Research Article

Oscillation Criteria for Second-Order Neutral Delay Dynamic Equations with Mixed Nonlinearities

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This paper is concerned with some oscillation criteria for the second order neutral delay dynamic equations with mixed nonlinearities of the form $(r(t)u(t))^{\Delta} + q(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) + \sum_{i=1}^{n}q_i(t)|x(\tau_i(t))|^{\alpha_i-1}x(\tau_i(t)) = 0$, where $t \in \mathbb{T}$ and $u(t) = |(x(t) + p(t)x(\delta(t)))^{\Delta}|^{\alpha-1}(x(t) + p(t)x(\delta(t)))^{\Delta}$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha > \alpha_{m+1} > \cdots > \alpha_n > 0$. Further the results obtained here generalize and complement to the results obtained by Han et al. (2010) . Examples are provided to illustrate the results.

1. Introduction

Since the introduction of time scale calculus by Stefan Hilger in 1988, there has been great interest in studying the qualitative behavior of dynamic equations on time scales, see, for example, [1–3] and the references cited therein. In the last few years, the research activity concerning the oscillation and nonoscillation of solutions of ordinary and neutral dynamic equations on time scales has been received considerable attention, see, for example, [4–8] and the references cited therein. Moreover the oscillatory behavior of solutions of second order differential and dynamic equations with mixed nonlinearities is discussed in [9–16].

In 2004, Agarwal et al. [5] have obtained some sufficient conditions for the oscillation of all solutions of the second order nonlinear neutral delay dynamic equation

$$\left(r(t)\left(\left(y(t) + p(t)y(t-\tau)\right)^{\Delta}\right)^{\gamma}\right)^{\Delta} + f\left(t, y(t-\delta)\right) = 0 \qquad (1.1)$$

on time scale $\mathbb{T}$, where $t \in \mathbb{T}$, γ is a quotient of odd positive integers such that $\gamma \geq 1$, $r(t)$, $p(t)$ are real valued rd-continuous functions defined on $\mathbb{T}$ such that $r(t) > 0$, $0 \leq p(t) < 1$, and $f(t, u) \geq q(t)|u|^{\gamma}$.

In 2009, Tripathy [17] has considered the nonlinear neutral dynamic equation of the form

$$\left(r(t) \left(\left(y(t) + p(t)y(t - \tau) \right)^{\Delta} \right)^{\gamma} \right)^{\Delta} + q(t) \left| y(t - \delta) \right|^{\gamma} \operatorname{sgn} y(t - \delta) = 0, \quad t \in \mathbb{T}, \qquad (1.2)$$

where $\gamma > 0$ is a quotient of odd positive integers, $r(t)$, $q(t)$ are positive real valued rd-continuous functions on $\mathbb{T}$, $p(t)$ is a nonnegative real valued rd-continuous function on $\mathbb{T}$ and established sufficient conditions for the oscillation of all solutions of (1.2) using Ricatti transformation.

Saker et al. [18], Şahíner [19], and Wu et al. [20] established various oscillation results for the second order neutral delay dynamic equations of the form

$$\left(r(t) \left(\left(y(t) + p(t)y(\tau(t)) \right)^{\Delta} \right)^{\gamma} \right)^{\Delta} + f(t, y(\delta(t))) = 0, \quad t \in \mathbb{T}, \qquad (1.3)$$

where $0 \leq p(t) < 1$, $\gamma \geq 1$ is a quotient of odd positive integers, $r(t)$, $p(t)$ are real valued nonnegative rd-continuous functions on $\mathbb{T}$ such that $r(t) > 0$, and $f(t, u) \geq q(t)|u|^{\gamma}$.

In 2010, Sun et al. [21] are concerned with oscillation behavior of the second order quasilinear neutral delay dynamic equations of the form

$$\left(r(t) \left(z^{\Delta}(t) \right)^{\gamma} \right)^{\Delta} + q_1(t)x^{\alpha}(\tau_1(t)) + q_2(t)x^{\beta}(\tau_2(t)) = 0, \quad t \in \mathbb{T}, \qquad (1.4)$$

where $z(t) = x(t) + p(t)x(\tau_0(t))$, γ, α, β are quotients of odd positive integers such that $0 < \alpha < \gamma < \beta$ and $\gamma \geq 1$, $r(t)$, $p(t)$, $q_1(t)$, and $q_2(t)$ are real valued rd-continuous functions on $\mathbb{T}$.

Very recently, Han et al. [22] have established some oscillation criteria for quasilinear neutral delay dynamic equation

$$\left(r(t) \left| x^{\Delta}(t) \right|^{\gamma - 1} x^{\Delta} \right)^{\Delta} + q_1(t) \left| y(\delta_1(t)) \right|^{\alpha - 1} y(\delta_1(t)) + q_2(t) \left| y(\delta_2(t)) \right|^{\beta - 1} y(\delta_2(t)) = 0, \quad t \in \mathbb{T},$$

$$(1.5)$$

where $x(t) = y(t) + p(t)y(\tau(t))$, α, β, γ are quotients of odd positive integers such that $0 < \alpha < \gamma < \beta$, $r(t)$, $p(t)$, $q_1(t)$, and $q_2(t)$ are real valued rd-continuous functions on $\mathbb{T}$.

