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New extension of p-metric spaces with some fixed-point results on M-metric spaces

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Abstract

In this paper, we extend the p-metric space to an M-metric space, and we shall show that the definition we give is a real generalization of the p-metric by presenting some examples. In the sequel we prove some of the main theorems by generalized contractions for getting fixed points and common fixed points for mappings.

Keywords: fixed point; partial metric space

1 Introduction and preliminaries

In 1994, in [1] Matthews introduced the notion of a partial metric space and proved the contraction principle of Banach in this new framework. Next, many fixed-point theorems in partial metric spaces have been given by several mathematicians. Recently Haghi *et al.* published [2] a paper which stated that we should 'be careful on partial metric fixed point results' along with giving some results. They showed that fixed-point generalizations to partial metric spaces can be obtained from the corresponding results in metric spaces.

In this paper, we extend the p-metric space to an M-metric space, and we shall show that our definition is a real generalization of the p-metric by presenting some examples. In the sequel we prove some of the main theorems by generalized contractions for getting fixed points and common fixed points for mappings.

Definition 1.1 ([1], [3, Definition 1.1]) A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p1) $p(x,x) = p(y,y) = p(x,y) \iff x = y$,

(p2) $p(x,x) \le p(x,y)$,

(p3) $p(x,y) = p(y,x)$,

(p4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

Notation The following notation is useful in the sequel.

1. $m_{xy} := \min\{m(x,x), m(y,y)\}$,
2. $M_{xy} := \max\{m(x,x), m(y,y)\}$.

Now we want to extend Definition 1.1 as follows.

Definition 1.2 Let X be a nonempty set. A function $m : X \times X \to \mathbb{R}^+$ is called an m-metric if the following conditions are satisfied:

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,
(m2) $m_{xy} \leq m(x,y)$,
(m3) $m(x,y) = m(y,x)$,
(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

Then the pair (X,m) is called an M-metric space.

According to the above definition the condition (p1) in the definition of [1] changes to (m1), and (p2) is expressed for $p(x,x)$ where $p(y,y) = 0$ may become $p(y,y) \neq 0$. Thus we improve that condition by replacing it by $\min\{p(x,x),p(y,y)\} \leq p(x,y)$, and also we improve the condition (p4) extending it to the form of (m4). In the sequel we present an example that holds for the m-metric but not for the p-metric.

Remark 1.1 For every $x,y \in X$

1. $0 \leq M_{xy} + m_{xy} = m(x,x) + m(y,y)$,
2. $0 \leq M_{xy} - m_{xy} = |m(x,x) - m(y,y)|$,
3. $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$.

The next examples show that m^s and m^w are ordinary metrics.

Example 1.1 Let $X := [0,\infty)$. Then $m(x,y) = \frac{x+y}{2}$ on X is an m-metric.

Example 1.2 Let m be an m-metric. Put

1. $m^w(x,y) = m(x,y) - 2m_{xy} + M_{xy}$,
2. $m^s(x,y) = m(x,y) - m_{xy}$ when $x \neq y$ and $m^s(x,y) = 0$ if $x = y$.

Then m^w and m^s are ordinary metrics.

Proof If $m^w(x,y) = 0$, then

$$m(x,y) = 2m_{xy} - M_{xy}. \tag{1}$$

But from equation (1) and $m_{xy} \leq m(x,y)$ we get $m_{xy} = M_{xy} = m(x,x) = m(y,y)$, so by equation (1) we obtain $m(x,y) = m(x,x) = m(y,y)$ and therefore $x = y$. For the triangle inequality it is enough that we consider Remark 1.1 and (m4). □

Remark 1.2 For every $x,y \in X$

1. $m(x,y) - M_{xy} \leq m^w(x,y) \leq m(x,y) + M_{xy}$,
2. $(m(x,y) - M_{xy}) \leq m^s(x,y) \leq m(x,y)$.

In other words

$$\left|m^w(x,y) - m(x,y)\right| \leq M_{xy}, \qquad \left|m^s(x,y) - m(x,y)\right| \leq M_{xy}.$$

In the following example we present an example of an m-metric which is not a p-metric.

Example 1.3 Let $X = \{1,2,3\}$; define

$$m(1,1) = 1, \qquad m(2,2) = 9, \qquad m(3,3) = 5,$$
$$m(1,2) = m(2,1) = 10, \qquad m(1,3) = m(3,1) = 7, \qquad m(3,2) = m(2,3) = 7.$$

So m is an m-metric, but it is not p-metric.

Example 1.4 Let (X,d) be a metric space. Let $\phi : [0,\infty) \to [\phi(0),\infty)$ be a one to one and nondecreasing or strictly increasing mapping, with $\phi(0)$ defined such that

$$\phi(x+y) \leq \phi(x) + \phi(y) - \phi(0), \quad \forall x,y \geq 0.$$

Then $m(x,y) = \phi(d(x,y))$ is an m-metric.

