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Detection of continuous-time quaternion signals in additive noise

Jesús Navarro-Moreno*, Juan Carlos Ruiz-Molina, Antonia Oya and José M Quesada-Rubio

Abstract

Different kinds of quaternion signal detection problems in continuous-time by using a widely linear processing are dealt with. The suggested solutions are based on an extension of the Karhunen-Loève expansion to the quaternion domain which provides uncorrelated scalar real-valued random coefficients. This expansion presents the notable advantage of transforming the original four-dimensional eigen problem to a one-dimensional problem. Firstly, we address the problem of detecting a quaternion deterministic signal in quaternion Gaussian noise and a version of Pitcher's Theorem is given. Also the particular case of a general quaternion Wiener noise is studied and an extension of the Cameron-Martin formula is presented. Finally, the problem of detecting a quaternion random signal in quaternion white Gaussian noise is tackled. In such a case, it is shown that the detector depends on the quaternion widely linear estimator of the signal.

Keywords: Detection, Quaternion random signals, Series expansion

1 Introduction

Quaternion signals are of great relevance to applications in the area of statistical signal processing in which the received signal is composed of a certain number of random components since they account naturally for their correlated nature [1,2]. These are useful, for example, in studying communication, electromagnetics, seismology, acoustics, etc., problems frequently encountered in this area [3]. One of these problems where the application of the mathematical theory of quaternions has recently attracted significant attention is vector-sensor signals [4]. A vector-sensor array model uses an array of sensors whose output is a vector corresponding to the different magnitudes of the problem analyzed, i.e., it is a device that measures a complete physical vector quantity [5,6].

As is the case with complex-valued random signals, the suitable statistical processing for quaternions requires the augmented statistics to be considered, i.e., requires the operation on the quaternion and its involutions over the three pure unit quaternions in an orthogonal basis. This approach, called quaternion widely linear (QWL) processing, can lead to better performances than the traditional quaternion linear processing for multiple problems [1].

*Correspondence: jnavarro@ujaen.es
Department of Statistics and Operations Research, University of Jaén, Campus Las Lagunillas, 23071 Jaén, Spain

On the other hand, one classical approach to addressing the signal detection problem is via an appropriate series representation [7-10]. Series expansions enable us to bridge the gap between the continuous-time observation set and the discrete-time one in a straightforward manner. In fact, they provide a countable set of random coefficients with the same information content up to sets of measure zero as the observation process. If such random coefficients are uncorrelated, then they become an excellent tool to derive optimal detection structures. The Karhunen-Loève (KL) expansion is the most widely used because of its optimality properties in information compression [11]. This series representation has been recently extended to the quaternion domain by using augmented statistics [12]. The technique to derive the QKL expansion is based on the definition of a real-valued univariate stochastic signal whose second-order statistics match that of quaternion. This strategy avoids addressing a four-dimensional vectorial problem which notably simplifies the obtaining of the representation. Another advantage of such a series expansion is that the random coefficients take the form of scalar real-valued random variables.

This article handles the problem of detecting a quaternion signal corrupted by additive noise by means of the QKL expansion and following a WL processing. More specifically, two classes of quaternion signal detection problems are tackled. First, we study the detection of a quaternion deterministic signal in quaternion Gaussian noise. The main result is a version of the Pitcher's Theorem adapted to the quaternion domain. The particular case in which the noise is a general quaternion Wiener process is also analyzed and, as a consequence, a version of the well-known Cameron-Martin formula to the quaternion field is presented. Finally, we address the detection of quaternion random signals in quaternion white Gaussian noise (QWGN). In this case, we demonstrate that the log-likelihood ratio depends on the QWL estimator of the signal provided in [12].

The quaternion detection problem has been studied in the discrete-time setting previously. For example, the problem of detecting a polarized signal corrupted by unpolarized noise, in the Gaussian case, in terms of different types of properness was formulated in [13]. In [14] an efficient color-impulse detector for switching vector median filters based on the quaternion representation of color difference is presented. Our approach is different in that we formulate the problem in continuous-time and use the QKL expansion to extract the random coefficients.