Motivated by the above observation, in this paper we consider the following second order neutral delay dynamic equation with mixed nonlinearities of the form:

$$(r(t)u(t))^{\Delta} + q(t)|x(\tau(t))|^{\alpha - 1}x(\tau(t)) + \sum_{i=1}^{n} q_i(t)|x(\tau_i(t))|^{\alpha_i - 1}x(\tau_i(t)) = 0, \qquad (1.6)$$

where $\mathbb{T}$ is a time scale, $t \in \mathbb{T}$ and $u(t) = \left| (x(t) + p(t)x(\delta(t)))^{\Delta} \right|^{\alpha - 1} (x(t) + p(t)x(\delta(t)))^{\Delta}$, and this includes all the equations (1.1)–(1.5) as special cases.

By a proper solution of (1.6) on $[t_0, \infty)_{\mathbb{T}}$ we mean a function $x(t) \in C^1_{rd}[t_0, \infty)$, which has a property that $r(t)(x(t) + p(t)x(\tau(t)))^\alpha \in C^1_{rd}[t_0, \infty)$, and satisfies (1.6) on $[t_x, \infty)_{\mathbb{T}}$. For the existence and uniqueness of solutions of the equations of the form (1.6), refer to the monograph [2]. As usual, we define a proper solution of (1.6) which is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is known as nonoscillatory.

Throughout the paper, we assume the following conditions:

(C_1) the functions $\delta, \tau, \tau_i : \mathbb{T} \to \mathbb{T}$ are nondecreasing right-dense continuous and satisfy $\delta(t) \leq t, \tau(t) \leq t, \tau_i(t) \leq t$ with $\lim_{t \to \infty} \delta(t) = \infty$, $\lim_{t \to \infty} \tau(t) = \infty$, and $\lim_{t \to \infty} \tau_i(t) = \infty$ for $i = 1, 2, \ldots, n$;

(C_2) $p(t)$ is a nonnegative real valued rd-continuous function on $\mathbb{T}$ such that $0 \leq p(t) < 1$;

(C_3) $r(t), q(t)$ and $q_i(t), i = 1, 2, \ldots, n$ are positive real valued rd-continuous functions on $\mathbb{T}$ with $r^\Delta(t) \geq 0$;

(C_4) $\alpha, \alpha_i, i = 1, 2, \ldots, n$ are positive constants such that $\alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha > \alpha_{m+1} > \cdots > \alpha_n > 0$ $(n > m \geq 1)$.

We consider the two possibilities

$$\int_{t_0}^\infty \frac{1}{r^{1/\alpha}(s)} \Delta s = \infty, \tag{1.7}$$

$$\int_{t_0}^\infty \frac{1}{r^{1/\alpha}(s)} \Delta s < \infty. \tag{1.8}$$

Since we are interested in the oscillatory behavior of the solutions of (1.6), we may assume that the time scale $\mathbb{T}$ is not bounded above, that is, we take it as $[t_0, \infty)_{\mathbb{T}} = \{t \geq t_0 : t \in \mathbb{T}\}$.

The paper is organized as follows. In Section 2, we present some oscillation criteria for (1.6) using the averaging technique and the generalized Riccati transformation, and in Section 3, we provide some examples to illustrate the results.

2. Oscillation Results

We use the following notations throughout this paper without further mention:

$$d_+(t) = \max\{0, d(t)\}, \qquad d_-(t) = \max\{0, -d(t)\}$$

$$Q(t) = q(t)(1 - p(\tau(t)))^\alpha, \qquad Q_i(t) = q_i(t)(1 - p(\tau_i(t)))^{\alpha_i}, \quad i = 1, 2, 3, \ldots, n, \tag{2.1}$$

$$\kappa(t) = \frac{\sigma(t)}{t}, \qquad \beta(t) = \frac{\tau(t)}{\sigma(t)}, \qquad \beta_i(t) = \frac{\tau_i(t)}{\sigma(t)}, \qquad z(t) = x(t) + p(t)x(\delta(t)).$$

In this section, we obtain some oscillation criteria for (1.6) using the following lemmas. Lemma 2.1 is an extension of Lemma 1 of [13].

Lemma 2.1. *Let α_i, $i = 1, 2, \ldots, n$ be positive constants satisfying*

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha > \alpha_{m+1} > \cdots > \alpha_n > 0. \tag{2.2}$$

Then there is an n-tuple $(\eta_1, \eta_2, \ldots, \eta_n)$ satisfying

$$\sum_{i=1}^{n} \alpha_i \eta_i = \alpha \tag{2.3}$$

which also satisfies either

$$\sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1, \tag{2.4}$$

or

$$\sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1. \tag{2.5}$$

In the following results we use the Keller's Chain rule [1] given by

$$(y^{\alpha}(t))^{\Delta} = \alpha y^{\Delta}(t) \int_0^1 \left[h y^{\sigma}(t) + (1-h)y(t) \right]^{\alpha-1} dh, \tag{2.6}$$

where y is a positive and delta differentiable function on $\mathbb{T}$.

