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Research Article

Unifying View on Min-Max Fairness, Max-Min Fairness, and Utility Optimization in Cellular Networks

Holger Boche,[1,2] Marcin Wiczanowski,[1] and Slawomir Stanczak[2]

[1] *Heinrich Hertz Chair for Mobile Communications, Faculty of Electrical Engineering and Computer Science (EECS), Berlin University of Technology, Einsteinufer 25, 10587 Berlin, Germany*
[2] *German-Sino Lab for Mobile Communications (MCI), Fraunhofer Institute for Telecommunications, Einsteinufer 37, 10587 Berlin, Germany*

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We are concerned with the control of *quality of service* (QoS) in wireless cellular networks utilizing linear receivers. We investigate the issues of fairness and total performance, which are measured by a utility function in the form of a weighted sum of link QoS. We disprove the common conjecture on incompatibility of min-max fairness and utility optimality by characterizing network classes in which both goals can be accomplished concurrently. We characterize power and weight allocations achieving min-max fairness and utility optimality and show that they correspond to saddle points of the utility function. Next, we address the problem of the difference between min-max fairness and max-min fairness. We show that in general there is a (fairness) gap between the performance achieved under min-max fairness and under max-min fairness. We characterize the network class for which both performance values coincide. Finally, we characterize the corresponding network subclass, in which both min-max fairness and max-min fairness are achievable by the same power allocation.

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1. INTRODUCTION

In concurrent wireless cellular networks the data links already outnumber traditional voice connections. Moreover, the importance of data links is going to increase within future wireless standards. The data links serviced within one cell have in general different priorities and requirements in terms of the perceived user QoS (*quality of service*). The problem of optimal service of such heterogeneous multiuser traffic is nowadays the dominant design problem on and above the second layer of the communication stack.

On the one side, the traffic heterogeneity forces the network operator to service the links with higher QoS expectations with the corresponding higher priority. On the other side, some notion of fundamental fairness in link service has to be maintained, so that even the users associated with the lowest priority links are kept satisfied. Hence, due to the constrained power and bandwidth resources in the network, the operator has to find the best possible trade-off between (a suitable notion of) fairness and the efficiency of overall QoS provision.

There is some degree of freedom in nominating an appropriate notion of network fairness. However, the usual and best established fairness notion is the notion which is referred to in this work as min-max fairness and corresponds to ideal social fairness in the behavioral and economic science [1]. In our framework, min-max fairness is the notion of fairness which implies that the worst link QoS in the network is maximally improved [2]. Such goal is achieved by the classical power control for CDMA (*code division multiple access*) networks [3–5]. Hereby, the total power is minimized, while the worst ratio of the link QoS and the corresponding link QoS requirement is optimized and takes value one at the optimum [6–10]. Some considerations on the min-max fair service in multihop wireless networks can be also found in [11, 12].

The overall network performance can be measured by a utility function, which is, in the cellular case, the function of all link QoS in the cell. The best established and most intuitive form of a utility function is the weighted sum, with weights expressing the traffic or link priorities. The weighted sum as the performance measure originates from

the optimization of bandwidth sharing schemes in wired networks [13–19]. In the wireless case the weighted sum objective is used both in the multihop context [20] and in the cellular context [21–23]. The weighted sum optimization is not always of purely heuristic nature. When link QoS parameters correspond to link data rates and weights express the buffer occupancies on the corresponding links, the optimization of the weighted sum of link QoS leads to the largest stability region of the network [24].

In this work we address the problem of the interdependence between min-max fairness and utility optimality in cellular networks. To the best of our knowledge this work is the first analytic approach to this problem for cellular networks (see [19] for the corresponding results in the context of high-speed wireless medium access). An analogous problem was however addressed in a number of recent works concerning wired networks. In the wired case, a common conjecture had been originally that min-max fairness and optimization of the utility value are two incompatible goals. This was prompted by some network examples, for example, in [15, 16, 18]. The authors in [25] disproved the general incompatibility conjecture, by giving some network topology examples, for which min-max fairness is achievable concurrently with utility optimality.

As the first fundamental step we characterize the network class for which a min-max fair allocation exists. We then show that in some cellular networks min-max fairness and utility optimality can be achieved concurrently. We characterize the class of networks for which it is possible in terms of the interference situation, by using matrix-theoretic and combinatorial arguments. We further characterize power and weight allocations combining min-max fairness and utility-optimality in such networks. We prove the interpretation of such allocations as saddle points of the utility function as a function of powers and weights. This in particular mirrors the fairness utility trade-off, as it implies that the utility optimum achieved together with min-max fairness is the worst-case utility optimum among all utility-optimal power and weight allocations. Next, we address the problem of the difference between min-max fairness and max-min fairness. Our results show that in general there is a nonzero difference in performance between the approach of maximal improvement of the worst link QoS (min-max fairness) and the approach of maximal degradation of the best link QoS (max-min fairness). We characterize a special class of networks for which such performance gap is zero, that is, for which min-max fairness and max-min fairness achieve equal performance. Finally we prove that for some class of networks, there exist power allocations, which concurrently achieve min-max fairness and max-min fairness.

We present the system model in Section 2. Next, in Section 3 we introduce in short the fundamentals of fairness and utility optimization. In Section 4 we address the problem of concurrently achieving min-max fairness and utility optimum in a special class of networks. Section 5 provides the generalization of the results from Section 4 to arbitrary networks and characterizes the cases of existence of allocations combining min-max fairness and utility optimality. In Section 6 we prove that any min-max fair and utility-optimal power and weight allocation represents a saddle point of the utility function, as a function of weights and powers. In Section 7 we address the problem of the gap between min-max fairness and max-min fairness performance. We characterize there the classes of networks for which both notions achieve the same performance and for which there exist allocations achieving both notions concurrently. We conclude the work in Section 8. Some necessary background knowledge is placed in the appendices.

2. SYSTEM MODEL

We consider a single-cell cellular network with K links denoted by indices $1 \le k \le K$. The results presented hold both for the uplink (multiple access) and the downlink (broadcast) case. The transmit powers allocated to the links are grouped in the power vector $\mathbf{p} = (p_1, \ldots, p_K)$. Any power vector is assumed to be included in the set[1] $\mathcal{P} \subseteq \mathbb{R}_+^K$, $\mathcal{P} \neq \varnothing$ of feasible power vectors, referred to as the power region. In the real world downlink, the power region is likely to be constrained by the transmit sum power P of the base station, that is, $\mathcal{P} = \{\mathbf{p} \ge 0 : \|\mathbf{p}\|_1 \le P\}$, while in the real world uplink the link (or batch of links) of each node k is likely to be constrained by the corresponding node transmit power limit $\widehat{p_k}$, that is, $\mathcal{P} = \{\mathbf{p} \ge 0 : \mathbf{p} \le \hat{\mathbf{p}}\}$.

Some remarks on the power region

All the results in the work are independent of the form of the power region. Precisely, the considered optimization problems over $\mathcal{P}$ easily follow to be equivalent to optimization problems over $\mathbb{R}_+^K$. Thus, in the entire work we can assume $\mathcal{P} = \mathbb{R}_+^K$ without loosing the link to the real world networks with constrained power budgets. As a consequence of the equivalence to the optimization problems over $\mathbb{R}_+^K$, one can show that the constraint qualification holds for any optimization problem considered in this work [26]. Hence, for simplicity of formulation, the requirement of satisfied constraint qualification is omitted in each statement which needs this assumption.

We assume the receivers in the cell to be single-user receivers. We choose the link SIR (*signal-to-interference ratio*) as the function characterizing the link signal at the receiver output. Denoting each link SIR as γ_k, $1 \le k \le K$, we can write

$$\gamma_k = \gamma_k(\mathbf{p}) = \frac{p_k}{\sum_{i=1}^K V_{ki} p_i} = \frac{p_k}{(\mathbf{V}\mathbf{p})_k}, \quad 1 \le k \le K. \tag{1}$$

To exclude "pathological" interference scenarios, we make a nonrestrictive assumption that $\sum_{i=1}^K V_{ki} p_i > 0$, $1 \le k \le K$, for some $\mathbf{p} \in \mathcal{P}$. Each interference coefficient $V_{kl} \ge 0$ models the interference influence of the lth link signal on the kth link

[1] As usual, $\mathbb{R}_+^K$ denotes the K-dimensional nonnegative orthant and $\mathbb{R}_{++}^K$ is its interior, that is, the K-dimensional positive orthant.

receiver, $k \neq l$. The resulting interference matrix $\mathbf{V}$, which describes the interference coupling within the network, is defined as

$$(\mathbf{V})_{kl} = \begin{cases} V_{kl} & k \neq l, \\ 0 & k = l, \end{cases} \quad 1 \leq k,\, l \leq K. \tag{2}$$

Independently of the system realization, all factors V_{kl} include the influence of channels. In particular linear receiver systems, the factors V_{kl} depend additionally on other factors, for example, on aperiodic cross-correlations of sequences in the CDMA case [3], on beamforming type and beamforming filter coefficients in the MISO (multiple-input single-output) downlink case [27], on spatial receiver type and spatial filter coefficients in the SIMO (single-input multiple-output) case [28]. The interference matrix is nonnegative and we denote its spectral radius as $\rho(\mathbf{V})$ and its left and right Perron-Frobenius eigenvectors (PF eigenvectors) as $\mathbf{l} = \mathbf{l}(\mathbf{V})$ and $\mathbf{r} = \mathbf{r}(\mathbf{V})$, respectively. Note that we do not assume here the normalization of the PF eigenvectors to $\|\mathbf{r}\|_2 = \|\mathbf{l}\|_2 = 1$ in general. Vectors $\mathbf{l}$, $\mathbf{r}$ are included in the left and right PF eigenmanifolds, which we denote as $\mathcal{L} = \mathcal{L}(\mathbf{V}) = \{\mathbf{x} \neq 0 : \mathbf{V}^T\mathbf{x} = \rho(\mathbf{V})\mathbf{x}\}$ and $\mathcal{R} = \mathcal{R}(\mathbf{V}) = \{\mathbf{x} \neq 0 : \mathbf{V}\mathbf{x} = \rho(\mathbf{V})\mathbf{x}\}$, respectively, where $\mathcal{L}, \mathcal{R} \subseteq \mathbb{R}_+^K$ is obvious from the nonnegativity of $\mathbf{V}$ [29].

Some remarks on the SIR model

The link SIR can be considered to take the role of the usual SINR (*signal-to-interference-and-noise ratio*) function in the case when at each receiver $1 \leq k \leq K$ the *multiple access interference* (MAI) power, or simply interference power, $\sum_{i=1}^K V_{ki} p_i$, dominates the variance σ_k^2 of the Gaussian noise perceived at the output of the receiver. Thus, the SIR model can correspond to an asymptotic SINR model in the regime of high received powers (both the received own link powers and the interference powers). On the other side, the use of the SIR model is justified in networks, which utilize transceivers with especially low-noise figures, since then the received noise variance at each receiver output is likely to be low in relation to the corresponding MAI power. Low-noise figure can be expected in specialized transceiver designs with high-end components. Finally, the use of SIR model for network optimization purposes might be suitable in the case when the noise variances σ_k^2, $1 \leq k \leq K$, or the noise figures of all receivers $1 \leq k \leq K$ are not known to the network control unit (which is usually at the base station). In such case the assumption $\sigma_k^2 = 0$, $1 \leq k \leq K$, which gives rise to the SIR model, is one of the options how the network control unit can handle the lack of the noise knowledge in power control. The SIR-based considerations constitute a significant part within the established theory of power control, see, for example, [6, 9] and references therein.

We group the link QoS parameters of interest, for example, the data rate, the bit error rate under some fixed code, and so forth, in the QoS vector $\mathbf{q} = (q_1, \ldots, q_K)$. We assume each link QoS parameter to be associated with the corresponding link SIR by the relation

$$q_k = q_k(\gamma_k) = F\left(\frac{1}{\gamma_k}\right), \tag{3}$$

where $F : \mathbb{R}_{++} \to I \subseteq \mathbb{R}$ is an increasing, continuously differentiable bijection. Clearly, from the increase of F follows the decrease of the QoS-SIR function $q_k(\gamma_k)$. It is further easy to see with (1) that this implies the decrease of the resulting QoS-power function $q_k(\gamma_k(\mathbf{p})) = F((\mathbf{V}\mathbf{p})_k/p_k)$ in the corresponding link power p_k, $1 \leq k \leq K$. The introduced dependence (3) is special, but applies to any QoS parameter which is expressible as a monotone function of the SIR. For instance, the function $F(x) = -B\log(1 + x^{-1})$, with B as the system bandwidth, gives rise to $-q_k(\gamma) = B\log(1+\gamma)$, which is the data rate in Gaussian channel.[2] Similarly, the function $F(x) = cx^a$, with $a \in \mathbb{N}_+$ and some system-dependent constant c, corresponds to $q_k(\gamma) = c/\gamma^a$, which is the channel-averaged bit error rate (slope) in fading Gaussian channel under receiver diversity a.

Due to bijectivity of functions (1) and (3), the power region $\mathcal{P}$ characterizes one-to-one the set of achievable QoS vectors. We denote such set as $\mathcal{Q}_F = \{\mathbf{q}(\mathbf{p}) = (q_1(\mathbf{p}), \ldots, q_K(\mathbf{p})) : \mathbf{p} \in \mathcal{P}\}$, and refer to it as the QoS region.

3. FAIRNESS AND UTILITY

The optimization of an aggregated utility and ensuring some notion of fairness among the links are intuitively incompatible goals. However, depending on the fairness and utility definition, further strong relations between both goals can be recognized.

3.1. Min-max fairness and proportional fairness

The analysis of fairness issues in networks has its origin in the framework of wired networks [2, 13, 14]. Although we are free to define specialized notions of fairness for particular networks of interest, two fundamental fairness principles are established. These principles give rise to the majority of related fairness notions applicable to different network types (wired/wireless), different network topologies (cellular/ad-hoc networks), and different QoS parameters (e.g., the end-to-end delay in multihop ad hoc networks or data rate in cellular networks).

The first fairness principle is referred to in this work as min-max fairness and consists in making the worst QoS parameter (of a route, link, etc.) as good as possible. In wired networks the min-max fair equilibrium of QoS parameters is the one at which no QoS parameter q_i can be improved without degradation of any QoS parameter q_j, $j \neq i$, which is

[2] The sign of the considered QoS parameters has to be chosen so that $q_k(\gamma_k)$, $1 \leq k \leq K$, are decreasing, since we consider minimization problems in the remainder. Hence, QoS parameters being nondecreasing functions of SIR have to be taken with the minus sign.

already inferior to q_i [13–18, 25]. The same definition translates usually to the case of wireless multihop ad hoc networks, when the QoS parameters are associated with routes (end-to-end QoS) [11, 12].

