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Hybrid iterative scheme for a generalized equilibrium problems, variational inequality problems and fixed point problem of a finite family of κ_i -strictly pseudocontractive mappings

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Abstract

In this article, by using the S -mapping and hybrid method we prove a strong convergence theorem for finding a common element of the set of fixed point problems of a finite family of κ_i -strictly pseudocontractive mappings and the set of generalized equilibrium defined by Ceng et al., which is a solution of two sets of variational inequality problems. Moreover, by using our main result we have a strong convergence theorem for finding a common element of the set of fixed point problems of a finite family of κ_i -strictly pseudocontractive mappings and the set of solution of a finite family of generalized equilibrium defined by Ceng et al., which is a solution of a finite family of variational inequality problems.

Keywords: κ -strict pseudo contraction mapping, α -inverse strongly monotone, generalized equilibrium problem, variational inequality, the S -mapping

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . A mapping T of H into itself is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. We denote by $F(T)$ the set of fixed points of T (i.e., $F(T) = \{x \in H : Tx = x\}$). Goebel and Kirk [1] showed that $F(T)$ is always closed convex, and also nonempty provided T has a bounded trajectory.

Recall the mapping T is said to be κ -strict pseudo-contraction if there exist $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in D(T). \quad (1.1)$$

Note that the class of κ -strict pseudo-contractions strictly includes the class of nonexpansive mappings, that is T is nonexpansive if and only if T is 0-strict pseudo-contraction. If $\kappa = 1$, T is said to be *pseudo-contraction mapping*. T is *strong pseudo-contraction* if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contraction. In a real Hilbert space H (1.1) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2} \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in D(T). \quad (1.2)$$

T is pseudo-contraction if and only if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 \quad \forall x, y \in D(T).$$

T is strong pseudo-contraction if there exists a positive constant $\lambda \in (0, 1)$

$$\langle Tx - Ty, x - y \rangle \leq (1 - \lambda)\|x - y\|^2 \quad \forall x, y \in D(T)$$

The class of κ -strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contraction mappings and class of strong pseudo-contraction mappings is independent of the class of κ -strict pseudo-contraction.

A mapping A of C into H is called *inverse-strongly monotone*, see [2] if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$.

The equilibrium problem for G is to determine its equilibrium points, i.e., the set

$$EP(G) = \{x \in C : G(x, y) \geq 0, \quad \forall y \in C\}. \tag{1.3}$$

Given a mapping $T : C \rightarrow H$, let $G(x, y) = \langle Tx, y - x \rangle$ for all $x, y \in C$. Then, $z \in EP(G)$ if and only if $\langle Tz, y - z \rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Let $A : C \rightarrow H$ be a nonlinear mapping. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0 \tag{1.4}$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$.

In 2005, Combettes and Hirstoaga [3] introduced some iterative schemes of finding the best approximation to the initial data when $EP(G)$ is nonempty and proved strong convergence theorem.

Also in [3] Combettes and Hiratoaga, following [4] define $S_r : H \rightarrow C$ by

$$S_r(x) = \{z \in C : G(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \forall y \in C\}. \tag{1.5}$$

they proved that under suitable hypotheses G , S_r is single-valued and firmly nonexpansive with $F(S_r) = EP(G)$.

Numerous problems in physics, optimization, and economics reduce to find a element of $EP(G)$ (see, e.g., [5-16])

Let $CB(H)$ be the family of all nonempty closed bounded subsets of H and $\mathcal{H}(\cdot, \cdot)$ be the Hausdorff metric on $CB(H)$ defined as

$$\mathcal{H}(U, V) = \max \left\{ \sup_{u \in U} d(u, V), \sup_{v \in V} d(U, v) \right\}, \quad \forall U, V \in CB(H),$$

where $d(u, V) = \inf_{v \in V} d(u, v)$, $d(U, v) = \inf_{u \in U} d(u, v)$, and $d(u, v) = \|u - v\|$.

Let C be a nonempty closed convex subset of H . Let $\phi : C \rightarrow \mathbb{R}$ be a real-valued function, $T : C \rightarrow CB(H)$ a multivalued mapping and $\Phi : H \times C \times C \rightarrow \mathbb{R}$ an equilibrium-like function, that is, $\Phi(w, u, v) + \Phi(w, v, u) = 0$ for all $(w, u, v) \in H \times C \times C$ which satisfies the following conditions with respect to the multivalued map $T : C \rightarrow CB(H)$.

(H1) For each fixed $v \in C$, $(\omega, u) \mapsto \Phi(\omega, u, v)$ is an upper semicontinuous function from $H \times C$ to \mathbb{R} , that is, for $(\omega, u) \in H \times C$, whenever $\omega_n \rightarrow \omega$ and $u_n \rightarrow u$ as $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \Phi(\omega_n, u_n, v) \leq \Phi(\omega, u, v);$$

(H2) For each fixed $(w, v) \in H \times C$, $u \mapsto \Phi(w, u, v)$ is a concave function;

(H3) For each fixed $(w, u) \in H \times C$, $v \mapsto \Phi(w, u, v)$ is a convex function.

In 2009, Ceng et al. [17] introduced the following generalized equilibrium problem (GEP) as follows:

$$(GEP) \begin{cases} \text{Find } u \in C \text{ and } w \in T(u) \text{ such that} \\ \Phi(w, u, v) + \varphi(v) - \varphi(u) \geq 0, \forall v \in C. \end{cases} \quad (1.6)$$

The set of such solutions $u \in C$ of (GEP) is denote by $(GEP)_s(\Phi, \phi)$.

In the case of $\phi \equiv 0$ and $\Phi(w, u, v) \equiv G(u, v)$, then $(GEP)_s(\Phi, \phi)$ is denoted by $EP(G)$. By using Nadler's theorem they introduced the following algorithm:

Let $x_1 \in C$ and $w_1 \in T(x_1)$, there exists sequences $\{w_n\} \subseteq H$ and $\{x_n\}, \{u_n\} \subseteq C$ such that

$$\begin{cases} w_n \in T(x_n), \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi(w_n, u_n, v) + \varphi(v) - \varphi(u_n) + \frac{1}{r_n} \langle u_n - x_n, v - u_n \rangle \geq 0, \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n, \quad n = 1, 2, \dots \end{cases} \quad (1.7)$$

They proved a strong convergence theorem of the sequence $\{x_n\}$ generated by (1.7) as follows:

Theorem 1.1. (See [17]) *Let C be a nonempty, bounded, closed, and convex subset of a real Hilbert space H and let $\phi : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T : C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with constant μ , $\Phi : H \times C \times C \rightarrow \mathbb{R}$ be an equilibrium-like function satisfying (H1)-(H3) and S be a nonexpansive mapping of C into itself such that $F(S) \cap (GEP)_s(\Phi, \varphi) \neq \emptyset$. Let f be a contraction of C into itself and let $\{x_n\}, \{w_n\}$, and $\{u_n\}$ be sequences generated by (1.7), where $\{\alpha_n\} \subseteq [0,1]$ and $\{r_n\} \subseteq (0, \infty)$ satisfy*

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n &= 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ \liminf_{n \rightarrow \infty} r_n &> 0 \text{ and } \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{aligned}$$

If there exists a constant $\lambda > 0$ such that

$$\Phi(w_1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi(w_2, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\lambda \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2 \quad (1.8)$$

for all $(r_1, r_2) \in \Xi \times \Xi, (x_1, x_2) \in C \times C$ and $w_i \in T(x_i), i = 1, 2$, where $\Xi = \{r_n : n \geq 1\}$, then for $\hat{x} = P_{F(S) \cap (GEP)_s(\Phi, \varphi)} f(\hat{x})$, there exists $\hat{w} \in T(\hat{x})$ such that (\hat{x}, \hat{w}) is a solution of (GEP) and

$$x_n \rightarrow \hat{x}, w_n \rightarrow \hat{w} \text{ and } u_n \rightarrow \hat{x} \text{ as } n \rightarrow \infty.$$

In 2011, Kangtunyakarn [18] proved the following theorem for strict pseudocontractive mapping in Hilbert space by using hybrid method as follows:

Theorem 1.2. *Let C be a nonempty closed convex subset of a Hilbert space H . Let F and G be bifunctions from $C \times C$ into \mathbb{R} satisfying (A_1) - (A_4) , respectively. Let $A : C \rightarrow H$ be a α -inverse strongly monotone mapping and let $B : C \rightarrow H$ be a β -inverse strongly monotone mapping. Let $T : C \rightarrow C$ be a κ -strict pseudo-contraction mapping with $\mathbb{F} = F(T) \cap EP(F, A) \cap EP(G, B) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C = C_1$ and*

$$\begin{cases} F(u_n, u) + (Ax_n, u - u_n) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0, & \forall u \in C, \\ G(v_n, v) + (Bx_n, v - v_n) + \frac{1}{s_n} \langle v - v_n, v_n - x_n \rangle \geq 0, & \forall v \in C, \\ z_n = \delta_n u_n + (1 - \delta_n) v_n \\ \gamma_n = \alpha_n z_n + (1 - \alpha_n) T z_n \\ C_{n+1} = \{z \in C_n : \|\gamma_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \quad (1.9)$$

where $\{\alpha_n\}_{n=0}^\infty$ is sequence in $[0, 1]$, $r_n \in [a, b] \subset (0, 2\alpha)$ and $s_n \in [c, d] \subset (0, 2\beta)$ satisfy the following condition:

- (i) $\lim_{n \rightarrow \infty} \delta_n = \delta \in (0, 1)$
- (ii) $0 \leq \kappa \leq \alpha_n < 1, \quad \forall n \geq 1$

Then x_n converges strongly to $P_{\mathbb{F}} x_1$.