Lemma 2.2 (see [23]). *Let $f(u) = Bu - Au^{(\alpha+1)/\alpha}$, where $A > 0$ and B are constants, γ is a positive integer. Then f attains its maximum value on $\mathbb{R}$ at $u* = (B^{\gamma}/A^{\gamma+1})^{\gamma}$, and*

$$\max_{u \in \mathbb{R}} f = f(u*) = \frac{\gamma^{\gamma}}{(\gamma+1)^{(\gamma+1)}} \frac{B^{\gamma+1}}{A^{\gamma}}. \tag{2.7}$$

Lemma 2.3. *Assume that (1.7) holds. If $x(t)$ is an eventually positive solution of (1.6), then there exists a $T \in [t_0, \infty)_{\mathbb{T}}$ such that $z(t) > 0$, $z^{\Delta}(t) > 0$, and $(r(t)(z^{\Delta}(t))^{\alpha})^{\Delta} < 0$ for $t \in [T, \infty)_{\mathbb{T}}$. Moreover one obtains*

$$x(t) \geq (1 - p(t))z(t), \quad t \geq t_1. \tag{2.8}$$

Since the proof of Lemma 2.3 is similar to that of Lemma 2.1 in [6], we omit the details.

Lemma 2.4. *Assume that (1.7) and*

$$\int_{t_0}^{\infty} \tau^{\alpha}(s)Q(s)\Delta s = \infty \tag{2.9}$$

hold. If $x(t)$ is an eventually positive solution of (1.6)*, then*

$$z^{\Delta\Delta}(t) < 0, \qquad z(t) \geq t z^{\Delta}(t), \tag{2.10}$$

and $z(t)/t$ is strictly decreasing.

Proof. From Lemma 2.3, we have $\left(r(t)(z^{\Delta}(t))^{\alpha} \right)^{\Delta} < 0$ and

$$\left(r(t)\left(z^{\Delta}(t) \right)^{\alpha} \right)^{\Delta} = r^{\Delta}(t)\left(z^{\Delta}(t) \right)^{\alpha} + r(\sigma(t))\left(\left(z^{\Delta}(t) \right)^{\alpha} \right)^{\Delta}. \tag{2.11}$$

Since $r^{\Delta}(t) \geq 0$, we have $\left((z^{\Delta}(t))^{\alpha} \right)^{\Delta} < 0$. Now using the Keller's Chain rule, we find that

$$0 < \left(\left(z^{\Delta}(t) \right)^{\alpha} \right)^{\Delta} = \alpha z^{\Delta\Delta}(t) \int_0^1 \left(h z^{\Delta}(t) + (1-h)z^{\Delta}(t) \right)^{\alpha-1} dh \tag{2.12}$$

or $z^{\Delta\Delta}(t) < 0$. Let $Z(t) := z(t) - t z^{\Delta}(t)$. Clearly $Z^{\Delta}(t) = -\sigma(t)z^{\Delta\Delta}(t) > 0$. We claim that there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $Z(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume the contrary, then $Z(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$. Therefore,

$$\left(\frac{z(t)}{t} \right)^{\Delta} = \frac{t z^{\Delta}(t) - z(t)}{t\sigma(t)} = -\frac{Z(t)}{t\sigma(t)} > 0, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{2.13}$$

which implies that $z(t)/t$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. Pick $t_2 \in [t_1, \infty)_{\mathbb{T}}$ so that $\tau(t) \geq \tau(t_2)$ and $\tau_i(t) \geq \tau_i(t_2)$ for $t \geq t_2$. Then $z(\tau(t))/\tau(t) \geq z(\tau(t_2))/\tau(t_2) =: d > 0$, and $z(\tau(t))/\tau(t) \geq z(\tau(t_2))/\tau(t_2) =: d_i > 0$, so that $z(\tau(t)) > \tau(t)$ for $t \geq t_2$.

Using the inequality (2.8) in (1.6), we have that

$$\left(r(t)\left(z^{\Delta}(t) \right)^{\alpha} \right)^{\Delta} + Q(t)z^{\alpha}(\tau(t)) + \sum_{i=1}^{n} Q_i(t)z^{\alpha_i}(\tau_i(t)) \leq 0. \tag{2.14}$$

Now by integrating from t_2 to t, we have

$$r(t)\left(z^{\Delta}(t) \right)^{\alpha} - r(t_2)\left(z^{\Delta}(t_2) \right)^{\alpha} + \int_{t_2}^{t} \left[Q(s)z^{\alpha}(\tau(s)) + \sum_{i=1}^{n} Q_i(s)z^{\alpha_i}(\tau_i(s)) \right] \Delta s \leq 0, \tag{2.15}$$

which implies that

$$\begin{aligned}
r(t_2)z^{\Delta}(t_2) &\geq r(t)z^{\Delta}(t) + \int_{t_2}^{t} \left[Q(s)z^{\alpha}(\tau(s)) + \sum_{i=1}^{n} Q_i(s)z^{\alpha_i}(\tau_i(s)) \right] \Delta s \\
&> d^{\alpha} \int_{t_2}^{t} Q(s)\tau^{\alpha}(s)\Delta s + \sum_{i=1}^{n} d_i^{\alpha_i} \int_{t_2}^{t} Q_i(s)\tau_i^{\alpha_i}(s)\Delta s
\end{aligned} \tag{2.16}$$

which contradicts (2.4). Hence there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $Z(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Consequently,

$$\left(\frac{z(t)}{t}\right)^{\Delta} = \frac{tz^{\Delta}(t) - z(t)}{t\sigma(t)} = -\frac{Z(t)}{t\sigma(t)} < 0, \quad t \in [t_1, \infty)_{\mathbb{T}}, \tag{2.17}$$

and we have that $z(t)/t$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. $\qquad\square$