Some remarks on denoting the fairness as min-max

The fairness principle referred to here as min-max fairness is equivalent to the notion of max-min fairness in the references and in the majority of related literature. Nevertheless, we chose here a different convention to comply with the fact that the problem of ensuring this notion of fairness (i.e., maximally improving the worst QoS parameter) takes the min-max form. This problem form is actually caused by our assumption that the QoS parameter in (3) is an increasing function of inverse SIR, and thus a decreasing function of the corresponding resource (transmit power). Consequently, it is desired to minimize each QoS parameter and the worst parameter value is the maximal one. The difference in fairness results precisely from the fact that the majority of references assumes the increase of the QoS parameter as the function of the corresponding resource. Hence, the desired optimization principle there is of max-min type.

The formulation of the problem of ensuring min-max fairness as an optimization problem is prohibited in wired networks by the network topology constraints, and precisely by the existence of so-called bottleneck links [15, 16, 25]. Similarly, in considerations of end-to-end QoS in wireless multihop ad hoc networks such formulation is prohibited by the natural constraints on the routing policy [12]. For the considered cellular network model with minimum per-link service requirements $\mathbf{q}^{\text{req}}$, we are able to formulate the min-max criterion in the obvious form

$$\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} \frac{q_k(\mathbf{p})}{q_k^{\text{req}}} = \inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} \frac{F((\mathbf{V}\mathbf{p})_k/p_k)}{F(1/\gamma_k^{\text{req}})}, \tag{4}$$

where $\gamma_k^{\text{req}} = 1/F^{-1}(q_k^{\text{req}})$, $1 \le k \le K$, are the SIR requirements (see [5] for the special case $q_k = 1/\gamma_k$). The incorporation of link-specific requirements/weights in (4) lets us refer to the fairness notion arising from (4) as the weighted min-max fair one. This parallels the fairness definition in [12] with respect to end-to-end QoS. The pure min-max fairness neglects unequal per-link requirements and corresponds to the special case $\mathbf{q}^{\text{req}} = c\mathbf{1}$, $\mathbf{1} := (1,\dots,1)$, $c > 0$. In the behavioral and economic science such notion parallels ideal social fairness [1]. The (pure) min-max fairness is analyzed in the remainder.

In the following proposition we provide a simple extension of the Collatz-Wielandt min-max formula for the Perron root. The Collatz-Wielandt formulae are two characterizations, in min-max and max-min problem forms, of the spectral radius of a nonnegative matrix. For the basics we refer here to [29]. The proposition is fundamental for all the characterizations in the remainder.

Proposition 1. *For any interference matrix* $\mathbf{V}$ *and any increasing bijection* F*, one has*

$$\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V})), \tag{5}$$

where $F((\mathbf{V}\mathbf{r})_i/r_i) = F(\rho(\mathbf{V}))$, $1 \le i \le K$ *whenever* $\mathbf{r} > 0$.

Since Proposition 1 is essential for the considerations in Section 7, we defer the proof of it to Section 7, where the proposition is proven. With increasing F, the optimization approach (5) is interpretable as improving the worst link QoS parameter as much as possible. In analogy, we can think of a goal of degrading the best link QoS performance as much as possible. This can be formulated as $\sup_{\mathbf{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F((\mathbf{V}\mathbf{p})_k/p_k)$. In analogy, it is intuitive to refer to such optimization approach as to ensuring max-min fairness. (Notice that the notion of max-min fairness introduced here should not be confused with the notion of max-min fairness used in the given references. The latter notion corresponds to the notion of min-max fairness in this paper; see the remarks given above.) One is tempted to ask if (or when) the notions of min-max fairness and max-min fairness coincide. This problem is in the focus of Section 7.

It may misleadingly appear that any solution to (5) is a min-max fair allocation. This is not always the case. Precisely, the following subtlety has to be accounted for. By the definition of the infimum it follows from (5) that for any accuracy $\epsilon > 0$, there exists a power vector $\mathbf{p}(\epsilon) > 0$, which is ϵ-near the solution, precisely $F((\mathbf{V}\mathbf{p}(\epsilon))_k/p_k(\epsilon)) \le F(\rho(\mathbf{V}))+\epsilon$. If the accuracy is increased according to $\epsilon \to 0$, the existence of some link subset $\mathcal{K} \subset \{1,\dots,K\}$, such that $\mathbf{p}(0) = \lim_{\epsilon\to 0}\mathbf{p}(\epsilon) = \mathbf{r}$ with $r_k = 0$, $k \in \mathcal{K}$, cannot be excluded in general. This means that although the link SIR values $\gamma_k(\mathbf{r})$, $k \in \mathcal{K}$, are positive and finite at the optimum of (5), they in fact represent the limits of ratios with numerator and denominator both approaching zero. In other words, the links $k \in \mathcal{K}$ are practically shut off, while their associated SIR values are formally positive. Consequently, we cannot speak of $\gamma_k(\mathbf{r})$, $1 \le k \le K$, as of an achieved tuple of SIRs in the network and consequently, any allocation $\mathbf{r}$ with zero components cannot be regarded as a valid allocation in real world networks. Due to this fact, in [30, 31] such SIR tuples, which are given by (1) under not (strictly) positive power vectors, are referred to as ineffective. Clearly, when $\mathbf{r} > 0$ exists, then no such difficulty is encountered and $\mathbf{r}$ is implied by (5) to be valid and min-max fair. Hence, we can summarize as follows.

Observation 1. The infimum in (5) is attained if and only if there exists some right PF eigenvector $\mathbf{r} > 0$. In such case $\mathbf{r}$ is a min-max fair allocation.

Observation 2. Any right PF eigenvector $\mathbf{r}$, which does not satisfy $\mathbf{r} > 0$, is not a valid allocation.

Remark 1. In the context of nonvalid allocations $\mathbf{r}$, it is important to notice that an allocation $\mathbf{r} > 0$ is always valid, regardless how small its elements are. This is a consequence of

the multiplicative homogeneity of the SIR function, that is, $\mathbf{V}\mathbf{p}_k/p_k = c\mathbf{V}\mathbf{p}_k/cp_k$, $c > 0$. Thus, an arbitrarily small allocation $\mathbf{r} > 0$ is equivalent in terms of the SIR to a suitably upscaled allocation $c\mathbf{r} > 0$ (within $\mathcal{P}$). In other words, in considerations relying on the SIR model, the relations of link powers within an allocation are sufficient to determine the resulting SIR tuple.

For completeness, we have to address in short the second fairness principle, which was introduced in [13] and is referred to as proportional fairness. This notion was established originally for wired networks, but is meanwhile well understood also in the wireless context. The proportional fair equilibrium $\mathbf{q}^{\mathrm{pf}}$ of QoS parameters is the one at which the difference to any other QoS vector $\mathbf{q}$ measured in the aggregated proportional change is nonnegative.[3] Precisely, with our model $\mathbf{q}^{\mathrm{pf}}$ being a proportional fair QoS vector if $\sum_{k=1}^{K}((q_k - q_k^{\mathrm{pf}})/q_k^{\mathrm{pf}}) \geq 0$, $\mathbf{q} \in \mathcal{Q}_F$. Interestingly, proportional fairness corresponds to the optimum of a specific utility function (Section 3.2) with logarithmic QoS parameters [32, 33]. The motivation for the formulation of the proportional fairness principle was the observed significant utility inefficiency (emphatic preferential treatment of small network flows [13, 14]) of a min-max fair allocation in wired networks. The conclusion of Section 6 is an analogy of this behavior.

3.2. Utility optimization

Complying with the established terminology, we refer to the (global) utility as to the aggregation of link- or route-specific utilities. The optimization of utility of this form is a usual bandwidth/rate sharing approach for wired networks [13–16, 18, 25] and one of possible scheduling approaches in wireless multihop ad hoc networks [12, 20]. In both cases the single utilities are associated with routes from different sources. Clearly, in our context of cellular networks, the single utilities are associated with links and correspond simply to link QoS parameters (see also [21–23] and references therein). The arising utility optimization problem takes then the form

$$\inf_{\mathbf{p} \in \mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right), \quad \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K) \in \mathcal{A}, \tag{6}$$

with $\boldsymbol{\alpha}$ as the vector of link priority/weight factors from the set of weight vectors

$$\mathcal{A} := \{\boldsymbol{\alpha} \geq 0 : \|\boldsymbol{\alpha}\|_1 = 1\}. \tag{7}$$

The utility-based scheduling approach (6), which aims at the optimization of some global performance measure, stands in opposition to the traditional power control approach, which aims at the most power-efficient achievement of minimum required QoS for each link. The latter approach is well understood and extensively studied in a huge framework, see, for example, [3–10] and references therein. The traffic type for which the utility-based scheduling is favorable is sometimes referred to illustratively as *elastic*, since no fixed per-link requirements have to be accounted for.

It is worth noting that there is a specific form of the utility optimization problem, which is sometimes of special interest. This is the case when the weights in the utility are chosen as linear functions of buffer occupancies on the source nodes of the corresponding links (cellular case), or routes (multihop case), and the QoS parameters express the capacity of the corresponding links/routes. It was shown originally in [34] (see also [35–37]) that the optimization of such utility provides the largest stability region of the network. Hereby, the size of the stability region of the network can be seen, in broad terms, as a measure of robustness of the network with respect to arrival rates of bursty traffic on the physical layer [38].

3.3. The trade-off of min-max fairness and utility optimality

For particular wired networks, min-max fairness and utility optimality of bandwidth sharing schemes were shown in [16, 18, 19] to be incompatible goals. However, such incompatibility is in general strongly topology-dependent. This follows from [25], where the corresponding conditions for compatibility/incompatibility were stated and some examples of min-max fair and utility-optimal schemes were constructed. A kind of similar incompatibility was observed in [12] in the context of wireless multihop ad hoc networks. To the best of our knowledge, the trade-off between min-max fairness and utility optimality has not been studied yet for cellular networks.

We restrict our analysis to the following class of functions F.

Definition 1. Given some interference matrix $\mathbf{V}$, the function F is included in the class $\mathcal{E}(\mathbf{V})$ if and only if the problem (6) is well defined for any $\boldsymbol{\alpha} \in \mathcal{A}$ and all locally optimal power allocations in the problem (6) are also globally optimal.

Definition 1 indicates that the class $\mathcal{E}(\mathbf{V})$ is the class of QoS parameters, which allows for efficient online utility optimization, since for $F \in \mathcal{E}(\mathbf{V})$ locally converging iterative methods applied to (6) exhibit global convergence.[4] Given some $\mathbf{V}$, a complete characterization of the class $\mathcal{E}(\mathbf{V})$ remains an open question. However, for the cases of individual per-link power constraints (usually as in the uplink) and sum power constraint (usually as in the downlink) the characterization of a specific subclass of $\mathcal{E}(\mathbf{V})$ follows from [30, 39, 40]. The following proposition is a modified restatement of the results from [39], [40, Theorem 3], and [30, Lemma 2].[5]

[3] Clearly, in the case of QoS parameters increasing in service quality the nonnegativity condition has to be replaced by the nonpositivity condition.

[4] Under some nonrestrictive technical conditions [26].

[5] The proposition is slightly modified compared to the references, since in [39, 40] a different SIR-QoS relation $q_k = F(\gamma_k)$ is analyzed.

Proposition 2. *Let the class of increasing, continuously differentiable functions F be defined as $\mathcal{F} := \{F : G(q) := 1/F^{-1}(q)$ is log-convex$\}$. Then,*

(i) *$F \in \mathcal{F}$ if and only if $F_e(x) := F(e^{-x})$ is convex,*
(ii) *for any $\mathbf{V}$ such that the solution to (6) exists for any $\boldsymbol{\alpha} \in \mathcal{A}$, one has $\mathcal{F} \subset \mathcal{E}(\mathbf{V})$,*
(iii) *$\mathcal{Q}_F$ is a convex set.*

Subclass $\mathcal{F}$ includes a number of functions of great use for QoS considerations. Two prominent members of $\mathcal{F}$ are the following.

(i) $F(x) = cx^a$, $a \in \mathbb{N}_+$, $c > 0$, giving rise to the QoS parameter $q_k(\gamma) = c/\gamma^a$, which is the channel-averaged bit error rate in fading Gaussian channel under receiver diversity a.
(ii) $F(x) = B\log(x)$, with B as the system bandwidth, giving rise to the QoS parameter $-q_k(\gamma) = B\log(\gamma)$, which is the approximation of the data rate in Gaussian channel for large γ.

4. MIN-MAX FAIR AND UTILITY OPTIMAL ALLOCATION: THE UNIQUENESS CASE

We first concentrate on so-called entirely interference-coupled networks. These are networks with a specific form of coupling of links by interference. The coupling of links is in such case described by an irreducible interference matrix. Let the interference graph be defined as a $\mathbf{V}$-dependent directed graph on the node set $\{1,\ldots,K\}$, which has an edge (i,j) whenever $V_{ij} > 0$. Then, irreducibility of $\mathbf{V}$ is equivalent to the property that any pair of nodes in the corresponding interference graph is joined by a path [31, 41]. For the interpretation of irreducibility in terms of the canonical form of $\mathbf{V}$ see Appendix A.1.

For an entirely coupled network there exists a unique power and weight allocation, which combines min-max fairness and utility optimality. This is shown in the following proposition.

Proposition 3. *For an irreducible interference matrix $\mathbf{V}$, let $F \in \mathcal{E}(\mathbf{V})$ and $\mathbf{w} = (w_1,\ldots,w_K)$, $w_k := r_k l_k$, $1 \le k \le K$. Then the following are true.*

(i) $\mathbf{r}, \mathbf{l} > 0$, and $\mathbf{r}, \mathbf{l}$ are unique up to a scaling constant.

(ii) $\mathbf{r} = \arg\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F((\mathbf{V}\mathbf{p})_k/p_k)$ if and only if $\boldsymbol{\alpha} = \mathbf{w}$.

(iii) The equality

$$\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V})) \tag{8}$$

is satisfied if and only if $\boldsymbol{\alpha} = \mathbf{w}$, with $\mathbf{w}$ unique in $\mathcal{A}$.

Proof. (i) Follows directly from the properties of nonnegative irreducible matrices [29].

