies

$$t_1(1 + t_3) = t_3(1 + t_2);$$

therefore there is no set K of the type (9) for which (47) may hold. The set associated with such measure μ is in fact of *degenerate Type 1* (Definition 4).

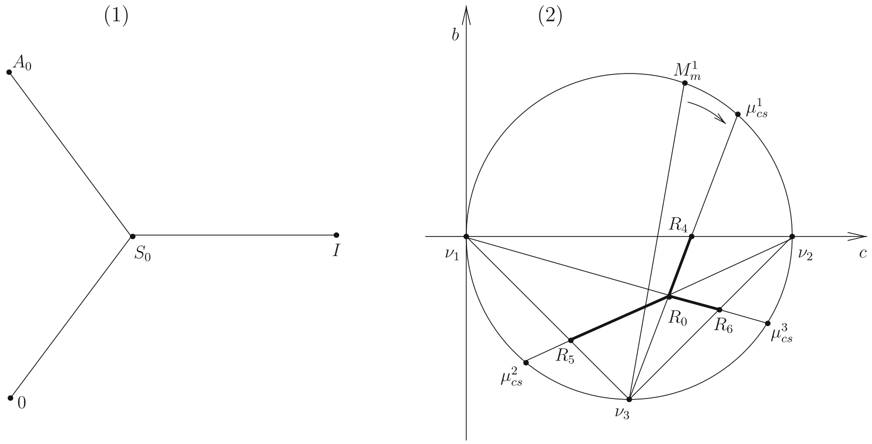

Fig. 4. (1) The set K_0. (2) The set $T_{cs}(K_0)$ on the (c, b)-plane

Proposition 3. *Let $\theta \in (0,1)^3$, $d = 3$, and let $\mu \in Y_e(\theta)$ be supported on three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in S^2$:*

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r} .$$

Suppose that

$$\phi_1 \in (0, \pi), \quad \phi_2 \in (\pi, \tfrac{3}{2}\pi), \quad \phi_3 \in (\tfrac{3}{2}\pi, 2\pi), \quad \textit{and} \quad t_1(1+t_3) = t_3(1+t_2).$$

Then

$$\mu \in Y^H(\theta) \Longleftrightarrow M_{cs} \in [R_0, R_4) \cup [R_0, R_5) \cup [R_0, R_6) \Longleftrightarrow \theta_2 I + \theta_3 A_0 \in \{0, I, A_0\}_S^{qc},$$

where the matrix A_0 is defined as follows:

$$A_0 = -G^{-1}\mathrm{diag}(t_1, t_2, t_3)G, \quad G = (\xi_1 \,|\xi_2|\, \xi_3)^{-1}.$$

Proof. Set $K_0 = \{0, I, A_0\}$ and $S_0 = G^{-1}\mathrm{diag}(0, 1, -t_3)G$. Observe first that S_0 is rank-2 connected with each of the three matrices $0, I, A_0$ and that the set $T(K_0)$ defined by (12) is given in this case by the union of three segments:

$$T(K_0) = [0, S_0] \cup [I, S_0] \cup [A_0, S_0]$$

[see Fig. 4(1)]. According to Definition 4, K_0 is a set of *degenerate Type 1*. Moreover it is easily checked that

$$\rho((0, S_0] \cup (I, S_0] \cup (A_0, S_0]) = [R_0, R_4) \cup [R_0, R_5) \cup [R_0, R_6), \quad \rho(S_0) = R_0,$$

with the mapping ρ defined by (55). Using the function $\mathscr{T}^+$ introduced in Section 3 and arguing as for the sets of *Type 1*, one can show that every point outside $T(K_0)$ does not belong to $(K_0)_S^{qc}$. Then, using the algorithm from Lemma 9 and proceeding as in the proof of Lemma 8, one checks that the lower bound for $Q_S^\theta F(B_0)$,

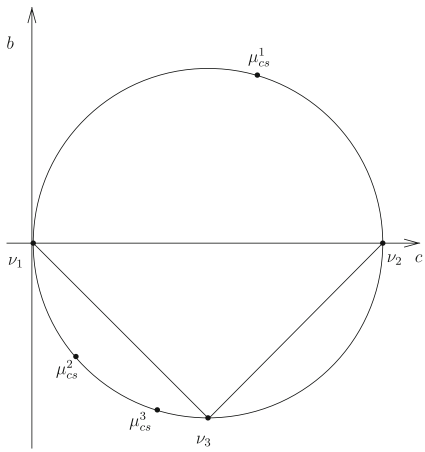

Fig. 5. The case of two normalized masses lying on the same arch

with $K = K_0$ and $B_0 = \theta_2 I + \theta_3 A_0$, is zero and is delivered *only* by the given measure μ. Therefore if $\mu \in Y^H(\theta)$ then $M_{cs} \in [R_0, R_4) \cup [R_0, R_5) \cup [R_0, R_6)$.

Now let $M_{cs} \in [R_0, R_5) \cup [R_0, R_6)$. A way to prove that $\mu \in Y^H(\theta)$ is to use an approximation argument. Consider a sequence of points M^1_m on the circle C such that $M^1_m \to \mu^1_{cs}$ as $m \to \infty$ [see Fig. 4(2)]. By Theorem 7 it follows that for every m the measure μ^m corresponding to the split $M^1_m, \mu^2_{cs}, \mu^3_{cs}$ is an H-measure. By construction $\mu^m \overset{*}{\rightharpoonup} \mu$ and therefore $\mu \in Y^H(\theta)$ by closedness of the H-measures, see (B.3). If $M_{cs} \in [R_0, R_4)$ then one introduces a similar perturbation around the point μ^2_{cs} or μ^3_{cs} and proceeds as before. We have thus proved that

$$\mu \in Y^H(\theta) \Longleftrightarrow M_{cs} \in [R_0, R_4) \cup [R_0, R_5) \cup [R_0, R_6)$$

and hence $(K_0)^{qc}_S = [0, S_0] \cup [I, S_0] \cup [A_0, S_0]$. $\square$

Remark 10. The case when *all* the points μ^r_{cs} lie on the *same* circular segment (that is either $\nu_2\nu_1$, or $\nu_1\nu_3$ or $\nu_3\nu_2$) is clearly *not* associated with an H-measure: the projection on the cross-section of the total mass of the measure is then outside the triangle $\nu_1\nu_2\nu_3$, which must not be the case.

The next result describes the case when two of the normalized masses lie on the same arch. Fig. 5 represents a measure with one normalized mass on the arch $\nu_1\nu_2$ and the other two masses on the same arch $\nu_1\nu_3$.