Theorem 2.5. *Assume that condition* (1.7) *holds. Let* $(\eta_1, \eta_2, \ldots, \eta_n)$ *be n-tuple satisfying* (2.3) *of Lemma 2.1. Furthermore one assumes that there exist positive delta differentiable function* $\rho(t)$ *and a nonnegative delta differentiable function* $\phi(t)$ *such that*

$$\limsup_{t \to \infty} \int_{t_1}^{t} \rho(\sigma(s)) \left[Q^*(s) - \phi^{\Delta}(s) - \frac{(\rho^{\Delta}(s))_+}{\rho^{\sigma}(s)} \phi(s) - \frac{\kappa^{\alpha^2}(s)}{(\alpha+1)^{\alpha+1}} \frac{r(s)(\rho^{\Delta}(s))_+^{\alpha+1}}{(\rho(\sigma(s)))^{\alpha+1}} \right] \Delta s = \infty, \tag{2.18}$$

for all sufficiently large t_1 *where* $Q^*(t) = Q(t)\beta^{\alpha}(t) + \eta \prod_{i=1}^{n} Q_i^{\eta_i}(t)\beta_i^{\alpha_i \eta_i}(t)$, *and* $\eta = \prod_{i=1}^{n} \eta_i^{-\eta_i}$. *Then every solution of* (1.6) *is oscillatory.*

Proof. Suppose that there is a nonoscillatory solution $x(t)$ of (1.6). We assume that $x(t)$ is an eventually positive for $t \geq t_0$ (since the proof for the case $x(t) < 0$ eventually is similar). From the definition of $z(t)$ and Lemma 2.3, there exists $t_1 \geq t_0$ such that, for $t \geq t_1$,

$$z(t) > 0, \quad z(\delta(t)) > 0, \quad z(\tau(t)) > 0, \quad z(\tau_i(t)) > 0, \quad z^{\Delta}(t) > 0, \quad \left(r(t)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} \leq 0. \tag{2.19}$$

Define

$$w(t) = \rho(t) \left[\frac{r(t)(z^{\Delta}(t))^{\alpha}}{z^{\alpha}(t)} + \phi(t) \right], \quad t \geq t_1. \tag{2.20}$$

Then from (2.19), we have $w(t) > 0$ and

$$w^{\Delta}(t) = \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho(\sigma(t)) \left[\frac{r(t)(z^{\Delta}(t))^{\alpha}}{z^{\alpha}(t)} \right]^{\Delta} + \rho(\sigma(t))\phi^{\Delta}(t)$$

$$\leq \frac{\rho^{\Delta}(t)}{\rho(t)} w(t) + \rho(\sigma(t)) \frac{\left(r(t)(z^{\Delta}(t))^{\alpha}\right)^{\Delta}}{z^{\alpha}(\sigma(t))} \tag{2.21}$$

$$- \rho(\sigma(t)) \frac{r(t)(z^{\Delta}(t))^{\alpha}(z^{\alpha}(t))^{\Delta}}{z^{\alpha}(t)z^{\alpha}(\sigma(t))} + \rho(\sigma(t))\phi^{\Delta}(t).$$

From Keller's chain rule, we have, from Lemma 2.1,

$$(z^\alpha(t))^\Delta \geq \begin{cases} \alpha z^{\alpha-1}(t) z^\Delta(t), & \alpha \geq 1, \\ \alpha z^{\alpha-1}(\sigma(t)) z^\Delta(t), & 0 < \alpha < 1. \end{cases} \tag{2.22}$$

Using (2.22) and the definition of $\kappa(t)$ in (2.21), we obtain

$$\begin{aligned} w^\Delta(t) \leq &-\frac{\rho(\sigma(t))}{z^\alpha(\sigma(t))}\left[Q(t)z^\alpha(\tau(t)) + \sum_{i=1}^{n} Q_i(t)z^{\alpha_i}(\tau_i(t))\right] + \frac{\rho^\Delta(t)}{\rho(t)}w(t) \\ &-\frac{\alpha\rho(\sigma(t))}{r^{1/\alpha}(t)}\frac{1}{\kappa^\alpha(t)}\left|\frac{w(t)}{\rho(t)} - \phi(t)\right|^{(\alpha+1)/\alpha} + \rho(\sigma(t))\phi^\Delta(t). \end{aligned} \tag{2.23}$$

From Lemma 2.4, we see that $z(t)/t$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$, and therefore

$$\frac{z(\tau_i(t))}{\tau_i(t)} \geq \frac{z(\sigma(t))}{\sigma(t)} \tag{2.24}$$

or

$$\frac{z(\tau_i(t))}{z(\sigma(t))} \geq \frac{\tau_i(t)}{\sigma(t)}, \tag{2.25}$$

since $\tau_i(t) \leq \sigma(t)$ for all $i = 1, 2, \ldots, n$. Using (2.25) in (2.23), we have

$$\begin{aligned} w^\Delta(t) \leq &-\rho(\sigma(t))\left[Q(t)\beta^\alpha(t) + \sum_{i=1}^{n} Q_i(t)\beta_i^{\alpha_i}(t)z^{\alpha_i-\alpha}(\sigma(t))\right] + \frac{\rho^\Delta(t)}{\rho(t)}w(t) \\ &-\frac{\alpha\rho(\sigma(t))}{r^{1/\alpha}(t)}\frac{1}{\kappa^\alpha(t)}\left|\frac{w(t)}{\rho(t)} - \phi(t)\right|^{(\alpha+1)/\alpha} + \rho(\sigma(t))\phi^\Delta(t). \end{aligned} \tag{2.26}$$

