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Research Article

Regularization and Iterative Methods for Monotone Variational Inequalities

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We provide a general regularization method for monotone variational inequalities, where the regularizer is a Lipschitz continuous and strongly monotone operator. We also introduce an iterative method as discretization of the regularization method. We prove that both regularization and iterative methods converge in norm.

1. Introduction

Variational inequalities (VIs) have widely been studied (see the monographs [1–3]). A monotone variational inequality problem (VIP) is stated as finding a point x^* with the following property:

$$x^* \in C, \quad \langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and A is a monotone operator in H with domain $\operatorname{dom}(A) \supset C$.

Recall that A is monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in \operatorname{dom}(A). \tag{1.2}$$

A typical example of monotone operators is the subdifferential of a proper convex lower semicontinuous function.

Variational inequality problems are equivalent to fixed point problems. As a matter of fact, x^* solves VIP (1.1) if and only if x^* solves the following fixed point problem (FPP), for any $\gamma > 0$,

$$x^* = P_C(I - \gamma A)x^*, \tag{1.3}$$

where P_C is the metric (or nearest point) projection from H onto C; namely, for each $x \in H$, $P_C x$ is the unique point in C with the property

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}. \tag{1.4}$$

The equivalence between VIP (1.1) and FPP (1.3) is an immediate consequence of the following characterization of P_C:

$$\text{Given } x \in H \text{ and } z \in C; \text{then } z = P_C x \Longleftrightarrow \langle x - z, y - z\rangle \le 0, \quad \forall y \in C. \tag{1.5}$$

The dual VIP of (1.1) is the following VIP:

$$x^* \in C, \quad \langle Ax, x - x^*\rangle \ge 0, \quad x \in C. \tag{1.6}$$

The following equivalence between the dual VIP (1.6) and the primal VIP (1.1) plays a useful role in our regularization in Section 2.

Lemma 1.1 (cf. [4]). *Assume that $A : C \to H$ is monotone and weakly continuous along segments (i.e., $A((1-t)x + ty) \to Ax$ weakly as $t \to 0$ for $x, y \in C$), then the dual VIP* (1.6) *is equivalent to the primal VIP* (1.1).

To guarantee the existence and uniqueness of a solution of VIP (1.1), one has to impose conditions on the operator A. The following existence and uniqueness result is well known.

Theorem 1.2. *If A is Lipschitz continuous and strongly monotone, then there exists one and only one solution to VIP* (1.1).

However, if A fails to be Lipschitz continuous or strongly monotone, then the result of the above theorem is false in general. We will assume that A is Lipschitz continuous, but do not assume strong monotonicity of A. Thus, VIP (1.1) is ill-posed and regularization is needed; moreover, a solution is often sought through iteration methods.

In the special case where A is of the form $A = I - T$, with T being a nonexpansive mapping, regularization and iterative methods for VIP (1.1) have been investigated in literature; see, for example, [5–19]; work related to variational inequalities of monotone operators can be found in [20–25], and work related to iterative methods for nonexpansive mappings can be found in [26–33].

The aim of this paper is to provide a regularization and its induced iteration method for VIP (1.1) in the general case. The paper is structured as follows. In the next section we present a general regularization method for VI (1.1) with the regularizer being a Lipschitz continuous and strongly monotone operator. In Section 3, by discretizing the implicit method of the regularization obtained in Section 2, we introduce an iteration process and prove its strong convergence. In the final section, Section 4, we apply the results obtained in Sections 2 and 3 to a convex minimization problem.

2. Regularization

Since VIP (1.1) is usually ill-posed, regularization is necessary, towards which we let $B : H \to H$ be a Lipschitz continuous, everywhere defined, strongly monotone, and single-valued operator. Consider the following regularized variational inequality problem:

$$x_\varepsilon \in C, \quad \langle Ax_\varepsilon + \varepsilon Bx_\varepsilon, x - x_\varepsilon \rangle \geq 0, \quad x \in C. \tag{2.1}$$

Since $A + \varepsilon B$ is strongly monotone, VI (2.1) has a unique solution which is denoted by $x_\varepsilon \in C$. Indeed, VI (2.1) is equivalent to the fixed point equation

$$x_\varepsilon = P_C\bigl(I - \gamma(A + \varepsilon B)\bigr)x_\varepsilon \equiv T_\varepsilon x_\varepsilon, \tag{2.2}$$

where $T_\varepsilon = P_C(I - \gamma(A + \varepsilon B)) \equiv P_C(I - \gamma F_\varepsilon)$, with $F_\varepsilon = A + \varepsilon B$.