From motivation of (1.7) and (1.9), we define the following algorithm as follows:

Algorithm 1.3. *Let $T_i, i = 1, 2, \dots, N$, be κ_i -pseudo contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ and let S_n be the S -mappings generated by T_1, T_2, \dots, T_N and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$ where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0, 1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, \dots, N - 1, \kappa < c \leq \alpha_1^{n,N} \leq 1, \kappa \leq \alpha_3^{n,N} \leq d < 1, \kappa \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1, 2, \dots, N$. Let $x_1 \in C = C_1$ and $w_1^1 \in T(x_1), w_1^2 \in D(x_1)$, there exists sequence $\{w_n^1\}, \{w_n^2\} \in H$ and $\{x_n\}, \{u_n\}, \{v_n\} \subseteq C$ such that*

$$\begin{cases} w_n^1 \in T(x_n), \quad \|w_n^1 - w_{n+1}^1\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}(T(x_n), T(x_{n+1})), \\ w_n^2 \in D(x_n), \quad \|w_n^2 - w_{n+1}^2\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}(D(x_n), D(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n} \langle u_n - x_n, u - u_n \rangle \geq 0, \quad \forall u \in C, \\ \Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n} \langle v_n - x_n, v - v_n \rangle \geq 0, \quad \forall v \in C, \\ z_n = \delta_n P_C(I - \lambda A)u_n + (1 - \delta_n) P_C(I - \eta B)v_n, \\ \gamma_n = \alpha_n z_n + (1 - \alpha_n) S_n z_n, \\ C_{n+1} = \{z \in C_n : \| \gamma_n - z \| \leq \| x_n - z \| \}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1. \end{cases} \quad (1.10)$$

where $D, T : C \rightarrow CB(H)$ are \mathcal{H} -Lipschitz continuous with constant μ_1, μ_2 , respectively, $\Phi_1, \Phi_2 : H \times C \times C \rightarrow \mathbb{R}$ are equilibrium-like functions satisfying $(H1)$ - $(H3)$, $A :$

$C \rightarrow H$ is a α -inverse strongly monotone mapping and $B : C \rightarrow H$ is a β -inverse strongly monotone mapping.

In this article, we prove under some control conditions on $\{\delta_n\}$, $\{\alpha_n\}$, $\{s_n\}$, and $\{r_n\}$ that the sequence $\{x_n\}$ generated by (1.7) converges strongly to $P_{\mathbb{F}}x_1$ where $\mathbb{F} = \bigcap_{i=1}^N F(T_i) \cap (GEP)_s(\Phi_1, \varphi_1) \cap (GEP)_s(\Phi_2, \varphi_2) \cap F(G_1) \cap F(G_2)$, $G_1, G_2 : C \rightarrow C$ are defined by $G_1(x) = P_C(x - \lambda Ax)$, $G_2(x) = P_C(x - \eta Bx)$, $\forall x \in C$ and $P_{\mathbb{F}}x_1$ is solution of the following system of variational inequality:

$$\begin{cases} \langle Ax^*, x - x^* \rangle \geq 0, \\ \langle Bx^*, x - x^* \rangle \geq 0. \end{cases}$$

2 Preliminaries

In this section, we need the following lemmas and definition to prove our main result.

Let C be a nonempty closed convex subset of H . Then for any $x \in H$, there exists a unique nearest point in C , denoted by P_Cx , such that

$$\|x - P_Cx\| \leq \|x - y\|, \text{ for all } y \in C.$$

The following lemma is a property of P_C .

Lemma 2.1. (See [19].) *Given $x \in H$ and $y \in C$. Then $P_Cx = y$ if and only if there holds the inequality*

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.2. (See [20]) *Let C be a closed convex subset of a strictly convex Banach space E . Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C . Suppose $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by*

$$S(x) = \sum_{n=1}^{\infty} \lambda_n T_n x$$

for $x \in C$ is well defined, nonexpansive and $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$ hold.

The following lemma is well known.

Lemma 2.3. *Let H be Hilbert space, C be a nonempty closed convex subset of H . Let $T : C \rightarrow C$ be a κ -strictly pseudo-contractive, then the fixed point set $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

In 2009, Kangtunyakarn and Suantai [21] introduced the S -mapping generated by a finite family of κ -strictly pseudo contractive mappings and real numbers as follows:

Definition 2.1. Let C be a nonempty convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i -strict pseudo-contractions of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. We define the mapping $S : C \rightarrow C$ as follows:

$$\begin{aligned}
 U_0 &= I \\
 U_1 &= \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I \\
 U_2 &= \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I \\
 U_3 &= \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I \\
 &\vdots \\
 &\vdots \\
 &\vdots \\
 U_{N-1} &= \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I \\
 S = U_N &= \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
 \end{aligned}
 \tag{2.1}$$

This mapping is called *S-mapping* generated by T_1, \dots, T_N and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 2.4. (See [21]) Let C be a nonempty closed convex subset of real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of κ -strict pseudo contraction mapping of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1, 2, 3, \dots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \dots, N - 1$ and $\alpha_1^N \in (\kappa, 1), \alpha_3^N \in [\kappa, 1), \alpha_2^j \in [\kappa, 1)$ for all $j = 1, 2, \dots, N$. Let S be the mapping generated by T_1, \dots, T_N and $\alpha_1, \alpha_2, \dots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a nonexpansive mapping.

Lemma 2.5. (See [22]) Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \rightarrow C$ be a self-mapping of C . If S is a κ -strict pseudo-contraction mapping, then S satisfies the Lipschitz condition

$$\|Sx - Sy\| \leq \frac{1 + \kappa}{1 - \kappa} \|x - y\|, \quad \forall x, y \in C.$$

We prove the following lemma by using the concept of the *S-mapping* as follows:

Lemma 2.6. Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T_i, i = 1, 2, \dots, N$ be κ_i strictly pseudo-contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ and let $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}), \alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$ such that $\alpha_i^{n,j} \rightarrow \alpha_i^j \in [0, 1]$ as $n \rightarrow \infty$ for $i = 1, 3$ and $j = 1, 2, 3, \dots, N$. For every $n \in \mathbb{N}$, let S and S_n be the *S-mapping* generated by T_1, T_2, \dots, T_N and $\alpha_1, \alpha_2, \dots, \alpha_N$ and T_1, T_2, \dots, T_N and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, respectively. Then $\lim_{n \rightarrow \infty} \|S_n x_n - Sx_n\| = 0$ for every bounded sequence $\{x_n\}$ in C .

Proof. Let $\{x_n\}$ be bounded sequence in C , U_k and $U_{n,k}$ be generated by T_1, T_2, \dots, T_N and $\alpha_1, \alpha_2, \dots, \alpha_N$ and T_1, T_2, \dots, T_N and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$, respectively. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}
 \|U_{n,1}x_n - U_1x_n\| &= \left\| \alpha_1^{n,1}T_1x_n + (1 - \alpha_1^{n,1})x_n - \alpha_1^1T_1x_n - (1 - \alpha_1^1)x_n \right\| \\
 &= \left\| \alpha_1^{n,1}T_1x_n - \alpha_1^{n,1}x_n - \alpha_1^1T_1x_n + \alpha_1^1x_n \right\| \\
 &= \left\| (\alpha_1^{n,1} - \alpha_1^1)T_1x_n - (\alpha_1^{n,1} - \alpha_1^1)x_n \right\| \\
 &= \left| \alpha_1^{n,1} - \alpha_1^1 \right| \|T_1x_n - x_n\|
 \end{aligned} \tag{2.2}$$