Now let $u_i(t) = (1/\eta_i)Q_i(t)\beta_i^{\alpha_i}z^{\alpha_i-\alpha}(\sigma(t))$, $i = 1, 2, \ldots, n$. Then (2.26) becomes

$$\begin{aligned} w^\Delta(t) \leq &-\rho(\sigma(t))\left[Q(t)\beta^\alpha(t) + \sum_{i=1}^{n} \eta_i u_i(t)\right] + \frac{\rho^\Delta(t)}{\rho(t)}w(t) \\ &-\frac{\alpha\rho(\sigma(t))}{r^{1/\alpha}(t)}\frac{1}{\kappa^\alpha(t)}\left|\frac{w(t)}{\rho(t)} - \phi(t)\right|^{(\alpha+1)/\alpha} + \rho(\sigma(t))\phi^\Delta(t). \end{aligned} \tag{2.27}$$

By Lemma 2.1 and using the arithmetic-geometric inequality $\sum_{i=1}^{n} \eta_i u_i \geq \prod_{i=1}^{n} \eta_i^{u_i}$ in (2.27), we obtain

$$
\begin{aligned}
w^{\Delta}(t) \leq &-\rho(\sigma(t))\left[Q^*(t) - \phi^{\Delta}(t)\right] + \frac{\rho^{\Delta}(t)}{\rho(t)}w(t) \\
&- \frac{\alpha\rho(\sigma(t))}{r^{1/\alpha}(t)}\frac{1}{\kappa^{\alpha}(t)}\left|\frac{w(t)}{\rho(t)} - \phi(t)\right|^{(\alpha+1)/\alpha} + \rho(\sigma(t))\phi^{\Delta}(t)
\end{aligned}
\tag{2.28}
$$

or

$$
\begin{aligned}
w^{\Delta}(t) \leq &-\rho(\sigma(t))\left[Q^*(t) - \phi^{\Delta}(t)\right] + \left(\rho^{\Delta}(t)\right)_{+}\phi(t) \\
&+ \left(\rho^{\Delta}(t)\right)_{+}\left|\frac{w(t)}{\rho(t)} - \phi(t)\right| - \frac{\alpha\rho(\sigma(t))}{r^{1/\alpha}(t)}\frac{1}{\kappa^{\alpha}(t)}\left|\frac{w(t)}{\rho(t)} - \phi(t)\right|^{(\alpha+1)/\alpha}, \quad t \geq t_1.
\end{aligned}
\tag{2.29}
$$

Set $\gamma = \alpha$, $A = ((\alpha\rho(\sigma(t)))/(r(1/\alpha)(t)))(1/\kappa^{\alpha}(t))$, $B = (\rho^{\Delta}(t))_{+}$, and $u(t) = |(w(t)/\rho(t)) - \phi(t)|$ and applying Lemma 2.2 to (2.29), we have

$$
w^{\Delta}(t) \leq -\rho(\sigma(t))\left[Q^*(t) - \phi^{\Delta}(t)\right] + \left(\rho^{\Delta}(t)\right)_{+}\phi(t) + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(t)\left(\rho^{\Delta}(t)\right)_{+}^{\alpha+1}}{(\rho(\sigma(t)))^{\alpha}}\kappa^{\alpha^2}(t).
\tag{2.30}
$$

Now integrating (2.30) from t_1 to t, we obtain

$$
\int_{t_1}^{t} \rho^{\sigma}(s)\left[Q^*(s) - \phi^{\Delta}(s) - \frac{(\rho^{\Delta}(s))_{+}}{\rho^{\sigma}(s)}\phi(s) - \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(s)(\rho^{\Delta}(s))_{+}^{\alpha+1}}{(\rho(\sigma(s)))^{\alpha+1}}\kappa^{\alpha^2}(s)\right]\Delta s \leq w(t_1),
\tag{2.31}
$$

which leads to a contradiction to condition (2.18). The proof is now complete. $\qquad\square$

By different choices of $\rho(t)$ and $\phi(t)$, we obtain some sufficient conditions for the solutions of (1.6) to be oscillatory. For instance, $\rho(t) = 1$, $\phi(t) = 1$ and $\rho(t) = t$, $\phi(t) = 1/t$ in Theorem 2.5, we obtain the following corollaries:

Corollary 2.6. *Assume that* (1.7) *holds. Furthermore assume that, for all sufficiently large T, for $T \geq t_0$,*

$$
\limsup_{t\to\infty}\int_{T}^{\infty} Q^*(s)\Delta s = \infty,
\tag{2.32}
$$

where $Q^*(t)$ is as in Theorem 2.5. Then every solution of (1.6) *is oscillatory.*

Corollary 2.7. *Assume that* (1.7) *holds. Furthermore assume that, for all sufficiently large T, for $T \geq t_0$,*

$$\limsup_{t \to \infty} \int_T^\infty \left[\sigma(s)Q^*(s) - \frac{r(t)(\sigma(t))^{\alpha^2 - \alpha}}{t^{\alpha^2}} \right] \Delta s = \infty, \tag{2.33}$$

where $Q^*(t)$ is as in Theorem 2.5. Then every solution of (1.6) *is oscillatory.*

Next we establish some Philos-type oscillation criteria for (1.6).