(ii) With $F \in \mathcal{E}(\mathbf{V})$ a power vector solves equation (6) if and only if it satisfies the Karush-Kuhn-Tucker (KKT) conditions for equation (6). From the definition of $\mathcal{P}$, the property $\gamma_k(c\mathbf{p}) = \gamma_k(\mathbf{p})$, $c > 0$, and bijectivity of F follows $\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F((\mathbf{V}\mathbf{p})_k/p_k) = \min_{\mathbf{p}\in\mathbb{R}_+^K} \sum_{k=1}^{K} \alpha_k F((\mathbf{V}\mathbf{p})_k/p_k)$. Hence, the KKT conditions for (6) correspond to the gradient set to zero, which yields

$$\sum_{\substack{j=1\\ j\neq k}}^{K} \alpha_j F'\left(\frac{(\mathbf{V}\mathbf{p})_j}{p_j}\right)\frac{V_{jk}}{p_j} = \alpha_k F'\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right)\frac{(\mathbf{V}\mathbf{p})_k}{p_k^2}, \quad 1 \le k \le K. \tag{9}$$

With the definition $\boldsymbol{\beta}(\boldsymbol{\alpha},\mathbf{p}) := (\alpha_1/p_1, \alpha_2/p_2, \ldots, \alpha_K/p_K)$ we can write (9) in an equivalent matrix form

$$(\mathbf{F}'(\mathbf{p})\mathbf{V})^T \boldsymbol{\beta}(\boldsymbol{\alpha},\mathbf{p}) = \mathbf{F}'(\mathbf{p})\boldsymbol{\Gamma}^{-1}(\mathbf{p})\boldsymbol{\beta}(\boldsymbol{\alpha},\mathbf{p}), \tag{10}$$

with $\mathbf{F}'(\mathbf{p}) := \operatorname{diag}(F'((\mathbf{V}\mathbf{p})_1/p_1),\ldots,F'((\mathbf{V}\mathbf{p})_K/p_K))$ and $\boldsymbol{\Gamma}(\mathbf{p}) := \operatorname{diag}(p_1/(\mathbf{V}\mathbf{p})_1,\ldots,p_K/(\mathbf{V}\mathbf{p})_K)$. By the definition of the right PF eigenvector we can write

$$\frac{r_k}{(\mathbf{V}\mathbf{r})_k} = \frac{1}{\rho(\mathbf{V})}, \quad 1 \le k \le K. \tag{11}$$

Hence, with the definitions of $\mathbf{F}'$ and $\boldsymbol{\Gamma}$, setting $\mathbf{p} = \mathbf{r}$ in the optimality condition (10) yields (for (11)),

$$\mathbf{V}^T \boldsymbol{\beta}(\boldsymbol{\alpha},\mathbf{r}) = \rho(\mathbf{V})\boldsymbol{\beta}(\boldsymbol{\alpha},\mathbf{r}). \tag{12}$$

This implies immediately $\boldsymbol{\beta}(\boldsymbol{\alpha},\mathbf{r}) = \mathbf{l}$ which, by the definition, is equivalent to $\boldsymbol{\alpha} = \mathbf{w}$ and completes the proof of the *if* part of (ii). For the *only if* part assume by contradiction that $\mathbf{r}$ satisfies the KKT conditions for some $\boldsymbol{\alpha} \neq \mathbf{w}$. This means that (12) is satisfied for some $\boldsymbol{\beta}(\boldsymbol{\alpha},\mathbf{r}) \neq \mathbf{l}$, which is a contradiction and completes the proof of (ii). (iii) From part (ii), the fact that $\|\mathbf{w}\|_1 = 1$ (since $\mathbf{w} \in \mathcal{A}$ by definition), and (11), we have

$$\begin{aligned}\min_{\mathbf{p}\in\mathcal{P}} \sum_{k=1}^{K} w_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) &= \sum_{k=1}^{K} w_k F\left(\frac{(\mathbf{V}\mathbf{r})_k}{r_k}\right)\\ &= \sum_{k=1}^{K} w_k F(\rho(\mathbf{V})) = F(\rho(\mathbf{V})).\end{aligned} \tag{13}$$

The uniqueness of $\mathbf{w}$ in $\mathcal{A}$ follows directly from its definition and the uniqueness property (i). To show that $\mathbf{w}$ is the only vector in $\mathcal{A}$ satisfying (13), assume by contradiction that (8) is satisfied for some $\boldsymbol{\alpha} \neq \mathbf{w}$. Then, by (11) and $\boldsymbol{\alpha} \in \mathcal{A}$ we have that $\mathbf{r}$ is still a minimizer. This further yields with (ii) that $\boldsymbol{\alpha} = \mathbf{w}$, which is a contradiction and completes the proof of (iii). □

The obvious part (i) of the proposition means that for entirely interference-coupled networks the min-max fair allocation exists and is unique (up to a scaling constant). Part (ii) says that a min-max fair allocation is utility optimal for the specific weight vector $\mathbf{w}$, corresponding to component-wise product of PF eigenvectors of the interference matrix. Such weighting is unique in the normalized class $\mathcal{A}$ due to the uniqueness of the eigenvectors of an irreducible matrix. Moreover, the min-max fair allocation is strictly utility

suboptimal for any other weight vector. Precisely, we have from part (ii),

$$\sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{r})_k}{r_k}\right) > \min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right), \quad \boldsymbol{\alpha} \neq \mathbf{w}. \tag{14}$$

Summarizing, we can state what follows.

Observation 3. Under entire interference coupling in the network, the power and weight allocation $(\mathbf{r}, \mathbf{w})$ combines utility optimality and min-max fairness, and any other power and weight allocation in $\{\mathbf{v} : \|\mathbf{v}\|_1 = c\} \times \mathcal{A}$, for any $c > 0$, is either not min-max fair or utility suboptimal, or both.

From the practical point of view it has to be noted that the uniqueness of the min-max fair and utility optimal weight and power allocation in $\{\mathbf{v} : \|\mathbf{v}\|_1 = c\} \times \mathcal{A}$ is a disadvantage. This is because to achieve fairness and utility optimality at least approximatively, it is necessary that the weights of links be determined by some vector in a sufficiently small neighborhood of a specific unique vector $\mathbf{w}$. If however there is a degree of freedom in choosing the weights for the links (and thus the optimization over the weight vectors can be taken into account), Observation 3 becomes interesting also from the view of practical power and weight control.

5. MIN-MAX FAIR AND UTILITY OPTIMAL ALLOCATION: THE GENERAL CASE

The characterization from Proposition 3 does not hold if the network is not entirely interference-coupled. For such case, even the existence of a min-max fair allocation is not ensured, since some $\tilde{\mathbf{r}} \in \mathcal{R}$, $\tilde{\mathbf{r}} > 0$, may not exist (Observation 1) [29]. In a general network, not necessarily entirely interference-coupled, the existence of interference-decoupled link pairs is allowed. Equivalently, the corresponding interference graph may include some pair of nodes which is not joined by a path [41]. In terms of the representation of $\mathbf{V}$ in the canonical form, this means that the network can be partitioned into two or more subnetworks which are entirely interference-coupled in themselves and, in general, interfere with each other (see Appendix A).

The characterization of the trade-off of min-max fairness and utility optimality, which generalizes Proposition 3 to the case of arbitrary networks, is as follows.

Proposition 4. *Let $F \in \mathcal{E}(\mathbf{V})$ and $\mathcal{W} := \{\hat{\mathbf{w}} = (\hat{w}_1, \ldots, \hat{w}_K) \in \mathcal{A} : \hat{w}_k = \hat{r}_k \hat{l}_k, \hat{\mathbf{r}} = (\hat{r}_1, \ldots, \hat{r}_K) \in \mathcal{R}, \hat{\mathbf{l}} = (\hat{l}_1, \ldots, \hat{l}_K) \in \mathcal{L}\}$. Then, the following are true.*

(i) *For any $\tilde{\mathbf{r}} \in \mathcal{R}$, $\tilde{\mathbf{r}} = \arg\inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F((\mathbf{V}\mathbf{p})_k/p_k)$ if and only if $\boldsymbol{\alpha} \in \mathcal{W}$.*

(ii) *The equality*

$$\inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V})) \tag{15}$$

is satisfied if and only if $\boldsymbol{\alpha} \in \mathcal{W}$.

Proof. (i) The proof is a straightforward generalization of the proof of Proposition 3(ii), with $\mathbf{r}$ replaced by any $\tilde{\mathbf{r}} \in \mathcal{R}$, due to the nonuniqueness of PF eigenvectors for general matrices $\mathbf{V}$. (ii) Construct a matrix $\mathbf{V}_\epsilon = \mathbf{V} + \epsilon \mathbf{1}\mathbf{1}^T$, $\epsilon > 0$. From the construction follows that $\mathbf{V}_\epsilon$ is irreducible for any $\epsilon > 0$ (because it is positive for any $\epsilon > 0$). We have

$$\frac{(\mathbf{V}_\epsilon \mathbf{p})_k}{p_k} = \frac{(\mathbf{V}\mathbf{p})_k}{p_k} + \epsilon \frac{\|\mathbf{p}\|_1}{p_k}, \quad \mathbf{p} \in \mathcal{P},\ 1 \le k \le K. \tag{16}$$

From the increase of F we have $F((\mathbf{V}_\epsilon \mathbf{p})_k/p_k) \ge F((\mathbf{V}\mathbf{p})_k/p_k)$, $1 \le k \le K$. Let $\tilde{\mathbf{w}}(\epsilon) \in \mathcal{A}$ be some parameterized vector. Since $\mathcal{A}$ is compact, there exist sequences $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \epsilon_n = 0$ and

$$\lim_{n\to\infty} \|\tilde{\mathbf{w}}(\epsilon_n) - \tilde{\mathbf{w}}\| = 0 \tag{17}$$

for some vector $\tilde{\mathbf{w}} \in \mathcal{A}$. Choose any such sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$. With continuity of the spectral radius as a function of matrix elements, Proposition 3(iii), and the increase of F it follows then

$$\begin{aligned} F(\rho(\mathbf{V})) &= \lim_{n\to\infty} F(\rho(\mathbf{V}_{\epsilon_n})) \\ &= \lim_{n\to\infty} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k(\epsilon_n) F\left(\frac{(\mathbf{V}_{\epsilon_n}\mathbf{p})_k}{p_k}\right) \\ &\ge \lim_{n\to\infty} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k(\epsilon_n) F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \\ &= \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right). \end{aligned} \tag{18}$$

On the other side we can also write

$$\begin{aligned} &\inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k(\epsilon_n) F\left(\frac{(\mathbf{V}_{\epsilon_n}\mathbf{p})_k}{p_k}\right) \\ &\quad = \inf_{\mathbf{p}\in\mathcal{P}_{++}} \left(\sum_{k=1}^{K} (\tilde{w}_k(\epsilon) - \tilde{w}_k) F\left(\frac{(\mathbf{V}_{\epsilon_n}\mathbf{p})_k}{p_k}\right) \right. \\ &\qquad + \sum_{k=1}^{K} \tilde{w}_k \left(F\left(\frac{(\mathbf{V}_{\epsilon_n}\mathbf{p})_k}{p_k}\right) - F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \right) \\ &\qquad \left. + \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \right). \end{aligned} \tag{19}$$

The first two sums on the right-hand side of (19) can be upper bounded using the Cauchy-Schwarz inequality and the bounds disappear with $n \to \infty$ due to (16) and (17). Hence, for the limit transition we get

$$\begin{aligned} F(\rho(\mathbf{V})) &= \lim_{n\to\infty} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k(\epsilon_n) F\left(\frac{(\mathbf{V}_{\epsilon_n}\mathbf{p})_k}{p_k}\right) \\ &\le \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right). \end{aligned} \tag{20}$$

Inequalities (20) and (18) together imply now $F(\rho(\mathbf{V})) = \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k F((\mathbf{V}\mathbf{p})_k/p_k)$ for $\tilde{\mathbf{w}} \in \mathcal{W}$. The if and only

if property in (ii) parallels the if and only if property in Proposition 3(iii). Thus, the proof of the if and only if property is analogous to the corresponding proof in Proposition 3(iii). □

Hence, one can say that the characterization of the trade-off for entirely coupled networks translates to the general network case, except the uniqueness property. Thus, Propositions 3 and 4 can be summarized as follows. Whenever a min-max fair allocation (i.e., a PF eigenvector $\tilde{\mathbf{r}} \in \mathcal{R}$, $\tilde{\mathbf{r}} > 0$) exists, then any such allocation remains utility optimal for specific weight vectors constituting set $\mathcal{W}$. Moreover, for any weight vector not in $\mathcal{W}$ any min-max fair allocation, if existent, remains strictly utility suboptimal, that is,

$$\sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{r})_k}{r_k}\right) > \inf_{\mathbf{p} \in \mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right), \quad \boldsymbol{\alpha} \notin \mathcal{W}. \tag{21}$$

In the particular case of entire interference coupling, the sets $\mathcal{W}$ and $\{\mathbf{v} : \|\mathbf{v}\|_1 = c\} \cap \mathcal{R}$, $c > 0$, become singletons so that the min-max fair power and weight allocation exists and is unique on $\{\mathbf{v} : \|\mathbf{v}\|_1 = c\} \cap \mathcal{A}$, $c > 0$. Hence, together with Observation 1, we can extend Observation 3 as follows.

Observation 4. Any power and weight allocation $(\tilde{\mathbf{r}}, \tilde{\mathbf{w}})$, satisfying $\tilde{\mathbf{r}} \in \mathcal{R} \cap \mathbb{R}^K_{++}$ and $\tilde{\mathbf{w}} \in \mathcal{W}$, combines utility optimality and min-max fairness. Whenever $\hat{\mathbf{r}} \in \mathcal{R}$ and $\hat{\mathbf{r}} \notin \mathbb{R}^K_{++}$, then $(\hat{\mathbf{r}}, \hat{\mathbf{w}})$ is not a power and weight allocation. Whenever $\hat{\mathbf{r}} \notin \mathcal{R}$ or $\hat{\mathbf{w}} \notin \mathcal{W}$, then the power and weight allocation $(\hat{\mathbf{r}}, \hat{\mathbf{w}})$ either does not achieve min-max fairness or is utility suboptimal, or both.

The nonuniqueness of the power and weight allocation $(\hat{\mathbf{r}}, \hat{\mathbf{w}}) \in \mathcal{R}^K_{++} \times \mathcal{W}$ makes Observation 4 practically more relevant than Observation 3. In the restricted case of entirely coupled networks, fairness and utility optimality is approximatively achievable under a power and weight allocation from a neighborhood of $(\hat{\mathbf{r}}, \hat{\mathbf{w}})$, which is unique in $\{\mathbf{v} : \|\mathbf{v}\|_1 = c\} \times \mathcal{A}$ (Observation 3). As is implied by Observation 4, in the general case of interference coupling, to achieve this goal it suffices to choose a power and weight allocation from the neighborhood of the entire set $\in \mathcal{R}^K_{++} \times \mathcal{W}$. Thus, in the general case it is more likely that some weight vector from the neighborhood of $\mathcal{W}$ is suitable for the link priorities on hand. If this is the case, the choice of a power vector from the neighborhood of the set $\in \mathcal{R}^K_{++}$ allows for the achievement of fairness and utility optimality concurrently.

5.1. Existence of a min-max fair allocation

Recall from Section 4 that in entirely coupled networks a min-max fair allocation exists and is additionally unique. In this section we characterize the class of all networks, including in particular the class of entirely coupled networks, for which a min-max fair allocation is existent. The characterization is in terms of the canonical form of the interference matrix. The result is a straightforward consequence of [31, Theorem 3], which can be restated for our purposes in the following equivalent form. (In the remainder we denote by $\mathcal{I}$ and $\mathcal{M}$ the sets of isolated and maximal diagonal blocks of an interference matrix. See Appendix A for the definitions of isolation, maximality, and other issues related to the canonical form.)