To analyze more details of VI (2.1) (or its equivalent fixed point equation (2.2)), we need to impose more assumptions on the operators A and B. Assume that A and B are Lipschitz continuous with Lipschiz constants L_1, L_2, respectively. We also assume that B is β-strongly monotone; namely, there is a constant $\beta > 0$ satisfying the property

$$\langle Bx_1 - Bx_2, x_1 - x_2 \rangle \geq \beta \|x_1 - x_2\|^2, \quad x_1, x_2 \in H. \tag{2.3}$$

Lemma 2.1. *If γ is chosen in such a way that*

$$0 < \gamma < \frac{2\varepsilon\beta}{(L_1 + \varepsilon L_2)^2}, \tag{2.4}$$

then T_ε is a contraction with contraction coefficient

$$\sqrt{1 - \gamma\Bigl[2\varepsilon\beta - \gamma(L_1 + \varepsilon L_2)^2\Bigr]} < 1. \tag{2.5}$$

Moreover, if

$$0 < \gamma < \frac{2\varepsilon\beta}{(L_1 + \varepsilon L_2)^2 + (\varepsilon^2/4)}, \tag{2.6}$$

then

$$\sqrt{1-\gamma\left[2\varepsilon\beta-\gamma(L_1+\varepsilon L_2)^2\right]} \le 1-\frac{1}{2}\beta\varepsilon\gamma; \tag{2.7}$$

hence, T_ε is a $(1-(1/2)\beta\varepsilon\gamma)$-contraction.

Proof. Noticing that F_ε is $(L_1+\varepsilon L_2)$-Lipschitzian and $\varepsilon\beta$-strongly monotone, we deduce that, for $x, y \in H$,

$$\begin{aligned}
\|T_\varepsilon x - T_\varepsilon y\|^2 &= \|P_C(I-\gamma F_\varepsilon)x - P_C(I-\gamma F_\varepsilon)y\|^2 \\
&\le \|(I-\gamma F_\varepsilon)x-(I-\gamma F_\varepsilon)y\|^2 \\
&= \|(x-y)-\gamma(F_\varepsilon x - F_\varepsilon y)\|^2 \\
&= \|x-y\|^2 - 2\gamma\langle x-y, F_\varepsilon x - F_\varepsilon y\rangle + \gamma^2\|F_\varepsilon x - F_\varepsilon y\|^2 \\
&\le \left(1-\gamma\left[2\varepsilon\beta-\gamma(L_1+\varepsilon L_2)^2\right]\right)\|x-y\|^2.
\end{aligned} \tag{2.8}$$

It turns out that if γ satisfies (2.4), then T_ε is a contraction with coefficient given by the left side of (2.5).

Finally, it is straightforward that (2.7) holds provided that γ satisfies (2.6). □

Below we always assume that γ satisfies (2.6) so that T_ε is a $(1-(1/2)\beta\varepsilon\gamma)$-contraction from C into itself. Therefore, for such a choice of γ, T_ε has a unique fixed point in C which is denoted as x_ε whose asymptotic behavior when $\varepsilon \to 0$ is given in the following result.

Theorem 2.2. *Assume that*

(a) *$A : C \to H$ is monotone on C and weakly continuous along segments in C (i.e., $A((1-t)x+ty) \to Ax$ weakly as $t \to 0$ for $x, y \in C$),*

(b) *B is β-monotone on H,*

(c) *the solution set S of VI* (1.1) *is nonempty.*

For $\varepsilon \in (0,1)$, let x_ε be the unique solution of the regularized VIP (2.1)*. Then, as $\varepsilon \to 0$, x_ε converges in norm to a point ξ in S which is the unique solution of the VIP*

$$\xi \in S, \quad \langle B\xi, x-\xi\rangle \ge 0, \quad \forall x \in S. \tag{2.9}$$

Therefore, if one takes B to be the identity operator, then the regularized solution (x_ε) of the corresponding regularized VIP (2.1) *converges in norm to the minimal norm point of the solution set S.*

To prove Theorem 2.2, we first prove the boundedness of the net (x_ε).