Theorem 2.8. *Assume that* (1.7) *holds. Suppose that there exists a function $H \in C_{\rm rd}(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} \equiv \{ (t, s)/t, s \in [t_0, \infty)_{\mathbb{T}} \text{ and } t > s \}$ such that*

$$H(t,t) = 0, \quad t \geq t_0, \quad H(t,s) \geq 0, \quad t > s \geq 0, \tag{2.34}$$

and H has a nonpositive continuous Δ-partial derivative H^{Δ_s} with respect to the second variable such that

$$H^{\Delta_s}(\sigma(t), s) + H(\sigma(t), \sigma(s)) \frac{\rho^\Delta(s)}{\rho(s)} = \frac{h(t,s)}{\rho(s)} (H(\sigma(t), \sigma(s)))^{\alpha/(\alpha+1)}, \tag{2.35}$$

and for all sufficiently large T,

$$\limsup_{t \to \infty} \frac{1}{H(\sigma(t), T)} \int_T^t \left[\rho^\sigma(s)Q^*(s) - \frac{(h(t,s))^{\alpha+1}r(s)}{(\alpha+1)^{\alpha+1}(\rho^\sigma(s))^\alpha} \right] \Delta s = \infty, \tag{2.36}$$

where $Q^*(t)$ is same as in Theorem 2.5. Then every solution of (1.6) *is oscillatory.*

Proof. We proceed as in the proof of Theorem 2.5 and define $w(t)$ by (2.20). Then $w(t) > 0$ and satisfies (2.28) for all $t \in [t_1, \infty)_{\mathbb{T}}$. Multiplying (2.28) by $H(\sigma(t), \sigma(s))$ and integrating, we obtain

$$\int_{t_1}^t H(\sigma(t), \sigma(s))\rho^\sigma(s) \left[Q^*(s) - \phi^\Delta(t) \right] \Delta s$$

$$\leq -\int_{t_1}^t H(\sigma(t), \sigma(s))w^\Delta(s)\Delta s + \int_{t_1}^t H(\sigma(t), \sigma(s)) \frac{\rho^\Delta(t)}{\rho(s)} w(t)\Delta s \tag{2.37}$$

$$-\int_{t_1}^t H(\sigma(t), \sigma(s)) \frac{\alpha\rho(\sigma(t))}{r^{1/\alpha}(t)\rho^{(\alpha+1)/\alpha}(t)} \frac{1}{\kappa^\alpha(t)} \left| \frac{w(t)}{\rho(t)} - \phi(t) \right|^{(\alpha+1)/\alpha} \Delta s.$$

Using the integration by parts formula, we have

$$
\int_{t_1}^{t} H(\sigma(t), \sigma(s)) w^{\Delta}(s) \Delta s = H(t,s) w(s) \big|_{t_1}^{t} - \int_{t_1}^{t} H^{\Delta_s}(\sigma(t), s) w(s) \Delta s
$$

$$
= -H(t, t_1) w(t_1) - \int_{t_1}^{t} H^{\Delta_s}(\sigma(t), s) w(s) \Delta s. \tag{2.38}
$$

Substituting (2.38) into (2.37), we obtain

$$
\int_{t_1}^{t} H(\sigma(t), \sigma(s)) \rho^{\sigma}(s) \left[Q^{*}(s) - \phi^{\Delta}(t) \right] \Delta s
$$

$$
\leq H(t, t_1) w(t_1)
$$

$$
+ \int_{t_1}^{t} \left[H^{\Delta_s}(\sigma(t), s) + H(\sigma(t), \sigma(s)) \frac{\rho^{\Delta}(t)}{\rho(s)} \right] w(s) \Delta s
$$

$$
- \int_{t_1}^{t} H(\sigma(t), \sigma(s)) \frac{\alpha \rho(\sigma(t))}{r^{1/\alpha}(t) \rho^{(\alpha+1)/\alpha}(t)} \frac{1}{\kappa^{\alpha}(t)} \left| \frac{w(t)}{\rho(t)} - \phi(t) \right|^{(\alpha+1)/\alpha} \Delta s. \tag{2.39}
$$

From (2.35) and (2.39), we have

$$
\int_{t_1}^{t} H(\sigma(t), \sigma(s)) \rho^{\sigma}(s) Q(t, s) \Delta s
$$

$$
\leq H(t, t_1) w(t_1)
$$

$$
+ \int_{t_1}^{t} \frac{h(t, s)}{\rho(s)} H^{\alpha/(\alpha+1)}(\sigma(t), \sigma(s)) w(s) \Delta s
$$

$$
- \int_{t_1}^{t} H(\sigma(t), \sigma(s)) \frac{\alpha \rho(\sigma(t))}{r^{1/\alpha}(t) \rho^{(\alpha+1)/\alpha}(t)} \frac{1}{\kappa^{\alpha}(t)} \left| \frac{w(t)}{\rho(t)} - \phi(t) \right|^{(\alpha+1)/\alpha} \Delta s \tag{2.40}
$$