Proposition 5. *Let $\{\mathbf{V}^{(n)}\}_{n\in\mathcal{I}}$ and $\{\mathbf{V}^{(m)}\}_{m\in\mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\mathbf{V}$, respectively. Matrix $\mathbf{V}$ has a right* PF *eigenvector $\tilde{\mathbf{r}} \in \mathcal{R}$ satisfying $\tilde{\mathbf{r}} > 0$ if and only if $\mathcal{I} = \mathcal{M}$.*

The isolation property of some diagonal block in $\mathbf{V}$ is equivalent to the isolation of the corresponding subnetwork from the interference from other subnetworks (Appendix A). Analogously, the nonisolated blocks correspond to subnetworks which include some nodes which perceive interference from some nodes in other subnetworks. Since the distinguished subnetworks are entirely interference-coupled in itself, we can interpret Proposition 5 as follows.

Observation 5. A min-max fair allocation exists for any network with interference matrix $\mathbf{V}$ such that

(i) the interference matrix $\mathbf{V}^{(n)}$ of each interference-isolated and entirely coupled subnetwork $n \in \mathcal{I}$ satisfies $\rho(\mathbf{V}^{(n)}) = \rho(\mathbf{V})$,

(ii) the interference matrix $\mathbf{V}^{(m)}$ of each entirely coupled subnetwork $m \in \{1, \ldots, K\} \setminus \mathcal{I}$ perceiving interference from some other entirely coupled subnetwork satisfies $\rho(\mathbf{V}^{(m)}) < \rho(\mathbf{V})$. For any network violating either (i) or (ii) no min-max fair allocation exists.

It is clear that the values of spectral radii $\rho(\mathbf{V}^{(n)})$, $1 \le n \le N$, are determined solely by the interference coupling, so that the fulfillment of the conditions (i), (ii) in Observation 5 cannot be influenced by link powers and weights. Thus, except the fact that we know that $\rho(\mathbf{V}^{(n)}) = \rho(\mathbf{V})$ for some n, the prediction of the probability that (i) and (ii) are satisfied in a real world network requires some assumptions on the distribution of the interference coefficients in the entire network. Under some specific assumptions, the probability that (i) and (ii) are satisfied might be quantified by means of the general results on eigenvalue distribution of random matrices (e.g., with [42]). This is however a topic for a separate treatment and cannot be addressed in this work. This remark holds also for all the results in the remainder which concern the relations of spectral radii of interference matrices of subnetworks.

It is worth pointing out an interesting relation between the min-max fair allocation for the entire network and for its entirely interference-coupled subnetworks. Denote the left and right eigenvectors of the nth diagonal block of the interference matrix $\mathbf{V}$ as $\mathbf{l}^{(n)}$ and $\mathbf{r}^{(n)}$, respectively, and notice that both are unique up to a scaling constant due to the irreducibility of each diagonal block. From the eigenvalue equation for the canonical form of $\mathbf{V}$ it is then easy to see that the eigenvectors $\mathbf{l}^{(n)}$, $\mathbf{r}^{(n)}$ of any isolated and maximal diagonal block $\mathbf{V}^{(n)}$ (if existent) correspond to the projections of any $\tilde{\mathbf{l}} \in \mathcal{L}$ and $\tilde{\mathbf{r}} \in \mathcal{R}$, respectively, on the subspace with

dimensions restricted to the diagonal block $\mathbf{V}^{(n)}$. Precisely,

$$\begin{aligned}(\tilde{r}_{k_1(n)}, \tilde{r}_{k_1(n)+1}, \ldots, \tilde{r}_{k_M(n)}) &= \mathbf{r}^{(n)}, \quad \tilde{\mathbf{r}} \in \mathcal{R}, \\ (\tilde{l}_{k_1(n)}, \tilde{l}_{k_1(n)+1}, \ldots, \tilde{l}_{k_M(n)}) &= \mathbf{l}^{(n)}, \quad \tilde{\mathbf{l}} \in \mathcal{L},\end{aligned} \tag{22}$$

whenever the diagonal block of $\mathbf{V}^{(n)}$ is isolated and maximal, and corresponds to the components $k_1(n) \le l \le k_M(n)$, with $1 \le k_1(n), k_M(n) \le K$ in the matrix $\mathbf{V}$. We can interpret this property as follows.

Observation 6. Let the network satisfy (i) and (ii) in Observation 5. Then, any min-max fair allocation for an entirely interference-coupled and interference-isolated subnetwork corresponds to the restriction of the min-max fair allocation for the entire network to such subnetwork.

Clearly, the eigenvalue equation implies also that the projection property (22) cannot hold for nonisolated diagonal blocks of $\mathbf{V}$.

5.2. Existence of a positive weight allocation

The set $\mathcal{W}$ of utility optimal and min-max fair weight allocations is in general not guaranteed to include positive weight allocations. In fact, even for networks satisfying (i), (ii) in Observation 5, the existence of $\tilde{\mathbf{l}} \in \mathcal{L}, \tilde{\mathbf{l}} > 0$ is not ensured, so that the construction of $\tilde{\mathbf{w}} \in \mathcal{W}$, such that $\tilde{\mathbf{w}} > 0$, may be prevented. Therefore, the characterization of the class of networks for which a positive utility optimal and min-max fair weight allocation exists is here of interest. It is clear from the construction of $\mathcal{W}$ that such class must be included in the class of networks having some $\tilde{\mathbf{r}} \in \mathcal{R}$, $\widetilde{\mathrm{ver}} > 0$, which is characterized in Proposition 5. The corresponding characterization follows straightforwardly from [41] or, equivalently, from [31, Theorems 3 and 4].

Proposition 6. *Let $\{\mathbf{V}^{(m)}\}_{m\in\mathcal{M}}$ be the set of maximal diagonal blocks in the canonical form of the interference matrix $\mathbf{V}$. Matrix $\mathbf{V}$ has right and left PF eigenvectors $\tilde{\mathbf{l}} \in \mathcal{R}$, $\tilde{\mathbf{l}} \in \mathcal{L}$ satisfying $\tilde{\mathbf{l}}, \tilde{\mathbf{r}} > 0$ if and only if it is block-irreducible and $\mathcal{M} = \{1, \ldots, N\}$.*

The existence of positive left and right PF eigenvectors following from above proposition makes the construction of a weight vector $\tilde{\mathbf{w}} \in \mathcal{W} \cap \mathbb{R}^K_{++}$ possible. Proposition 6 characterizes a subclass of interference matrices from Proposition 5, for which $\mathcal{I} = \mathcal{M} = \{1, \ldots, N\}$, that is, for which no nonisolated diagonal blocks exist. We can interpret Proposition 6 as follows.

Observation 7. A positive utility optimal and min-max fair weight allocation exists for any network with interference matrix $\mathbf{V}$ such that

(i) the network consists of a number of entirely interference coupled and pairwise interference-isolated subnetworks,

(ii) the interference matrix $\mathbf{V}^{(n)}$ of each entirely coupled subnetwork satisfies $\rho(\mathbf{V}^{(n)}) = \rho(\mathbf{V})$. For any network violating either (i) or (ii), no positive utility optimal and min-max fair weight allocation exists.

Obviously, the entirely interference-coupled networks are the trivial case of networks satisfying (i), (ii) in Observation 7, as they formally consist of one entirely interference-coupled subnetwork.

Some remarks on the role of block irreducibility for utility optimization

The networks with the properties characterized in Observation 7 (i.e., with interference matrices characterized in Proposition 6) play a specific role not only in terms of the trade-off between min-max fairness and utility optimality. Such networks have also a specific property of the QoS region, which we describe here briefly. As a slight difference to Proposition 6 and Observation 7, the discussion below concerns a weighted interference matrix.

From [31] we know that the QoS region $\mathcal{Q}_F$ can be represented alternatively as

$$\mathcal{Q}_F = \left\{\mathbf{q} = \left(F\left(\frac{1}{\gamma_1}\right), \ldots, F\left(\frac{1}{\gamma_K}\right)\right) : \rho(\mathbf{\Gamma V}) \le 1\right\}, \tag{23}$$

with $\mathbf{\Gamma} := \mathrm{diag}(\gamma_1, \ldots, \gamma_K)$. From the normal form of the interference matrix we have further

$$\rho(\mathbf{\Gamma V}) = \max_{1\le n\le N} \rho\left(\mathbf{\Gamma}^{(n)}\mathbf{V}^{(n)}\right), \tag{24}$$

where the diagonal components of $\mathbf{\Gamma}^{(n)}$ are γ_l, with $k_1(n) \le l \le k_M(n)$ as the interval of components corresponding to the diagonal block $\mathbf{V}^{(n)}$. Consequently it follows that $\mathcal{Q}_F = \times_{n=1}^N \mathcal{Q}_F^{(n)}$, with $\mathcal{Q}_F^{(n)} = \{\mathbf{q}^{(n)} = (F(1/\gamma_{k_1(n)}), \ldots, F(1/\gamma_{k_M(n)})) : \rho(\mathbf{\Gamma}^{(n)}\mathbf{V}^{(n)}) \le c(n)\}$, $1 \le n \le N$, where for the constant $c(n)$ we have $c(n) \le 1$, $1 \le n \le N$, due to (23) and (24). In other words, QoS region of the network is the Cartesian product of QoS regions of entirely coupled subnetworks. By the one-to-one correspondence $\mathbf{q}(\mathbf{p})$ (on $\{\mathbf{p} : \|\mathbf{p}\|_1 = c\}$, $c > 0$) we can get the link between the utility optimization in the form (6) and the utility optimization with $\mathcal{Q}_F$ as the optimization domain. Precisely, we have

$$\begin{aligned}\min_{\mathbf{q}\in\mathcal{Q}_F} \sum_{k=1}^K \alpha_k q_k &= \sum_{n=1}^N \min_{\mathbf{q}(n)\in\mathcal{Q}_F^{(n)}} \sum_{l=k_1(n)}^{k_M(n)} \alpha_l q_l \\ &= \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^K \alpha_k F\left(\frac{(\mathbf{Vp})_k}{p_k}\right), \quad \boldsymbol{\alpha} \in \mathcal{A}.\end{aligned} \tag{25}$$

Assume now $\boldsymbol{\alpha} > 0$ and notice that the minimum of the partial objective $\sum_{l=k_1(n)}^{k_M(n)} \alpha_l q_l$ is achieved on the boundary of the QoS region $\mathcal{Q}_F^{(n)}$, $1 \le n \le N$. Consequently, whenever there exists some subnetwork n, such that $c(n) < 1$, the corresponding partial objective $\sum_{l=k_1(n)}^{k_M(n)} \alpha_l q_l$ achieves a value which is strictly suboptimal compared to the case when $c(n) = 1$ holds for subnetwork n. Consequently, the optimal partial utility values in all subnetworks, and hence the overall optimal network utility value, are achievable exactly

in the case when all weighted subnetwork interference matrices $\mathbf{\Gamma}^{(n)}\mathbf{V}^{(n)}$, $1 \le n \le N$, correspond to maximal diagonal blocks of $\mathbf{\Gamma V}$, that is,

$$\rho(\mathbf{\Gamma}^{(n)}\mathbf{V}^{(n)}) = 1, \quad 1 \le n \le N. \tag{26}$$

In other words, in some sense the farthest boundary part of the QoS region $\mathcal{Q}_F$ is achievable in the utility optimization exactly when (26) is true.

6. THE TRADE-OFF BETWEEN MIN-MAX FAIRNESS AND UTILITY OPTIMALITY AS A SADDLE POINT

In the last section we showed that the power and weight allocations of the form $(\tilde{\mathbf{r}}, \tilde{\mathbf{w}})$, $\tilde{\mathbf{r}} \in \mathcal{R}$, $\tilde{\mathbf{w}} \in \mathcal{W}$, combine min-max fairness and utility optimality. In this section we assume that the link weights are variables and study the problems of minimization/maximization of utility over weight vectors from the set $\mathcal{A}$. This approach is followed in order to illustrate the relation of the power and weight allocation combining fairness and utility optimality with general power and weight allocations. In this way we are able to characterize the mechanism of the trade-off occurring under combination of fairness and utility optimality. Precisely, we prove that such trade-off has the interpretation of a saddle point of the utility function as a function of power and weight allocations. For this purpose we need to consider two problem forms, a min-max problem and a max-min problem.

6.1. The min-max problem

Consider first the problem of utility optimization for a worst-case weight vector. In such case we have the following property.

Lemma 1. *Let* $\mathbf{V}$ *be any interference matrix and let* $F \in \mathcal{E}(\mathbf{V})$. *Then*

$$\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{\boldsymbol{\alpha}\in\mathcal{A}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{Vp})_k}{p_k}\right) = F(\rho(\mathbf{V})), \tag{27}$$

with $\tilde{\mathbf{r}} = \arg\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{\boldsymbol{\alpha}\in\mathcal{A}} \sum_{k=1}^{K} \alpha_k F((\mathbf{Vp})_k/p_k)$, $\tilde{\mathbf{r}} \in \mathcal{R}$. *If* $\mathbf{V}$ *is irreducible, then* $\mathbf{r} > 0$ *is the unique (up to a scaling constant) vector satisfying*

$$\mathbf{r} = \arg \min_{\mathbf{p}\in\mathcal{P}_{++}} \max_{\boldsymbol{\alpha}\in\mathcal{A}} \sum_{k=1}^{K} \alpha_k F((\mathbf{Vp})_k/p_k). \tag{28}$$

Proof. It is clear that $\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{\boldsymbol{\alpha}\in\mathcal{A}} \sum_{k=1}^{K} \alpha_k F((\mathbf{Vp})_k/p_k) = \inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F((\mathbf{Vp})_k/p_k)$, $\boldsymbol{\alpha} \in \mathcal{A}$. With Proposition 1 it follows further that

$$\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\mathbf{Vp})_k}{p_k}\right) = F\left(\frac{(\mathbf{V}\tilde{\mathbf{r}})_k}{\tilde{r}_k}\right) = F(\rho(\mathbf{V})), \quad \tilde{\mathbf{r}} \in \mathcal{R}. \tag{29}$$

By Proposition 3(i) in the special case of irreducible $\mathbf{V}$ there is an up to a scaling constant unique vector $\mathbf{r} > 0$, and the proof is completed. □

Lemma 1 characterizes the right PF eigenvectors of $\mathbf{V}$ as those which optimize the utility function for the worst-case vector of weights. Equivalently, the min-max fair allocation $\tilde{\mathbf{r}} \in \mathcal{R}$, $\tilde{\mathbf{r}} > 0$, (which exists whenever the interference matrix $\mathbf{V}$ satisfies (i), (ii) in Observation 5) is the optimal power vector when a weight vector in $\mathcal{A}$ is chosen which yields the largest value of the utility. For entirely coupled networks the lemma shows that given a worst-case weight vector, the utility optimum is achieved under a min-max fair allocation and under no other allocation.

6.2. The max-min problem

In what follows we denote the utility function as a function of powers and weights as

$$U : \mathcal{P} \times \mathcal{A} \longrightarrow J \subseteq \mathbb{R}, \qquad U(\mathbf{p}, \boldsymbol{\alpha}) = \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{Vp})_k}{p_k}\right) \tag{30}$$

and additionally

$$U_{\mathbf{p}} : \mathcal{A} \longrightarrow J \subseteq \mathbb{R}, \qquad U_{\mathbf{p}}(\boldsymbol{\alpha}) = \min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{Vp})_k}{p_k}\right). \tag{31}$$

For the utility function (31) we have first the following insight.