and for $k \in \{2, 3, \dots, N\}$, by using Lemma 2.5, we obtain

$$\begin{aligned}
 \|U_{n,k}x_n - U_kx_n\| &= \left\| \alpha_1^{n,k}T_kU_{n,k-1}x_n + \alpha_2^{n,k}U_{n,k-1}x_n + \alpha_3^{n,k}x_n \right. \\
 &\quad \left. - \alpha_1^kT_kU_{k-1}x_n - \alpha_2^kU_{k-1}x_n - \alpha_3^kx_n \right\| \\
 &= \left\| \alpha_1^{n,k}T_kU_{n,k-1}x_n + \alpha_3^{n,k}x_n - \alpha_1^kT_kU_{k-1}x_n - \alpha_3^kx_n \right. \\
 &\quad \left. + \alpha_2^{n,k}U_{n,k-1}x_n - \alpha_2^kU_{k-1}x_n \right\| \\
 &= \left\| \alpha_1^{n,k}T_kU_{n,k-1}x_n - \alpha_1^{n,k}T_kU_{k-1}x_n + \alpha_1^{n,k}T_kU_{k-1}x_n \right. \\
 &\quad \left. - \alpha_1^kT_kU_{k-1}x_n + (\alpha_3^{n,k} - \alpha_3^k)x_n + \alpha_2^{n,k}U_{n,k-1}x_n - \alpha_2^kU_{k-1}x_n \right\| \\
 &= \left\| \alpha_1^{n,k}(T_kU_{n,k-1}x_n - T_kU_{k-1}x_n) + (\alpha_1^{n,k} - \alpha_1^k)T_kU_{k-1}x_n \right. \\
 &\quad \left. + (\alpha_3^{n,k} - \alpha_3^k)x_n + \alpha_2^{n,k}U_{n,k-1}x_n - \alpha_2^kU_{k-1}x_n \right\| \\
 &= \left\| \alpha_1^{n,k}(T_kU_{n,k-1}x_n - T_kU_{k-1}x_n) + (\alpha_1^{n,k} - \alpha_1^k)T_kU_{k-1}x_n \right. \\
 &\quad \left. + (\alpha_3^{n,k} - \alpha_3^k)x_n + \alpha_2^{n,k}U_{n,k-1}x_n - \alpha_2^kU_{k-1}x_n \right. \\
 &\quad \left. + \alpha_2^{n,k}U_{k-1}x_n - \alpha_2^kU_{k-1}x_n \right\| \\
 &= \left\| \alpha_1^{n,k}(T_kU_{n,k-1}x_n - T_kU_{k-1}x_n) + (\alpha_1^{n,k} - \alpha_1^k)T_kU_{k-1}x_n \right. \\
 &\quad \left. + (\alpha_3^{n,k} - \alpha_3^k)x_n + \alpha_2^{n,k}(U_{n,k-1}x_n - U_{k-1}x_n) \right. \\
 &\quad \left. + (\alpha_2^{n,k} - \alpha_2^k)U_{k-1}x_n \right\| \\
 &\leq \alpha_1^{n,k} \|T_kU_{n,k-1}x_n - T_kU_{k-1}x_n\| + \left| \alpha_1^{n,k} - \alpha_1^k \right| \|T_kU_{k-1}x_n\| \\
 &\quad + \left| \alpha_3^{n,k} - \alpha_3^k \right| \|x_n\| + \alpha_2^{n,k} \|U_{n,k-1}x_n - U_{k-1}x_n\| \\
 &\quad + \left| \alpha_2^{n,k} - \alpha_2^k \right| \|U_{k-1}x_n\| \\
 &= \alpha_1^{n,k} \|T_kU_{n,k-1}x_n - T_kU_{k-1}x_n\| + \left| \alpha_1^{n,k} - \alpha_1^k \right| \|T_kU_{k-1}x_n\| \\
 &\quad + \alpha_2^{n,k} \|U_{n,k-1}x_n - U_{k-1}x_n\| + \left| 1 - \alpha_1^{n,k} - \alpha_3^{n,k} - 1 \right| \\
 &\quad + \left| \alpha_1^k + \alpha_3^k \right| \|U_{k-1}x_n\| + \left| \alpha_3^{n,k} - \alpha_3^k \right| \|x_n\| \\
 &\leq \alpha_1^{n,k} \frac{1 + \kappa}{1 - \kappa} \|U_{n,k-1}x_n - U_{k-1}x_n\| + \left| \alpha_1^{n,k} \right. \\
 &\quad \left. - \alpha_1^k \right| \|T_kU_{k-1}x_n\| + \alpha_2^{n,k} \|U_{n,k-1}x_n - U_{k-1}x_n\| + \left(\left| \alpha_1^k - \alpha_1^{n,k} \right| \right. \\
 &\quad \left. + \left| \alpha_3^{n,k} - \alpha_3^k \right| \right) \|U_{k-1}x_n\| + \left| \alpha_3^{n,k} - \alpha_3^k \right| \|x_n\| \\
 &\leq \frac{1 + \kappa}{1 - \kappa} \|U_{n,k-1}x_n - U_{k-1}x_n\| + \left| \alpha_1^{n,k} - \alpha_1^k \right| \|T_kU_{k-1}x_n\| \\
 &\quad + \frac{1 - \kappa}{1 - \kappa} \|U_{n,k-1}x_n - U_{k-1}x_n\| + \left(\left| \alpha_1^k - \alpha_1^{n,k} \right| \right. \\
 &\quad \left. + \left| \alpha_3^{n,k} - \alpha_3^k \right| \right) \|U_{k-1}x_n\| + \left| \alpha_3^{n,k} - \alpha_3^k \right| \|x_n\| \\
 &\leq \frac{2}{1 - \kappa} \|U_{n,k-1}x_n - U_{k-1}x_n\| + \left| \alpha_1^{n,k} - \alpha_1^k \right| (\|T_kU_{k-1}x_n\| + \|U_{k-1}x_n\|) \\
 &\quad + \left| \alpha_3^{n,k} - \alpha_3^k \right| (\|U_{k-1}x_n\| + \|x_n\|).
 \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned}
 \|S_n x_n - Sx_n\| &= \|U_{n,N}x_n - U_Nx_n\| \\
 &\leq \frac{2}{1-\kappa} \|U_{n,N-1}x_n - U_{N-1}x_n\| + \left| \alpha_1^{n,N} - \alpha_1^N \right| (\|T_N U_{N-1}x_n\| \\
 &\quad + \|U_{N-1}x_n\|) + \left| \alpha_3^{n,N} - \alpha_3^N \right| (\|U_{N-1}x_n\| + \|x_n\|) \\
 &\leq \frac{2}{1-\kappa} \left(\frac{2}{1-\kappa} \|U_{n,N-2}x_n - U_{N-2}x_n\| + \left| \alpha_1^{n,N-1} \right. \right. \\
 &\quad \left. \left. - \alpha_1^{N-1} \right| (\|T_{N-1} U_{N-2}x_n\| \right. \\
 &\quad \left. + \|U_{N-2}x_n\|) + \left| \alpha_3^{n,N-1} - \alpha_3^{N-1} \right| (\|U_{N-2}x_n\| + \|x_n\|) \right) \\
 &\quad + \left| \alpha_1^{n,N} - \alpha_1^N \right| (\|T_N U_{N-1}x_n\| + \|U_{N-1}x_n\|) \\
 &\quad + \left| \alpha_3^{n,N} - \alpha_3^N \right| (\|U_{N-1}x_n\| + \|x_n\|) \\
 &= \left(\frac{2}{1-\kappa} \right)^2 \|U_{n,N-2}x_n - U_{N-2}x_n\| + \sum_{j=N-1}^N \left(\frac{2}{1-\kappa} \right)^{N-j} \left| \alpha_1^{n,j} \right. \\
 &\quad \left. - \alpha_1^j \right| (\|T_j U_{j-1}x_n\| + \|U_{j-1}x_n\|) \\
 &\quad + \sum_{j=N-1}^N \left(\frac{2}{1-\kappa} \right)^{N-j} \left| \alpha_3^{n,j} - \alpha_3^j \right| (\|U_{j-1}x_n\| + \|x_n\|) \\
 &\quad \cdot \\
 &\quad \cdot \\
 &\quad \cdot \\
 &\leq \left(\frac{2}{1-\kappa} \right)^{N-1} \|U_{n,1}x_n - U_1x_n\| + \sum_{j=2}^N \left(\frac{2}{1-\kappa} \right)^{N-j} \left| \alpha_1^{n,j} \right. \\
 &\quad \left. - \alpha_1^j \right| (\|T_j U_{j-1}x_n\| + \|U_{j-1}x_n\|) + \sum_{j=2}^N \left(\frac{2}{1-\kappa} \right)^{N-j} \left| \alpha_3^{n,j} - \alpha_3^j \right| (\|U_{j-1}x_n\| + \|x_n\|) \\
 &= \left(\frac{2}{1-\kappa} \right)^{N-1} \left| \alpha_1^{n,1} - \alpha_1^1 \right| \|T_1 x_n - x_n\| + \sum_{j=2}^N \left(\frac{2}{1-\kappa} \right)^{N-j} \left| \alpha_1^{n,j} \right. \\
 &\quad \left. - \alpha_1^j \right| (\|T_j U_{j-1}x_n\| + \|U_{j-1}x_n\|) \\
 &\quad + \sum_{j=2}^N \left(\frac{2}{1-\kappa} \right)^{N-j} \left| \alpha_3^{n,j} - \alpha_3^j \right| (\|U_{j-1}x_n\| + \|x_n\|).
 \end{aligned}$$

This together with the assumption $\alpha_i^{n,j} \rightarrow \alpha_i^j$ as $n \rightarrow \infty$ ($i = 1, 3, j = 1, 2, \dots, N$), we can conclude that

$$\lim_{n \rightarrow \infty} \|S_n x_n - Sx_n\| = 0.$$

Lemma 2.7. (See [23]) *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $S : C \rightarrow C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 2.8. (See [24]) *Let C be a closed convex subset of H . Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$, if $\{x_n\}$ is such the $\omega(x_n) \subset C$ and satisfy the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n \in \mathbb{N}.$$

Then $x_n \rightarrow q$, as $n \rightarrow \infty$.

Definition 2.2. *A multivalued map $T : C \rightarrow CB(H)$ is say to be \mathcal{H} -Lipschitz continuous if there exists a constant $\mu > 0$ such that*

$$\mathcal{H}(T(u) - T(v)) \leq \mu \|u - v\|, \quad \forall u, v \in C,$$

where $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$.

Lemma 2.9. (Nadler's theorem, see [25]) Let $(X, \|\cdot\|)$ be a normed vector space and $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$. If $U, V \in CB(X)$, then for any given $\epsilon > 0$ and $u \in U$, there exists $v \in V$ such that

$$\|u - v\| \leq (1 + \epsilon)\mathcal{H}(U, V).$$

Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\phi: C \rightarrow H$ be a real-valued function, $T: C \rightarrow CB(H)$ be a multivalued map and $\Phi: H \times C \times C \rightarrow \mathbb{R}$ be an equilibrium-like function.

