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Common fixed points for some generalized multivalued nonexpansive mappings in uniformly convex metric spaces

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Abstract

Abkar and Eslamian (Nonlinear Anal. TMA, **74**, 1835-1840, 2011) prove that if K is a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ is a single-valued quasi-nonexpansive mapping and $T : K \to KC(K)$ is a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0, 1)$ such that t and T commute weakly, then there exists a point $z \in K$ such that $z = t(z) \in T(z)$. In this paper, we extend this result to the general setting of uniformly convex metric spaces. Nevertheless, condition (E) of T can be weakened to the strongly demiclosedness of $I - T$.

Keywords: generalized multivalued nonexpansive mapping, commuting mapping, common fixed point, uniformly convex metric space

1 Introduction

Let K be a nonempty subset of a CAT(0) space (X, d) (see Bridson and Haefliger [[1] for more details on this space). A mapping $t : K \to X$ is said to be *nonexpansive* if

$$d(t(x), t(y)) \leq d(x, y) \text{ for all } x, y \in K.$$

A point $x \in K$ is called a *fixed point* of t if $x = t(x)$. We shall denote by Fix(t) the set of fixed points of t: The mapping t is said to be *quasi-nonexpansive* if

$$d(t(x), y) \leq d(x, y) \text{ for all } x \in K \text{ and } y \in \text{Fix}(t).$$

Fixed point theory in CAT(0) spaces was first studied by Kirk [2,3]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed and many of papers have appeared (see e.g., [4-14] and the references therein). It is worth mentioning that fixed point theorems in CAT(0) spaces (specially in $\mathbb{R}$ - trees) can be applied to graph theory, biology, and computer science (see e.g., [15-20]).

In 2005, Dhompongsa et al. [6] obtained a common fixed point result for a commuting pair of single-valued and multivalued nonexpansive mappings in CAT(0) spaces. Shahzad and Markin [14] studied an invariant approximation problem and provided sufficient conditions for the existence of $z \in K \subseteq X$ such that $d(z, y) = \text{dist}(y, K)$ and z

$= t(z) \in T(z)$ where $y \in X$, t and T are commuting nonexpansive mappings on K. Shahzad [21] also obtained common fixed point and invariant approximation results for t and T, which are weakly commuting.

In 2008, Suzuki [22] introduced a condition on mappings, which is weaker than nonexpansiveness and stronger than quasi-nonexpansivemess and called it condition (C). He proved some interesting fixed point theorems and convergence theorems for such mappings in Banach spaces setting. Motivated by Suzuki's results, Garcia-Falset et al. [23] introduced two generalizations of condition (C), namely, conditions (E) and (C_λ) and studied both the existence of fixed points and the asymptotic behavior of mappings satisfying such conditions. Recently, Abkar and Eslamian [4] proved that if K is a nonempty bounded closed convex subset of a complete CAT(0) space X, $t : K \to K$ is a single-valued quasi-nonexpansive mapping and $T : K \to KC(K)$ is a multivalued mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0, 1)$ such that t and T are weakly commuting, then there exists a point $z \in K$ such that $z = t(z) \in T(z)$. In this paper, we extend this result to the general setting of uniformly convex metric spaces in the sense of Goebel and Reich [24]. Nevertheless, condition (E) of T can be weakened to the strongly demiclosedness of $I - T$. Since uniformly convex Banach spaces and CAT(0) spaces are uniformly convex metric spaces, then our results extend and improve the results in [4,25,26,23,27] and many others.

2 Preliminaries

Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t), c(t')) = |t - t'|$ for all t, $t' \in [0, l]$. The image $c([0, l])$ of c is called a *geodesic segment* joining x and y. When it is unique this geodesic segment is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic space* if every two points of X are joined by a geodesic path, and X is said to be *uniquely geodesic* if there is exactly one geodesic path joining x and y for each $x, y \in X$. A point $z \in X$ belongs to the geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that

$$d(z, x) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y),$$

and we will write $z = (1 - t)x \oplus ty$ for simplicity. A subset K of X is said to be *convex* if K includes every geodesic segment joining any two of its points.