Theorem 8. *Let $\theta \in (0,1)^3$, $d = 3$ and let $\mu \in Y_e(\theta)$ be supported on three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in S^2$:*

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r} .$$

Assume that the points $\mu^1_{cs}, \mu^2_{cs}, \mu^3_{cs}$ are pairwise distinct and that $\mu^r_{cs} \neq \nu_i$ for all $r, i = 1, 2, 3$. If two and only two of the normalized masses $\mu^1_{cs}, \mu^2_{cs}, \mu^3_{cs}$ lie on the same circular segment, then $\mu \notin Y^H(\theta)$.

Proof. Let $A = -G^{-1}\text{diag}(t_1, t_2, t_3)G$, with $G = (\xi_1\,|\xi_2|\,\xi_3)^{-1}$. It can then be easily checked that the set $K = \{0, I, A\}$ is of *Type 2* and does not contain any rank-2 connections. We now proceed as in the previous cases. We study problem (6) for $A_1 = 0$, $A_2 = I$, $A_3 = A$ and, again using the algorithm of Lemma 9, we find that the lower bound $L(\theta)$ for $Q_S^\theta F(\theta_2 I + \theta_3 A)$ is zero and is delivered *only* by the given measure μ. Since by Theorem 3 $K_S^{qc} = K$, necessarily $Q_S^\theta F(\theta_2 I + \theta_3 A) > 0$; therefore μ is *not* an H-measure. □

Remark 11. Observe that the result of Theorem 8 does not extend to all the limit cases when one of the normalized masses coincides with one of the basic points. For example, if in Fig. 5 the point μ_{cs}^1 merges with ν_2, it is easy to check that this corresponds to the degenerate *Type 2* case displayed on Fig. 2(2), see Remark 5. The corresponding three-point measures then *are* H-measures, although they are not extremal being convex combinations of two two-point measures related to (ν_2, μ_{cs}^2) and (ν_2, μ_{cs}^3). One can see that then M_{cs} could be any point inside the intersection of triangles $\nu_2\mu_{cs}^2\mu_{cs}^3$ and $\nu_1\nu_2\nu_3$. On the other hand, if $\mu_{cs}^2 = \nu_1$, this corresponds to the other limit of *Type 2* with $K_S^{qc} = [A_1, A_2] \cup A_3$ and no non-trivial H-measures (the case $\mu_{cs}^3 \to \nu_3$ corresponds to a limit *Type 1* case and is covered by Theorem 7).

Theorem 9. *Let $\theta \in (0, 1)^3$, $d = 3$, and let $\mu \in Y_e(\theta)$ be supported on three linearly independent vectors $\xi_1, \xi_2, \xi_3 \in S^2$:*

$$\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r}.$$

Assume that $\mu_{cs}^r \neq \nu_i$ for all $r, i = 1, 2, 3$. If μ_{cs}^1 and μ_{cs}^2 lie on different arches and $\mu_{cs}^2 = \mu_{cs}^3$, then $\mu \notin Y^H(\theta)$.

Proof. We again associate to μ a set K for which μ is the (only) extremizing measure in (B.14) and delivers a zero lower bound. Such set is again given by $K = \{0, I, A\}$, where $A = -G^{-1}\text{diag}(t_1, t_2, t_3)G$. By assumption we have $t_2 = t_3$ and therefore the set K satisfies the assumptions of Theorem 4. Then $K_S^{qc} = K$ and μ is not an H-measure. □

Remark 12. Notice that the conclusions of Theorem 9 do not extend to the case when one normalized mass coincides with a basic point. Indeed it is easy to check that if μ_{cs}^1 coincides with a basic point and $\mu_{cs}^2 = \mu_{cs}^3$ lie on the opposite arch (for example $\mu_{cs}^1 = \nu_2$ and $\mu_{cs}^2 = \mu_{cs}^3 \in \nu_1\nu_3$), or if $\mu_{cs}^2 = \mu_{cs}^3$ merge with a basic point and μ_{cs}^1 lies on the opposite arch (for example $\mu_{cs}^2 = \mu_{cs}^3 = \nu_1$ and $\mu_{cs}^1 \in \nu_2\nu_3$), then the corresponding measure *is* an H-measure for all M_{cs} on the segment $\mu_{cs}^1\mu_{cs}^2 \cap \nu_1\nu_2\nu_3$. In both cases, those are in fact *not* extremal points of $Y^H(\theta)$, being convex combinations of two other H-measures supported in two points each, (ξ_1, ξ_2) and (ξ_1, ξ_3), respectively. The latter are in correspondence with sets K containing one rank-2 connection, with a non-trivial S-quasiconvex hull, as described in Theorem 5 (vii-1).

Remark 13. The results in Theorems 7, 8 and 9 provide full characterization of certain extremal H-measures supported in (no more than) three linearly independent directions. This appears sufficient for purposes of full resolution of the problem of characterizing the quasiconvex hulls for three solenoidal wells. However the above results *do not* imply a full characterization of the three-phase H-measures $Y^H(\theta)$ themselves: while the latter are fully determined by their extremal points, there may exist additional extremal points of $Y^H(\theta)$ supported in more than three points, therefore *not* being extremal points of the (fully characterized) superset $Y(\theta)$. There may also be additional three-point supported extremal H-measures which are *not* extremal for $Y(\theta)$, that is such that at least one of μ^r is not extremal ($\mu^r \notin C$). We sketch below an argument establishing the existence of such "extra" extremal H-measures supported in both *four* points and three points.

Consider H-measures supported in three Dirac masses according to Theorem 7, that is associated with sets of *Type 1*, with fixed $\bar{\mu}^r_{cs}$, and linearly independent ξ_r (for example $\xi_r = e_r$), $r = 1, 2, 3$. This could be achieved for a range of volume fractions θ, in particular such that $M_{cs}(\theta)$ is well inside the triangle $R_1R_2R_3$, see Fig. 3. Select such $\theta = \theta^{(0)}$ and let the corresponding extremal H-measures be $\bar{\mu}^{(0)} \in Y^H(\theta^{(0)})$. By continuity, there exists $\Delta > 0$ such that the above property is held for all $|\theta - \theta^{(0)}| < \Delta$. Select on the circle C one more "cross-sectional mass" $\bar{\mu}^4_{cs}$ such that for $\theta = \theta^{(0)}$ there *does not* exist an H-measure corresponding to $\{\bar{\mu}^r_{cs}, r = 1, 2, 4\}$, which is clearly possible by Theorem 7 and let $\xi_4 \neq \xi_r$, $r = 1, 2$ such that ξ_1, ξ_2, ξ_4 are linearly independent (this includes in particular the case of $\xi_4 = \xi_3$). Hence for the corresponding three-point (ξ_1, ξ_2, ξ_4) Borel measure $\bar{\mu}$, extremal for $Y(\theta^{(0)})$, $\bar{\mu} \notin Y^H(\theta^{(0)})$. For $0 < t < 1$, the Borel measures $\bar{\mu}(t) := (1 - t)\bar{\mu}^{(0)} + t\bar{\mu}$ are, hence, supported in *four* points if $\xi_4 \neq \xi_3$ and still in three points for $\xi_4 = \xi_3$. We argue that at least for small enough positive t those *are* H-measures; therefore the one corresponding to the maximal value of such t ($t = t_0$, $0 < t_0 < 1$) can only be an extremal H-measure supported in four (or three if $\xi_4 = \xi_3$) points. To establish this, notice that since $\bar{\mu}^4_{cs} \in C$ is extremal, $\bar{\mu}^4_{cs} = m \otimes m$ for some $m \in \mathbb{R}^2$, $|m| = 1$. Let $\theta^{(1)} := \theta^{(0)} + \Delta m/2, \theta^{(2)} := \theta^{(0)} - \Delta m/2$ and let the corresponding extremal H-measures be $\bar{\mu}^{(1)} \in Y^H(\theta^{(1)})$, $\bar{\mu}^{(2)} \in Y^H(\theta^{(2)})$, respectively (hence all supported in the same ξ_r with the same $\bar{\mu}^r_{cs}$, $r = 1, 2, 3$). Then "mix" these two H-measures in equal volume fractions via a lamination in layers perpendicular to ξ_4. The "mixing formula" for H-measures (see, for example [17,47], [39, Section 6(a) (6.4)]) produces the following new H-measure $\bar{\mu}^{(12)} \in Y^H(\theta^{(0)})$:

$$\bar{\mu}^{(12)} = \frac{1}{2}\bar{\mu}^{(1)} + \frac{1}{2}\bar{\mu}^{(2)} + \frac{\Delta^2}{4}\bar{\mu}^4_{cs}\delta_{\xi_4}.$$

This is clearly an H-measure supported in the four (respectively, three if $\xi_4 = \xi_3$) points. Such a measure can only be a convex combination of $\bar{\mu}^{(0)}$ and $\bar{\mu}$ and hence $\bar{\mu}^{(12)} = \bar{\mu}(t^*)$, for some $0 < t^* < 1$. By convexity and closedness, there exists "maximal" t_0, $t_0 \geqq t^* > 0$ such that $\bar{\mu}(t) \in Y^H(\theta^{(0)})$ if and only if $t \in [0, t_0]$ (since $\bar{\mu}(1) = \bar{\mu} \notin Y^H(\theta^{(0)})$, $t_0 < 1$). Hence $\bar{\mu}(t_0)$ is an extremal H-measure supported in four (respectively three) points, that is $\bar{\mu}(t_0) \in Y^H_e(\theta)\backslash Y_e(\theta)$.

7. Last part of Theorem 5 and a brief summary

In the present section we complete the proof of Theorem 5 and give a summary of the main results of the paper.

Proposition 4. *Let* $K = \{0, I, A\}$ *where* $\det(A) \neq 0$ *and* $\det(A - I) \neq 0$. *Assume that one of the following conditions is satisfied:*

(i) *K is a set of Type 3;*
(ii) *A is diagonalizable and the plane formed by K contains only two distinct rank-2 directions (hence one of them is rank-1), and the corresponding affine rank-2 lines through* 0, *I and A do not intersect at points inside K^c;*
(iii) *A is not diagonalizable.*

Then $K_S^{qc} = K$.

Proof. We consider problem (6) for the given set K and show that the lower bound $L(\theta)$ defined by (B.14) is strictly positive for all values of the volume fractions θ, implying that the quasiconvex hull is trivial.

Assume (i). Then the matrix A is diagonalizable and has three distinct real eigenvalues. Using the algorithm of Lemma 9, one can see that $L(\theta)$ could be zero only if the extremizing measure in (B.14) had all the three normalized masses on the same circular segment, either $\nu_1\nu_2$, or $\nu_1\nu_3$ or $\nu_3\nu_2$. Since this cannot be the case (see Remark 10), the lower bound is strictly positive.

Assume (ii). Then the matrix A is diagonalizable and has two distinct real eigenvalues, one of multiplicity two. As in case (i), one can see that $L(\theta)$ could be zero only if the extremizing measure had normalized masses on the same circular segment, except that in this case two of them merge. Again, this cannot be the case.

Now assume (iii). If A has one real eigenvalue and two complex (hence complex conjugate), then the function ψ defined by (B.16), see also (51), (52), vanishes for a single value of ϕ and therefore the lower bound is strictly positive. If A has two distinct eigenvalues, one of which has algebraic multiplicity two but geometric multiplicity one, or if A has one eigenvalue of algebraic multiplicity three but geometric multiplicity two, then the lower bound may be zero but is delivered by a two-point supported measure. On the other hand, two-point measures are not H-measures. The latter can be shown for example via the Šverák's incompatibility result[3] for three gradient wells [41], see [8, Section 5], [39, Section 7(a)]; or by reformulating Theorem 4 above in an equivalent H-measure setting, appropriately exploiting again the equivalence Proposition 2.

Finally, if A has a single eigenvalue of algebraic multiplicity three but geometric multiplicity one, then the lower bound is strictly positive (with ψ vanishing again for a single value of ϕ). □

Summary. The results presented in Section 6 provide characterization of all extremal three-point H-measures of the form $\mu(\xi) = \sum_{r=1}^{3} \mu^r \delta_{\xi_r}$ when ξ_1, ξ_2, ξ_3 are

[3] Remark in passing that the Šverák's result itself could be re-derived via a straightforward adjustment of the approach in Sections 3 and 4 above.

linearly independent vectors and the associated points μ^r_{cs} on the (c, b)-plane lie on the circular segments $\nu_2\nu_1$, $\nu_1\nu_3$, $\nu_3\nu_2$, *one* on each segment, including possibly the endpoints, Fig. 3. The only extremal H-measures in this class are those described by Theorem 7, including the limit cases as discussed in Remark 7 and Proposition 3.

Theorems 8 and 9 complete full characterization of the extremal three point H-measures, within the extremal points $Y_e(\theta)$ of the superset $Y(\theta)$, supported on three arbitrary linearly independent directions: except for the limit cases described in Remarks 11 and 12, all other measures in the set $Y_e(\theta)$ are *not* H-measures.

On the other hand, the presented analysis allows us to fully solve the problem of the S-quasiconvexification for three arbitrary solenoidal wells (Theorem 5). The conclusion is that a non-trivial quasiconvex hull can only emerge in the situation as in Fig. 1(1), that is when there are three separate rank-two directions in the plane formed by $K = \{A_1, A_2, A_3\}$ *and* the mutual position of A_1, A_2 and A_3 on this plane is such that an inner triangle is formed, including the limit cases. Then, according to Corollary 2, $K^{qc}_S = T(K)$. In all other cases $K^{qc}_S = K$, unless K contains rank-2 connections, as catalogized in Theorem 5.