or

$$
\int_{t_1}^{t} H(\sigma(t), \sigma(s)) Q(t, s) \Delta s
$$

$$
\leq H(t, t_1) w(t_1)
$$

$$
+ \int_{t_1}^{t} \frac{h(t, s)}{\rho(s)} H^{(\alpha+1)/\alpha}(\sigma(t), \sigma(s)) \left| \frac{w(s)}{\rho(s)} - \phi(s) \right| \Delta s
$$

$$
- \int_{t_1}^{t} H(\sigma(t), \sigma(s)) \frac{\alpha \rho(\sigma(t))}{r^{1/\alpha}(t) \rho^{(\alpha+1)/\alpha}(t)} \frac{1}{\kappa^{\alpha}(t)} \left| \frac{w(t)}{\rho(t)} - \phi(t) \right|^{(\alpha+1)/\alpha} \Delta s. \tag{2.41}
$$

where $Q(t, s) = [\rho^{\sigma}(s) Q(t, s) - (h(t, s)) / (\rho(s) H^{(1/\alpha)}(\sigma(t), \sigma(s))) \phi(s)]$.

By setting $B = (h(t,s)/\rho(s))H^{(\alpha+1)/\alpha}(\sigma(t),\sigma(s))$ and $A = ((\alpha\rho(\sigma(t)))/(r^{1/\alpha}(t)\rho^{(\alpha+1)/\alpha}(t)))(1/\kappa^\alpha(t))$ in Lemma 2.2, we obtain

$$
\int_{t_1}^{t} H(\sigma(t),\sigma(s))\left[\rho^\sigma(s)Q(t,s) - \frac{h^{\alpha+1}(t,s)r(s)\kappa^{\alpha^2}(t)}{(\alpha+1)^{(\alpha+1)}\rho^\alpha(\sigma(s))H(\sigma(t),\sigma(s))}\right]\Delta s
$$
$$
\leq H(t,t_1)w(t_1),
$$
(2.42)

which contradicts condition (2.35). This completes the proof. $\qquad\square$

Finally in this section we establish some oscillation criteria for (1.6) when the condition (1.8) holds.

Theorem 2.9. *Assume that* (1.8) *holds and* $\lim_{t\to\infty} p(t) = p < 1$. *Let* $(\eta_1, \eta_2, \ldots, \eta_n)$ *be n-tuple satisfying* (2.3) *of Lemma 2.1. Moreover assume that there exist positive delta differentiable functions* $\rho(t)$ *and* $\theta(t)$ *such that* $\theta^\Delta(t) \geq 0$ *and a nonnegative function* $\phi(t)$ *with condition* (2.30) *for all* $t \geq t_1$. *If*

$$
\int_{t_0}^{\infty}\left(\frac{1}{\theta(s)r(s)}\int_{t_0}^{s}\theta(\sigma(v))\overline{Q}(v)\Delta v\right)^{1/\alpha}\Delta s = \infty,
$$
(2.43)

where $\overline{Q}(t) = Q(t) + \sum_{i=1}^{n} Q_i(t)$ *holds, then every solution of* (1.6) *either oscillates or converges to zero as* $t \to \infty$.

Proof. Assume to the contrary that there is a nonoscillatory solution $x(t)$ such that $x(t) > 0$, $x(\delta(t)) > 0$, $x(\tau(t)) > 0$, and $x(\tau_i(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ for some $t_1 \geq t_0$. From Lemma 2.3 we can easily see that either $z^\Delta(t) > 0$ eventually or $z^\Delta(t) < 0$ eventually.

If $z^\Delta(t) > 0$ eventually, then the proof is the same as in Theorem 2.5, and therefore we consider the case $z^\Delta(t) < 0$.

If $z^\Delta(t) < 0$ for sufficiently large t, it follows that the limit of $z(t)$ exists, say a. Clearly $a \geq 0$. We claim that $a = 0$. Otherwise, there exists $M > 0$ such that $z^\alpha(\tau(t)) \geq M$ and $z^{\alpha_i}(\tau_i(t)) \geq M$, $i = 1, 2, \ldots, n$, $t \in [t_1, \infty)_{\mathbb{T}}$. From (1.6) we have

$$
\left(r(t)\left(z^\Delta(t)\right)^\alpha\right)^\Delta \leq -M\left[Q(t) + \sum_{i=1}^{n} Q_i(t)\right] = -M\overline{Q}(t).
$$
(2.44)

Define the supportive function

$$
u(t) = \theta(t)r(t)\left(z^\Delta(t)\right)^\alpha, \quad t \in [t_1, \infty)_{\mathbb{T}},
$$
(2.45)

and we have

$$u^{\Delta}(t) = \theta^{\Delta}(t)r(t)\left(z^{\Delta}(t)\right)^{\alpha} + \theta(\sigma(t))\left(r(t)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta}$$

$$\leq \theta(\sigma(t))\left(r(t)\left(z^{\Delta}(t)\right)^{\alpha}\right)^{\Delta} \tag{2.46}$$

$$= -M\theta(\sigma(t))\overline{Q}(t).$$

Now if we integrate the last inequality from t_1 to t, we obtain

$$u(t) \leq u(t_1) - M \int_{t_1}^{t} \theta(\sigma(s))\overline{Q}(s)\Delta s \tag{2.47}$$

or

$$\left(z^{\Delta}(t)\right)^{\alpha} \leq -M\frac{1}{\theta(t)r(t)} \int_{t_1}^{t} \theta(\sigma(s))\overline{Q}(s)\Delta s. \tag{2.48}$$