Definition 2.1 In a geodesic space (X, d), the metric $d : X \times X \to \mathbb{R}$ is said to be *convex* if for any $x, y, z \in X$, one has

$$d(x, (1 - t)y \oplus tz) \le (1 - t)d(x, y) + td(x, z) \text{ for all } t \in [0, 1].$$

A geodesic space (X, d) with convex metric is called *uniformly convex* ([24]) if for any $r > 0$ and $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that if $a, x, y \in X$ with $d(x, a) \le r$, $d(y, a) \le r$ and $d(x, y) \ge \varepsilon r$, then

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) \le (1 - \delta)r.$$

A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0, 2]$ is called *a modulus of uniform convexity* (see [28]). Notice that this definition of uniform convex metric spaces is weaker than the one used in [29] because this modulus of convexity does depend on the two variables r and ε while it is assumed to depend only on ε in [29]. The mapping δ is called *monotone* if for every fix ε it decreases with respect to r. It is called *lower semi-continuous from the right* if for every fix ε it is lower semi-continuous from the right with respect to r. Throughout this paper, we assume that all uniformly convex metric spaces have monotone or lower semi-continuous from the right moduli of uniform convexity.

Recall that a subset K of a metric space X is said to be (uniquely) *proximinal* if each point $x \in X$ has a (unique) nearest point in K. In [28], Kohlenbach and Leustean prove that every decreasing sequence of nonempty bounded closed convex subsets of a complete uniformly convex metric space has nonempty intersection. As a consequence of this, we obtain the following result (see [[30], page 4] for the proof).

Proposition 2.2 *Every nonempty closed convex subset of a complete uniformly convex metric space is uniquely proximinal.*

The following result is proved by Kaewcharoen and Panyanak [30] in a uniformly convex metric space in the sense of [29]. We can state the result in the sense of Definition 2.1 because the proof is similar to the one given in [30].

Lemma 2.3 *Let K be a convex subset of a uniformly convex metric space and $t : K \to K$ a quasi-nonexpansive mapping whose fixed point set is nonempty. Then $Fix(t)$ is closed and convex.*

The following method and results deal with the concept of asymptotic centers. Let (X, d) be a metric space and let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* of $\{x_n\}$ is given by

$$r(\{x_n\}) := \inf\{r(x, \{x_n\}) : x \in X\},$$

and the *asymptotic center* of $\{x_n\}$ is the set

$$A(\{x_n\}) := \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

The following result is proved in [31].

Lemma 2.4 *If X be a complete uniformly convex metric space and $\{x_n\}$ be a bounded sequence in X, then $A(\{x_n\})$ is a singleton.*

A bounded sequence $\{x_n\}$ is called *regular* if $r(\{x_n\}) = r(\{x_{n_k}\})$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. It is known that in a Banach space, every bounded sequence always has a regular subsequence (see e.g., [[32], Lamma 15.2]). Since the proof has a metric nature, we can conclude that every bounded sequence in a uniformly convex metric space has a regular subsequence.

Definition 2.5 [22] Let K be a nonempty subset of a metric space (X, d). A mapping $T : K \to X$ is said to satisfy condition (C) if for each $x, y \in K$,

$$\frac{1}{2}d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le d(x, y).$$

In [23], Garcia-Falset et al. introduce two generalizations of condition (C) as follows.

Definition 2.6 Let K be a nonempty subset of a metric space (X, d). A mapping $t : K \to X$ is said to satisfy condition (C_λ) for some $\lambda \in (0, 1)$ if for each $x, y \in K$,

$$\lambda d(x, t(x)) \le d(x, y) \text{ implies } d(t(x), t(y)) \le d(x, y).$$

Let $\mu \geq 1$. We say that $t : K \to X$ satisfies condition (E_μ) if for each $x, y \in K$,

$$d(x, t(y)) \leq \mu d(x, t(x)) + d(x, y)$$

We say that t satisfies condition (E) if t satisfies (E_μ) for some $\mu \geq 1$.

Let (X, d) be a geodesic space. We denote by 2^X the family of nonempty subsets of X, by $P(X)$ the family of nonempty proximinal subsets of X, by $K(X)$ the family of nonempty compact subsets of X, and by $KC(X)$ the family of nonempty compact convex subsets of X: Let H be the Hausdorff metric with respect to d, that is,

$$H(A, B) := \max\left\{\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\right\}, \quad A, B \in 2^X,$$

where $\operatorname{dist}(a, B) := \inf\{d(a, b) : b \in B\}$ is the distance from the point a to the set B: The *metric projection (or nearest point mapping)* P_B onto B is the mapping

$$P_B(x) = \{b \in B : d(x, b) = \operatorname{dist}(x, B)\}, \text{ for every } x \in X.$$