The article is organized as follows. In the following section, we summarize some basic concepts about quaternion and outline the QKL expansion. In Section 3, we are concerned with the detection of a completely known quaternion signal in quaternion Gaussian noise. We obtain the expression of the general log-likelihood ratio and then, some particular cases are studied. The detection of quaternion random signals in QWGN is addressed in Section 4. The results obtained are first stated and then proved rigorously in an Appendix 1. Finally, a section of Conclusions ends this article.

2 Preliminaries

We use boldfaced uppercase letters to denote matrices, boldfaced lowercase letters for column vector, and lightfaced lowercase letters for scalar quantities. Superscripts $(\cdot)^*$, $(\cdot)^{\mathrm{T}}$, and $(\cdot)^{\mathrm{H}}$ represent quaternion (or complex) conjugate, transpose, and Hermitian (i.e., transpose and quaternion conjugate), respectively. All the random variables considered are assumed with zero-mean. Consider a quaternion $q = q_1 + q_2\mathrm{i} + q_3\mathrm{j} + q_4\mathrm{k}$, where q_1, q_2, q_3, q_4 are real random variables and $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are the imaginary units. The conjugate of a quaternion is defined as $q^* = q_1 - q_2\mathrm{i} - q_3\mathrm{j} - q_4\mathrm{k}$ and the norm of a quaternion is $\|q\| = \sqrt{qq^*} = \sqrt{q^*q} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$. Denote $q^{\eta} = -\eta q \eta$, $\eta = \mathrm{i}, \mathrm{j}, \mathrm{k}$, the three perpendicular quaternion involutions, i.e.,

$$\begin{aligned} q^{\mathrm{i}} &= q_1 + q_2\mathrm{i} - q_3\mathrm{j} - q_4\mathrm{k} \\ q^{\mathrm{j}} &= q_1 - q_2\mathrm{i} + q_3\mathrm{j} - q_4\mathrm{k} \\ q^{\mathrm{k}} &= q_1 - q_2\mathrm{i} - q_3\mathrm{j} + q_4\mathrm{k} \end{aligned}$$

We define an augmented quaternion vector as $\boldsymbol{q} = [q, q^{\mathrm{i}}, q^{\mathrm{j}}, q^{\mathrm{k}}]^{\mathrm{T}}$. The second-order properties of $\boldsymbol{q}$ are fully specified by its augmented covariance matrix, $E[\boldsymbol{q}\boldsymbol{q}^{\mathrm{H}}]$ [1].

We now consider quaternions in a continuous-time setting. Given a quaternion random signal $q(t) = q_1(t) + q_2(t)\mathrm{i} + q_3(t)\mathrm{j} + q_4(t)\mathrm{k}$, with $t \in [0, T]$, a complete description of the second-order characteristics of $q(t)$ in the quaternion domain is attained by the augmented quaternion vector, $\boldsymbol{q}(t)$, or, equivalently, by the augmented correlation function, $\boldsymbol{R_q}(t,s) = E[\boldsymbol{q}(t)\boldsymbol{q}^{\mathrm{H}}(s)]$. Also, if $q_n(t)$, $n = 1, \ldots, 4$, are mean-square continuous signals, then an extension of the KL expansion to the quaternion field can be suggested [12]. This series representation presents two remarkable properties: the deterministic coefficients have the same structure as the augmented vector $\boldsymbol{q}(t)$ and the scalar random coefficients are real-valued and uncorrelated. Specifically, consider the real-valued random signal

$$x(t) = \begin{cases} q_1(t), & t \in [0, T] \\ q_2(-t), & t \in [-T, 0) \\ q_3(-t-T), & t \in [-2T, -T) \\ q_4(-t-2T), & t \in [-3T, -2T) \end{cases}$$

and let λ_n and $a_n(t)$ be the eigenvalues and eigenfunctions of its correlation function, respectively. Then, the augmented quaternion vector and its correlation function admit the following series representations [12]

$$\boldsymbol{q}(t) = \sum_{n=1}^{\infty} \boldsymbol{\varphi}(t)\varepsilon_n \quad (1)$$

$$\boldsymbol{R_q}(t,s) = \sum_{n=1}^{\infty} \beta_n \boldsymbol{\varphi}_n(t)\boldsymbol{\varphi}_n^{\mathrm{H}}(s) \quad (2)$$

where $\boldsymbol{\varphi}_n(t) = [\varphi_n(t), \varphi_n^{\mathrm{i}}(t), \varphi_n^{\mathrm{j}}(t), \varphi_n^{\mathrm{k}}(t)]^{\mathrm{T}}$ with

$$\varphi_n(t) = \frac{1}{2}(a_n(t) + a_n(-t)\mathrm{i} + a_n(-t-T)\mathrm{j} + a_n(-t-2T)\mathrm{k})$$

and $\varepsilon_n = \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t)\boldsymbol{q}(t)dt$ are real random variables such that $E[\varepsilon_n\varepsilon_m] = \beta_n\delta_{nm}$, with $\beta_n = 4\lambda_n$.