8. On applications of the *H*-measure results. The three well problem for linear elasticity

An attractive feature of the H-measure is that it is a purely geometric object, that is independent of the differential constraints. Hence the same H-measures are involved in characterizing the relaxation of problems with different differential constraints, in particular of associated quasiconvex hulls. Therefore, any progress in characterizing the H-measures can be potentially transferred from problems with one type of differential constraints to those with another. In this section we discuss the application of the results on the H-measures to the problem of characterizing the quasiconvex hull for three linear elastic wells.

The problem is formulated similarly to that in Sections 3 and 5 with $K = \{A_1, A_2, A_3\}$ and A_1, A_2 and A_3 being now three *symmetric* matrices in $\mathbb{M}^{d\times d}$ of given linearized "transformation strains". The divergence-free differential constraint for a field B, cf. (7), is in turn replaced by the requirement that B is a symmetrized gradient of a periodic displacement field u:

$$B(x) = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right), \quad u \in H^1_\sharp(Q, \mathbb{R}^d). \tag{56}$$

The multi-well energy is analogous to (4), being characterized more generally by a quadratic form generated by a positive definite elastic tensor $\mathbf{C}$ which would formally coincide with (4) for the special case of an isotropic tensor with Lamé constants $\lambda = 0$ and $\mu = 1/2$ (cf. [39, Section 7(b)]), resulting in $\mathbf{C} = I$ with I being the identity tensor. Notice that the exact choice of $\mathbf{C}$ does not affect the issue of characterizing the (linear elastic) quasiconvex hull K^{qc}_{le}, so there is no loss of generality in choosing $\mathbf{C} = I$ for this purpose. As before, $\eta = \sum_{i=1}^3 \theta_i A_i$, $\theta \in [0, 1]^3$, $\sum_{i=1}^3 \theta_i = 1$, is in K^{qc}_{le} if and only if $Q^\theta_{le} F(\eta) = 0$. Here the relaxed energy $Q^{\theta}_{le}F$ is defined by (6) where in the definition (7) for V the divergence-free constraint is replaced by (56).

The relaxed energy $Q^{\theta}_{le}F(\eta)$ can, in turn, be equivalently expressed in terms of minimization with respect to H-measures [17,39], namely (43) still holds with the same set of H-measures $Y^H(\theta)$ as before but $f^{ij}(\xi)$ requiring re-evaluation for the linear elasticity context. Specializing to the three-dimensional elasticity ($d = 3$), $f^{ij}(\xi)$ is as follows (cf. [39, Section 7(b)]):

$$f^{ij}(\xi) = \frac{1}{2} A^{kl}_i \Delta_{klpq}(\xi) A^{pq}_j. \tag{57}$$

Here A^{kl}_i denotes the (kl) components of the matrix A_i and summation is implied with respect to repeated indices, and

$$\Delta_{klpq}(\xi) := \frac{1}{2}\{T_{kp}(\xi)T_{lq}(\xi) + T_{kq}(\xi)T_{lp}(\xi)\}, \quad T_{kl}(\xi) := \delta_{kl} - \xi_k\xi_l.$$

Hence (57) can be equivalently re-written as follows:

$$f^{ij}(\xi) = \mathrm{Tr}\left[A_i T(\xi) A_j T(\xi)\right], \tag{58}$$

where $T(\xi) = I - \xi \otimes \xi$.

Further, the lower bound $L(\theta)$ for $Q^{\theta}_{le}F$, as computed in [39] is given by (B.14) with the same "universal" superset $Y(\theta)$.

Assuming further without loss of generality $A_1 = 0$, the linear elastic analog of (B.16) when restricted to the circle C parametrized as before by $\phi \in [0, 2\pi)$, see (B.13), can be computed, as in (51), and in the present case reads

$$\psi(a, b, c) = \psi(\phi) = \inf_{\xi \in S^2} \mathrm{Tr}\left[(e(\phi)T(\xi))^2\right], \tag{59}$$

where $e(\phi) := \sin(\phi/2)A_2 + \cos(\phi/2)A_3$. Denoting by $\nu_j(a, b, c) = \nu_j(\phi)$, $j = 1, 2, 3$, the eigenvalues of $e(\phi)$ and by k_1, k_2 and k_3 the components of ξ with respect to the (orthonormal) basis of the eigenvectors diagonalizing $e(\phi)$, (59) via a straightforward calculation reads:

$$\psi(a, b, c) = \inf_{k \in S^2} \left\{ \left(\nu_1 k_2^2 + \nu_2 k_1^2\right)^2 + \left(\nu_2 k_3^2 + \nu_3 k_2^2\right)^2 + \left(\nu_3 k_1^2 + \nu_1 k_3^2\right)^2 \right.$$
$$\left. + 2\nu_1^2 k_2^2 k_3^2 + 2\nu_2^2 k_3^2 k_1^2 + 2\nu_3^2 k_1^2 k_2^2 \right\}. \tag{60}$$

It is easy to see that $\psi(\phi) = 0$ if and only if at least one of the eigenvalues ν_j, $j = 1, 2, 3$ is zero and the two others are not of the same sign, that is

$$\nu_1 \leqq \nu_2 = 0 \leqq \nu_3. \tag{61}$$

In particular, for the case of strict inequalities ($\nu_1 < \nu_2 = 0 < \nu_3$) the zero minimum in (60) is achieved at exactly *two* different directions $k \in S^2$: $k_2 = 0$, $k_3 = \pm|\nu_3/\nu_1|^{1/2} k_1$, giving rise to two different locations on the sphere for the component extremal mass corresponding to such ϕ.

The condition of compatibility of two linear elastic matrices A_i and A_j is known to be of similar type: one of the eigenvalues of $(A_i - A_j)$ must be zero and the two others must not be of the same sign, see for example [2]. Hence, for pairwise compatible wells $\psi(0) = \psi(\pi) = \psi(3\pi/2) = 0$, in which case $K_{le}^{qc} = K^c$, for example [2]. Therefore, it remains to consider the cases when the wells are *not* pairwise compatible. We assume without loss of generality that A_2 and $A_1 = 0$ are incompatible, that is upon diagonalization, $A_2 =$ diag $(\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1\alpha_2 > 0$. We then argue that the equation $\psi(\phi) = 0$ does not have more than three solutions for ϕ (within the range $[0, 2\pi)$) unless, in the chosen basis, $\alpha_3 = 0$ and $A_3^{k3} = A_3^{3k} = 0$, $k = 1, 2, 3$. The latter corresponds to two-dimensional linear elasticity, for which the quasiconvex hull is known and is in particular trivial in the case of pairwise incompatible wells, see for example [39, Section 7(b)]. For the former assertion, a necessary condition for $\psi(\phi) = 0$ is $\det e(\phi) = 0$. The latter equation does not have more than three solutions: if $\det e(\phi) = 0$ then either $\phi = \pi$ implying $\det A_2 = 0$ or $\det(A_3 + t A_2) = 0$ which is (a nontrivial) at most cubic equation in $t := \tan(\phi/2)$. From these values of ϕ those failing (61) should be excluded further, and as a result we end up with no more than three values of t such that $\psi(\phi) = 0$.