We now state the multivalued analogs of conditions (E) and (C_λ) in the following manner:

Definition 2.7 A multivalued mapping $T : K \to 2^X$ is said to satisfy condition (C_λ) for some $\lambda \in (0, 1)$ if for each $x, y \in K$,

$$\lambda \operatorname{dist}(x, T(x)) \leq d(x, y) \text{ implies } H(T(x), T(y)) \leq d(x, y).$$

For $\mu \geq 1$. We say that T satisfies condition (E_μ) if for each $x, y \in K$,

$$\operatorname{dist}(x, T(y)) \leq \mu \operatorname{dist}(x, T(x)) + d(x, y).$$

We say that T satisfies condition (E) if T satisfies (E_μ) for some $\mu \geq 1$.

There are other kinds of conditions (C_λ) and (E) for multivalued mappings defined by Razani and Salahifard [12] for the case of $\lambda > \frac{1}{2}$ in CAT(0) spaces.

They are studied in uniformly convex metric spaces by Espinola et al. [26]. We call them here the condition (C'_λ) (and (E′) respectively.

Definition 2.8 A multivalued mapping $T : K \to 2^X$ is said to satisfy condition (C'_λ) if for each $x, y \in K$ and $u_x \in T(x)$ such that

$$\lambda d(x, u_x) \leq d(x, y),$$

there exists $u_y \in T(y)$ such that

$$d(u_x, u_y) \leq d(x, y).$$

Let $\mu \geq 1$. We say that T satisfies condition (E'_μ) if for each $x, y \in K$ and $u_x \in T(x)$, there exists $u_y \in T(y)$ such that

$$d(x, u_y) \leq \mu d(x, u_x) + d(x, y).$$

We say that T satisfies condition (E′) if T satisfies (E'_μ) for some $\mu \geq 1$.

We note from Definitions 2.7 and 2.8 that if T takes compact values, then the condition (C_λ) (resp. (E)) implies condition (C'_λ) (resp. (E′)). On the other hand, Espinola et al. [26] prove that the condition $(C'_{\frac{1}{2}})$ implies condition (E'_3). By a slightly modification of the proof of Lemma 3.2 in [26], we can show that the condition $(C_{\frac{1}{2}})$ implies condition (E_3). However, it is unknown that condition (C'_λ) (resp. (C_λ)) implies condition (E′) (resp. (E)) for $\lambda > \frac{1}{2}$ even in the case of single-valued mappings (see [[23], Remark 4]).

Recall that a single-valued mapping $t : K \to K$ and a multivalued mapping $T : K \to 2^X$ are said to be *commuting* ([33]) if $t(y) \in T(t(x))$ for all $x \in K$ and $y \in T(x)$. T and t are said to be *weakly commuting* if $t(\partial_K T(x)) \subseteq T(t(x)))$ for all $x \in K$, where $\partial_K A$ denotes the relative boundary of $A \subseteq K$. By using the fact that condition $(C'_{\frac{1}{2}})$ implies condition (E'_3), Espinola et al. [26] ensure the existence of common fixed points for a pair of mappings t and T that are commuting and satisfying condition $(C'_{\frac{1}{2}})$ in uniformly convex metric spaces. We observe that their method can not be extended to the case of $\lambda > \frac{1}{2}$ because, in this case, we do not know if condition (C'_λ) implies condition (E′). However, if we assume, in addition, that the mapping $I - T$ is strongly demiclosed, then we can extend Espinola et al.'s result to the case of $\lambda > \frac{1}{2}$.

Definition 2.9 [23] Let $T : K \to 2^X$, we say that $I - T$ is *strongly demi-closed* if for every sequence $\{x_n\}$ in K which converges to $x \in K$ and such that $\lim_n \operatorname{dist}(x_n, T(x_n)) = 0$, we have $x \in T(x)$.

Notice that for every continuous mapping $T : K \to 2^X$, $I - T$ is strongly demiclosed but the converse is not true (see [[23], Example 5]). The following proposition shows that condition (E′) implies the strongly demiclosedness of $I - T$ but converse is not true (see [[23], Example 2]).