A potential application of the QKL expansion is found in the problem of estimating the quaternion signal $q(t)$ in additive QWGN [12]. The solution provided is optimal in the minimum mean-squared error sense and is derived following a QWL processing. For that, consider the observation quaternion process of the form

$$z(t) = \int_0^t q(s)ds + w_0(t), \quad t \in [0, T]$$

being $w_0(t)$ a quaternion $\mathbb{Q}$-proper[a] Wiener process with parameter r_0 and uncorrelated with $q(s)$. Thus, the QWL estimator of $q(t)$, $\hat{q}_{\mathrm{QWL}}(t)$, is given by

$$\hat{q}_{\mathrm{QWL}}(t) = \int_0^T h_1(t,s)dz(s) + \int_0^T h_2(t,s)dz^{\mathrm{i}}(s) + \int_0^T h_3(t,s)dz^{\mathrm{j}}(s) + \int_0^T h_4(t,s)dz^{\mathrm{k}}(s), \ t \in [0,T] \quad (3)$$

with

$$h_1(t,s) = \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n + r_0}\varphi_n(t)\varphi_n^*(s),$$

$$h_2(t,s) = \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n + r_0}\varphi_n(t)\varphi_n^{\mathrm{i}*}(s)$$

$$h_3(t,s) = \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n + r_0}\varphi_n(t)\varphi_n^{\mathrm{j}*}(s),$$

$$h_4(t,s) = \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n + r_0}\varphi_n(t)\varphi_n^{\mathrm{k}*}(s)$$

3 Detection of quaternion deterministic signals in quaternion Gaussian noise

Our first objective is to study the problem of detection

$$\begin{aligned} \mathcal{H}_0 &: z(t) = v(t), \quad t \in [0,T] \\ \mathcal{H}_1 &: z(t) = x(t) + v(t), \quad t \in [0,T] \end{aligned} \quad (4)$$

with $x(t)$ a quaternion continuous completely known signal and $v(t)$ a quaternion mean-square continuous Gaussian noise. Denote $\mathcal{P}_0$ and $\mathcal{P}_1$ the probability measures corresponding to $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. According to Grenander's Theorem one way of computing likelihood ratios for continuous-time observation models is first to reduce the observation signal to an equivalent observation sequence, and then looking for the limit of the likelihood ratio for the truncated sequence. An alternative, somewhat more practical, representation of the likelihood ratio for problem (4) is provided by Pitcher's Theorem. This result suggests a simpler and more efficient implementation of the corresponding signal detection system. In the particular case of Gaussian white noise the representation of the optimum detection statistic obtained is known as the Cameron-Martin formula. In the next result, we give an extension of Pitcher's Theorem to the quaternion domain.

Theorem 3.1. *Suppose that there exists a quaternion function $g(t)$ with components of bounded variation such that*

$$\boldsymbol{x}(t) = \int_0^T \boldsymbol{R}_{\boldsymbol{v}}(t,s)d\boldsymbol{g}(s), \quad t \in [0,T] \quad (5)$$

then the detection problem (4) *is not singular ($\mathcal{P}_0 \equiv \mathcal{P}_1$) and the log-likelihood ratio test is given by*

$$\log \frac{d\mathcal{P}_1}{d\mathcal{P}_0}(z) = \int_0^T z^*(t)dg(t) + \int_0^T z^{\mathrm{i}*}(t)dg^{\mathrm{i}}(t) + \int_0^T z^{\mathrm{j}*}(t)dg^{\mathrm{j}}(t) + \int_0^T z^{\mathrm{k}*}(t)dg^{\mathrm{k}}(t) - \Delta_1 \quad (6)$$

with

$$\Delta_1 = \frac{1}{2}\int_0^T \left[\int_0^T \boldsymbol{R}_{\boldsymbol{v}}(t,s)d\boldsymbol{g}(s)\right]^H d\boldsymbol{g}(t) \quad (7)$$