Lemma 2. *Let* $\mathbf{V}$ *be any irreducible interference matrix and let* $F \in \mathcal{E}(\mathbf{V})$. *Then,* $U_{\mathbf{p}}$ *is strictly concave.*

Proof. Function $U_{\mathbf{p}}$ is concave by definition, due to the properties of the minimum function [43]. Assume now by contradiction that $U_{\mathbf{p}}$ is not strictly concave. Hence, there exist $\boldsymbol{\alpha}^{(1)}, \boldsymbol{\alpha}^{(2)}$, $\boldsymbol{\alpha}^{(1)} \ne \boldsymbol{\alpha}^{(2)}$ such that

$$\begin{aligned} &U_{\mathbf{p}}((1-t)\boldsymbol{\alpha}^{(1)} + t\boldsymbol{\alpha}^{(2)}) \\ &\quad = (1-t)U_{\mathbf{p}}(\boldsymbol{\alpha}^{(1)}) + tU_{\mathbf{p}}(\boldsymbol{\alpha}^{(2)}), \quad \text{for some } t \in (0,1). \end{aligned} \tag{32}$$

As a first case assume that

(i) if $\mathbf{p}^{(1)} = \arg\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k^{(1)} F((\mathbf{Vp})_k/p_k)$ and $\mathbf{p}^{(2)} = \arg\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k^{(2)} F((\mathbf{Vp})_k/p_k)$, then $\mathbf{p}^{(1)} \ne \mathbf{p}^{(2)}$. Let $\mathbf{p}(t) := \arg\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} ((1-t)\alpha_k^{(1)} + t\alpha_k^{(2)}) F((\mathbf{Vp})_k/p_k)$. Then,

$$\begin{aligned} &U_{\mathbf{p}}((1-t)\boldsymbol{\alpha}^{(1)} + t\boldsymbol{\alpha}^{(2)}) \\ &\quad = \sum_{k=1}^{K} ((1-t)\alpha_k^{(1)} + t\alpha_k^{(2)}) F\left(\frac{(\mathbf{Vp}(t))_k}{p_k(t)}\right) \\ &\quad = (1-t) \sum_{k=1}^{K} \alpha_k^{(1)} F\left(\frac{(\mathbf{Vp}(t))_k}{p_k(t)}\right) \\ &\qquad + t \sum_{k=1}^{K} \alpha_k^{(2)} F\left(\frac{(\mathbf{Vp}(t))_k}{p_k(t)}\right). \end{aligned} \tag{33}$$

Hence, (32) and (33) together imply that

$$\mathbf{p}(t) = \arg\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k^{(1)} F((\mathbf{V}\mathbf{p})_k/p_k) \tag{34}$$

and $\mathbf{p}(t) = \arg\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k^{(2)} F((\mathbf{V}\mathbf{p})_k/p_k)$. This contradicts (i) and completes the proof under assumption (i). Assume now the complementary case:

(ii) there exists $\tilde{\mathbf{p}} \in \mathcal{P}$, such that

$$\begin{aligned}\tilde{\mathbf{p}} &= \arg\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k^{(1)} F((\mathbf{V}\mathbf{p})_k/p_k)\\ &= \arg\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k^{(2)} F((\mathbf{V}\mathbf{p})_k/p_k).\end{aligned} \tag{35}$$

With (1) and (3), and the assumption $\boldsymbol{\alpha}^{(1)} \neq \boldsymbol{\alpha}^{(2)}$, it follows that (ii) corresponds to the vertex property of the point $\mathbf{q}(\tilde{\mathbf{p}})$, which is on the boundary of the QoS region $\mathcal{Q}_F$. This implies that at the point $\mathbf{q}(\tilde{\mathbf{p}})$ the Frechet derivative is not defined. The boundary of the QoS region $\mathcal{Q}_F$ can be bijectively mapped, by means of the componentwise mapping F, onto the boundary of the manifold $\{\boldsymbol{\Gamma}(\mathbf{p}) = \operatorname{diag}(\gamma_1(\mathbf{p}),\ldots,\gamma_K(\mathbf{p})) : \mathbf{p} \in \mathcal{P}\}$. Such boundary is known to be representable as the manifold $\{\boldsymbol{\Gamma}(\mathbf{p}) : \rho(\boldsymbol{\Gamma}(\mathbf{p})\mathbf{V}) = 1\}$ [31]. Since the spectral radius is a smooth function of matrix elements and F is continuously Frechet differentiable by our assumptions, the boundary of $\mathcal{Q}_F$ must be Frechet differentiable. This contradicts the existence of a vertex on the boundary of $\mathcal{Q}_F$. Hence, condition (ii) is never satisfied and the proof is completed. □

With the above lemma we can provide a max-min characterization, which is complementary to the max-min characterization from Lemma 1. For clarity, we split the presentation into the one for entirely interference-coupled networks only and the one generalizing it to arbitrary networks.

Proposition 7. *Let* $\mathbf{V}$ *be an irreducible interference matrix and let* $F \in \mathcal{E}(\mathbf{V})$. *Then*

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V})), \tag{36}$$

and $\tilde{\boldsymbol{\alpha}} = \arg\max_{\boldsymbol{\alpha}\in\mathcal{A}} \min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F((\mathbf{V}\mathbf{p})_k/p_k)$ *if and only if* $\tilde{\boldsymbol{\alpha}} = \mathbf{w}$, *with* $\mathbf{w} = (w_1,\ldots,w_K)$, $w_k = l_k r_k$, $1 \le k \le K$, *which is unique in* $\mathcal{A}$.

Proof. It is clear that

$$\sum_{k=1}^{K} \alpha_k F((\mathbf{V}\mathbf{p})_k/p_k) \le \max_{1\le k\le K} F((\mathbf{V}\mathbf{p})_k/p_k), \tag{37}$$

$\mathbf{p} \in \mathcal{P}$, $\boldsymbol{\alpha} \in \mathcal{A}$. This yields with Proposition 1 that

$$\min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \le \min_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V})), \tag{38}$$

where we can take the minimum instead of the infimum in (5), since $\mathbf{r} > 0$ due to irreducibility. Inequality (38) is further equivalent to

$$U_{\mathbf{p}}(\boldsymbol{\alpha}) \le F(\rho(\mathbf{V})), \quad \boldsymbol{\alpha} \in \mathcal{A}. \tag{39}$$

By Lemma 2, function $U_{\mathbf{p}}$ is strictly concave under irreducibility of $\mathbf{V}$, and thus has a unique maximum. With the definition of $U_{\mathbf{p}}$ and Proposition 3(iii) it follows then with (39),

$$\begin{aligned}\max_{\boldsymbol{\alpha}\in\mathcal{A}} U_{\mathbf{p}}(\boldsymbol{\alpha}) &= \max_{\boldsymbol{\alpha}\in\mathcal{A}} \min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right)\\ &= \min_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{\alpha}_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V})),\end{aligned} \tag{40}$$

if and only if $\tilde{\boldsymbol{\alpha}} = \mathbf{w}$, with vector $\mathbf{w} = (w_1,\ldots,w_K)$, $w_k = r_k l_k$, $1 \le k \le K$, which is unique in $\mathcal{A}$. This completes the proof. □

The proposition states that for entirely coupled networks the weight vector $\mathbf{w}$, unique in $\mathcal{A}$, is the one for which the optimum utility value achieved is the worst possible, that is, largest. Moreover, for any other weight vector the achieved optimum utility value is smaller, that is, the optimal utility performance is better. At this point notice a crucial but subtle difference between the min-max and max-min considerations in Lemma 1 and Proposition 7, respectively. Precisely, between utility optimality under worst-case weights (Lemma 1) and worst-case weights for the utility optimum (Proposition 7).

The generalization of Proposition 7 to arbitrary networks is as follows.

Proposition 8. *Let* $\mathbf{V}$ *be any interference matrix and let* $F \in \mathcal{E}(\mathbf{V})$. *Then*

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V})), \tag{41}$$

with $\tilde{\boldsymbol{\alpha}} = \arg\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F((\mathbf{V}\mathbf{p})_k/p_k)$ *if and only if* $\tilde{\boldsymbol{\alpha}} \in \mathcal{W} := \{\hat{\mathbf{w}} = (\hat{w}_1,\ldots,\hat{w}_K) \in \mathcal{A} : \hat{w}_k = \hat{r}_k\hat{l}_k, \hat{\mathbf{r}} = (\hat{r}_1,\ldots,\hat{r}_K) \in \mathcal{R}, \hat{\mathbf{l}} = (\hat{l}_1,\ldots,\hat{l}_K) \in \mathcal{L}\}$.

Proof. As in the proof of Proposition 4 we construct an irreducible (since positive) matrix $\mathbf{V}_\epsilon = \mathbf{V} + \epsilon\mathbf{1}\mathbf{1}^T$, $\epsilon > 0$. Hence, (16) is true and implies with the increase of F that

$$\sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \le \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}_\epsilon\mathbf{p})_k}{p_k}\right), \quad \boldsymbol{\alpha} \in \mathcal{A},\ \mathbf{p} \in \mathcal{P}. \tag{42}$$

This further implies with Proposition 7,

$$\begin{aligned}&\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right)\\ &\quad\le \max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}_\epsilon\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V}_\epsilon)).\end{aligned} \tag{43}$$

The left-hand side of (43) does not depend on ϵ. Hence, taking the limit of both sides of (43) by letting $\epsilon \to 0$ yields with continuity of the spectral radius as a function of matrix elements

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \le F(\rho(\mathbf{V})). \tag{44}$$

From Proposition 4(ii) we further have

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F((\mathbf{V}\mathbf{p})_k/p_k) \ge F(\rho(\mathbf{V})). \tag{45}$$

Hence, together with (44) the equality

$$\begin{aligned}\max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \\ = \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{\alpha}_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V}))\end{aligned} \tag{46}$$

for some $\tilde{\boldsymbol{\alpha}} \in \mathcal{A}$ is true. By Proposition 4(ii) again, it follows that the equality holds if and only if $\tilde{\boldsymbol{\alpha}} \in \mathcal{W}$, which completes the proof. □

The generalization of Proposition 7 by Proposition 8 is analogous to the generalization of Proposition 3 by Proposition 4. It implies that for arbitrary networks any weight vector from the specific set $\mathcal{W}$, which is a singleton under irreducibility of the interference matrix, makes the achieved optimum utility value the worst among all weight vectors in $\mathcal{A}$. The optimum utility value achievable under any weight vector from outside of $\mathcal{W}$ is superior to the one achieved for $\tilde{\boldsymbol{\alpha}} \in \mathcal{W}$.

6.3. The saddle point conclusion

With Proposition B.2 in the appendix it is now easily seen that the min-max and max-min relations from Lemma 1 and Propositions 7, 8 describe together a saddle point of the utility function as a function of weight and power vectors. We can formulate the following corollary.

Corollary 1. *Let* $\mathbf{V}$ *be any interference matrix and let* $F \in \mathcal{E}(\mathbf{V})$. *Then, any vector pair* $(\tilde{\mathbf{r}}, \tilde{\mathbf{w}})$ *from the set* $\mathcal{S} := \{(\hat{\mathbf{r}}, \hat{\mathbf{w}}) \in \mathcal{P} \times \mathcal{A} : \hat{\mathbf{r}} \in \mathcal{R}, \hat{\mathbf{w}} \in \mathcal{W}\}$, *with* $\mathcal{W}$ *defined as in Propositions 4 and 8, is a saddle point of the utility* U *from* (30) *and one has*

$$\begin{aligned}\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{\boldsymbol{\alpha}\in\mathcal{A}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \\ = F(\rho(\mathbf{V})) = \max_{\boldsymbol{\alpha}\in\mathcal{A}} \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right).\end{aligned} \tag{47}$$

If the interference matrix is irreducible, then $\mathcal{S}$ *is a singleton representing a unique saddle point of* U.

It is an easy consequence of Corollary 1 and Observation 4 that the set of min-max fair and utility-optimal power and weight allocations corresponds to the subset $\mathcal{S} \cap (\mathbb{R}^K_{++} \times \mathcal{A})$ of the set of saddle points $\mathcal{S}$. Recall that such subset is nonempty if and only if the network pertains to a class of networks satisfying (i), (ii) in Observation 5.

The saddle point property is a compact illustration of the trade-off between fairness in the min-max sense and utility optimality. It shows that the min-max fair allocation $\tilde{\mathbf{r}} \in \mathcal{R}$, $\tilde{\mathbf{r}} > 0$ (if existent), optimizes the utility under the penalty that the worst possible weight vector is chosen. Consequently, for any nonworst-case choice of the weight vector the optimum performance in terms of the utility is better than the one achieved by min-max fair allocation. On the other side, any weight vector $\tilde{\mathbf{w}} \in \mathcal{W}$ corresponding to the weight allocation achieving min-max fairness and utility-optimality has the property of yielding the worst-case utility performance among the utility optima achieved under any weight vector. Thus, under any choice $\tilde{\mathbf{w}} \notin \mathcal{W}$, the achieved optimal utility performance is better than under the choice $\tilde{\mathbf{w}} \in \mathcal{W}$. These features can be expressed compactly by the chain inequality

$$\begin{aligned}\inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \\ \le \sum_{k=1}^{K} \alpha_k F\left(\frac{(\mathbf{V}\tilde{\mathbf{r}})_k}{\tilde{r}_k}\right) \le \inf_{\mathbf{p}\in\mathcal{P}_{++}} \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \\ = \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\mathbf{V}\tilde{\mathbf{r}})_k}{\tilde{r}_k}\right) \le \sum_{k=1}^{K} \tilde{w}_k F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right), \quad (\tilde{\mathbf{r}}, \tilde{\mathbf{w}}) \in \mathcal{S}.\end{aligned} \tag{48}$$

The chain inequality (48) contains the relation

$$\begin{aligned}\sum_{k=1}^{K} \alpha_k F((\mathbf{V}\tilde{\mathbf{r}})_k/\tilde{r}_k) \le \sum_{k=1}^{K} \tilde{w}_k F((\mathbf{V}\tilde{\mathbf{r}})_k/\tilde{r}_k) \\ \le \sum_{k=1}^{K} \tilde{w}_k F((\mathbf{V}\mathbf{p})_k/p_k),\end{aligned} \tag{49}$$

which is, by Definition B.1 in the appendix, another verification of the saddle point property of any vector pair $(\tilde{\mathbf{r}}, \tilde{\mathbf{w}}) \in \mathcal{S}$.

The saddle point property can be summarized as follows.

Observation 8. For any power and weight allocation $(\tilde{\mathbf{r}}, \tilde{\mathbf{w}})$, $\tilde{\mathbf{r}} > 0$ combining utility optimality and min-max fairness, the following are true.

(i) The min-max fair allocation $\tilde{\mathbf{r}} > 0$ yields the optimal utility value under the worst-case choice of the weight vector.