To solve the GEP, let us assume that the equilibrium-like function $\Phi: H \times C \times C \rightarrow \mathbb{R}$ satisfies the following conditions with respect to the multivalued map $T: C \rightarrow CB(H)$.

(H1) For each fixed $v \in C$, $(\omega, u) \mapsto \Phi(\omega, u, v)$ is an upper semicontinuous function from $H \times C$ to \mathbb{R} , that is, for $(\omega, u) \in H \times C$, whenever $\omega_n \rightarrow \omega$ and $u_n \rightarrow u$ as $n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \Phi(\omega_n, u_n, v) \leq \Phi(\omega, u, v);$$

(H2) For each fixed $(w, v) \in H \times C$, $u \mapsto \Phi(w, u, v)$ is a concave function;

(H3) For each fixed $(w, u) \in H \times C$, $v \mapsto \Phi(w, u, v)$ is a convex function.

Theorem 2.10. (See [17]) Let C be a nonempty, bounded, closed, and convex subset of a real Hilbert space H , and let $\phi: C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex functional. Let $T: C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with constant μ , and $\Phi: H \times C \times C \rightarrow \mathbb{R}$ be an equilibrium-like function satisfying (H1)-(H3). Let $r > 0$ be a constant. For each $x \in C$, take $w_x \in T(x)$ arbitrarily and define a mapping $T_r: C \rightarrow C$ as follows:

$$T_r(x) = \left\{ u \in C : \Phi(w_x, u, v) + \phi(v) - \phi(u) + \frac{1}{r} \langle u - x, v - u \rangle \geq 0, \quad \forall v \in C \right\}.$$

Then, there hold the following:

(a) T_r is single-valued;

(b) T_r is firmly nonexpansive (that is, for any $u, v \in C$, $\|T_r u - T_r v\|^2 \leq \langle T_r u - T_r v, u - v \rangle$) if

$$\Phi(w_1, T_r(x_1), T_r(x_2)) + \Phi(w_2, T_r(x_2), T_r(x_1)) \leq 0,$$

for all $(x_1, x_2) \in C \times C$ and all $w_i \in T(x_i)$, $i = 1, 2$;

(c) $F(T_r) = (GEP)_s(\Phi, \phi)$

(d) $(GEP)_s(\Phi, \phi)$ is closed and convex.

Lemma 2.11. (See [26]) Let C be a nonempty closed convex subset of a Hilbert space H and let $G: C \rightarrow C$ be defined by

$$G(x) = P_C(x - \lambda Ax), \quad \forall x \in C,$$

with $\forall \lambda > 0$. Then $x^* \in VI(C, A)$ if and only if $x^* \in F(G)$.

3 Main results

In this section, we prove a strong convergence theorem of the sequence $\{x_n\}$ generated by (1.10) to $P_{\mathbb{F}}x_1$.

Theorem 3.1. *Let C be a nonempty bounded, closed, and convex subset of Hilbert space H and let ϕ_1, ϕ_2 be a lower semicontinuous and convex function. Let $D, T : C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with constant μ_1, μ_2 , respectively, $\Phi_1, \Phi_2 : H \times C \times C \rightarrow \mathbb{R}$ be equilibrium-like functions satisfying (H1) - (H3). Let $A : C \rightarrow H$ be a α -inverse strongly monotone mapping and $B : C \rightarrow H$ be a β -inverse strongly monotone mapping, let $T_i, i = 1, 2, \dots, N$, be κ_i -pseudo contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ with $\mathbb{F} = \bigcap_{i=1}^N F(T_i) \cap (GEP)_s(\Phi_1, \phi_1) \cap (GEP)_s(\Phi_2, \phi_2) \cap F(G_1) \cap F(G_2)$, where $G_1, G_2 : C \rightarrow C$ are defined by $G_1(x) = P_C(x - \lambda Ax)$, $G_2(x) = P_C(x - \eta Bx)$, $\forall x \in C$. Let S_n be the S -mappings generated by T_1, T_2, \dots, T_N and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$ where $\alpha_j^{(n)} = (\alpha_1^{nj}, \alpha_2^{nj}, \alpha_3^{nj}) \in I \times I \times I, I = [0, 1], \alpha_1^{nj} + \alpha_2^{nj} + \alpha_3^{nj} = 1$ and $\kappa < a \leq \alpha_1^{nj}, \alpha_3^{nj} \leq b < 1$ for all $j = 1, 2, \dots, N - 1, \kappa < c \leq \alpha_1^{n,N} \leq 1, \kappa \leq \alpha_3^{n,N} \leq d < 1, \kappa \leq \alpha_2^{nj} \leq e < 1$ for all $j = 1, 2, \dots, N$ and let $\{x_n\}, \{u_n\}, \{v_n\}, \{w_n^1\}$, and $\{w_n^2\}$ be sequences generated by (1.10), where $\{\alpha_n\}$ is a sequence in $[0, 1], r, \lambda \in [a, b] \subset (0, 2\alpha)$ and $s, \eta \in [c, d] \subset (0, 2\beta)$, for every $n \in \mathbb{N}$ and suppose the following conditions hold:*

- (i) $\lim_{n \rightarrow \infty} \delta_n = \delta \in (0, 1)$,
- (ii) $0 \leq \kappa \leq \alpha_n < 1, \forall n \geq 1$,
- (iii) $\sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty$, for all $j \in \{1, 2, 3, \dots, N\}$.
- (iv) There exists λ_1, λ_2 such that

$$\begin{cases} \Phi_1(w_1^1, T_{r_1}(x_1), T_{r_2}(x_2)) + \Phi_1(w_2^1, T_{r_2}(x_2), T_{r_1}(x_1)) \leq -\lambda_1 \|T_{r_1}(x_1) - T_{r_2}(x_2)\|^2 \text{ and} \\ \Phi_2(w_1^2, T_{s_1}(x_1), T_{s_2}(x_2)) + \Phi_2(w_2^2, T_{s_2}(x_2), T_{s_1}(x_1)) \leq -\lambda_2 \|T_{s_1}(x_1) - T_{s_2}(x_2)\|^2 \end{cases} \quad (3.1)$$

for all $(r_1, r_2) \in \Theta \times \Theta, (s_1, s_2) \in \Xi \times \Xi, w_i^1 \in T(x_i)$ and $w_i^2 \in D(x_i)$, for $i = 1, 2$ where $\Theta = \{r_n : n \geq 1\}$ and $\Xi = \{s_n : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$ which is a solution of (3.2):

$$\begin{cases} \langle Ax^*, x - x^* \rangle \geq 0, \\ \langle Bx^*, x - x^* \rangle \geq 0. \end{cases} \quad (3.2)$$

Proof. From (3.1) for every $r \in \Theta$, we have

$$\Phi_1(w_1^1, T_r(x_1), T_r(x_2)) + \Phi_1(w_2^1, T_r(x_2), T_r(x_1)) \leq -\lambda_1 \|T_r(x_1) - T_r(x_2)\|^2 \leq 0, \quad (3.3)$$

for all $(x_1, x_2) \in C \times C$ and $w_i^1 \in T(x_i), i = 1, 2$.

Similarly, for every $s \in \Xi$, we have

$$\Phi_2(w_1^2, T_s(x_1), T_s(x_2)) + \Phi_2(w_2^2, T_s(x_2), T_s(x_1)) \leq -\lambda_2 \|T_s(x_1) - T_s(x_2)\|^2 \leq 0. \quad (3.4)$$

for all $(x_1, x_2) \in C \times C$ and $w_i^2 \in D(x_i), i = 1, 2$. From (3.3) and (3.4), we have Theorem 2.10 hold.

It is easy to see that $I - \lambda A$ and $I - \eta B$ are nonexpansive mapping. Indeed, since A is a α -inverse strongly monotone mapping with $\lambda \in (0, 2\alpha)$, we have

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda \langle x - y, Ax - Ay \rangle + \lambda^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\alpha\lambda \|Ax - Ay\|^2 + \lambda^2 \|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned}$$

Thus $(I - \lambda A)$ is nonexpansive, so is $I - \eta B$. Since

$$\Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n} \langle u_n - x_n, u - u_n \rangle \geq 0, \quad \forall u \in C,$$

and Theorem 2.10, we have $u_n = T_{r_n}x_n$. Since

$$\Phi_2(w_n^2, v_n, v) + \varphi_2(v) - \varphi_2(v_n) + \frac{1}{s_n} \langle v_n - x_n, v - v_n \rangle \geq 0, \quad \forall v \in C,$$

and Theorem 2.10, we have $v_n = T_{s_n}x_n$. Let $z \in \mathbb{F}$, again by Theorem 2.10, we have $z = T_{r_n}z = T_{s_n}z = P_C(I - \lambda A)z = P_C(I - \eta B)z$. From nonexpansiveness of $\{T_{r_n}\}$, $\{T_{s_n}\}$, $\{I - \lambda A\}$, and $\{I - \eta B\}$, we have

$$\begin{aligned} \|z_n - z\| &= \|\delta_n(P_C(I - \lambda A)u_n - z) + (1 - \delta_n)(P_C(I - \eta B)v_n - z)\| \\ &\leq \delta_n \|P_C(I - \lambda A)u_n - z\| + (1 - \delta_n) \|P_C(I - \eta B)v_n - z\| \\ &\leq \delta_n \|T_{r_n}x_n - z\| + (1 - \delta_n) \|T_{s_n}x_n - z\| \\ &\leq \|x_n - z\|. \end{aligned} \tag{3.5}$$