Remark 1. *From* (5) *and* (7) *we have the following alternative representation for* Δ_1

$$\Delta_1 = \frac{1}{2}\left[\int_0^T x^*(t)dg(t) + \int_0^T x^{\mathrm{i}*}(t)dg^{\mathrm{i}}(t) + \int_0^T x^{\mathrm{j}*}(t)dg^{\mathrm{j}}(t) + \int_0^T x^{\mathrm{k}*}(t)dg^{\mathrm{k}}(t)\right]$$

Remark 2. *If the quaternion function $g(t)$ is differentiable with respect to $p(t) = dg(t)/dt$, then equation (5) becomes*

$$\boldsymbol{x}(t) = \int_0^T \boldsymbol{R}_{\boldsymbol{v}}(t,s)\boldsymbol{p}(s)ds, \quad t \in [0,T]$$

and the first term of (6),

$$\int_0^T z^*(t)p(t)dt + \int_0^T z^{\mathrm{i}*}(t)p^{\mathrm{i}}(t)dt + \int_0^T z^{\mathrm{j}*}(t)p^{\mathrm{j}}(t)dt + \int_0^T z^{\mathrm{k}*}(t)p^{\mathrm{k}}(t)dt$$

3.1 Particular case: the general quaternion Wiener process

Following the classical strategy, the detection problem of a deterministic signal $x(t)$ in additive QWGN is formulated of the form [9]

$$\mathcal{H}_0 : z(t) = w_0(t), \quad t \in [0, T]$$
$$\mathcal{H}_1 : z(t) = \int_0^t x(s)ds + w_0(t), \quad t \in [0, T]$$

with $x(t)$ a known continuous quaternion signal and $w_0(t)$ is the quaternion $\mathbb{Q}$-proper Wiener process with parameter r_0 defined in the previous section.

The four-dimensional structure of a quaternion allows us to give a more general definition of a quaternion Wiener process in a similar way to [15]. Next, we introduce this new process and afterwards, we tackle the detection problem for this type of process.

Definition 3.1. *The general quaternion Wiener process is defined as a quaternion $\{w(t), t \in [0, T]\}$ such that its augmented correlation function $\boldsymbol{R}_{\boldsymbol{w}}(t, s)$ is of the form*

$$\boldsymbol{R}_{\boldsymbol{w}}(t,s) = \int_0^t \boldsymbol{A}(\tau)d\tau, \qquad 0 \le t \le s \le T \tag{8}$$

where $\boldsymbol{A}(t) = \boldsymbol{C}(t)\boldsymbol{C}^H(t)$, with the quaternion matrix $\boldsymbol{C}(t)$ having the particular form

$$\boldsymbol{C}(t) = \begin{pmatrix} b(t) & c(t) & d(t) & e(t) \\ b^{\mathrm{i}}(t) & c^{\mathrm{i}}(t) & d^{\mathrm{i}}(t) & e^{\mathrm{i}}(t) \\ b^{\mathrm{j}}(t) & c^{\mathrm{j}}(t) & d^{\mathrm{j}}(t) & e^{\mathrm{j}}(t) \\ b^{\mathrm{k}}(t) & c^{\mathrm{k}}(t) & d^{\mathrm{k}}(t) & e^{\mathrm{k}}(t) \end{pmatrix}$$

and being $b(t), c(t), d(t)$, and $e(t)$ quaternion continuous functions.

Remark 3. *If $\boldsymbol{C}(t) = \sqrt{r_0}\boldsymbol{I}_{4\times 4}$, then we get the quaternion $\mathbb{Q}$-proper Wiener process $w_0(t)$.*

Using this new concept, we consider the detection problem with the hypotheses of the form

$$\mathcal{H}_0 : z(t) = w(t), \quad t \in [0, T]$$
$$\mathcal{H}_1 : z(t) = \int_0^t x(s)ds + w(t), \quad t \in [0, T]$$

with $x(t)$ a known continuous quaternion signal and $w(t)$ the general quaternion Wiener process. Denoting $y(t) = \int_0^t x(s)ds$ and considering its augmented vector $\boldsymbol{y}(t)$ then, the generalized Pitcher's equation (5) for this case is

$$\boldsymbol{y}(t) = \int_0^T \boldsymbol{R}_{\boldsymbol{w}}(t,s)d\boldsymbol{g}(s), \quad t \in [0, T] \tag{9}$$

In the following result we solve equation (9) and give an explicit expression for $\boldsymbol{g}(s)$.