Further, $\eta \in K_{le}^{qc}$ if and only if $L(\theta) = 0$ *and* a minimizing measure in $Y(\theta)$ is an H-measure. The previous reasoning assures that, as in the divergence-free case, the total mass could generically only be split in a no more than a single triple of extremal masses corresponding to $\psi(\phi_r) = 0$, $r = 1, 2, 3$. Since for each of such ϕ_r there are generically two corresponding "minimizing" directions on the sphere, $\xi_r^{(1)}$ and $\xi_r^{(2)}$, the minimizing measures could be supported in *up to six* Dirac masses.

On the other hand, the results of this paper, in particular of Section 6, provide full characterization of H-measures supported in no more than three (linearly independent) points. The most interesting case of three solutions $\psi(\phi_r) = 0$, $r = 1, 2, 3$ corresponds to the situation when the plane (A_1, A_2, A_3) contains three "linear elastically compatible" directions, see [2] and [39, Section 7(b)] for such examples. The results of Section 6 are directly applicable to characterize the "inner bound" for K_{le}^{qc}, namely in the *Type 1* case when the "internal triangle" is formed, Fig. 1, implying $T(K) \subset K_{le}^{qc}$, cf. [2].

However, for establishing the outer bounds, some further developments are required to eliminate (or otherwise) the possibility of the minimizing H-measure being supported in four to six points. For example, our results in Section 6 ensure that in the *Type 1* case the exterior to $T(K)$ is *not* realized by any extremal point of the superset $Y(\theta)$, that is by (generically) any extremal measure supported in a triple of points $\xi_1^{(k)}$, $\xi_2^{(l)}$ and $\xi_3^{(m)}$, where $(k, l, m) \in \{1, 2\}^3$ (note that one of those triples may be co-planar, which is the case at least for diagonal matrices by direct inspection, cf Remark 9). This argument on its own does not, however, eliminate the possibility of an H-measure being a convex combination of those points. This poses an interesting open problem, whose resolution would possibly require further developments of the ideas of harmonic analysis akin to [24] and/or other ideas. It is also possible that the arguments based on the linear elastic analog of rank-2 convexity implying (linear elastic) quasiconvexity may fail. In this context, counter examples of Milton [23, Section 31.9] of linear elastic mixtures that cannot be mimicked by sequential lamination via an elastic analog of the Šverák's counterexample [42], cf. Remark 9, may be relevant.

9. Discussion

We have essentially established that the key semicontinuity result, Theorem 1, has two seemingly different but fundamentally equivalent implications: full characterization of quasiconvex hulls for three solenoidal wells in dimension three, and complete resolution of the problem whether or not the extremal points of the superset $Y(\theta)$ are (extremal) three-phase H-measures (with the exception of the degenerate case of measures supported in three co-planar directions). In this work we confined ourselves to characterizing the zero sets for the relaxed energy $Q_S F$; however, the approach could equally be extended for establishing the optimality, or otherwise, of the H-measure lower bound (B.20) for $Q_S F$ with equal quadratic wells, cf. [13,39] (remark in passing that the analysis becomes substantially harder for the case of unequal elastic moduli even for two wells, $N = 2$, see for example [3,37] for recent progress).

The presented results could be generalized further in various ways. The semicontinuity Theorem 1 can be generalized both from the two-dimensional planes to three-dimensional subspaces of diagonal matrices for $d = 3$, and for $d > 3$, with further counter-examples analogous to [42] for some higher dimensions, see [32]. It should also be possible to describe a rather general class of differential constraints $\mathscr{A}$ where on one hand the constant rank condition does not hold (making the classical results [9,29] inapplicable) but on the other hand the appropriate semicontinuity result still holds via an appropriate application of the Haar wavelet estimates.

The most recent extension in [19] of the interpolatory estimates between Haar projections and Riesz transforms for arbitrary p-growth ($1 < p < \infty$) allows almost immediately generalizing the presented result correspondingly. It also widens the scope for further applications, for example for improving further the bounds for nonlinear composites, cf. [11, Section 6], including the "harder" case of $p > 2$, cf. [44]. Remark in passing that whenever the bounds happen to be non-optimal our construction allows, at least in principle, for *quantifying* this non-optimality, with the potential application for the bounds improvement, et cetera. Likewise, the elimination of the points of $Y_e(\theta)$ from the H-measures yields also, in principle, elimination of a quantifiable neighborhood of such points, leading in effect to additional restrictions on the H-measures.

For N-phase H-measures with arbitrary N, the structure of the extremal points of $Y(\theta)$ was studied in [13] where a direct analog of Proposition 5 was established, with the extremal points supported in (no more than) $l(N) = N(N-1)/2$ Dirac masses, and the realizability of some of those by sequential lamination was also discussed, which is in the direction of generalization of the sufficient condition in [39, Proposition 6.1] for arbitrary N. However, for generalizing the necessary conditions of the present approach, the $l(N)$ directions have to be linearly independent, which requires $d \geqq N(N-1)/2$ (with lower dimensions for the points in

$Y_e(\theta)$ supported in less than $l(N)$ directions). In particular, for $N = 2$ the approach works for any dimension ($l(2) = 1$), and for $N = 3$ for $d \geqq 3$. Possible further generalizations would require additional modifications of the presented ideas (for example from harmonic analysis, cf. [19]) and/or other ideas. We also expect the new H-measure results to be applicable for characterizing the $\mathscr{A}$-quasiconvex hulls for rather generic differential constraints.

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Appendix A. Proof of Lemma 1

We may assume that $M = 0$ and $N = G^{-1}$ since, as already remarked, shift by a matrix and left-multiplication by an invertible matrix do not play any role (with the latter for example keeping the divergence-free property). Let $B_0 \in K_S^{qc}$ and $G \in GL(3, \mathbb{R})$. By definition there exists a sequence of Q-periodic L^2 equi-integrable divergence free matrix fields $\{B_h\}$ which satisfy

$$\operatorname{dist}(B_h, K) \to 0 \quad \text{locally in measure,} \quad \text{and} \quad \fint_Q B_h = B_0.$$

We introduce the new variable y in $\mathbb{R}^3$ given by

$$y = G^T x$$

and define the sequence $\{\bar{B}_h\}$ in the following way:

$$\bar{B}_h(y) := G^{-1} B_h(G^{-T} y) G. \tag{A.1}$$

Then one can check that B_h is still divergence free in $\mathbb{R}^3$, and

$$\operatorname{dist}(\bar{B}_h, \bar{K}) \to 0 \quad \text{locally in measure.} \tag{A.2}$$