By (3.5), we have

$$\begin{aligned} \|\gamma_n - z\| &= \|\alpha_n(z_n - z) + (1 - \alpha_n)(S_n z_n - z)\| \\ &\leq \alpha_n \|z_n - z\| + (1 - \alpha_n) \|S_n z_n - z\| \\ &\leq \|z_n - z\| \leq \|x_n - z\|. \end{aligned} \tag{3.6}$$

Next, we show that C_n is closed and convex for every $n \in \mathbb{N}$. It is obvious that C_n is closed. In fact, we know that, for $z \in C_n$

$$\|\gamma_n - z\| \leq \|x_n - z\| \text{ is equivalent to } \|\gamma_n - x_n\|^2 + 2\langle \gamma_n - x_n, x_n - z \rangle \leq 0.$$

So, we have that $\forall z_1, z_2 \in C_n$ and $t \in (0,1)$, it follows that

$$\begin{aligned} \|\gamma_n - x_n\|^2 + 2\langle \gamma_n - x_n, x_n - (tz_1 + (1-t)z_2) \rangle \\ &= t(2\langle \gamma_n - x_n, x_n - z_1 \rangle + \|\gamma_n - x_n\|^2) \\ &\quad + (1-t)(2\langle \gamma_n - x_n, x_n - z_2 \rangle + \|\gamma_n - x_n\|^2) \\ &\leq 0, \end{aligned}$$

then, we have C_n is convex. By Theorem 2.10 and Lemma 2.3, we conclude that \mathbb{F} is closed and convex. This implies that $P_{\mathbb{F}}$ is well defined. Next, we show that $\mathbb{F} \subset C_n$ for every $n \in \mathbb{N}$. Putting $q \in \mathbb{F}$, by (3.6), it is easy to see that $q \in C_n$, then we have $\mathbb{F} \subset C_n$ for all $n \in \mathbb{N}$. Since $x_n = P_{C_n}x_1$, for every $w \in C_n$ we have

$$\|x_n - x_1\| \leq \|w - x_1\|, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

In particular, we have

$$\|x_n - x_1\| \leq \|P_{\mathbb{F}}x_1 - x_1\|. \tag{3.8}$$

Since C is bounded, we have $\{x_n\}$ is bounded, so are $\{u_n\}$, $\{v_n\}$, $\{z_n\}$, and $\{y_n\}$. Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$ and $x_n = P_{C_n}x_1$, we have

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1} \rangle \\ &\leq -\|x_n - x_1\|^2 + \|x_n - x_1\| \|x_1 - x_{n+1}\|, \end{aligned}$$

it implies that

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|.$$

Hence, we have $\lim_{n \rightarrow \infty} \|x_n - x_1\|$ exists. Since

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_n + x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1} \rangle + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2, \end{aligned} \tag{3.9}$$

it implies that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+1}\| = 0. \tag{3.10}$$

Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|,$$

by (3.10), we have

$$\lim_{n \rightarrow \infty} \|y_n - x_{n+1}\| = 0. \tag{3.11}$$

Since

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

by (3.10) and (3.11), we have

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{3.12}$$

Next, we show that

$$\lim_{n \rightarrow \infty} \|z_n - S_n z_n\| = 0. \tag{3.13}$$

By definition of y_n , we have

$$y_n - z_n = (1 - \alpha_n)(S_n z_n - z_n). \tag{3.14}$$

Claim that

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \tag{3.15}$$

Putting $M_n = P_C(I - \lambda A)u_n$ and $N_n = P_C(I - \eta B)v_n$, we have

$$\|z_n - x_n\| \leq \delta_n \|M_n - x_n\| + (1 - \delta_n) \|N_n - x_n\|. \tag{3.16}$$

Let $z \in \mathbb{F}$. Since T_{r_n} is firmly nonexpansive mapping and $T_{r_n}x_n = u_n$, we have

$$\begin{aligned} \|z - u_n\|^2 &= \|T_{r_n}z - T_{r_n}x_n\|^2 \\ &\leq \langle T_{r_n}z - T_{r_n}x_n, z - x_n \rangle \\ &= \frac{1}{2} (\|u_n - z\|^2 + \|x_n - z\|^2 - \|u_n - x_n\|^2). \end{aligned}$$

Hence

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.17}$$

Since T_{s_n} is firmly nonexpansive mapping and $T_{s_n}x_n = v_n$, by using the same method as (3.17), we have

$$\|v_n - z\|^2 \leq \|x_n - z\|^2 - \|v_n - x_n\|^2. \tag{3.18}$$

By nonexpansiveness of S_n and (3.17), (3.18), we have

$$\begin{aligned} \|\gamma_n - z\|^2 &\leq \|z_n - z\|^2 \\ &\leq \delta_n \|u_n - z\|^2 + (1 - \delta_n) \|v_n - z\|^2 \\ &\leq \delta_n (\|x_n - z\|^2 - \|u_n - x_n\|^2) + (1 - \delta_n) (\|x_n - z\|^2 - \|v_n - x_n\|^2) \\ &= \|x_n - z\|^2 - \delta_n \|u_n - x_n\|^2 - (1 - \delta_n) \|v_n - x_n\|^2, \end{aligned}$$

it implies that

$$\begin{aligned} \delta_n \|u_n - x_n\|^2 &\leq \|x_n - z\|^2 - \|\gamma_n - z\|^2 - (1 - \delta_n) \|v_n - x_n\|^2 \\ &\leq \|x_n - z\|^2 - \|\gamma_n - z\|^2 \\ &\leq (\|x_n - z\| + \|\gamma_n - z\|) \|x_n - \gamma_n\|, \end{aligned}$$

by (3.12) and condition (i), we have

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \tag{3.19}$$

By using the same method as (3.19), we have

$$\lim_{n \rightarrow \infty} \|v_n - x_n\| = 0. \tag{3.20}$$

Since

$$\begin{aligned} \|\gamma_n - z\|^2 &\leq \alpha_n \|z_n - z\|^2 + (1 - \alpha_n) \|z_n - z\|^2 \\ &\leq \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) \|z_n - z\|^2 \\ &\leq \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) (\delta_n \|M_n - z\|^2 \\ &\quad + (1 - \delta) \|N_n - z\|^2) \end{aligned} \tag{3.21}$$

Claim that

$$\lim_{n \rightarrow \infty} \|Au_n - Az\| = \lim_{n \rightarrow \infty} \|Bv_n - Bz\| = 0.$$

By nonexpansiveness of P_C , we have

$$\begin{aligned}
 \|y_n - z\|^2 &\leq \|z_n - z\|^2 \\
 &\leq \delta_n \|P_C(I - \lambda A)u_n - P_C(I - \lambda A)z\|^2 \\
 &\quad + (1 - \delta_n) \|P_C(I - \eta B)v_n - P_C(I - \eta B)z\|^2 \\
 &\leq \delta_n \|(I - \lambda A)u_n - (I - \lambda A)z\|^2 + (1 - \delta_n) \|(I - \eta B)v_n - (I - \eta B)z\|^2 \\
 &\leq \delta_n \|(u_n - \lambda Au_n - (z - \lambda Az))\|^2 + (1 - \delta_n) \|(v_n - \eta Bv_n - (z - \eta Bz))\|^2 \\
 &= \delta_n \|(u_n - z) - \lambda(Au_n - Az)\|^2 + (1 - \delta_n) \|(v_n - z) - \eta(Bv_n - Bz)\|^2 \\
 &= \delta_n (\|u_n - z\|^2 + \lambda^2 \|Au_n - Az\|^2 - 2\lambda \langle u_n - z, Au_n - Az \rangle) \\
 &\quad + (1 - \delta_n) (\|v_n - z\|^2 + \eta^2 \|Bv_n - Bz\|^2 - 2\eta \langle v_n - z, Bv_n - Bz \rangle) \\
 &\leq \delta_n (\|u_n - z\|^2 + \lambda^2 \|Au_n - Az\|^2 - 2\lambda \alpha \|Au_n - Az\|^2) \\
 &\quad + (1 - \delta_n) (\|v_n - z\|^2 + \eta^2 \|Bv_n - Bz\|^2 - 2\eta \beta \|Bv_n - Bz\|^2) \\
 &\leq \delta_n (\|x_n - z\|^2 + \lambda(\lambda - 2\alpha) \|Au_n - Az\|^2) \\
 &\quad + (1 - \delta_n) (\|x_n - z\|^2 + \eta(\eta - 2\beta) \|Bv_n - Bz\|^2) \\
 &= \|x_n - z\|^2 - \delta_n \lambda (2\alpha - \lambda) \|Au_n - Az\|^2 \\
 &\quad - (1 - \delta_n) \eta (2\beta - \eta) \|Bv_n - Bz\|^2,
 \end{aligned}$$

it follows that

$$\begin{aligned}
 \delta_n \lambda (2\alpha - \lambda) \|Au_n - Az\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\
 &\quad - (1 - \delta_n) \eta (2\beta - \eta) \|Bv_n - Bz\|^2 \\
 &\leq (\|x_n - z\| + \|y_n - z\|) \|y_n - x_n\|,
 \end{aligned} \tag{3.22}$$

by conditions (i), (ii), $\lambda \in (0, 2\alpha)$ and (3.12), it implies that

$$\lim_{n \rightarrow \infty} \|Au_n - Az\| = 0. \tag{3.23}$$

By using the same method as (3.23), we have

$$\lim_{n \rightarrow \infty} \|Bv_n - Bz\| = 0. \tag{3.24}$$

By nonexpansiveness of T_{r_n} , we have

$$\begin{aligned}
 \|M_n - z\|^2 &= \|P_C(u_n - \lambda Au_n) - P_C(z - \lambda Az)\|^2 \\
 &\leq \langle (u_n - \lambda Au_n) - (z - \lambda Az), M_n - z \rangle \\
 &= \frac{1}{2} (\|(u_n - \lambda Au_n) - (z - \lambda Az)\|^2 + \|M_n - z\|^2 - \|(u_n - \lambda Au_n) \\
 &\quad - (z - \lambda Az) - (M_n - z)\|^2) \\
 &\leq \frac{1}{2} (\|u_n - z\|^2 + \|M_n - z\|^2 - \|(u_n - M_n) - \lambda(Au_n - Az)\|^2) \\
 &= \frac{1}{2} (\|T_{r_n} x_n - T_{r_n} z\|^2 + \|M_n - z\|^2 - \|u_n - M_n\|^2 \\
 &\quad + 2\lambda \langle u_n - M_n, Au_n - Az \rangle - \lambda^2 \|Au_n - Az\|^2) \\
 &\leq \frac{1}{2} (\|x_n - z\|^2 + \|M_n - z\|^2 - \|u_n - M_n\|^2 + 2\lambda \langle u_n - M_n, Au_n - Az \rangle \\
 &\quad - \lambda^2 \|Au_n - Az\|^2).
 \end{aligned}$$