Corollary 3.2. *Suppose that the quaternion matrix $\boldsymbol{A}(t)$ given in (8) has inverse for $t \in [0, T]$ then*

$$\boldsymbol{g}(t) = \begin{cases} \boldsymbol{g}(T) - \boldsymbol{A}^{-1}(t)\boldsymbol{x}(t), & 0 \le t < T \\ \boldsymbol{g}(T), & t = T \end{cases} \tag{10}$$

where $\boldsymbol{g}(T) = \left[g, g^{\mathrm{i}}, g^{\mathrm{j}}, g^{\mathrm{k}}\right]^T$ with g arbitrary.

Remark 4. *In the particular case that we have the quaternion $\mathbb{Q}$-proper Wiener process $w_0(t)$ with parameter r_0, then we obtain the extension of the well-known Cameron-Martin formula to the quaternion domain, which is given by*

$$\log \frac{d\mathcal{P}_1}{d\mathcal{P}_0}(z) = \frac{1}{r_0}\left[\int_0^T x^*(t)dz(t) + \int_0^T x^{\mathrm{i}*}(t)dz^{\mathrm{i}}(t) + \int_0^T x^{\mathrm{j}*}(t)dz^{\mathrm{j}}(t) + \int_0^T x^{\mathrm{k}*}(t)dz^{\mathrm{k}}(t)\right] - \frac{2}{r_0}\int_0^T \|x(t)\|^2 dt$$

3.2 Simulation example

In order to illustrate the performance of the proposed detector we consider the model (4) with the quaternion signal $x(t)$ of the form

$$x(t) = 6t - 1 - 6t^2\mathrm{i} - (1 - 6t + 6t^2)\mathrm{j} - (1 - 6t + 6t^2)\mathrm{k}, \quad t \in [0, 1]$$

and the quaternion noise $v(t) = v_1(t) + v_2(t)\mathrm{i} + v_3(t)\mathrm{j} + v_4(t)\mathrm{k}$ the one given in the example of [12], i.e., $\{v_n(t), t \in [0, 1]\}$, $n = 1, \ldots, 4$, are Gaussian processes with $v_3(t) = v_1(t) + v_2(t) + w_1(t)$ and $v_4(t) = v_3(t) + w_2(t)$, $w_1(t)$ and $w_2(t)$ real-valued independent Gaussian processes and also independent of $v_1(t)$ and $v_2(t)$. Moreover, $E[v_1(t)v_1(s)] = f(t)f(s)$, $E[v_2(t)v_2(s)] = g(t)g(s)$, $E[v_1(t)v_2(s)] = f(t)g(s)$, $E[w_1(t)w_1(s)] = d(t)d(s)$ and $E[w_2(t)w_2(s)] = h(t)h(s)$ with $f(t) = 1 - 6t$, $g(t) = 6t^2$, $d(t) = 2t - 1$ and $h(t) = 20t^3 - 30t^2 + 12t - 1$.

In Figure 1, we show the detection probability versus the false-alarm probability by using the Neyman-Pearson criterion.

4 Detection of quaternion random signals in QWGN

So far we have considered quaternion deterministic signals. However, there are other situations in which the quaternion signals have a stochastic nature. In this framework, we study the detection problem of a quaternion