Proof (m1), (m2), and (m3) are clear. For (m4) we have

$$\begin{aligned}
\phi\bigl(d(x,y)\bigr) &\leq \phi\bigl(d(x,z) + d(z,y)\bigr) \\
&\leq \phi\bigl(d(x,z)\bigr) + \phi\bigl(d(z,y)\bigr) - \phi(0), \\
\bigl(\phi\bigl(d(x,y)\bigr) - \phi(0)\bigr) &\leq \bigl(\phi\bigl(d(x,z)\bigr) - \phi(0)\bigr) + \bigl(\phi\bigl(d(z,y)\bigr) - \phi(0)\bigr), \\
\bigl(m(x,y) - m_{xy}\bigr) &\leq \bigl(m(x,z) - m_{xz}\bigr) + \bigl(m(z,y) - m_{zy}\bigr).
\end{aligned}$$ □

Example 1.5 Let (X,d) be a metric space. Then $m(x,y) = ad(x,y) + b$ where $a, b > 0$ is an m-metric, because we can put $\phi(t) = at + b$.

Remark 1.3 According to Example 1.5, by the Banach contraction

$$\exists k \in [0,1), \quad m(Tx,Ty) \leq km(x,y), \quad \text{for all } x,y \in X,$$

we have

$$m(Tx,Ty) = ad(Tx,Ty) + b \leq kad(x,y) + kb \quad \Rightarrow \quad d(Tx,Ty) \leq kd(x,y) + \frac{b(k-1)}{a},$$

which does not imply the ordinary Banach contraction

$$\exists k \in [0,1), \quad d(Tx,Ty) \leq kd(x,y), \quad \text{for all } x,y \in X,$$

for all self-maps T on X. Thus, this states that even if the m-metric m and the ordinary metric d have the same topology, the Banach contraction of the m-metric does not imply the Banach contraction of the ordinary metric d.

Lemma 1.1 *Every p-metric is an m-metric.*

Proof Let m be a p-metric. It is enough that we consider the following cases:

1. $m(x,x) = m(y,y) = m(z,z)$,
2. $m(x,x) < m(y,y) < m(z,z)$,

3. $m(x,x) = m(y,y) < m(z,z)$,
4. $m(x,x) = m(y,y) > m(z,z)$,
5. $m(x,x) < m(y,y) = m(z,z)$,
6. $m(x,x) > m(y,y) = m(z,z)$.

For example, to prove (2), we have

$$m(x,y) \leq m(x,z) + m(z,y) - m(z,z),$$

$$m(x,y) \leq m(x,z) + m(z,y) - m(y,y),$$

$$m(x,y) - m(x,x) \leq m(x,z) - m(x,x) + m(z,y) - m(y,y),$$

$$m(x,y) - m_{x,y} \leq m(x,z) - m_{x,z} + m(z,y) - m_{z,y}.$$

□

2 Topology for *M*-metric space

It is clear that each m-metric p on X generates a T_0 topology τ_m on X. The set

$$\{B_m(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_m(x,\varepsilon) = \{y \in X : m(x,y) < m_{x,y} + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$, forms a base of τ_m.

Definition 2.1 Let (X,m) be a m-metric space. Then:

1. A sequence $\{x_n\}$ in a M-metric space (X,m) converges to a point $x \in X$ if and only if

$$\lim_{n\to\infty} \bigl(m(x_n,x) - m_{x_n,x}\bigr) = 0. \tag{2}$$

2. A sequence $\{x_n\}$ in a M-metric space (X,m) is called an m-Cauchy sequence if

$$\lim_{n,m\to\infty} \bigl(m(x_n,x_m) - m_{x_n,x_m}\bigr), \qquad \lim_{n,m\to\infty} (M_{x_n,x_m} - m_{x_n,x_m}) \tag{3}$$

exist (and are finite).

3. An M-metric space (X,m) is said to be complete if every m-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_m, to a point $x \in X$ such that

$$\Bigl(\lim_{n\to\infty} \bigl(m(x_n,x) - m_{x_n,x}\bigr) = 0 \ \&\ \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0\Bigr).$$

Lemma 2.1 *Let (X,m) be a m-metric space. Then:*

1. *$\{x_n\}$ is an m-Cauchy sequence in (X,m) if and only if it is a Cauchy sequence in the metric space (X,m^w).*
2. *An M-metric space (X,m) is complete if and only if the metric space (X,m^w) is complete. Furthermore,*

$$\lim_{n\to\infty} m^w(x_n,x) = 0 \quad \Longleftrightarrow \quad \Bigl(\lim_{n\to\infty} \bigl(m(x_n,x) - m_{x_n,x}\bigr) = 0,\ \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0\Bigr).$$

Likewise the above definition holds also for m^s.