Hence, we have

$$\begin{aligned} \|M_n - z\|^2 \leq & \|x_n - z\|^2 - \|u_n - M_n\|^2 + 2\lambda \langle u_n - M_n, Au_n - Az \rangle \\ & - \lambda^2 \|Au_n - Az\|^2. \end{aligned} \quad (3.25)$$

By using the same method as (3.25), we have

$$\|N_n - z\|^2 \leq \|x_n - z\|^2 - \|v_n - N_n\|^2 + 2\eta \langle v_n - N_n, Bv_n - Bz \rangle - \eta^2 \|Bv_n - Bz\|^2. \quad (3.26)$$

Substitute (3.25) and (3.26) in (3.21), we have

$$\begin{aligned} \|\gamma_n - z\|^2 \leq & \alpha_n \|x_n - z\|^2 + (1 - \alpha_n)(\delta_n \|M_n - z\|^2 \\ & + (1 - \delta_n) \|N_n - z\|^2) \\ \leq & \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) (\delta_n (\|x_n - z\|^2 - \|u_n - M_n\|^2 \\ & + 2\lambda \langle u_n - M_n, Au_n - Az \rangle - \lambda^2 \|Au_n - Az\|^2) + (1 + \delta_n) (\|x_n - z\|^2 - \|v_n - N_n\|^2 \\ & + 2\eta \langle v_n - N_n, Bv_n - Bz \rangle - \eta^2 \|Bv_n - Bz\|^2)) \\ \leq & \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) (\delta_n \|x_n - z\|^2 - \delta_n \|u_n - M_n\|^2 \\ & + 2\lambda \delta_n \langle u_n - M_n, Au_n - Az \rangle + (1 - \delta_n) \|x_n - z\|^2 - (1 - \delta_n) \|v_n - N_n\|^2 \\ & + 2\eta (1 - \delta_n) \langle v_n - N_n, Bv_n - Bz \rangle) \\ = & \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) (\|x_n - z\|^2 - \delta_n \|u_n - M_n\|^2 \\ & + 2\lambda \delta_n \langle u_n - M_n, Au_n - Az \rangle - (1 - \delta_n) \|v_n - N_n\|^2 \\ & + 2\eta (1 - \delta_n) \langle v_n - N_n, Bv_n - Bz \rangle) \\ = & \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 - (1 - \alpha_n) \delta_n \|u_n - M_n\|^2 \\ & + 2(1 - \alpha_n) \lambda \delta_n \langle u_n - M_n, Au_n - Az \rangle - (1 - \delta_n) (1 - \alpha_n) \|v_n - N_n\|^2 \\ & + 2\eta (1 - \delta_n) (1 - \alpha_n) \langle v_n - N_n, Bv_n - Bz \rangle \\ = & \|x_n - z\|^2 - (1 - \alpha_n) \delta_n \|u_n - M_n\|^2 \\ & + 2(1 - \alpha_n) \lambda \delta_n \langle u_n - M_n, Au_n - Az \rangle - (1 - \delta_n) (1 - \alpha_n) \|v_n - N_n\|^2 \\ & + 2\eta (1 - \delta_n) (1 - \alpha_n) \langle v_n - N_n, Bv_n - Bz \rangle, \end{aligned}$$

it implies that

$$\begin{aligned} (1 - \alpha_n) \delta_n \|u_n - M_n\|^2 \leq & \|x_n - z\|^2 - \|\gamma_n - z\|^2 \\ & + 2(1 - \alpha_n) \lambda \delta_n \langle u_n - M_n, Au_n - Az \rangle - (1 - \delta_n) (1 - \alpha_n) \|v_n - N_n\|^2 \\ & + 2\eta (1 - \delta_n) (1 - \alpha_n) \langle v_n - N_n, Bv_n - Bz \rangle \\ \leq & (\|x_n - z\| + \|\gamma_n - z\|) \|\gamma_n - x_n\| \\ & + 2(1 - \alpha_n) \lambda \delta_n \langle u_n - M_n, Au_n - Az \rangle \\ & + 2\eta (1 - \delta_n) (1 - \alpha_n) \langle v_n - N_n, Bv_n - Bz \rangle, \end{aligned} \quad (3.27)$$

from (3.12), (3.23), (3.24) and conditions (i) and (ii), we have

$$\lim_{n \rightarrow \infty} \|u_n - M_n\| = 0. \quad (3.28)$$

By using the same method as (3.28), we have

$$\lim_{n \rightarrow \infty} \|v_n - N_n\| = 0. \quad (3.29)$$

By (3.19) and (3.28), we have

$$\lim_{n \rightarrow \infty} \|M_n - x_n\| = 0. \quad (3.30)$$

By (3.20) and (3.29), we have

$$\lim_{n \rightarrow \infty} \|N_n - x_n\| = 0. \tag{3.31}$$

From (3.16), (3.30) and (3.31), we have

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \tag{3.32}$$

From (3.12) and (3.32), we have

$$\lim_{n \rightarrow \infty} \|\gamma_n - z_n\| = 0. \tag{3.33}$$

From (3.14), (3.33) and condition (i), we have (3.13).

Let $a \in (0,1)$, by (3.10) there exists $N_0 \in \mathbb{N}$ such

$$\|x_{n+1} - x_n\| < a^n, \forall n \geq N_0. \tag{3.34}$$

Thus, for any number $n, p \in \mathbb{N}, p > 0$, we have

$$\|x_{n+p} - x_n\| \leq \sum_{k=n}^{n+p-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+p-1} a^k \leq \frac{a^n}{1-a}. \tag{3.35}$$

Since $a \in (0,1)$, we have $\lim_{n \rightarrow \infty} a^n = 0$. By (3.35), we have $\{x_n\}$ is Cauchy sequence in Hilbert space. Let $\lim_{n \rightarrow \infty} x_n = x^*$. Since $T : C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with constant μ_1 and (1.10), we have

$$\|w_n^1 - w_{n+1}^1\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}(T(x_n), T(x_{n+1})) \leq \left(1 + \frac{1}{n}\right) \mu_1 \|x_{n+1} - x_n\|. \tag{3.36}$$

By (3.34) and for any number $n, p \in \mathbb{N}, p > 0$, we have

$$\begin{aligned} \|w_{n+p}^1 - w_n^1\| &\leq \sum_{k=n}^{n+p-1} \|w_{k+1}^1 - w_k^1\| \\ &\leq \sum_{k=n}^{n+p-1} \left(1 + \frac{1}{k}\right) \mu_1 \|x_{k+1} - x_k\| \\ &\leq \sum_{k=n}^{n+p-1} 2\mu_1 a^k \\ &\leq 2\mu_1 \frac{a^n}{1-a}. \end{aligned} \tag{3.37}$$

Since $a \in (0,1)$, we have $\lim_{n \rightarrow \infty} a^n = 0$. By (3.37), we have $\{w_n^1\}$ is cauchy sequence in Hilbert space. Let $\lim_{n \rightarrow \infty} w_n^1 = w_1^*$. Next, we will prove that $w_1^* \in T(x^*)$. Since $w_n^1 \in T(x_n)$, we have

$$\begin{aligned}
 d(w_n^1, T(x^*)) &\leq \max \left\{ d(w_n^1, T(x^*)), \sup_{w_1 \in T(x^*)} d(T(x_n), w_1) \right\} \\
 &\leq \max \left\{ \sup_{z \in T(x_n)} d(z, T(x^*)), \sup_{w_1 \in T(x^*)} d(T(x_n), w_1) \right\} \\
 &= \mathcal{H}(T(x_n), T(x^*)).
 \end{aligned}
 \tag{3.38}$$

Since

$$\begin{aligned}
 d(w_1^*, T(x^*)) &\leq \|w_1^* - w_n^1\| + d(w_n^1, T(x^*)) \\
 &\leq \|w_1^* - w_n^1\| + \mathcal{H}(T(x_n), T(x^*)) \\
 &\leq \|w_1^* - w_n^1\| + \mu_1 \|x_n - x^*\|,
 \end{aligned}$$

by $\lim_{n \rightarrow \infty} x_n = x^*$ and $\lim_{n \rightarrow \infty} w_n^1 = w_1^*$, we have $d(w_1^*, T(x^*)) = 0$, this implies that $w_1^* \in T(x^*)$. By using the same method as above, we have $\lim_{n \rightarrow \infty} w_n^2 = w_2^*$ and $w_2^* \in D(x^*)$.