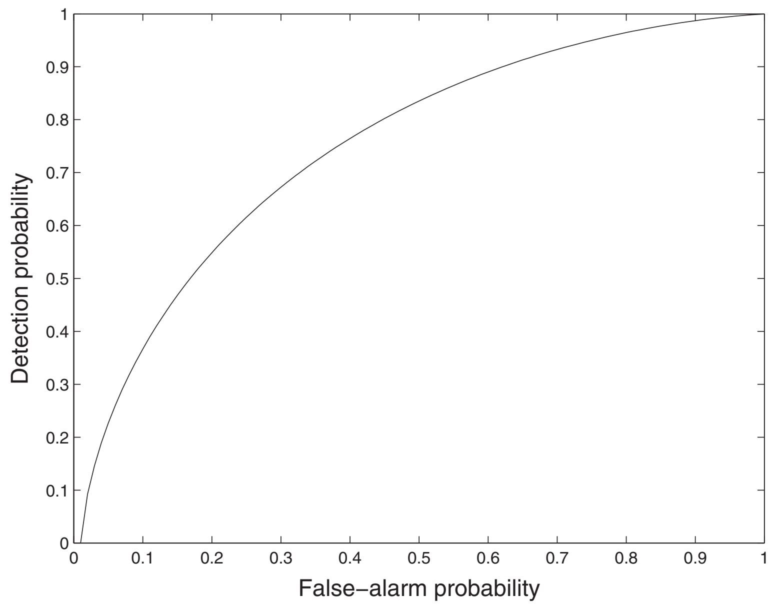

Figure 1 Detection probability versus the false-alarm probability.

random signal in additive QWGN, i.e, we consider the hypotheses pair

$$\begin{aligned} \mathcal{H}_0 &: z(t) = w_0(t), \quad t \in [0, T] \\ \mathcal{H}_1 &: z(t) = \int_0^t x(s)ds + w_0(t), \quad t \in [0, T] \end{aligned} \tag{11}$$

with $x(t)$ a mean-square continuous quaternion random signal and $w_0(t)$ the quaternion $\mathbb{Q}$-proper Wiener process with parameter r_0. Suppose also that $x(t)$ is independent of $w_0(t)$.

Theorem 4.1. *$\mathcal{P}_0 \equiv \mathcal{P}_1$ and the log-likelihood ratio is*

$$\begin{aligned} \log \frac{d\mathcal{P}_1}{d\mathcal{P}_0}(z) =& \frac{1}{2r_0} \left[\int_0^T \hat{x}^*_{QWL}(t)dz(t) + \int_0^T \hat{x}^{\mathrm{i}*}_{QWL}(t)dz^{\mathrm{i}}(t) \right. \\ & \left. + \int_0^T \hat{x}^{\mathrm{j}*}_{QWL}(t)dz^{\mathrm{j}}(t) + \int_0^T \hat{x}^{\mathrm{k}*}_{QWL}(t)dz^{\mathrm{k}}(t) \right] \\ & - \frac{1}{2} \sum_{n=1}^{\infty} \log \left(1 + \frac{\beta_n}{r_0} \right) \end{aligned} \tag{12}$$

where $\hat{x}_{QWL}(t)$ is the QWL estimator of $x(t)$ given in (3) and β_n and $\boldsymbol{\varphi}_n(t)$ are the eigenvalues and eigenfunctions of $\boldsymbol{R_x}(t,s)$, respectively.

5 Conclusions

Different quaternion detectors obtained from augmented statistics have been presented. Although we have avoided dealing with a four-dimensional eigen problem by introducing the signal $x(t)$, we have to solve a unidimensional eigen problem which can be very involved in practice. In those cases where a closed-form solution of the eigen problem is not available, a numerical method of solution can be used, as for example, the Rayleigh-Ritz method [16]. This numerical procedure allows us to solve operator equations approximately and thus, to obtain suboptimum detectors for the Gaussian detection problems addressed which converge to the optimum ones. To this end, we can use an approximate QKL expansion for quaternion signals based on the approximate eigenvalues and eigenfunctions obtained from the application of the Rayleigh-Ritz method.

Finally, we would like to give an outlook to the possible extensions of the results provided in this work. For instance, in the problem of detecting a random signal in white Gaussian noise it is well-known the estimator-correlator representation of the log-likelihood ratio, which depends on the causal estimator of the signal. Our future goal will be the extension of this closed form for the detector to the quaternion domain. On the other hand, the application of the methodology proposed in the field of Reproducing Kernel Hilbert Spaces could allows us to find an interesting solution for the discrimination problem between two quaternion random signals.