Lemma 2.3. *Assume that A is monotone on C. Assume conditions (b) and (c) in Theorem 2.2. Then (x_ε) is bounded; indeed, for any $x^* \in S$,*

$$\|x^* - x_\varepsilon\| \leq \frac{1}{\beta}\|Bx^*\|, \quad \forall \varepsilon \in (0,1). \tag{2.10}$$

Proof. We have (2.1) holds for all $x \in C$. In particular, for $x^* \in S$, we have

$$\langle Ax_\varepsilon + \varepsilon Bx_\varepsilon, x^* - x_\varepsilon \rangle \geq 0. \tag{2.11}$$

It turns out that

$$\langle Ax_\varepsilon, x^* - x_\varepsilon \rangle + \varepsilon \langle Bx_\varepsilon, x^* - x_\varepsilon \rangle \geq 0. \tag{2.12}$$

Since A is monotone and B is β-strongly monotone, we have

$$\begin{gathered} \langle Ax^*, x^* - x_\varepsilon \rangle \geq \langle Ax_\varepsilon, x^* - x_\varepsilon \rangle, \\ \langle Bx^*, x^* - x_\varepsilon \rangle \geq \langle Bx_\varepsilon, x^* - x_\varepsilon \rangle + \beta \|x^* - x_\varepsilon\|^2. \end{gathered} \tag{2.13}$$

Substituting them into (2.12) we obtain

$$\varepsilon\beta \|x^* - x_\varepsilon\|^2 \leq \langle Ax^*, x^* - x_\varepsilon \rangle + \varepsilon \langle Bx^*, x^* - x_\varepsilon \rangle. \tag{2.14}$$

However, since $x^* \in S$, $\langle Ax^*, x^* - x_\varepsilon \rangle \leq 0$. We therefore get from (2.14) that

$$\|x^* - x_\varepsilon\|^2 \leq \frac{1}{\beta} \langle Bx^*, x^* - x_\varepsilon \rangle. \tag{2.15}$$

Now (2.10) follows immediately from (2.15). □

Proof of Theorem 2.2. Since (x_ε) is bounded by Lemma 2.3, the set of weak limit points as $\varepsilon \to 0$ of the net (x_ε), $\omega_w(x_\varepsilon)$, is nonempty. Pick a $\xi \in \omega_w(x_\varepsilon)$ and let (ε_n) be a null sequence in the interval $(0,1)$ such that $x_{\varepsilon_n} \to \xi$ weakly as $n \to \infty$. We first show that $\xi \in S$. To see this we use the equivalent dual VI of (2.1):

$$x_\varepsilon \in C, \quad \langle Ax + \varepsilon Bx, x - x_\varepsilon \rangle \geq 0, \quad x \in C. \tag{2.16}$$

Thus, we have, for all $x \in C$ and n,

$$\langle Ax + \varepsilon_n Bx, x - x_{\varepsilon_n} \rangle \geq 0. \tag{2.17}$$

Taking the limit as $n \to \infty$ yields that

$$\langle Ax, x - \xi \rangle \geq 0, \quad \forall x \in C. \tag{2.18}$$

It turns out that $\xi \in S$.

We next prove that the sequence $\{x_{\varepsilon_n}\}$ actually converges to ξ strongly. Replacing in (2.15) x^* with ξ gives

$$\|\xi - x_{\varepsilon_n}\|^2 \leq \frac{1}{\beta}\langle B\xi, \xi - x_{\varepsilon_n}\rangle, \quad x \in C. \tag{2.19}$$

Now it is straightforward from (2.19) that the weak convergence to ξ of $\{x_{\varepsilon_n}\}$ implies strong convergence to ξ of $\{x_{\varepsilon_n}\}$.

The relation (2.15) particularly implies that, for $\varepsilon > 0$,

$$\langle Bx^*, x^* - x_\varepsilon \rangle, \quad x^* \in S, \tag{2.20}$$

which in turns implies that every point $\xi \in \omega_w(x_\varepsilon) \subset S$ solves the VIP

$$\xi \in S, \quad \langle Bx^*, x^* - \xi \rangle \geq 0, \quad \forall x^* \in S, \tag{2.21}$$

or equivalently, the VIP

$$\xi \in S, \quad \langle B\xi, x^* - \xi \rangle \geq 0, \quad \forall x^* \in S. \tag{2.22}$$

However, since B is strongly monotone, the solution to VIP (2.22) is unique. This has shown that the unique solution ξ of VIP (2.22) is the strong limit of the net $\{x_\varepsilon\}$.

Finally, if B is the identity operator, then VIP (2.22) is reduced to

$$\langle \xi, x^* - \xi \rangle \geq 0, \quad \forall x^* \in S. \tag{2.23}$$

This is equivalent to

$$\|\xi\|^2 \leq \langle x^*, \xi \rangle, \quad \forall x^* \in S, \tag{2.24}$$

which immediately implies that $\|\xi\| \leq \|x^*\|$ for all $x^* \in S$ and hence ξ is the minimal norm of S. □

Remark 2.4. In Theorem 2.2, we have proved that if the solution set S of VIP (1.1) is nonempty, then the net (x_ε) of the solutions of the regularized VIPs (2.1) is bounded (and hence converges in norm). The converse is indeed also true; that is, the boundedness of the net (x_ε) implies that the solution set S of VIP (1.1) is nonempty. As a matter of fact, suppose that (x_ε) is bounded and $M > 0$ is a constant such that $\|x_\varepsilon\| \leq M$ for all $\varepsilon \in (0, 1)$.