Let $\omega(x_n)$ be the set of all weakly ω -limit of $\{x_n\}$. We shall show that $\omega(x_n) \subset \mathbb{F}$. Since $\{x_n\}$ is bounded, then $\omega(x_n) \neq \emptyset$. Let $q \in \omega(x_n)$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converge weakly to q . Since $\{x_n\}$ is a Cauchy sequence in Hilbert space, we have $x_{n_i} \rightarrow q$ as $\{i \rightarrow \infty\}$, it implies that $x_n \rightarrow q$ as $n \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} x_n = x^*$ and $\lim_{n \rightarrow \infty} x_n = q$, we have $x^* = q$, then we have $w_1^* \in T(q)$ and $w_2^* \in D(q)$. From (3.19) and $x_n \rightarrow q$ as $n \rightarrow \infty$, we have $u_n \rightarrow q$ as $n \rightarrow \infty$.

By $u_n = T_{r_n} x_n$, we have

$$\Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n} \langle u_n - x_n, u - u_n \rangle \geq 0, \quad \forall u \in C,$$

by (3.19), (H1) and lower semicontinuity of ϕ_1 , we have

$$\Phi_1(w_1^*, q, u) + \varphi_1(u) - \varphi_1(q) \geq 0, \quad \forall u \in C,$$

then, we have

$$q \in (GEP)_s(\Phi_1, \varphi_1).
 \tag{3.39}$$

By using the same method as (3.39), we have

$$q \in (GEP)_s(\Phi_2, \varphi_2).
 \tag{3.40}$$

Since $\kappa < a \leq \alpha_1^{n_j}, \alpha_3^{n_j} \leq b < 1$ for all $j = 1, 2, \dots, N-1$, $\kappa < c \leq \alpha_1^{n_N} \leq 1$, $\kappa \leq \alpha_3^{n_N} \leq d < 1$ and $\kappa \leq \alpha_2^{n_j} \leq e < 1$ for all $j = 1, 2, \dots, N$. Without loss of generality, we may assume $\alpha_1^{n_{i_j}} \rightarrow \alpha_1^j \in (\kappa, 1)$ as $i \rightarrow \infty$, $\alpha_3^{n_{i_j}} \rightarrow \alpha_3^j \in (\kappa, 1)$ and $\alpha_2^{n_{i_j}} \rightarrow \alpha_2^j \in (\kappa, 1)$ as $i \rightarrow \infty$, $\forall j = 1, 2, \dots, N$.

Let S be S -mapping generated by T_1, T_2, \dots, T_N . and $\beta_1, \beta_2, \dots, \beta_N$, where $\beta_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$. By Lemma 2.4, we have S is nonexpansive and $F(S) = \bigcap_{i=1}^N F(T_i)$.

By Lemma 2.6, we have

$$\lim_{k \rightarrow \infty} \|S_{n_i} z_{n_i} - Sz_{n_i}\| = 0. \tag{3.41}$$

By (3.13) and (3.41), we have

$$\lim_{n \rightarrow \infty} \|z_{n_i} - Sz_{n_i}\| = 0. \tag{3.42}$$

Since $x_{n_i} \rightharpoonup q$ as $i \rightarrow \infty$ and (3.32), we have $z_{n_i} \rightharpoonup q$ as $i \rightarrow \infty$. By $z_{n_i} \rightharpoonup q$ as $i \rightarrow \infty$, (3.42) and Lemma 2.7, we have

$$q \in \bigcap_{i=1}^N F(T_i). \tag{3.43}$$

Next, we define $Q : C \rightarrow C$ by

$$Qx = \delta P_C(I - \lambda A)x + (1 - \delta)P_C(I - \eta B)x. \tag{3.44}$$

By Lemma 2.2, we have

$$F(Q) = F(P_C(I - \lambda A)) \cap (P_C(I - \eta B)) = F(G_1) \cap F(G_2). \tag{3.45}$$

From (3.44), we have

$$\begin{aligned} \|Qx_n - x_n\| &\leq \|Qx_n - z_n\| + \|z_n - x_n\| \\ &\leq \|\delta P_C(I - \lambda A)x_n + (1 - \delta)P_C(I - \eta B)x_n \\ &\quad - \delta_n P_C(I - \lambda A)u_n - (1 - \delta_n)P_C(I - \eta B)v_n\| + \|z_n - x_n\| \\ &= \|\delta P_C(I - \lambda A)x_n - \delta P_C(I - \lambda A)u_n + \delta P_C(I - \lambda A)u_n \\ &\quad + (1 - \delta)P_C(I - \eta B)x_n - (1 - \delta)P_C(I - \eta B)v_n + (1 - \delta)P_C(I - \eta B)v_n \\ &\quad - \delta_n P_C(I - \lambda A)u_n - (1 - \delta_n)P_C(I - \eta B)v_n\| + \|z_n - x_n\| \\ &\leq \delta \|P_C(I - \lambda A)x_n - P_C(I - \lambda A)u_n\| + |\delta - \delta_n| \|P_C(I - \lambda A)u_n\| \\ &\quad + (1 - \delta) \|P_C(I - \eta B)x_n - P_C(I - \eta B)v_n\| + |(1 - \delta) - (1 - \delta_n)| \|P_C(I - \eta B)v_n\| \\ &\quad + \|z_n - x_n\| \\ &\leq \delta \|x_n - u_n\| + |\delta - \delta_n| \|P_C(I - \lambda A)u_n\| \\ &\quad + (1 - \delta) \|x_n - v_n\| + |\delta_n - \delta| \|P_C(I - \eta B)v_n\| \\ &\quad + \|z_n - x_n\| \end{aligned}$$

by condition (i), (3.19), (3.20), and (3.32), we have

$$\lim_{n \rightarrow \infty} \|Qx_n - x_n\| = 0. \tag{3.46}$$

Since $x_{n_i} \rightharpoonup q$ as $i \rightarrow \infty$ and Lemma 2.7, we have

$$q \in F(Q) = F(G_1) \cap F(G_2). \tag{3.47}$$

From (3.39), (3.40), (3.43), and (3.47), we have $q \in F$. Hence $\omega(x_n) \subset F$. Hence, by Lemma 2.8 and (3.8), it implies that $\{x_n\}$ converges strongly to $P_F x_1$. This completes the proof.

Remark 3.2. If we take $T \equiv D$, $w_n^1 = w_n^2$, $u_n = v_n \forall n \in \mathbb{N}$, $\Phi_1 \equiv \Phi_2$ and $\phi_1 = \phi_2$, then the Algorithm 1.3 reduces to the following algorithm:

$$\begin{cases} w_n^1 \in T(x_n), \|w_n^1 - w_{n+1}^1\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}(T(x_n), T(x_{n+1})), \\ \Phi_1(w_n^1, u_n, u) + \varphi_1(u) - \varphi_1(u_n) + \frac{1}{r_n} \langle u_n - x_n, u - u_n \rangle \geq 0, \forall u \in C, \\ z_n = P_C(I - \lambda A)u_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n)S_n z_n, \\ C_{n+1} = \{z \in C_{n+1} : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \forall n \geq 1, \end{cases} \quad (3.48)$$

under the same conditions of Theorem 3.1, we have the sequence $\{x_n\}$ generated by algorithm (3.48) converges strongly to $P_{\mathbb{F}}x_1$ where $\mathbb{F} = \bigcap_{i=1}^N F(T_i) \cap (GEP)_s(\Phi_1, \varphi_1) \cap F(G_1)$, where $G_1 : C \rightarrow C$ is defined by $G_1(x) = P_C(x - \lambda A x) \forall x \in C$ and $P_{\mathbb{F}}x_1$ is a solution of $\langle Ax^*, x - x^* \rangle \geq 0$

4 Application

In this section, by using our main result we prove a strong convergence theorem of the sequence $\{x_n\}$ generated by Algorithm 4.1 as follows:

Algorithm 4.1. Let $T_i, i = 1, 2, \dots, N$, be κ_i -pseudo contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ and let S_n be the S -mappings generated by T_1, T_2, \dots, T_N and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$ where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0, 1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, \dots, N - 1, \kappa < c \leq \alpha_1^{n,N} \leq 1, \kappa \leq \alpha_3^{n,N} \leq d < 1, \kappa \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1, 2, \dots, N$. Let $x_1 \in C = C_1$ and $w_1^i \in T^i(x_1)$, there exists sequence $\{w_n^i\} \in H$ and $\{x_n\}, \{u_n^i\} \subseteq C, \forall i = 1, 2, \dots, N$ such that

$$\begin{cases} w_n^i \in T^i(x_n), \|w_n^i - w_{n+1}^i\| \leq \left(1 + \frac{1}{n}\right) \mathcal{H}(T(x_n), T(x_{n+1})) \quad \forall i = 1, 2, \dots, N. \\ \Phi_i(w_n^i, u_n^i, u) + \varphi_i(u) - \varphi_i(u_n^i) + \frac{1}{r_n^i} \langle u_n^i - x_n, u - u_n^i \rangle \geq 0, \quad \forall u \in C, \quad \forall i = 1, 2, \dots, N. \\ z_n = \sum_{i=1}^N \delta_n^i P_C(I - \lambda_i A_i)u_n^i, \quad \text{where } \sum_{i=1}^N \delta_n^i = 1, \\ y_n = \alpha_n z_n + (1 - \alpha_n)S_n z_n, \\ C_{n+1} = \{z \in C_{n+1} : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \forall n \geq 1. \end{cases} \quad (4.1)$$

The following result can be obtained from Theorem 3.1. We, therefore, omit the proof.