Appendix 1

Proof of Theorem 3.1

From (1) and (2), $\boldsymbol{v}(t)$ and $\boldsymbol{R}_{\boldsymbol{v}}(t,s)$ admit the series representations

$$\boldsymbol{v}(t) = \sum_{n=1}^{\infty} \boldsymbol{\varphi}_n(t)\varepsilon_n \tag{13}$$

$$\boldsymbol{R}_{\boldsymbol{v}}(t,s) = \sum_{n=1}^{\infty} \beta_n \boldsymbol{\varphi}_n(t)\boldsymbol{\varphi}_n^{\mathrm{H}}(s) \tag{14}$$

where $\varepsilon_n = \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t)\boldsymbol{v}(t)dt$. Thus, taking (5) and (14) into account, we get

$$\boldsymbol{x}(t) = \sum_{n=1}^{\infty} \beta_n \boldsymbol{\varphi}_n(t)\left[\int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(s)d\boldsymbol{g}(s)\right] = \sum_{n=1}^{\infty} \boldsymbol{\varphi}_n(t)\chi_n \tag{15}$$

with $\chi_n = \beta_n \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(s)d\boldsymbol{g}(s)$. Then, to study the continuous-time problem (4) we can consider the equivalent discrete problem[b]

$$\begin{aligned} \mathcal{H}_0 &: \varsigma_n = \varepsilon_n, \quad n = 1, 2, \ldots \\ \mathcal{H}_1 &: \varsigma_n = \chi_n + \varepsilon_n, \quad n = 1, 2, \ldots \end{aligned} \tag{16}$$

On the other hand, since $\boldsymbol{R}_{\boldsymbol{v}}(t,s)$ is a continuous function we have that $2\Delta_1 < \infty$. Likewise, from (14)

$$2\Delta_1 = \int_0^T \left[\int_0^T \boldsymbol{R}_{\boldsymbol{v}}(t,s)d\boldsymbol{g}(s)\right]^{\mathrm{H}} d\boldsymbol{g}(t) = \sum_{n=1}^{\infty} \frac{\chi_n^2}{\beta_n} \tag{17}$$

Hence, applying Grenander's Theorem [9] to (16) we obtain that $\mathcal{P}_0 \equiv \mathcal{P}_1$ and

$$\log \frac{d\mathcal{P}_1}{d\mathcal{P}_0}(z) = \sum_{n=1}^{\infty} \frac{\chi_n \varsigma_n}{\beta_n} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{\chi_n^2}{\beta_n} \tag{18}$$

Now, from (13) and (15) we have that $z(t) = \sum_{n=1}^{\infty} \boldsymbol{\varphi}_n(t)\varsigma_n$ and thus,

$$\int_0^T \boldsymbol{z}^{\mathrm{H}}(t)d\boldsymbol{g}(t) = \sum_{n=1}^{\infty} \varsigma_n \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t)d\boldsymbol{g}(t) = \sum_{n=1}^{\infty} \frac{\chi_n \varsigma_n}{\beta_n} \tag{19}$$

Finally, from (18), (19), and (17) we demonstrate (6).

Proof of Corollary 3.2

Consider the Hermitian matrix $\boldsymbol{M}(t) = \int_0^t \boldsymbol{A}(\tau)d\tau$ then, taking (8) into account, it follows that (9) is equivalent to

$$\begin{aligned} \int_0^t \boldsymbol{x}(s)ds &= \int_0^t \boldsymbol{M}(s)d\boldsymbol{g}(s) + \boldsymbol{M}(t)\int_t^T d\boldsymbol{g}(s) \\ &= \int_0^t \boldsymbol{M}(s)d\boldsymbol{g}(s) + \boldsymbol{M}(t)(\boldsymbol{g}(T) - \boldsymbol{g}(t)) \end{aligned} \tag{20}$$

Thus, integrating by parts (20), we get

$$\begin{aligned} \int_0^t \boldsymbol{x}^{\mathrm{H}}(s)ds &= \boldsymbol{g}^{\mathrm{H}}(t)\boldsymbol{M}(t) - \int_0^t \boldsymbol{g}^{\mathrm{H}}(s)d\boldsymbol{M}(s) + (\boldsymbol{g}^{\mathrm{H}}(T) \\ &\quad - \boldsymbol{g}^{\mathrm{H}}(t))\boldsymbol{M}(t) \\ &= -\int_0^t \boldsymbol{g}^{\mathrm{H}}(s)d\boldsymbol{M}(s) + \boldsymbol{g}^{\mathrm{H}}(T)\boldsymbol{M}(t) \end{aligned} \tag{21}$$