If $G \in GL(3, \mathbb{Q})$, that is G is rational-valued and invertible, then there exists a positive integer l such that $\bar{B}_h$ is periodic with periodicity cube $(0, 2l\pi)^3$, which can be re-scaled back to a 2π-periodic field $\bar{B}_h(yl)$ completing the proof. If G has irrational entries, we can employ the following standard construction, cf. for example [14]. Decompose $\bar{B}_h$ in the following way:

$$\bar{B}_h(y) = G^{-1}B_0G + G^{-1}(B_h - B_0)(G^{-T}y)G.$$

Notice then that $G^{-1}(B_h - B_0)(G^{-T}y)G$ is divergence free and periodic with periodicity cell $\bar{Q} := G^T Q$ and zero mean in $\bar{Q}$. Therefore there exists a sequence of $\bar{Q}$-periodic matrix fields $\{V_h\} \subset H^1_{\mathrm{loc}}(\mathbb{R}^3)$ bounded in $L^2(\bar{Q})$ such that

$$G^{-1}(B_h - B_0)(G^{-T}y)G = \mathrm{Curl}\, V_h(y). \tag{A.3}$$

[The curl-operation is understood above as acting on each row of the matrix $V_h(y)$. The representation (A.3) can be derived by, for example, directly adopting the argument in [14, page 7] to $\bar{Q}$-periodic functions.] Now let $\{L_h\}$ be an increasing sequence of positive numbers such that $L_h \to \infty$ as $h \to \infty$ and define

$$\varphi_h(y) := \min\{1, \mathrm{dist}(y, \partial Q_h)\} \quad \text{for} \quad y \in Q_h$$

where $Q_h = (0, 2\pi L_h)^3$ and extend φ_h periodically to the whole $\mathbb{R}^3$. Next set

$$\widehat{B}_h(y) = G^{-1}B_0G + \frac{1}{L_h}\mathrm{Curl}\,(\varphi_h(L_h y)V_h(L_h y))$$

and observe that the sequence $\{\widehat{B}_h\}$ is divergence-free Q-periodic, L^2_{loc}-equi -integrable and satisfies (A.2) since, in particular,

$$\int_Q |V_h(L_h y) \wedge \nabla\varphi_h(L_h y)|^2 \,\mathrm{d}y \leqq \frac{1}{L_h^3}\int_{Q_h \cap \{\nabla\varphi_h \neq 0\}} |V_h(y)|^2 \,\mathrm{d}y$$
$$\leqq C\frac{|\det G|^{-1}}{L_h}\|V_h\|^2_{L^2(G^T Q)} \to 0\,.$$

□

Appendix B. H-measures of characteristic functions

Appendix B.1. Definition and basic properties

The following is the definition of H-measures associated with periodic microgeometries of H-measures, sufficient for the purposes of the present work. For a general construction, involving functions that need not be periodic or characteristic, see, for example [12,47].

We denote by $\hat{\chi}_j(k)$, $k \in \mathbb{Z}^d$, the Fourier coefficients for the Q-periodic functions χ_j:

$$\hat{\chi}_j(k) := \fint_Q \chi_j(x)e^{-ik\cdot x}\mathrm{d}x.$$

For every $\chi \in I(\theta)$, see (5), we call H-measure generated by χ the matrix-valued measure $\mu = (\mu_{ij})_{i,j}$ defined as follows:

$$\mu_{ij} = \operatorname{Re} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \hat{\chi}_i(k) \overline{\hat{\chi}_j(k)} \delta_{k/|k|} , \quad 1 \leqq i, j \leqq N , \tag{B.1}$$

where $\delta_{k/|k|}$ denotes a unit Dirac mass at the point $\xi = k/|k|$ on the unit sphere S^{d-1} and $k \neq 0$ has integer components. Introduce the notation

$$\int_{S^{d-1}} \varphi(\xi) \mathrm{d}\mu_{ij}(\xi)$$

to denote integration of a continuous function φ, $\varphi \in C(S^{d-1})$, with respect to the measure μ_{ij}.
The set of all possible H-measures for a given θ, which we will denote by $Y^H(\theta)$, is characterized by including all weak* limits of (B.1), that is all Borel matrix-valued measures μ_{ij} such that there exists a sequence of measures μ_{ij}^m of the form (B.1) for some $\chi^m \in I(\theta)$, $m = 1, 2, \ldots$, and $\mu_{ij}^m \overset{*}{\rightharpoonup} \mu_{ij}$, that is

$$\int_{S^{d-1}} \varphi_{ij}(\xi) \mathrm{d}\mu_{ij}^m(\xi) \longrightarrow \int_{S^{d-1}} \varphi_{ij}(\xi) \mathrm{d}\mu_{ij}(\xi) \tag{B.2}$$

as $m \to \infty$ for all $\varphi_{ij} \in C(S^{d-1})$. So

$$Y^H(\theta) = \left\{ \mu_{ij} : \exists\, \mu_{ij}^m \text{ of the form (B.1) and } \mu_{ij}^m \overset{*}{\rightharpoonup} \mu_{ij} \text{ as } m \to \infty \right\}. \tag{B.3}$$

It can be checked (see, for example [17,39]) that the H-measures satisfy the following properties:

$$\mu_{ij} = \mu_{ji} \quad \text{and} \quad \sum_{i=1}^{N} \mu_{ij} = 0 , \quad 1 \leqq j \leqq N , \tag{B.4}$$

$$\int_{S^{d-1}} \mathrm{d}\mu_{ij}(\xi) = \delta_{ij}\theta_i - \theta_i\theta_j \;=: M_{ij}(\theta) \quad \text{(no summation in } i, j), \tag{B.5}$$

$$\begin{aligned} \mu_{ij}(\xi) = \mu_{ij}(-\xi), \quad \text{that is} &\int_{S^{d-1}} f(-\xi) \mathrm{d}\mu_{ij}(\xi) \\ &= \int_{S^{d-1}} f(\xi) \mathrm{d}\mu_{ij}(\xi), \quad \forall f \in C(S^{d-1}), \end{aligned} \tag{B.6}$$

$$\sum_{i,j=1}^{N} \int_{S^{d-1}} \varphi_i(\xi)\varphi_j(\xi) \mathrm{d}\mu_{ij}(\xi) \geqq 0 \quad \text{for any } \varphi_j \in C(S^{d-1}), \quad j = 1, \ldots, N. \tag{B.7}$$

We denote the set of all Borel measures on the unit sphere[4] S^{d-1} subject to restrictions (B.4)–(B.7) by $Y(\theta)$:

$$Y(\theta) = \left\{ \mu = (\mu_{ij})_{i,j=1}^{N} : \text{(B.4)} - \text{(B.7) hold} \right\}. \tag{B.8}$$

It follows that $Y^H(\theta) \subset Y(\theta)$. Both $Y(\theta)$ and $Y^H(\theta)$ are weak* closed infinite-dimensional convex sets in the space of matrix measures. Further, since (B.4)–(B.7) ensure that $Y(\theta)$ is bounded in $\left(C(S^{d-1})^{N\times N}\right)^*$, it is weak* compact and hence so is $Y^H(\theta)$. Hence, by Krein–Milman theorem, for example [5], both $Y^H(\theta)$ and $Y(\theta)$ are fully characterized by their extremal points.