(i′) If the network is entirely interference-coupled, then the min-max fair allocation $\tilde{\mathbf{r}} > 0$ is the unique (up to a scaling constant) allocation yielding the optimal utility value under the worst-case choice of the weight vector.

(ii) Under the weight vector $\tilde{\mathbf{w}}$ the worst-case of the optimum utility value is yielded.

(ii′) If the network is entirely interference-coupled, the weight vector $\tilde{\mathbf{w}}$ is the unique weight vector in $\mathcal{A}$ under which the worst-case of the optimum utility value is yielded.

In Figure 1 we illustrate the saddle point property of a min-max fair and utility-optimal power and weight

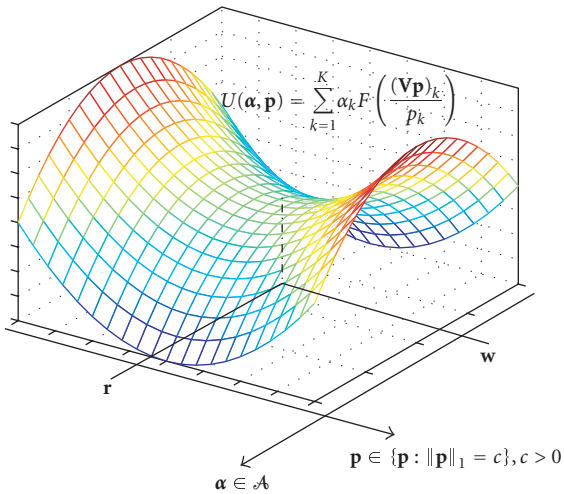

Figure 1: Figurative visualization of the saddle point property of the min-max fair and utility-optimal power and weight allocation. The two scalar dimensions represent the spaces of power allocations and weight allocations, respectively. The visualized saddle point is unique, as would be the case for entirely interference-coupled networks.

allocation. The visualization is figurative since the vector dimensions corresponding to the power vector and the weight vector are represented by two scalar dimensions.

7. THE FAIRNESS INEQUALITY

Up to now the focus of our considerations was on the notion of fairness in the min-max sense

$$\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right). \tag{50}$$

Under increasing function F, problem (50) can be interpreted as improving the worst link QoS performance as much as possible. An interesting question in this context is the relation to the problem

$$\sup_{\mathbf{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right), \tag{51}$$

which is in some way dual to finding the min-max fair allocation. Problem (51) can be interpreted as degrading the best link QoS performance as much as possible and hence, by analogy to (50) and to the notion of min-max fairness, we propose referring to such problem as ensuring max-min fairness, since the corresponding optimization problem takes a max-min form (recall the remarks on denoting fairness notions from Section 3.1: the max-min fairness defined by (51) is not equivalent to the usual max-min fairness in the literature.). From this interpretation of max-min fairness (51) it is apparent that the applicability of this fairness notion is in general limited. In fact, providing the maximal degradation of the best link QoS is intuitively not a notion of optimality desired in the network (it is rather a notion of a worst case).

However, the situation changes if the notions of min-max fairness and max-min fairness are related by some known deterministic relation. In particular, there is an interest in achieving max-min fairness (51) if it coincides, in terms of the achieved value or even in terms of the optimizers, with the notion of min-max fairness (50). In such case max-min fairness (51) is an alternative characterization/interpretation of the common notion of min-max fairness. Precisely this issue of coincidence is addressed in the remainder of this section.

From convex analysis we have that for any network, precisely for any interference matrix $\mathbf{V}$,

$$\sup_{\mathbf{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \le \inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right). \tag{52}$$

Inequality (52) suggests the following question. *For which class of networks (i.e., matrices* $\mathbf{V}$*) we have the following.*

Property i. The optimal values in problems (50) and (51) coincide.

However, even under equality of optimal values, the optimizers of (50) and (51) may not coincide. Therefore we are also interested in the answer to the following related question. *For which class of networks we have the following.*

Property i′. The optimal values in problems (50) and (51) coincide and there exists a positive allocation which solves both (50) and (51)?

By complementarity to Property i, a question is also: *for which class of networks we have the following.*

Property ii. The optimal value in problem (50) is larger than the optimal value in problem (51)?

Assume for a while that a min-max fair allocation exists, that is, the network satisfies conditions (i), (ii) in Observation 5. Then, the networks having Property i can be regarded as those having no gap between min-max and max-min fairness performance, or simply having no (or zero) *fairness gap*. In other words, the maximally degraded best link QoS performance exactly meets the performance of the maximally improved worst link QoS in such networks. For networks with no fairness gap which have a stronger Property i′ we are additionally free to choose between (50) and (51) as equivalent problem formulations. This provides an alternative in the design of online optimization routines. Depending on hardware constraints, signaling constraints and protocol type, the alternative formulation (51) may happen to be favorable in terms of implementation issues. On the other side, networks having Property ii can be interpreted as those with (nonzero) fairness gap. Thus, for such networks we know that the maximally degraded best link QoS performance is still superior to the maximally improved worst link QoS performance. In such networks, one cannot resort to (51) as an equivalent formulation of the min-max fairness problem for implementation purposes.

Some remarks on the maximally degraded best link QoS

Consider for a while max-min fairness in the SINR-based network model, that is, when the SIR expression (1) is replaced by $\gamma_k(\mathbf{p}) = p_k/(\mathbf{V}\mathbf{p}_k + \sigma_k^2)$, $1 \le k \le K$, with some additional nonzero background noise variance σ_k^2 on each link. Assuming (3) with increasing F, it can be deduced that the best link QoS performance is maximally degraded in the trivial case of all-zero power allocation $\mathbf{p} = \mathbf{0}$. When using the SIR model (1) however, this is no longer the case. Precisely, let some parameterized allocation $\mathbf{p}(\epsilon) \in \mathcal{P}$ converge to $\mathbf{p}(0) = \lim_{\epsilon\to 0}\mathbf{p}(\epsilon) = \mathbf{0}$. Then, all SIR values converge to finite values, each one representing a ratio of two values approaching zero. This is the same mechanism as the one described in Section 3.1 in the context of validity/nonvalidity of allocations. Consequently, we deduce that the optimal value in (51) is assumed by a max-min fair allocation which is in general not all-zero. In comparison with the SINR model this feature slightly contradicts the intuition. However, from the algorithmic point of view such feature may provide advantages compared to the SINR model and SINR-based power control. Precisely, if Property i′ is true, the already described degree of freedom occurs: the online optimization algorithms computing the min-max fair allocation can be designed to solve either of the two problems (50) or (51). Such degree of freedom cannot occur in the SINR-based power control.

7.1. The cases of zero and nonzero fairness gap

The first step towards the characterization of the network classes having zero and nonzero fairness gap is a simple lemma.

Lemma 3. *For any interference matrix* $\mathbf{V}$ *and any increasing bijection* F*, one has*

$$\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \ge F(\rho(\mathbf{V})). \tag{53}$$

Proof. Construct first a matrix $\mathbf{V}_\epsilon = \mathbf{V} + \epsilon\mathbf{1}\mathbf{1}^T$, $\epsilon > 0$, which is positive. Denote $\overline{f}(\mathbf{X},\mathbf{p}) := \max_{1\le k\le K}((\mathbf{X}\mathbf{p})_k/p_k)$. We have obviously $\overline{f}(\mathbf{V}_\epsilon,\mathbf{p}) \ge (\mathbf{V}_\epsilon\mathbf{p})_k/p_k$, $1 \le k \le K$, $\epsilon > 0$, for any $\mathbf{p} \in \mathcal{P}_{++}$. Hence, it follows $\overline{f}(\mathbf{V}_\epsilon,\mathbf{p})\mathbf{p} \ge \mathbf{V}_\epsilon\mathbf{p}$, $\epsilon > 0$, $\mathbf{p} \in \mathcal{P}_{++}$. Let $\mathbf{l}_\epsilon$ be the left PF eigenvector of $\mathbf{V}_\epsilon$ such that $\mathbf{l}_\epsilon^T(\mathbf{p})\mathbf{p} = 1$ (thus, $\mathbf{l}_\epsilon = \mathbf{l}_\epsilon(\mathbf{p})$, since in general $\mathbf{l}_\epsilon^T\mathbf{p} > 0$). Thus, we have

$$\begin{aligned}\overline{f}(\mathbf{V}_\epsilon,\mathbf{p}) &= \overline{f}(\mathbf{V}_\epsilon,\mathbf{p})\mathbf{l}_\epsilon^T\mathbf{p} \ge \mathbf{l}_\epsilon^T\mathbf{V}_\epsilon\mathbf{p} = \rho(\mathbf{V}_\epsilon)\mathbf{l}_\epsilon^T\mathbf{p}\\ &= \rho(\mathbf{V}_\epsilon), \quad \epsilon > 0,\ \mathbf{p} \in \mathcal{P}_{++}.\end{aligned} \tag{54}$$

Hence, with the definitions and the nondecrease of the spectral radius as a function of matrix elements it follows from (54),

$$\max_{1\le k\le K}\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k} + \frac{(\epsilon\mathbf{1}\mathbf{1}^T\mathbf{p})_k}{p_k}\right) \ge \rho(\mathbf{V}), \quad \epsilon > 0,\ \mathbf{p} \in \mathcal{P}_{++}. \tag{55}$$

Notice now that $F(\max_{1\le k\le K} x_k) = \max_{1\le k\le K} F(x_k)$, $1 \le k \le K$, due to the increase of F. Hence, transforming by F both sides of (55) and taking infimum over $\epsilon > 0$ and $\mathbf{p} \in \mathcal{P}_{++}$ of both sides of (55) yield $\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F((\mathbf{V}\mathbf{p})_k/p_k) \ge F(\rho(\mathbf{V}))$ and complete the proof. □

The lemma specifies a lower bound on the maximally degraded best link QoS performance. This bound can be shown to be tight in Proposition 1. This proposition and the subsequent results can be regarded as an extension of the known Collatz-Wieland characterization to the case of general nonnegative matrices [29].

Proposition 1 can now be proven, where $F((\mathbf{V}\tilde{\mathbf{r}})_i/\tilde{r}_i) = F(\rho(\mathbf{V}))$, $1 \le i \le K$ whenever $\tilde{\mathbf{r}} \in \mathcal{R}$, $\tilde{\mathbf{r}} > 0$.

Proof of Proposition 1. Let $\mathcal{S}(n) \subset \{1,\ldots,K\}$ denote the set of row/column indices corresponding to the nth diagonal block $\mathbf{V}^{(n)}$ in the normal form of $\mathbf{V}$. Let $\mathbf{p}(\lambda) \in \mathbb{R}_{++}^K$ be some vector associated with $\lambda > 0$, where $\lambda = \lambda(\epsilon)$ is a function of $\epsilon > 0$. The idea of the proof is the construction of a vector $\mathbf{p}(\lambda)$, which achieves $\max_{1\le k\le K} F((\mathbf{V}\mathbf{p}(\lambda))_k/p_k(\lambda)) = F(\rho(\mathbf{V}))$. Together with Lemma 3 this will yield the proof. Assume for the components of $\mathbf{p}(\lambda)$ that $p_k(\lambda) = \lambda r_k^{(n)}$, $k \in \mathcal{S}(n)$, whenever block $\mathbf{V}^{(n)}$ is maximal ($\mathbf{r}^{(n)}$ is the right eigenvector of the diagonal block $\mathbf{V}^{(n)}$ and we have $\mathbf{r}^{(n)} > 0$, $1 \le n \le N$, due to irreducibility of the diagonal blocks). Then we can write from the construction of the normal form of $\mathbf{V}$ for any maximal block $\mathbf{V}^{(n)}$,

$$\begin{aligned}\frac{(\mathbf{V}\mathbf{p}(\lambda))_k}{p_k(\lambda)} &= \frac{(\mathbf{V}^{(n)}\lambda\mathbf{r}^{(n)})_k}{\lambda r_k^{(n)}}\\ &\quad + \frac{\left(\sum_{m=1}^{n-1}\mathbf{V}^{(n,m)}\mathbf{p}^{(m)}(\lambda)\right)_k}{\lambda r_k^{(n)}}, \quad \lambda(\epsilon) > 0,\ k \in \mathcal{S}(n),\end{aligned} \tag{56}$$

where for any vector $\mathbf{p} \in \mathbb{R}_{++}^K$ the notation $\mathbf{p}^{(n)}$ means a vector in $\mathbb{R}_+^{|\mathcal{S}(n)|}$ with components $p_k^{(n)} = p_k$, $k \in \mathcal{S}(n)$. Define now $\mathbf{t}^{(n)}(\mathbf{p}) := \sum_{m=1}^{n-1}\mathbf{V}^{(n,m)}\mathbf{p}^{(m)}$, $1 \le n \le N$. Choose now $\lambda(\epsilon)$ such that $t_k^{(n)}(\mathbf{p}(\lambda))_k/\lambda r_k^{(n)} \le \epsilon$, $k \in \mathcal{S}(n)$, holds and transform both sides of (56) by F. Then we get from (56) due to maximality of block $\mathbf{V}^{(n)}$ and the increase of F,

$$\begin{aligned}F\left(\frac{(\mathbf{V}\mathbf{p}(\lambda))_k}{p_k(\lambda)}\right) &= F\left(\rho(\mathbf{V}) + \frac{t_k^{(n)}(\mathbf{p}(\lambda))}{\lambda r_k^{(n)}}\right)\\ &\le F(\rho(\mathbf{V}) + \epsilon), \quad \lambda(\epsilon) > 0,\ \epsilon > 0,\ k \in \mathcal{S}(n).\end{aligned} \tag{57}$$

Consider now nonmaximal blocks. For any nonmaximal block $\mathbf{V}^{(n)}$ assume for the components of $\mathbf{p}(\lambda)$ that $p_k(\lambda) = \tilde{r}_k$, $\tilde{\mathbf{r}} \in \mathcal{R}$, $k \in \mathcal{S}(n)$. Then, from the eigenvalue problem for the normal form of $\mathbf{V}$ follows for any nonmaximal block $\mathbf{V}^{(n)}$ that

$$\rho(\mathbf{V})\mathbf{p}^{(n)}(\lambda) = \mathbf{V}^{(n)}\mathbf{p}^{(n)}(\lambda) + \mathbf{t}^{(n)}(\mathbf{p}(\lambda)), \quad \lambda(\epsilon) > 0. \tag{58}$$