Lemma 2.2 *Assume that* $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$ *in an M-metric space* (X, m). *Then*

$$\lim_{n\to\infty}\big(m(x_n, y_n) - m_{x_n,y_n}\big) = m(x, y) - m_{xy}.$$

Proof We have

$$\big|\big(m(x_n, y_n) - m_{x_n,y_n}\big) - \big(m(x, y) - m_{x,y}\big)\big| \le \big(m(x_n, x) - m_{x_n,x}\big) + \big(m(y, y_n) - m_{y,y_n}\big).$$ □

From Lemma 2.2 we deduce the following lemma.

Lemma 2.3 *Assume that* $x_n \to x$ *as* $n \to \infty$ *in an M-metric space* (X, m). *Then*

$$\lim_{n\to\infty}\big(m(x_n, y) - m_{x_n,y}\big) = m(x, y) - m_{x,y},$$

for all $y \in X$.

Lemma 2.4 *Assume that* $x_n \to x$ *and* $x_n \to y$ *as* $n \to \infty$ *in an M-metric space* (X, m). *Then* $m(x, y) = m_{xy}$. *Furthermore, if* $m(x, x) = m(y, y)$, *then* $x = y$.

Proof By Lemma 2.2 we have

$$0 = \lim_{n\to\infty}\big(m(x_n, x_n) - m_{x_n,x_n}\big) = m(x, y) - m_{xy}.$$ □

Lemma 2.5 *Let* $\{x_n\}$ *be a sequence in an m-metric space* (X, m), *such that*

$$\exists r \in [0, 1), \quad m(x_{n+1}, x_n) \le rm(x_n, x_{n-1}), \quad \forall n \in \mathbb{N}. \tag{4}$$

Then

(A) $\lim_{n\to\infty} m(x_n, x_{n-1}) = 0$,
(B) $\lim_{n\to\infty} m(x_n, x_n) = 0$,
(C) $\lim_{m,n\to\infty} m_{x_m x_n} = 0$,
(D) $\{x_n\}$ *is an m-Cauchy sequence*.

Proof From equation (4) we have

$$m(x_n, x_{n-1}) \le rm(x_{n-1}, x_{n-2}) \le r^2 m(x_{n-2}, x_{n-3}) \le \cdots \le r^n m(x_0, x_1),$$

thus,

$$\lim_{n\to\infty} m(x_n, x_{n-1}) = 0,$$

which implies that (A) holds.
From (m2) and (A) we have

$$\lim_{n\to\infty} \min\big\{m(x_n, x_n), m(x_{n-1}, x_{n-1})\big\} = \lim_{n\to\infty} m_{x_n x_{n-1}} \le \lim_{n\to\infty} m(x_n, x_{n-1}) = 0.$$

That is, (B) holds.

Clearly, (C) holds, since $\lim_{n\to\infty} m(x_n, x_n) = 0$. □

Theorem 2.1 *The topology τ_m is not Hausdorff.*

Proof Let $x, y, z \in X$ be such that

$$a := m(x,x) < m(z,z) = \frac{a+b}{2} < b := m(y,y)$$

with

$$\frac{b}{2} < \frac{k}{2} < m(x,y) < M_{x,y} = b, \qquad r := 2m(x,y) - a - b > 0$$

and

$$\max\{m(x,z), m(z,y)\} \leq \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r};$$

without loss of generality we assume that for each $\varepsilon > 0$ we have $\varepsilon < r$. We want to show that the intersection of the following neighborhoods is not empty:

$$U_x = \{z \in X : m(x,z) - m_{xz} < \varepsilon\}, \qquad V_y = \{z \in X : m(y,z) - m_{yz} < \varepsilon\}.$$

To prove $z \in U_x$, we have

$$\begin{aligned} m(x,z) &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r}, \\ m(x,z) - m_{xz} &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r} - a \\ &< \bigl(2m(x,y) - k - a\bigr)\frac{\varepsilon}{r} \\ &< \bigl(2m(x,y) - a - b\bigr)\frac{\varepsilon}{r} = \varepsilon \end{aligned}$$

and for $z \in V_y$

$$\begin{aligned} m(y,z) &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r}, \\ m(x,z) - m_{yz} &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r} - \frac{a+b}{2} \\ &< \bigl(2m(x,y) - k\bigr)\frac{\varepsilon}{r} - \frac{a+b}{2}\frac{\varepsilon}{r} \\ &< \left(2m(x,y) - k - \frac{a+b}{2}\right)\frac{\varepsilon}{r} \\ &< \bigl(2m(x,y) - a - b\bigr)\frac{\varepsilon}{r} = \varepsilon, \end{aligned}$$

so we can find $x, y \in X$ such that for all nonempty neighborhoods U_x of x and V_y of y we have $U_x \cap V_y \neq \emptyset$. □

3 Fixed point results on M-metric space

Theorem 3.1 *Let (X,m) be a complete M-metric space and let $T:X\to X$ be a mapping satisfying the following condition:*

$$\exists k\in[0,1)\text{ such that } m(Tx,Ty)\le km(x,y)\quad\text{for all } x,y\in X. \tag{5}$$

Then T has a unique fixed point.