Proposition 2.10 *Let $K \in 2^X$. If $T : K \to P(X)$ satisfies condition (E′) then $I - T$ is strongly demiclosed.*

Proof. Let $\{x_n\}$ be a sequence in K such that $\lim_n \operatorname{dist}(x_n, T(xn)) = 0$ and $\lim_n x_n = x \in K$. Since $T(x_n)$ is proximinal, we can choose $y_n \in T(x_n)$ such that

$$d(x_n, y_n) = \operatorname{dist}(x_n, T(x_n)).$$

Since T satisfies (E'_μ) for some $\mu \geq 1$, there exists $z_n \in T(x)$ such that

$$d(x_n, z_n) \leq \mu d(x_n, y_n) + d(x_n, x).$$

This implies

$$\operatorname{dist}(x_n, T(x)) \leq \mu \operatorname{dist}(x_n, T(x_n)) + d(x_n, x).$$

By taking $n \to \infty$, we have $\lim_n \operatorname{dist}(x_n, T(x)) = 0$. Thus

$$\operatorname{dist}(x, T(x)) \leq d(x, x_n) + \operatorname{dist}(x_n, T(x)) \to 0 \text{ as } n \to \infty.$$

Therefore, $x \in T(x)$. ■

3 Main results

The following lemma is proved in [26] for the case of $\lambda > \frac{1}{2}$. As the same argument in the proof of [[26], Proposition 3.4], it is easy to see that the result also holds for $\lambda > \frac{1}{2}$.

Lemma 3.1 *Let X be a geodesic space with convex metric, K be a bounded convex subset of X and $T : K \to 2^K$. If T satisfies condition (C'_λ),then T has an approximate fixed point sequence in K:*

Theorem 3.2 *Let X be a complete uniformly convex metric space and K be a bounded closed convex subset of X: Suppose that $T : K \to K(K)$ satisfies condition $(C'_\lambda)$and $I - T$ is strongly demiclosed. Then T has a fixed point.*

Proof. By Lemma 3.1, T has an approximate fixed point sequence in K, say $\{x_n\}$. By passing to a subsequence, we may suppose that $\{x_n\}$ is regular. Let $A(\{x_n\}) = \{x\}$. We will show that x is a fixed point of T. Now, if $\liminf_n d(x_n, x) = 0$, again by passing to a subsequence, we may suppose that $\lim_n x_n = x \in K$. Since $I - T$ is strongly demiclosed, $x \in T(x)$. For the case that $\liminf_n d(x_n, x) > 0$, we let $\varepsilon = \frac{1}{2} \liminf_n d(x_n, x)$. For each $n \in \mathbb{N}$, we can choose $y_n \in T(x_n)$ such that

$$d(x_n, y_n) = \text{dist}(x_n, T(x_n)).$$

Notice that

$\lambda d(x_n, y_n) = \lambda d(x_n, T(x_n)) < \varepsilon < d(x_n, x)$ for sufficiently large $n \in \mathbb{N}$.

For such n, there exists $u_n \in T(x)$ such that $d(y_n, u_n) \le d(x_n, x)$. Since $T(x)$ is compact, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\lim_k u_{n_k} = u \in T(x)$. So

$$\begin{aligned} d(x_{n_k}, u) &\le d(x_{n_k}, y_{n_k}) + d(y_{n_k}, u_{n_k}) + d(u_{n_k}, u) \\ &\le d(x_{n_k}, T(x_{n_k})) + d(x_{n_k}, x) + d(u_{n_k}, u). \end{aligned}$$

This implies $\limsup_k d(x_{n_k}, u) \le \limsup_k d(x_{n_k}, x)$. Since $A(\{x_n\}) = \{x\}$, $x = u \in T(x)$. ■

Lemma 3.3 *Let K be a nonempty subset of a metric space X and $t : K \to K$ be a quasi-nonexpansive mapping whose fixed point set is nonempty. Suppose that $T : K \to 2^K$ is such that for every $x, y \in Fix(t)$, the set $P_{\partial_K T(y)}(x)$is a singleton. If t and T commute weakly, then $P_{\partial_K T(y)}(x) \in Fix(t)$for all $x, y \in Fix(t)$.*

Proof. Let $x, y \in Fix(t)$ and denote $P_{\partial_K T(y)}(x)$ by u. Since t and T commute weakly,

$$t(\partial_K T(y)) \subseteq T(t(y)) = T(y).$$

Thus $t(u) \in T(y)$: Hence

$$d(x, t(u)) = d(t(x), t(u)) \le d(x, u).$$

Thus $t(u) = u$ and the conclusion follows. ■

Theorem 3.4 *Let X be a complete uniformly convex metric space, K a bounded closed convex subset of X and $t : K \to K$ a quasi-nonexpansive mapping whose fixed point set is nonempty. Suppose that $T : K \to KC(K)$ satisfies condition (C'_λ)* and *$I - T$ is strongly demiclosed. If t and T commute weakly, then there exists $z \in K$ such that $z = t(z) \in T(z)$.*