Theorem 4.2. Let C be a nonempty bounded, closed, and convex subset of Hilbert space H and let $\phi_i : C \rightarrow \mathbb{R}$ be a lower semicontinuous and convex function, for all $i = 1, 2, \dots, N$. Let $T^i : C \rightarrow CB(H)$ be \mathcal{H} -Lipschitz continuous with constant $\mu_i, \Phi_i : H \times C \times C \rightarrow \mathbb{R}$ be equilibrium-like functions satisfying (H1)-(H3) and $A_i : C \rightarrow H$ be a α_i -inverse strongly monotone mappings $\forall i = 1, 2, \dots, N$ and let $T_i, i = 1, 2, \dots, N$, be κ_i -pseudo contraction mappings of C into itself and $\kappa = \max\{\kappa_i : i = 1, 2, \dots, N\}$ with $\mathbb{F} = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N (GEP)_s(\Phi_i, \varphi_i) \cap \bigcap_{i=1}^N F(G_i)$, where $G_i : C \rightarrow C$ is defined by $G_i(x) = P_C(x - \lambda_i A_i x) \forall x \in C, i = 1, 2, \dots, N$. Let S_n be the S -mappings generated by T_1, T_2, \dots, T_N and $\alpha_1^{(n)}, \alpha_2^{(n)}, \dots, \alpha_N^{(n)}$ where $\alpha_j^{(n)} = (\alpha_1^{n,j}, \alpha_2^{n,j}, \alpha_3^{n,j}) \in I \times I \times I, I = [0, 1], \alpha_1^{n,j} + \alpha_2^{n,j} + \alpha_3^{n,j} = 1$ and $\kappa < a \leq \alpha_1^{n,j}, \alpha_3^{n,j} \leq b < 1$ for all $j = 1, 2, \dots, N - 1, \kappa < c \leq \alpha_1^{n,N} \leq 1, \kappa \leq \alpha_3^{n,N} \leq d < 1, \kappa \leq \alpha_2^{n,j} \leq e < 1$ for all $j = 1, 2, \dots, N$ and let $\{x_n\}, \{u_n^i\}, \{w_n^i\}, \forall i = 1, 2, \dots, N$, be sequences generated by (4.1), where $\{\alpha_n\}$ is

a sequence in $[0,1]$, $r_n^i, \lambda_i \in [a, b] \subset (0, 2\alpha) \forall i = 1, 2, \dots, N$ and $n \in \mathbb{N}$ and suppose the following conditions hold:

$$(i) \lim_{n \rightarrow \infty} \delta_n^i = \delta^i \in (0, 1), \forall i = 1, 2, \dots, N.$$

$$(ii) 0 \leq \kappa \leq \alpha_n < 1, \forall n \geq 1,$$

$$(iii) \sum_{n=1}^{\infty} |\alpha_1^{n+1,j} - \alpha_1^{n,j}| < \infty, \sum_{n=1}^{\infty} |\alpha_3^{n+1,j} - \alpha_3^{n,j}| < \infty, \text{ for all } j \in \{1, 2, \dots, N\}.$$

(iv) There exists $\lambda^i, \forall i = 1, 2, \dots, N$ such that

$$\Phi_1 \left(w_1^i, T_{r_1^i}(x_1), T_{r_2^i}(x_2) \right) + \Phi_1 \left(w_2^i, T_{r_2^i}(x_2), T_{r_1^i}(x_1) \right) \leq -\lambda^i \|T_{r_1^i}(x_1) - T_{r_2^i}(x_2)\|^2, \quad (4.2)$$

for all $i = 1, 2, \dots, N$, $(r_1^i, r_2^i) \in \Theta^i \times \Theta^i, w_k^i \in T^i(x_k)$ for $k = 1, 2$ where $\Theta^i = \{r_n^i : n \geq 1\}$. Then $\{x_n\}$ converges strongly to $P_{\mathbb{F}}x_1$ and $P_{\mathbb{F}}x_1$ is a solutions of (4.3):

$$\langle A_i x^*, x - x^* \rangle \geq 0, \forall i = 1, 2, \dots, N. \quad (4.3)$$

Competing interests

The author declares that they have no competing interests.

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References

- Goebel, K, Kirk, WA: Topics in Metric Fixed Point Theory. In Cambridge Stud Adv Math, vol. 28, Cambridge University Press, Cambridge (1990)
- Iiduka, H, Takahashi, W: Weak convergence theorem by Cesàro means for nonexpansive mappings and inverse-strongly monotone mappings. *J Nonlinear Convex Anal.* **7**, 105–113 (2006)
- Combettes, PL, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. *J Nonlinear Convex Anal.* **6**(1):117–136 (2005)
- Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. *Math Student.* **63**(1-4):123–145 (1994)
- Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. *Math Stud.* **63**, 123–145 (1994)
- Moudafi, A: Weak convergence theorems for nonexpansive mappings and equilibrium problems. *J Nonlinear Convex Anal.* **9**, 37–43 (2008)
- He, H, Liu, S, Cho, YJ: An explicit method for systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings. *J Comput Appl Math.* **235**, 4128–4139 (2011). doi:10.1016/j.cam.2011.03.003
- Yao, Y, Cho, YJ, Liou, YC: Iterative algorithms for variational inclusions, mixed equilibrium problems and fixed point problems approach to optimization problems. *Cent Eur J Math.* **9**, 640–656 (2011). doi:10.2478/s11533-011-0021-3
- Yao, Y, Cho, YJ, Liou, YC: Algorithms of common solutions for variational inclusions, mixed equilibrium problems and fixed point problems. *Eur J Oper Res.* **212**, 242–250 (2011). doi:10.1016/j.ejor.2011.01.042
- Cho, YJ, Argyros, IK, Petrot, N: Approximation methods for common solutions of generalized equilibrium, systems of nonlinear variational inequalities and fixed point problems. *Comput Math Appl.* **60**, 2292–2301 (2010). doi:10.1016/j.camwa.2010.08.021
- Cho, YJ, Petrot, N: An Optimization problem related to generalized equilibrium and fixed point problems with applications. *Fixed Point Theory.* **11**, 237–250 (2010)
- Qin, X, Chang, Ss, Cho, YJ: Iterative methods for generalized equilibrium problems and fixed point problems with applications. *Nonlinear Anal: Real World Appl.* **11**, 2963–2972 (2010). doi:10.1016/j.nonrwa.2009.10.017
- Cho, YJ, Petrot, N: On the system of nonlinear mixed implicit equilibrium problems in Hilbert spaces. *J Inequal Appl* **2010**, 12 (2010). Article ID 437976
- Qin, X, Cho, YJ, Kang, SM: Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications. *Nonlinear Anal.* **72**, 99–112 (2010). doi:10.1016/j.na.2009.06.042
- Qin, X, Cho, YJ, Kang, SM: Convergence analysis on hybrid projection algorithms for equilibrium problems and variational inequality problems. *Math Model Anal.* **14**, 335–351 (2009). doi:10.3846/1392-6292.2009.14.335-351
- Cho, YJ, Kang, JI, Qin, X: Convergence theorems based on hybrid methods for generalized equilibrium problems and fixed point problems. *Nonlinear Anal.* **71**, 4203–4214 (2009). doi:10.1016/j.na.2009.02.106
- Ceng, LC, Ansari, QH, Yao, JC: Viscosity approximation methods for generalized equilibrium problems and fixed point problems. *J Glob Optim.* **43**, 487–502 (2009). doi:10.1007/s10898-008-9342-6
- Kangtunyakarn, A: Hybrid algorithm for finding common elements of the set of generalized equilibrium problems and the set of fixed point problems of strictly pseudocontractive mapping. *Fixed Point Theory and Applications* **2011**, 19 (2011). Article ID 274820. doi:10.1186/1687-1812-2011-19
- Takahashi, W: *Nonlinear Functional Analysis*. Yokohama Publishers, Yokohama (2000)
- Bruck, RE: Properties of fixed point sets of nonexpansive mappings in Banach spaces. *Trans Am Math Soc.* **179**, 251–262 (1973)

21. Kangtunyakarn, A, Suantai, S: Strong convergence of a new iterative scheme for a finite family of strict pseudo-contractions. *Comput Math Appl.* **60**, 680–694 (2010). doi:10.1016/j.camwa.2010.05.016
22. Marino, G, Xu, HK: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. *J Math Anal Appl.* **329**, 336–346 (2007). doi:10.1016/j.jmaa.2006.06.055
23. Browder, FE: Nonlinear operators and nonlinear equations of evolution in Banach spaces. *Proc Sympos Pure Math.* **18**, 78–81 (1976)
24. Matines-Yanes, C, Xu, HK: Strong convergence of the CQ method for fixed point iteration processes. *Nonlinear Anal.* **64**, 2400–2411 (2006). doi:10.1016/j.na.2005.08.018
25. Nadler, SB Jr: Multivalued contraction mappings. *Pacific J Math.* **30**, 475–488 (1969)
26. Takahashi, W: *Nonlinear and Convex Analysis*. Yokohama Publishers, Yokohama (2009)

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