Now, since $\boldsymbol{M}(t) = \int_0^t \boldsymbol{A}(s)ds$ we have that (21) is equal to

$$\begin{aligned} \int_0^t \boldsymbol{x}^{\mathrm{H}}(s)ds &= -\int_0^t \boldsymbol{g}^{\mathrm{H}}(s)\boldsymbol{A}(s)ds + \boldsymbol{g}^{\mathrm{H}}(T)\int_0^t \boldsymbol{A}(s)ds \\ &= \int_0^t (\boldsymbol{g}^{\mathrm{H}}(T) - \boldsymbol{g}^{\mathrm{H}}(s))\boldsymbol{A}(s)ds \end{aligned}$$

Then the solution of (9) is given by (10).

Proof of Theorem 4.1

Consider the random variables $\varepsilon_n = \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t)\boldsymbol{x}(t)dt$ and $w_n = \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t)d\boldsymbol{w_0}(t)$. Then $E[\varepsilon_n\varepsilon_m] = \beta_n\delta_{nm}$, $E[w_n w_m] = r_0\delta_{nm}$ and $E[\varepsilon_n w_m] = 0$, for all n and m.

The problem (11) is equivalent to the following problem

$$\begin{aligned} \mathcal{H}_0 &: \varsigma_n = w_n, \quad n = 1, 2, \ldots \\ \mathcal{H}_1 &: \varsigma_n = \varepsilon_n + w_n, \quad n = 1, 2, \ldots \end{aligned}$$

Unlike (16), ε_n and w_n are now both random variables. Thus, under $\mathcal{H}_0$, $\varsigma_n \sim N(0, r_0)$ and under $\mathcal{H}_1$, $\varsigma_n \sim N(0, \beta_n + r_0)$. From these conditions, it is shown [9] that $\mathcal{P}_0 \equiv \mathcal{P}_1$ and

$$\log \frac{d\mathcal{P}_1}{d\mathcal{P}_0}(z) = \frac{1}{2r_0}\sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n + r_0}\varsigma_n^2 - \frac{1}{2}\sum_{n=1}^{\infty} \log\left(1 + \frac{\beta_n}{r_0}\right) \tag{22}$$

On the other hand, the random variables ς_n take the form $\varsigma_n = \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t) d\boldsymbol{z}(t)$. This fact is immediate under $\mathcal{H}_0$ and, under $\mathcal{H}_1$, we have

$$\varsigma_n = \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t) d \int_0^t \boldsymbol{x}(s) ds + \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t) d\boldsymbol{w}_0(t)$$
$$= \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t) \boldsymbol{x}(t) dt + \int_0^T \boldsymbol{\varphi}_n^{\mathrm{H}}(t) d\boldsymbol{w}_0(t) = \varepsilon_n + w_n$$

Thus, the first term of (22) can be expressed in the following way

$$\frac{1}{2r_0} \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n + r_0} \varsigma_n^2 = \frac{1}{2r_0} \int_0^T \left[\int_0^T \sum_{n=1}^{\infty} \frac{\beta_n}{\beta_n + r_0} \boldsymbol{\varphi}_n(t) \boldsymbol{\varphi}_n^{\mathrm{H}}(s) d\boldsymbol{z}(s) \right]^{\mathrm{H}} \times d\boldsymbol{z}(t) \quad (23)$$

Finally, from (22), (23), and (3) we get (12).

Endnotes

[a]That is, the augmented correlation function of $w_0(t)$ is $\boldsymbol{R}_{\boldsymbol{w}_0}(t,s) = r_0 \min(t,s) \boldsymbol{I}_{4\times 4}$, where $\boldsymbol{I}_{4\times 4}$ is the four-dimensional identity matrix [17]. [b] Due to the random coefficients ε_n having the same information up to sets of measure zero as that of $\boldsymbol{v}(t)$.

Abbreviations

QWL: Quaternion widely linear; KL: Karhunen-Loève; QKL: Quaternion Karhunen-Loève; QWGN: Quaternion white Gaussian noise; WL: Widely linear.

Competing interests

The authors declare that they have no competing interests.

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