After restatement we get $\mathbf{p}^{(n)}(\lambda) = (\rho(\mathbf{V})\mathbf{I} - \mathbf{V}^{(n)})^{-1}\mathbf{t}^{(n)}(\mathbf{p}(\lambda))$, which implies with $\rho(\mathbf{V}^{(n)}) < \rho(\mathbf{V})$ and $\mathbf{t}^{(n)}(\mathbf{p}(\lambda)) \ge 0$ that $p_k^{(n)}(\lambda) = \tilde{r}_k > 0$, $\lambda(\epsilon) > 0$, $k \in \mathcal{S}(n)$ whenever block $\mathbf{V}^{(n)}$ is nonmaximal [29]. Hence, from componentwise division of

both sides of (58) by $p_k(\lambda)$ and transformation by increasing function F follows

$$F\left(\frac{(\mathbf{V}\mathbf{p}(\lambda))_k}{p_k(\lambda)}\right) = F\left(\rho(\mathbf{V}) - \frac{t_k^{(n)}(\mathbf{p}(\lambda))}{p_k(\lambda)}\right) \le F(\rho(\mathbf{V})), \quad \lambda(\epsilon) > 0,\ k \in \mathcal{S}(n), \tag{59}$$

for any nonmaximal block $\mathbf{V}^{(n)}$. Summarizing now (57) and (59) we have $\max_{1\le k\le K} F((\mathbf{V}\mathbf{p}(\lambda))_k/p_k(\lambda)) \le F(\rho(\mathbf{V}) + \epsilon)$, $\epsilon > 0$. Hence it must hold

$$\lim_{\epsilon\to 0} \max_{1\le k\le K} F((\mathbf{V}\mathbf{p}(\lambda(\epsilon)))_k/p_k(\lambda(\epsilon))) \le F(\rho(\mathbf{V})), \tag{60}$$

which together with Lemma 3 implies

$$\lim_{\epsilon\to 0} \max_{1\le k\le K} F((\mathbf{V}\mathbf{p}(\lambda(\epsilon)))_k/p_k(\lambda(\epsilon))) = \inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F((\mathbf{V}\mathbf{p})_k/p_k) = F(\rho(\mathbf{V})) \tag{61}$$

since the infimum over $\mathcal{P}_{++}$ equals the infimum over $\mathbb{R}^K_{++}$ due to $\gamma_k(\mathbf{p}) = \gamma_k(c\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^K_{++}$ $c > 0$. This proves (5). Further, for maximal diagonal blocks $\mathbf{V}^{(n)}$ we have by construction of $\mathbf{p}(\lambda)$ and irreducibility of the diagonal blocks in the normal form of $\mathbf{V}$ that $p_k(\lambda(0)) = \lim_{\epsilon\to 0} p_k(\lambda(\epsilon)) = \lim_{\epsilon\to 0} \lambda(\epsilon) r_k^{(n)} > 0$, $k \in \mathcal{S}(n)$. For nonmaximal diagonal blocks $\mathbf{V}^{(n)}$ we have $p_k(0) = \tilde{r}_k$, $k \in \mathcal{S}(n)$. Hence from the eigenvalue problem for the normal form of $\mathbf{V}$ follows $\mathbf{p}(\lambda) \in \mathcal{R}$. Consequently, we have $\mathbf{p}(\lambda(0)) > 0$ and we can then write $F((\mathbf{V}\mathbf{p}(0))_i/p_i(0)) = F(\rho(\mathbf{V}))$, $1 \le i \le K$, whenever $\tilde{r}_k > 0$, $k \in \mathcal{S}(n)$ for $\mathbf{V}^{(n)}$ nonmaximal. A sufficient condition for it is $\tilde{\mathbf{r}} \in \mathcal{R}$, $\tilde{\mathbf{r}} > 0$ (see Proposition 5). This completes the proof. □

Hence, as discussed in preceding sections, we have $\tilde{\mathbf{r}} \in \mathcal{R}$ as a vector solving (50) and $F(\rho(\mathbf{V}))$ as the optimum value in (50).

The following simple consequence of (52) follows by Proposition 1.

Corollary 2. *For any interference matrix* $\mathbf{V}$ *and any increasing bijection* F *holds*

$$\sup_{\mathbf{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \le F(\rho(\mathbf{V})). \tag{62}$$

To describe the network classes with zero and nonzero fairness gap it remains now to characterize the case in which $F(\rho(\mathbf{V}))$ is achieved, respectively not achieved, as an optimal value in (51). Strict inequality in (62) can be shown to be true for a particular network class.

Lemma 4. *Let* $\{\mathbf{V}^{(n)}\}_{n\in\mathcal{I}}$ *and* $\{\mathbf{V}^{(m)}\}_{m\in\mathcal{M}}$ *be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix* $\mathbf{V}$*, respectively. If there exists some* $n \in \mathcal{I}$ *such that* $n \notin \mathcal{M}$*, then for any increasing bijection* F*, one has*

$$\sup_{\mathbf{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) < F(\rho(\mathbf{V})). \tag{63}$$

Proof. Let $\mathbf{V}^{(n)}$ be such that $n \in \mathcal{I}$ and $n \notin \mathcal{M}$. Then from the construction of the normal form of $\mathbf{V}$ follows $(\mathbf{V}^{(n)}\mathbf{p})_k/p_k = (\mathbf{V}^{(n)}\mathbf{p}^{(n)})_k/p_k^{(n)}$, $\mathbf{p} \in \mathbb{R}^K_{++}$, $k \in \mathcal{S}(n)$, where $\mathcal{S}(n)$ and $\mathbf{p}^{(n)}$ are defined as in the proof of Proposition 1. Since the nonmaximal block $\mathbf{V}^{(n)}$ is irreducible it follows from the classical Collatz-Wielandt characterization and the increase of F that [29]

$$F\left(\sup_{\mathbf{p}\in\mathbb{R}^K_{++}} \min_{k\in\mathcal{S}(n)} \frac{(\mathbf{V}^{(n)}\mathbf{p}^{(n)})_k}{p_k^{(n)}}\right) = F(\rho(\mathbf{V}^{(n)})) < F(\rho(\mathbf{V})). \tag{64}$$

Clearly, with $\min_{1\le k\le K} (\mathbf{V}\mathbf{p})_k/p_k \le \min_{k\in\mathcal{S}(n)} (\mathbf{V}^{(n)}\mathbf{p}^{(n)})_k/p_k^{(n)}$, $\mathbf{p} \in \mathbb{R}^K_{++}$, $1 \le n \le N$, and using the increase of F five times, let us write

$$\begin{aligned}
&\sup_{\mathbf{p}\in\mathbb{R}^K_{++}} \min_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \\
&= \sup_{\mathbf{p}\in\mathbb{R}^K_{++}} F\left(\min_{1\le k\le K} \frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \\
&= F\left(\sup_{\mathbf{p}\in\mathbb{R}^K_{++}} \min_{1\le k\le K} \frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \le F\left(\sup_{\mathbf{p}\in\mathbb{R}^K_{++}} \min_{k\in\mathcal{S}(n)} \frac{(\mathbf{V}\mathbf{p}^{(n)})_k}{p_k^{(n)}}\right) \\
&= \sup_{\mathbf{p}\in\mathbb{R}^K_{++}} F\left(\min_{k\in\mathcal{S}(n)} \frac{(\mathbf{V}\mathbf{p}^{(n)})_k}{p_k^{(n)}}\right) = \sup_{\mathbf{p}\in\mathbb{R}^K_{++}} \min_{k\in\mathcal{S}(n)} F\left(\frac{(\mathbf{V}\mathbf{p}^{(n)})_k}{p_k^{(n)}}\right).
\end{aligned} \tag{65}$$

The inequalities (64) and (65) together give

$$\sup_{\mathbf{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F((\mathbf{V}\mathbf{p})_k/p_k) < F(\rho(\mathbf{V})), \tag{66}$$

since the infimum over $\mathcal{P}_{++}$ equals the infimum over $\mathbb{R}^K_{++}$ due to $\gamma_k(\mathbf{p}) = \gamma_k(c\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^K_{++}$, $c > 0$. This completes the proof. □

We can interpret the condition in Lemma 4 as the existence of some entirely coupled subnetwork which is interference-isolated and its interference matrix, say $\mathbf{V}^{(n)}$, satisfies $\rho(\mathbf{V}^{(n)}) < \rho(\mathbf{V})$ (Appendix A). By Lemma 4, networks having such property cannot pertain to the class with Property i and hence cannot pertain to its subclass with Property i′ as well. An immediate consequence from Lemma 4 is as follows.

Corollary 3. *Let* $\{\mathbf{V}^{(n)}\}_{n\in\mathcal{I}}$ *and* $\{\mathbf{V}^{(m)}\}_{m\in\mathcal{M}}$ *be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix* $\mathbf{V}$*, respectively. If for any increasing bijection* F*, one has*

$$\sup_{\mathbf{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V})), \tag{67}$$

then $\mathcal{I} \subseteq \mathcal{M}$.

With Proposition 1 and inequality (52) we see that Corollary 3 formulates a necessary condition for the inclusion of a network in the class with Property i and hence also

in its subclass with Property i′. This condition is precisely that the interference matrix, say $\mathbf{V}^{(n)}$, of any entirely coupled and interference-isolated subnetwork satisfies $\rho(\mathbf{V}^{(n)}) = \rho(\mathbf{V})$. The following lemma shows even more.

Lemma 5. *Let $\{\mathbf{V}^{(n)}\}_{n\in\mathcal{I}}$ and $\{\mathbf{V}^{(m)}\}_{m\in\mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\mathbf{V}$, respectively. If $\mathcal{I} \subseteq \mathcal{M}$, then for any increasing bijection F, equality (67) is satisfied.*

Proof. Let $\mathcal{S}(n)$, $1 \le n \le N$, and $\mathbf{p}^{(n)}$ (for any $\mathbf{p} \in \mathbb{R}^K_{++}$) be defined as in the proof of Proposition 1. Let $\mathbf{p}(\lambda) \in \mathbb{R}^K_{++}$ be some vector associated with $\lambda > 0$, where $\lambda = \lambda(\epsilon)$ is a function of $\epsilon > 0$. As in the proof of Proposition 1 the idea of the proof is the construction of a vector $\mathbf{p}(\lambda)$, which achieves equality (67). Let first $\mathbf{V}^{(n)}$, $n \in \mathcal{I}$. Assume for the components of $\mathbf{p}(\lambda)$ that $p_k(\lambda) = r_k^{(n)}$, $k \in \mathcal{S}(n)$, whenever $n \in \mathcal{I}$ ($\mathbf{r}^{(n)}$ is the right eigenvector of the diagonal block $\mathbf{V}^{(n)}$ and we have $\mathbf{r}^{(n)} > 0$, $1 \le n \le N$, due to irreducibility of diagonal blocks). From the construction of the normal form of $\mathbf{V}$ and the increase of F follows then

$$F\left(\frac{(\mathbf{V}\mathbf{p}(\lambda))_k}{p_k(\lambda)}\right) = F\left(\frac{(\mathbf{V}^{(n)}\mathbf{r}^{(n)})_k}{r_k^{(n)}}\right) = F(\rho(\mathbf{V})), \quad \lambda(\epsilon) > 0, k \in \mathcal{S}(n),\ n \in \mathcal{I}, \tag{68}$$

since $n \in \mathcal{M}$ by assumption.

Let now $\mathbf{V}^{(n)}$, $n \notin \mathcal{I}$ and consider first the case when $n \in \mathcal{M}$ additionally. Assume for the components of $\mathbf{p}(\lambda)$ that $p_k(\lambda) = r_k^{(n)}$, $k \in \mathcal{S}(n)$, whenever $n \notin \mathcal{I}$, $n \in \mathcal{M}$ and define $\mathbf{t}^{(n)}(\mathbf{p})$, $1 \le n \le N$, as in the proof of Proposition 1. Then, from the construction of the normal form of $\mathbf{V}$ and the increase of F follows again

$$\begin{aligned} F\left(\frac{(\mathbf{V}\mathbf{p}(\lambda))_k}{p_k(\lambda)}\right) &= F\left(\frac{(\mathbf{V}^{(n)}\mathbf{r}^{(n)})_k}{r_k^{(n)}} + \frac{t_k^{(n)}}{p_k(\lambda)}\right) \\ &= F\left(\rho(\mathbf{V}) + \frac{t_k^{(n)}}{p_k(\lambda)}\right) \ge F(\rho(\mathbf{V})), \\ &\lambda(\epsilon) > 0,\ k \in \mathcal{S}(n),\ n \notin \mathcal{I},\ n \in \mathcal{M}. \end{aligned} \tag{69}$$

For the remaining case $n \notin \mathcal{I}$, $n \notin \mathcal{M}$ assume for the components of $\mathbf{p}(\lambda)$ that $p_k(\lambda) = \lambda\tilde{r}_k$, $\tilde{\mathbf{r}} \in \mathcal{R}$, $k \in \mathcal{S}(n)$. Then, from the eigenvalue problem for the normal form of $\mathbf{V}$ follows

$$\rho(\mathbf{V})\mathbf{p}^{(n)}(\lambda) = \mathbf{V}^{(n)}\mathbf{p}^{(n)}(\lambda) + \mathbf{t}^{(n)}(\mathbf{p}(\lambda)), \quad \lambda(\epsilon) > 0,\ n \notin \mathcal{I},\ n \notin \mathcal{M}. \tag{70}$$

After restatement we get $\mathbf{p}^{(n)}(\lambda) = (\rho(\mathbf{V})\mathbf{I} - \mathbf{V}^{(n)})^{-1}\mathbf{t}^{(n)}(\mathbf{p}(\lambda))$, which implies with $n \notin \mathcal{M}$ (i.e., $\rho(\mathbf{V}^{(n)}) < \rho(\mathbf{V})$) and $\mathbf{t}^{(n)}(\mathbf{p}(\lambda)) \ge 0$ that $p_k^{(n)}(\lambda) = \lambda\tilde{r}_k > 0$, for any $\lambda > 0$, $\tilde{\mathbf{r}} \in \mathcal{R}$, $k \in \mathcal{S}(n)$ [29]. Choose now $\lambda(\epsilon)$ such that $t_k^{(n)}(\mathbf{p}(\lambda))_k/\lambda\tilde{r}_k \le \epsilon$, $k \in \mathcal{S}(n)$, $n \notin \mathcal{I}$, $n \notin \mathcal{M}$. Hence, from componentwise division of both sides of (70) by $p_k(\lambda)$ and transformation by increasing function F follows

$$F\left(\frac{(\mathbf{V}\mathbf{p}(\lambda))_k}{p_k(\lambda)}\right) = F\left(\rho(\mathbf{V}) - \frac{t_k^{(n)}(\mathbf{p}(\lambda))}{\lambda\tilde{r}_k}\right) \ge F(\rho(\mathbf{V}) - \epsilon), \quad \lambda(\epsilon) > 0,\ \epsilon > 0,\ k \in \mathcal{S}(n),\ n \notin \mathcal{I},\ n \notin \mathcal{M}. \tag{71}$$

Summarizing (68), (69), and (71) we have

$$\min_{1\le k\le K} F((\mathbf{V}\mathbf{p}(\lambda(\epsilon)))_k/p_k(\lambda(\epsilon))) \ge F(\rho(\mathbf{V}) - \epsilon), \quad \epsilon > 0. \tag{72}$$

Hence, it must hold

$$\lim_{\epsilon\to 0}\min_{1\le k\le K} F((\mathbf{V}\mathbf{p}(\lambda(\epsilon)))_k/p_k(\lambda(\epsilon))) \ge F(\rho(\mathbf{V})), \tag{73}$$

which together with Corollary 2 implies

$$\begin{aligned} &\lim_{\epsilon\to 0}\min_{1\le k\le K} F((\mathbf{V}\mathbf{p}(\lambda(\epsilon)))_k/p_k(\lambda(\epsilon))) \\ &\quad = \sup_{\mathbf{p}\in\mathcal{P}_{++}}\min_{1\le k\le K} F((\mathbf{V}\mathbf{p})_k/p_k) = F(\rho(\mathbf{V})) \end{aligned} \tag{74}$$

since the supremum over $\mathcal{P}_{++}$ equals the supremum over $\mathbb{R}^K_{++}$ due to $\gamma_k(\mathbf{p}) = \gamma_k(c\mathbf{p})$, $\mathbf{p} \in \mathbb{R}^K_{++}$, $c > 0$. This completes the proof. □

As a consequence of Corollary 3 and Lemma 5 we recognize the following necessary and sufficient condition for a network to have Property i.