Proof. Let $A = Fix(t)$, then by Lemma 2.3, A is closed and convex. By Proposition 2.2, the projection onto each compact and convex set is a singleton. By Lemma 3.3, $P_{\partial_K T(x)}(x) \in T(x) \cap A$ for all $x \in A$. We consider the mapping

$$F(\cdot) := T(\cdot) \cap A : A \to K(A).$$

We show that F satisfies condition (C'_λ) and $I - F$ is strongly demiclosed. Let and $x, y \in A$, $u_x \in T(x) \cap A$ such that

$$\lambda d(x, u_x) \le d(x, y).$$

Since T satisfies (C'_λ), there exists $v_y \in T(y)$ such that

$$d(u_x, v_y) \le d(x, y).$$

Let u_y stands for $P_{\partial_K T(y)}(u_x)$. According to Lemma 3.3, $u_y \in T(y) \cap A$. It is also clear that

$$d(u_x, u_y) \leq d(u_x, v_y) \leq d(x, y).$$

Thus, F satisfies (C'_λ). The strongly demiclosedness of $I - F$ follows from the one of $I - T$. By Theorem 3.2, there exists $z \in K$ such that $z \in F(z)$: Therefore $z = t(z) \in T(z)$. ■

It is known that if $t : K \to K$ has a fixed point, then both conditions (E) and (C_λ) imply the quasi-nonexpansiveness of t. This fact yields the following corollary.

Corollary 3.5 *Let X be a complete uniformly convex metric space, K a bounded closed convex subset of X, $t : K \to K$ a mapping satisfying condition (E) or (C_λ) for which Fix(t) is nonempty. Suppose that $T : K \to KC(K)$ satisfies condition (C'_λ)* and *$I - T$ is strongly demiclosed. If t and T commute weakly, then there exists $z \in K$ such that $z = t(z) \in T(z)$.*

By using the argument in the proof of Theorem 3.4, we can also obtain the following result that is an extension of [[4], Theorem 3.8]. Recall that a point $z \in X$ is said to be a *center* of the mapping $t : K \to X$ if for each $x \in K$, $d(z, t(x)) \leq d(z, x)$. The set of all centers of t is denoted by $Z(t)$.

Theorem 3.6 *Let X be a complete uniformly convex metric space, K a bounded closed convex subset of X, $t : K \to K$ a mapping. Suppose that $T : K \to KC(K)$ satisfies condition $(C'_\lambda)$and $I - T$ is strongly demiclosed. If t and T commute weakly, and $\emptyset \neq T(x) \cap$ Fix$(t) \subseteq Z(t)$, and Fix(t) is closed and convex, then there exists $z \in K$ such that $z = t(z) \in T(z)$.*

Finally, we show that the strongly demiclosedness of $I - T$ in Theorems 3.2 and 3.4 is necessary.

Example 3.7 Put $X = \mathbb{R}$ and $K = [-1/4, 1]$: Let t be the identity mapping on K and let T be the mapping on K defined by

$$T(x) = \begin{cases} 1 & x = 0, \\ -(1/3)x & x \in [-1/4, 0) \cup (0, 3/4), \\ 1 - x & x \in [3/4, 1] \end{cases}$$

It is easy to see that t and T commute. In [23], the authors prove that either

$$|T(x) - T(y)| \leq |x - y| \quad \text{or} \quad (3/4)\min\left\{|x - T(x)|, |y - T(y)|\right\} \geq |x - y|.$$

We now let $\varepsilon \in \left(0, \frac{1}{4}\right)$, then either

$$|T(x) - T(y)| \leq |x - y| \quad \text{or} \quad \left(\frac{3}{4} + \varepsilon\right)\min\left\{|x - T(x)|, |y - T(y)|\right\} > |x - y|.$$

This implies that T satisfies $(C_{\frac{3}{4}+\varepsilon})$ for all $\varepsilon \in (0, \frac{1}{4})$. Let $\{x_n\} = \{\frac{1}{n}\}$, then $\{x_n\}$ is an approximate fixed point sequence for T which converges to 0. But 0 is not a fixed point of T. This shows that $I - T$ is not strongly demiclosed. It is easy to see that T does not have a fixed point.

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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