Kohn (see [17], Theorem 6.4) has shown that for the case of two wells ($N = 2$) the conditions (B.4)–(B.7) are necessary and sufficient to characterize the whole set $Y^H(\theta)$, that is the sets $Y^H(\theta)$ and $Y(\theta)$ coincide for $N = 2$. In contrast, for $N > 2$, the above restrictions are generally insufficient (Kohn, personal communications; see also discussion in [39]). Therefore the set $Y^H(\theta)$ is strictly contained, at least in some cases, in $Y(\theta)$.

Appendix B.2. Description of the set $Y(\theta)$ for $N = 3$

The following proposition gives an explicit description of the set $Y_e(\theta)$ of extremal points of the superset $Y(\theta)$ for $N = 3$, following Smyshlyaev and Willis [39], see also [13]:

Proposition 5. *Let $N = 3$ and $\theta \in [0,1]^3$, $\sum_{i=1}^{3} \theta_i = 1$. Then $\mu \in Y_e(\theta)$ if and only if*

$$\mu = \sum_{r=1}^{3} m^r \otimes m^r \, \delta_{\xi_r}, \tag{B.9}$$

where $m^r \in \mathbb{R}^3$, $r = 1, 2, 3$, are such that $\sum_{i=1}^{3} m_i^r = 0$, $\sum_{r=1}^{3} m_i^r m_j^r = \delta_{ij}\theta_i - \theta_i\theta_j$, and $\xi_r \in S^{d-1}$, $r = 1, 2, 3$, are (counting $\pm\xi$ as one point), either

(i) *three different points, with either m^r, $r = 1, 2, 3$ linearly independent, or $m^3 = 0$ and m^1, m^2 linearly independent; or*
(ii) $\xi_1 = \xi_2 = \xi_3$.

Proof. This follows essentially directly from the proofs of [39, Proposition 5.1, Lemma 5.2 and Proposition 5.3], see also [13, Proposition 5.1][5] where this was made more explicit and generalized for arbitrary N. □

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[4] Note that the restriction (B.6) requires the measures to be distributed over the sphere symmetrically. Therefore we can always identify the opposite points $\pm\xi$ on the sphere (hence, in effect dealing with the projective space $\mathbb{R}P^{d-1}$ rather than S^{d-1}), as we will henceforth implicitly assume.

[5] The "if" condition is stated in [13] not entirely precisely, which does not however affect the subsequent applications.

Notice that the above "component masses" $\mu^r = m^r \otimes m^r$ lie on the boundary $\partial\mathscr{K}$ of the cone $\mathscr{K}$ of non-negative symmetric matrices in the three-dimensional subspace of $\mathbb{M}^{3\times 3}$ specified by (B.4). Proposition 5 hence means that the extremal measures are either supported in three or two points and then are such that the component masses $\mu^r \in \partial\mathscr{K}$, $r = 1, 2(, 3)$ are linearly independent, or are supported in a single point with total mass (B.5).

The restriction (B.4) implies that it is sufficient to consider three component scalar measures

$$a(\xi) := \mu_{22}(\xi), \quad b(\xi) := \mu_{23}(\xi) = \mu_{32}(\xi), \quad c(\xi) := \mu_{33}(\xi). \tag{B.10}$$

with the condition (B.7) requiring the matrix measures $\begin{pmatrix} a(\xi) & b(\xi) \\ b(\xi) & c(\xi) \end{pmatrix}$ to be non-negative. By (B.5) for any measure $\mu \in Y(\theta)$ the associated reduced 2×2 "total mass" is

$$M(\theta) = \begin{pmatrix} \theta_2(1-\theta_2) & -\theta_2\theta_3 \\ -\theta_2\theta_3 & \theta_3(1-\theta_3) \end{pmatrix}. \tag{B.11}$$

Every non-negative symmetric matrix $\mu = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ belongs to the convex cone $\mathscr{K}$ in the (a, b, c) space:

$$\mathscr{K} = \left\{ (a, b, c) \in \mathbb{R}^3 : a \geqq 0\,, \ c \geqq 0\,, \ ac - b^2 \geqq 0 \right\}.$$

Every matrix $\mu \neq 0$ belonging to $\mathscr{K}$ is uniquely characterized by its trace $\operatorname{tr}\mu = a + c > 0$ and its "projection" μ_{cs} on the cross-section $\mathscr{K}_{cs}$ of the unit trace:

$$\mu = (\operatorname{tr}\mu)\mu_{cs}\,.$$

The cross-section $\mathscr{K}_{cs}$ is described by the relations $a + c = 1$, $b^2 + (c - 1/2)^2 \leqq 1/4$ and so can be identified with a disc in the (c, b)-plane (see Fig. 6). The total mass $M(\theta)$ belongs to $\mathscr{K}$ and (B.11) implies that its projection M_{cs} on $\mathscr{K}_{cs}$ always lies inside the triangle defined in the (c, b)-plane by the points

$$\nu_1 = (0, 0), \quad \nu_2 = (1, 0), \quad \nu_3 = (1/2, -1/2),$$

by noticing that $M = \theta_1\theta_2\nu_1 + \theta_1\theta_3\nu_2 + 2\theta_2\theta_3\nu_3$. The latter "basic points" ν_r, $r = 1, 2, 3$, are themselves the projections of three special directions $n^r \otimes n^r$ on $\partial\mathscr{K}$ where in the original symmetric "full" form

$$n^1 = (-1, 1, 0)^T, \quad n^2 = (1, 0, -1)^T, \quad n^3 = (0, 1, -1)^T, \tag{B.12}$$

and play important role in the subsequent analysis.