Proof Let $x_0\in X$ and $x_n:=Tx_{n-1}$, so we have

$$m(x_n,x_{n-1})=m(Tx_{n-1},Tx_{n-2})\le km(x_{n-1},x_{n-2}) \tag{6}$$

and so (A), (B), (C), and (D) of Lemma 2.5 hold. By completeness of X we get $x_n\to x$ for some $x\in X$. Thus by equation (5) $m(Tx_n,Tx)\le km(x_n,x)\to 0$. Hence by (m2) $m_{Tx_n,Tx}\le m(Tx_n,Tx)\to 0$ so by equation (2) $Tx_n\to Tx$.

Contraction (5) implies that $m(x_n,Tx_n)\to 0$ and $m(Tx,Tx)<m(x,x)$. Since $m_{x_n,Tx_n}\to 0$, by Lemma 2.2, we get $m(x,Tx)=m_{x,Tx}=m(Tx,Tx)$.

On the other hand, by Lemma 2.2 and $x_n=Tx_{n-1}\to x$,

$$0=\lim_{n\to\infty}\bigl(m(x_n,Tx_n)-m_{x_n,Tx_n}\bigr)=\lim_{n\to\infty}\bigl(m(x_n,x_{n-1})-m_{x_n,Tx_n}\bigr)=m(x,x)-m_{x,Tx},$$

thus $m(x,x)=m(x,Tx)$. Since $m(x,Tx)=m_{x,Tx}=m(Tx,Tx)$ now by (m1) $x=Tx$. Uniqueness by the contraction (5) is clear. □

Theorem 3.2 *Let (X,m) be a complete M-metric space and let $T:X\to X$ be a mapping satisfying the following condition:*

$$\exists k\in\left[0,\frac{1}{2}\right)\text{ such that } m(Tx,Ty)\le k\bigl(m(x,Tx)+m(y,Ty)\bigr)\quad\text{for all } x,y\in X. \tag{7}$$

Then T has an unique fixed point.

Proof Let $x_0\in X$ and $x_n:=Tx_{n-1}$, so we have

$$\begin{aligned} m(x_n,x_{n-1}) &= m(Tx_{n-1},Tx_{n-2})\\ &\le k\bigl(m(x_{n-1},x_n)+m(x_{n-2},x_{n-1})\bigr). \end{aligned}$$

So

$$m(x_n,x_{n-1})\le rm(x_{n-2},x_{n-1}),$$

where $0\le r=\frac{k}{1-k}<1$.

By Lemma 2.5 and completeness of X, $x_n\to x$ for some $x\in X$. So

$$m(x_n,x)-m_{x_n,x}\to 0,\qquad M_{x_n,x}-m_{x_n,x}\to 0,$$

and since $m_{x_n,x} \to 0$, we have $m(x_n, x) \to 0$ and $M_{x_n,x} \to 0$. Therefore by Remark 1.1, $m(x,x) = 0 = m_{x,Tx}$;

$$m(x_{n+1}, Tx) = m(Tx_n, Tx) \leq k\big(m(x_n, x_{n+1}) + m(x, Tx)\big),$$

hence by $m(x_n, x_{n+1}) \to 0$

$$\limsup_{n\to\infty} m(x_{n+1}, Tx) = \limsup_{n\to\infty} m(Tx_n, Tx) \leq km(x, Tx).$$

On the other hand

$$m(x, Tx) - m_{x,Tx} \leq m(x, x_n) + m(x_n, Tx)$$

implies that

$$m(x, Tx) \leq \limsup_{n\to\infty}\big(m(x, x_n) + m(x_n, Tx)\big) \leq km(x, Tx),$$

because $m_{x,Tx} = 0$ and $m(x_n, x) \to 0$. So $m(x, Tx) = 0$. Now by contraction (7) we have $m(Tx, Tx) \leq 2km(x, Tx) = 0$, so $m(Tx, Tx) = 0 = m(x, x) = m(x, Tx)$, thus $x = Tx$ by (m1). □

The next theorem is still open.

Theorem 3.3 *Let (X, m) be a complete M-metric space and let $T : X \to X$ be a mapping satisfying the following condition*:

$$\exists k \in \left[0, \frac{1}{2}\right) \text{ such that } m(Tx, Ty) \leq k\big(m(x, Ty) + m(Tx, y)\big) \quad \text{for all } x, y \in X. \tag{8}$$

Then T has a unique fixed point.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have read and approved the final manuscript.

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