Once again integrate from t_1 to t to obtain

$$M^{1/\alpha} \int_{t_1}^{t} \left(\frac{1}{\theta(s)r(s)} \int_{t_1}^{s} \theta(\sigma(\xi))\overline{Q}(\xi)\Delta\xi\right)^{1/\alpha} \Delta s \leq z(t_1), \tag{2.49}$$

which contradicts condition (2.43). Therefore $\lim_{t\to\infty} z(t) = 0$, and there exists a positive constant c such that $z(t) \leq c$ and $x(t) \leq z(t) \leq c$. Since $x(t)$ is bounded, $\limsup_{t\to\infty} x(t) = x_1$ and $\liminf_{t\to\infty} x(t) = x_2$. Clearly $x_2 \leq x_1$. From the definition of $z(t)$, we find that $x_1 + px_2 \leq 0 \leq x_2 + px_1$; hence $x_1 \leq x_2$ and $x_1 = x_2 = 0$. This completes proof of the theorem. □

Remark 2.10. If $q_i(t) \equiv 0$, $i = 1, 2, \ldots, n$, or $\delta(t) = t - \delta$, $\tau(t) = t - \tau$, and $q_i(t) \equiv 0$, $i = 1, 2, \ldots, n$, then Theorem 2.5 reduces to a result obtained in [20] or [24], respecively. If $p(t) \equiv 0$, or $p(t) \equiv 0$, and $\alpha = 1$, or $p(t) \equiv 0$, and $\tau(t) = \tau_i(t) = t$, $i = 1, 2, \ldots, n$, then the results established here complement to the results of [5, 9, 15] respectively.

3. Examples

In this section, we illustrate the obtained results with the following examples.

Example 3.1. Consider the second order delay dynamic equation

$$\left(x(t) + \frac{1}{t^2}x(\delta(t))\right)^{\Delta\Delta} + \frac{\lambda_1}{t^{3/2}}x\left(\sqrt{t}\right) + \frac{\lambda_2}{t}x^{5/3}\left(\sqrt{t}\right) + \frac{\lambda_3}{t^2}x^{1/3}\left(\sqrt{t}\right) = 0, \tag{3.1}$$

for all $t \in [1, \infty)_{\mathbb{T}}$. Here $\alpha = 1$, $\alpha_1 = 1/3$, $\alpha_2 = 5/3$, $p(t) = 1/t^2$, $q(t) = \lambda_1/t^{3/2}$, $q_1(t) = \lambda_2/t$, and $q_2(t) = \lambda_3/t^2$. Then $\eta_1 = \eta_2 = 1/2$. By taking $\rho(t) = t$, and $\phi(t) = 0$, we obtain

$$
\begin{aligned}
\limsup_{t \to \infty} & \int_{t_1}^{t} \rho^{\sigma}(s) \left[Q^*(s) - \frac{1}{(\alpha + 1)^{\alpha+1}} \frac{r(s)\left(\rho^{\Delta}(s)\right)^{\alpha+1}}{\left(\rho^{\sigma}(s)\right)^{\alpha+1}} \right] \Delta s \\
& = \limsup_{t \to \infty} \int_{t_0}^{t} \left[\left(\frac{\lambda_1}{s}\left(1 - \frac{1}{s}\right) + \frac{\sqrt{\lambda_2 \lambda_3}}{s}\left(1 - \frac{1}{s}\right) \right) - \frac{1}{4\sigma(s)} \right] \Delta s \\
& \geq \limsup_{t \to \infty} \int_{t_0}^{t} \left(\left[\lambda_1 + \sqrt{\lambda_2 \lambda_3} - \frac{1}{4} \right] \frac{1}{s} - \frac{\lambda_1 + \sqrt{\lambda_2 \lambda_3}}{s^2} \right) \Delta s \\
& \to \infty \text{ if } \lambda_1 + \sqrt{\lambda_2 \lambda_3} > 1/4.
\end{aligned}
\tag{3.2}
$$

By Theorem 2.5, all solutions of (3.1) are oscillatory if $\lambda_1 + \sqrt{\lambda_2 \lambda_3} > 1/4$.

Example 3.2. Consider the second order neutral delay dynamic equation

$$
\left(\left(\left(x(t) + \frac{1}{2}x(\delta(t)) \right)^{\Delta} \right)^3 \right)^{\Delta} + \frac{\sigma^3(t)}{t^4}x^3\left(\frac{t}{2}\right) + \frac{\sigma(t)}{t^2}x^5\left(\frac{t}{3}\right) + \frac{\sigma(t)}{t^2}x^{1/3}\left(\frac{t}{3}\right) = 0, \tag{3.3}
$$

for all $t \in [1, \infty)_{\mathbb{T}}$. Here $r(t) = 1$, $p(t) = 1/2$, $q(t) = (\sigma^3(t))/t^4$, $\tau(t) = t/2$, $\tau_1(t) = \tau_2(t) = t/3$, $\alpha = 3$, $\alpha_1 = 5$, $\alpha_2 = 1/3$. From Corollary 2.6, every solution of (3.3) is oscillatory.

Acknowledgment

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