The boundary component masses $\mu^r = m^r \otimes m^r$ have cross-sections lying on the (c, b)-plane on the boundary circle $C := \{(c, b) : b^2 + (c - 1/2)^2 = 1/4\}$, that

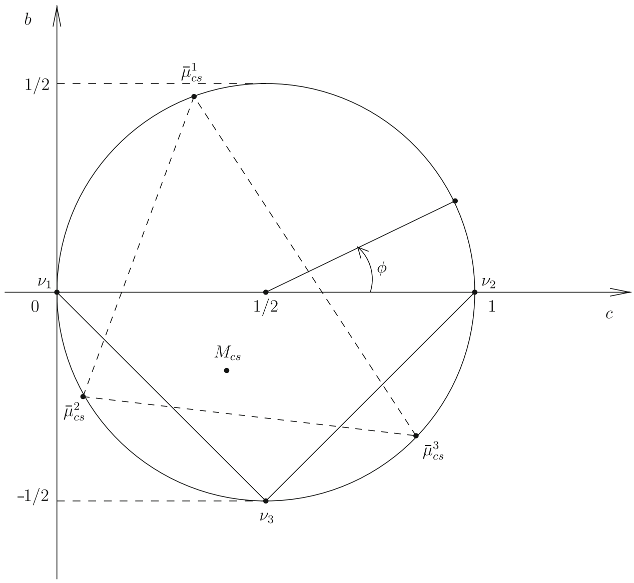

Fig. 6. The cross-section $\mathscr{K}_{cs}$ of the cone $\mathscr{K}$ on the (c, b)-plane

is are extremal for $\mathscr{K}_{cs}$ (equivalently, $\mu^r_{cs} = m^r_{cs} \otimes m^r_{cs}$, $m^r_{cs} \in \mathbb{R}^2$, $|m^r_{cs}| = 1$, that is $m^r_{cs} \in S^1$). Parametrize C by the angle $\phi \in [0, 2\pi)$, see Fig. 6:

$$\begin{cases} a = (1 - \cos\phi)/2 \; = \; \sin^2(\phi/2) \, (= \, 1 - c) \\ b = \frac{1}{2}\sin\phi \; = \sin(\phi/2)\cos(\phi/2) \\ c = (1 + \cos\phi)/2 \; = \; \cos^2(\phi/2). \end{cases} \tag{B.13}$$

On this way, for any $\mu \in Y_e(\theta)$, the points μ^r_{cs} can be identified by the angles ϕ_r, $r = 1, 2(, 3)$.

Now go back to the problem of evaluating $Q^\theta_S F$, via H-measures, see (42) for $N = 3$. Without knowing $Y^H(\theta)$, we instead minimize in (42) over the larger set $Y(\theta)$. This will lead to the precise evaluation of $Q^\theta_S F$ provided (one of) the minimizing measure(s) $\mu \in Y(\theta)$ turns out to be an H-measure, that is $\mu \in Y^H(\theta)$. Otherwise, it will provide a strict lower bound on $Q^\theta_S F$. With this aim we set

$$L(\theta) := \inf_{\mu \in Y(\theta)} \sum_{i,j=1}^{3} \int_{S^{d-1}} f^{ij}(\xi)\, \mathrm{d}\mu_{ij}(\xi)\,, \tag{B.14}$$

where f^{ij} is defined by (41) (hence, by construction, $Q^\theta_S F(B_0) \geqq L(\theta)$ where $B_0 = \sum_{i=1}^{3} \theta_i A_i$).

The next lemma gives an explicit formula for $L(\theta)$ clarifying further the role of the set $Y_e(\theta)$ in the minimization problem (B.14). It is assumed, without loss of generality, that $A_1 = 0$.

Lemma 9. [39] *Let $\theta \in (0,1)^3$ be given with $\sum_{i=1}^{3}\theta_i = 1$ and $A_1, A_2, A_3 \in \mathbb{M}^{m\times d}$ with $A_1 = 0$. Then the infimum in (B.14) is attained and the minimizing measure can be chosen in $Y_e(\theta)$. Moreover*

$$L(\theta) = (\mathrm{tr}\, M(\theta)) \min \sum_{r=1}^{3} \alpha_r \psi(\mu_{cs}^r), \tag{B.15}$$

where $\psi : \mathscr{K} \to \mathbb{R}$ is defined by

$$\psi(\mu) = \psi(a,b,c) := \min_{\xi\in S^{d-1}} \left\{ af^{22}(\xi) + 2bf^{23}(\xi) + cf^{33}(\xi) \right\}, \tag{B.16}$$

and the minimum in (B.15) is taken over all possible decompositions of $M_{cs}(\theta)$ into a convex combination of no more than three extremal masses $\mu_{cs}^r \in C$:

$$M_{cs} = \sum_{r=1}^{3} \alpha_r \mu_{cs}^r, \quad \alpha_r \geqq 0, \quad \sum_{r=1}^{3} \alpha_r = 1\,. \tag{B.17}$$

Proof. All the components of the proof can be found in [39] (in particular see Proposition 5.1 and 5.3 and Lemma 5.2 therein); see also [13, Theorem 5.2]. Namely, by the Krein–Milman theorem, the infimum of the linear continuous functional on the right hand side of (B.14) over (a weak* compact convex set) $Y(\theta)$ is achieved at its extremal point. Using $f^{ij}(\xi) = 0$ for either $i = 1$ or $j = 1$ (due to (41) and $A_1 = 0$), and Proposition 5, (B.14) is re-written as

$$L(\theta) = \min_{\mu\in Y_e(\theta)} \sum_{r=1}^{3} \left\{ a_r f^{22}(\xi_r) + 2b_r f^{23}(\xi_r) + c_r f^{33}(\xi_r) \right\}, \tag{B.18}$$

where $a_r = \left(m_2^r\right)^2$, $b_r = m_2^r m_3^r$ and $c_r = \left(m_3^r\right)^2$. Since for any $\mu \in Y_e(\theta)$, $\mu = \sum_{r=1}^{3} \alpha_r \mu_{cs}^r \delta_{\xi_r}$ and $\psi(\mu)$ is homogeneous of degree one, that is $\psi(t\mu) = t\psi(\mu)$ for any $t \geqq 0$, the result follows. □

Lemma 9, hence, suggests the following algorithm for computing $L(\theta)$:

(i) consider all possible decompositions of the cross-section $M_{cs}(\theta)$ of the total mass into the convex combination of at most three extremal matrices μ_{cs}^r, see (B.17).
(ii) for any split (B.17) choose ξ_r minimizing (B.16);

(iii) minimize with respect to all admissible splits of the form (B.17).

This procedure leads to finding (no more than) three critical points $\bar\mu_{cs}^1, \bar\mu_{cs}^2$ (, $\bar\mu_{cs}^3$) on C (see Fig. 6) with associated directions ξ_r, $r = 1, 2, 3$, such that the extremizing measure $\bar\mu$ is

$$\bar\mu = (\mathrm{tr}\, M) \sum_{i=1}^{3} \alpha_r \bar\mu_{cs}^r \delta_{\xi_r}, \quad \alpha_r \geqq 0\,, \quad \alpha_1 + \alpha_2 + \alpha_3 = 1\,. \tag{B.19}$$

Summarizing all the above,

Proposition 6. *The following lower bound holds:*

$$Q_S^{\theta} F(B_0) \geqq L(\theta). \tag{B.20}$$

All the minimizers of the right hand side are in the form (B.19)*. The equality is held in* (B.20) *if and only if at least one of such minimizers is an H-measure.*

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