Proposition 9. *Let $\{\mathbf{V}^{(n)}\}_{n\in\mathcal{I}}$ and $\{\mathbf{V}^{(m)}\}_{m\in\mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\mathbf{V}$, respectively. Then, the equality*

$$\sup_{\mathbf{p}\in\mathcal{P}_{++}}\min_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = F(\rho(\mathbf{V})) = \inf_{\mathbf{p}\in\mathcal{P}_{++}}\max_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \tag{75}$$

is satisfied for any increasing bijection F if and only if $\mathcal{I} \subseteq \mathcal{M}$.

Thus, for a network with interference matrix $\mathbf{V}$, the condition that any isolated diagonal block in the canonical form of $\mathbf{V}$ is maximal is a necessary and sufficient condition to have Property i for the network. Automatically we have also that the existence of an isolated diagonal block which is not maximal in the canonical form of the interference matrix is a necessary and sufficient condition for the corresponding network to have Property ii. The complete interpretation of Proposition 9 is as follows.

Observation 9. For a network with interference matrix $\mathbf{V}$ the following are true.

(i) The value of a maximally degraded best link QoS performance (max-min fairness performance) coincides with the value of a maximally improved worst link QoS performance (min-max fairness performance) exactly in the case when the interference matrix $\mathbf{V}^{(n)}$ of any entirely coupled and interference-isolated subnetwork $n \in \mathcal{I}$ satisfies $\rho(\mathbf{V}^{(n)}) = \rho(\mathbf{V})$.

(ii) The value of a maximally degraded best link QoS performance is greater than the value of a maximally improved worst link QoS performance exactly in the case when there exists some entirely coupled and interference-isolated subnetwork with interference matrix $\mathbf{V}^{(n)}$ satisfying $\rho(\mathbf{V}^{(n)}) < \rho(\mathbf{V})$.

7.2. The case of common optimizers

The question of the network class with Property i′ remains to be answered. It is precisely the question of description of the subclass of networks with zero fairness gap, for which some allocation achieves both min-max fairness in the sense of (50) and max-min fairness in the sense of (51). The following description of such class is possible with Proposition 1 and Lemma 5.

Proposition 10. *Let $\{\mathbf{V}^{(n)}\}_{n\in\mathcal{I}}$ and $\{\mathbf{V}^{(m)}\}_{m\in\mathcal{M}}$ be the sets of isolated and maximal diagonal blocks in the canonical form of the interference matrix $\mathbf{V}$, respectively.*

(i) *There exists some vector $\widetilde{\mathbf{p}} > 0$ satisfying*

$$\widetilde{\mathbf{p}} = \arg\inf_{\mathbf{p}\in\mathcal{P}_{++}} \max_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) = \arg\sup_{\mathbf{p}\in\mathcal{P}_{++}} \min_{1\le k\le K} F\left(\frac{(\mathbf{V}\mathbf{p})_k}{p_k}\right) \tag{76}$$

if and only if $\mathcal{I} = \mathcal{M}$.

(ii) *Moreover, $\widetilde{\mathbf{p}}$ satisfies (76) if and only if $\widetilde{\mathbf{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$.*

Proof. We prove the statements (i) and (ii) jointly in the circular manner. For the proof of the if part of (i), assume that some $\widetilde{\mathbf{p}} > 0$ satisfying (76) exists. By (76) it is implied that $\max_{1\le k\le K} F((\mathbf{V}\widetilde{\mathbf{p}})_k/\widetilde{p}_k) = \min_{1\le k\le K} F((\mathbf{V}\widetilde{\mathbf{p}})_k/\widetilde{p}_k)$. Hence, by (5) or (67) we have $F((\mathbf{V}\widetilde{\mathbf{p}})_k/\widetilde{p}_k) = F(\rho(\mathbf{V}))$, $1 \le k \le K$, and consequently $\widetilde{\mathbf{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$. By Proposition 9 and (76) we already know that $\mathcal{I} \subseteq \mathcal{M}$. Hence, assume by contradiction $\mathcal{I} \subset \mathcal{M}$, that is, there exists some nonisolated block $\mathbf{V}^{(n)}$ of $\mathbf{V}$, such that $\rho(\mathbf{V}^{(n)}) = \rho(\mathbf{V})$. With $\widetilde{\mathbf{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$ from above it follows for such block from the eigenvalue problem for the normal form of $\mathbf{V}$,

$$\rho(\mathbf{V})\widetilde{\mathbf{p}}^{(n)} = \mathbf{V}^{(n)}\widetilde{\mathbf{p}}^{(n)} + \sum_{m=1}^{n-1} \mathbf{V}^{(n,m)}\widetilde{\mathbf{p}}^{(m)}, \quad n \in \mathcal{M},\ n \notin \mathcal{I}, \tag{77}$$

with $\widetilde{p}_k^{(n)} := \widetilde{p}_k$, $k \in \mathcal{S}(n)$ ($\mathcal{S}(n)$ defined as in the proof of Proposition 1). Due to $n \notin \mathcal{I}$ at least one of the matrices $\mathbf{V}^{(n,m)}$, $1 \le m \le n-1$, is nonzero and nonnegative. Hence with $\widetilde{\mathbf{p}} > 0$ it follows from (77) that there exists some $k \in \mathcal{S}(n)$ such that $\rho(\mathbf{V})\widetilde{\mathbf{p}}_k^{(n)} > (\mathbf{V}^{(n)}\widetilde{\mathbf{p}}^{(n)})_k$, which implies further $\rho(\mathbf{V}^{(n)}) < \rho(\mathbf{V})$. This is a contradiction and the if part of (i) is proven. The next step to prove is that $\mathcal{I} = \mathcal{M}$ implies that there exists some $\widetilde{\mathbf{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$. But this follows from Proposition 5. The last step to show is the only if part of (ii), that any $\widetilde{\mathbf{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$ satisfies (76). But with $\widetilde{\mathbf{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$ we have (as before) $\max_{1\le k\le K} F((\mathbf{V}\widetilde{\mathbf{p}})_k/\widetilde{p}_k) = \min_{1\le k\le K} F((\mathbf{V}\widetilde{\mathbf{p}})_k/\widetilde{p}_k)$, which implies with (5) or (67) that (76) holds for $\widetilde{\mathbf{p}} \in \mathcal{R} \cap \mathcal{P}_{++}$. With this, the circle of three if relations is completed and (i), (ii) are proven. □

Hence, the class of networks for which min-max fairness and max-min fairness can be concurrently achieved by some allocation consists of networks for which the isolated diagonal blocks coincide with maximal diagonal blocks in the canonical form of their interference matrices. Consequently, whenever some maximal diagonal block in the canonical form of the interference matrix is not isolated, then there exists no allocation which is both min-max fair and max-min fair for the corresponding network. Similarly, the min-max fair and max-min fair allocation does not exist in the case when some isolated diagonal block is not maximal in the canonical form of the corresponding interference matrix. In both cases the networks satisfy Property i, but do not satisfy Property i′. The above can be interpreted as follows.

Observation 10. For a network with interference matrix $\mathbf{V}$ the following are true.

(i) An allocation which achieves both max-min fairness and min-max fairness exists, when any entirely coupled subnetwork with interference matrix $\mathbf{V}^{(n)}$ satisfies $\rho(\mathbf{V}^{(n)}) = \rho(\mathbf{V})$ exactly in the case when it is interference-isolated.

(ii) Whenever there exists some not interference-isolated entirely coupled subnetwork with interference matrix $\mathbf{V}^{(n)}$ satisfying $\rho(\mathbf{V}^{(n)}) = \rho(\mathbf{V})$, then an allocation achieving both max-min fairness and min-max fairness does not exist.

(iii) Whenever there exists some interference-isolated entirely coupled subnetwork with interference matrix $\mathbf{V}^{(n)}$ satisfying $\rho(\mathbf{V}^{(n)}) < \rho(\mathbf{V})$, then an allocation achieving both max-min fairness and min-max fairness does not exist.

Finally, notice the subtlety that there may exist some allocation which maximally improves the worst link QoS performance (in the sense of (50)) and at the same time maximally degrades the best link QoS performance (in the sense of (51)) without being both min-max fair and max-min fair. From the discussion in Section 3.1 we know that this is the case when both (50) and (51) are solved by an allocation with some zero components. Clearly, the class of networks for which such allocation exists is included in the class with Property i, but it includes the class with Property i′.

8. SUMMARY AND FINAL REMARKS

In this work we studied the interdependence between achieving min-max fairness and optimality of a weighted sum of QoS values in a wireless cellular network with single-user receivers. We characterized the networks for which a min-max fair allocation exists. We showed that the min-max fair allocation optimizes the utility function when specifically constructed vectors of weights are chosen. We characterized the class of networks, for which such weight vectors have all weights positive. Next, we proved that a min-max fair and utility-optimal allocation of powers and weights corresponds to a saddle point of the utility, as a function of powers and weights. Precisely, we showed that the min-max fair allocation optimizes the utility for a worst-case vector of weights, and that under such weight vector the utility-optimum is worse than under any other weight vector. The proven saddle point property is a compact analytical interpretation of the

trade-off between min-max fairness and utility optimality in cellular networks. Finally, we showed that the approach of ensuring min-max fairness, consisting in maximally improving the worst QoS, and the approach of ensuring max-min fairness, consisting in maximally degrading the best QoS, are in general not equivalent. We showed the existence of a gap in performance under both approaches and the difference in corresponding optimizers. We characterized network classes for which both notions coincide in terms of the achieved performance and for which an allocation achieving both goals exists.

We believe that the assumption of the SIR model, which gave rise to all the results in this work, is in numerous cases justified or inevitable (as described in Section 2). For such case a stringent analytic framework for the network optimization approaches of interest is needed. This work was intended to be a part of such framework. A continuation and extension of this work can be found in [44, 45], where similar issues are addressed in the case when the interference power is not a linear function of the transmit power allocation.

It is unfortunately not completely clear which of the proved interdependencies and features translate to the SINR-based network optimization, that is, when some nonzero background noise at each receiver is accounted for. It can be intuitively expected that some of the features proved in this work are qualitatively retained when the noise variance in the SINR model is small compared with the interference power at each receiver. Nevertheless, even if the theory of power control is relatively saturated, the joint framework of SINR-based fairness and utility optimization which addresses such questions, and parallels this work, is still an attractive research issue.

APPENDICES

A. CANONICAL FORM OF A NONNEGATIVE MATRIX

It is known that the spectrum and the eigenmanifolds are invariant with respect to the permutation of rows and columns of the matrix (the permutation effect on eigenmanifolds is merely the corresponding permutation of the dimensions) [29]. By such permutation, any nonnegative matrix $\mathbf{V}$ can be represented in the canonical form, referred also to as Frobenius normal form. The canonical form can be written as

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}^{(1)} & 0 & \cdots & 0 \\ \mathbf{X}^{(2,1)} & \mathbf{X}^{(2)} & \ddots & \vdots \\ \vdots & \vdots & \cdots & 0 \\ \mathbf{X}^{(N,1)} & \mathbf{X}^{(N,2)} & \cdots & \mathbf{X}^{(N)} \end{pmatrix}, \tag{A.1}$$

with the diagonal blocks $\mathbf{X}^{(n)}$, $1 \le n \le N$, as square irreducible matrices. We have $\rho(\mathbf{X}) = \max_{1\le m\le N} \rho(\mathbf{X}^{(m)})$, and any diagonal block $\mathbf{X}^{(n)}$ with $\rho(\mathbf{X}^{(n)}) = \rho(\mathbf{X})$ is referred to as *maximal*. Further we refer to a diagonal block $\mathbf{X}^{(n)}$ as an *isolated* one, if $\mathbf{X}^{(n,m)} = \mathbf{0}$, $1 \le m < n$. If all nondiagonal blocks in the normal form are identical to zero, the matrix is referred to as *block-irreducible*. Clearly, irreducibility is a special case of block-irreducibility with $N = 1$.

A.1. Interference interpretation of the canonical form

Interpret $\mathbf{X}$ as an interference matrix. Since the diagonal blocks $\mathbf{X}^{(n)}$, $1 \le n \le N$, in the canonical form are irreducible, they represent interference matrices of entirely interference-coupled link subsets, which we interpret as subnetworks. Clearly, N is then the maximal number of entirely coupled subnetworks, into which the network can be partitioned. The nondiagonal block $\mathbf{V}^{(n,m)}$ contains interference factors expressing the interference from links in the nth subnetwork perceived by the links in the mth subnetwork. The isolation of the diagonal block $\mathbf{V}^{(m)}$ means that the mth subnetwork does not perceive interference from other subnetworks and hence can be referred to as *interference-isolated*. Under block-irreducibility of $\mathbf{V}$ the network consists solely of interference-isolated subnetworks.

B. SADDLE POINT DEFINITION AND CONDITIONS

The saddle point is defined as follows (see, e.g., [26]).

Definition B.1. A vector $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is called a saddle point of a function $\phi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z} \subseteq \mathbb{R}$, if and only if $\phi(\tilde{\mathbf{x}}, \mathbf{y}) \le \phi(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \le \phi(\mathbf{x}, \tilde{\mathbf{y}})$, $\mathbf{x} \in \mathcal{X}$, $\mathbf{y} \in \mathcal{Y}$.

Instead of verifying the pair of inequalities in the definition, a saddle point can be identified by means of an equality, called sometimes minmax-maxmin equality. Such equality is a necessary and sufficient condition for the saddle point. See [46] for further reading.

Proposition B.2. *A vector $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ is a saddle point of function $\phi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z} \subseteq \mathbb{R}$, if and only if* $\inf_{\mathbf{x}\in\mathcal{X}} \sup_{\mathbf{y}\in\mathcal{Y}} \phi(\mathbf{x}, \mathbf{y}) = \sup_{\mathbf{y}\in\mathcal{Y}} \inf_{\mathbf{x}\in\mathcal{X}} \phi(\mathbf{x}, \mathbf{y})$ *and* $\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}\in\mathcal{X}} \sup_{\mathbf{y}\in\mathcal{Y}} \phi(\mathbf{x}, \mathbf{y})$, *and* $\tilde{\mathbf{y}} = \arg\max_{\mathbf{y}\in\mathcal{Y}} \inf_{\mathbf{x}\in\mathcal{X}} \phi(\mathbf{x}, \mathbf{y})$.

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