By Lemma 1.1, we have

$$x_\varepsilon \in C, \quad \langle Ax + \varepsilon Bx_\varepsilon, x - x_\varepsilon \rangle \geq 0, \quad x \in C. \tag{2.25}$$

Since (x_ε) is bounded, we can easily see that every weak cluster point ξ of the net (x_ε) solves the VIP

$$\xi \in C, \quad \langle Ax, x - \xi \rangle \geq 0, \quad x \in C. \tag{2.26}$$

This is the dual VI to the primal VI (2.1); hence ξ is a solution of VI (2.1) by Lemma 1.1.

3. Iterative Method

From the fixed point equation (2.2), it is natural to consider the following iteration method that generates a sequence $\{x_n\}$ according to the recursion:

$$x_{n+1} = P_C\big(x_n - \gamma_n(Ax_n + \varepsilon_n Bx_n)\big), \quad n = 0, 1, \ldots, \tag{3.1}$$

where the initial guess $x_0 \in C$ is selected arbitrarily, and $\{\gamma_n\}$ and $\{\varepsilon_n\}$ are two sequences of positive numbers in $(0,1)$. Put in another way, $x_{n+1} \in C$ is the unique solution in C of the following VIP:

$$\langle x_n - \gamma_n(Ax_n + \varepsilon_n Bx_n) - x_{n+1}, x - x_{n+1} \rangle \leq 0, \quad x \in C. \tag{3.2}$$

Theorem 3.1. *Assume that*

(a) *A is L_1-Lipschitz continuous and monotone on C,*

(b) *B is L_2-Lipschitz continuous and β-monotone on H,*

(c) *the solution set S of VI* (1.1) *is nonempty.*

Assume in addition that

(i) $0 < \gamma_n < \beta\varepsilon_n / ((L_1 + \varepsilon_n L_2)^2 + (\varepsilon_n^2/4))$,

(ii) $\varepsilon_n \to 0$ *as* $n \to \infty$,

(iii) $\sum_{n=1}^{\infty} \varepsilon_n \gamma_n = \infty$,

(iv) $\lim_{n\to\infty}(|\gamma_n - \gamma_{n-1}| + |\varepsilon_n\gamma_n - \varepsilon_{n-1}\gamma_{n-1}|)/(\varepsilon_n\gamma_n)^2 = 0$,

then the sequence $\{x_n\}$ generated by the algorithm (3.1) *converges in norm to the unique solution of VI* (2.9).

To prove Theorem 3.1, we need a lemma below.

Lemma 3.2 (cf. [20]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \beta_n)a_n + \beta_n\sigma_n, \quad n \geq 0, \tag{3.3}$$

where $\{\beta_n\}$ *and* $\{\sigma_n\}$ *are real sequences such that*

(i) $\beta_n \in (0,1)$ *for all* n*, and* $\sum_{n=1}^{\infty} \beta_n = \infty$*;*

(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$*,*

then $\lim_{n\to\infty} a_n = 0$*.*

Proof of Theorem 3.1. Let $T_n = P_C(I - \gamma_n F_n)$, where $F_n = A + \varepsilon_n B$. By assumption (i) and Lemma 2.1, T_n is a contraction and has a unique fixed point which is denoted by z_n. Moreover, by Theorem 2.2, $\{z_n\}$ converges in norm to the unique solution ξ of VI (2.9). Therefore, it suffices to prove that $\|x_{n+1} - z_n\| \to 0$ as $n \to \infty$.

To see this, observing that T_n is a $(1 - (1/2)\beta\varepsilon_n\gamma_n)$-contraction, we obtain

$$\begin{aligned}\|x_{n+1} - z_n\| &= \|T_n x_n - T_n z_n\| \\ &\leq \left(1 - \frac{1}{2}\beta\varepsilon_n\gamma_n\right)\|x_n - z_n\| \\ &\leq \left(1 - \frac{1}{2}\beta\varepsilon_n\gamma_n\right)\|x_n - z_{n-1}\| + \|z_n - z_{n-1}\|.\end{aligned} \tag{3.4}$$

However, we have

$$\begin{aligned}\|z_n - z_{n-1}\| &= \|T_n z_n - T_{n-1} z_{n-1}\| \\ &\leq \|T_n z_n - T_n z_{n-1}\| + \|T_n z_{n-1} - \mathrm{T}_{n-1} z_{n-1}\| \\ &\leq \left(1 - \frac{1}{2}\beta\varepsilon_n\gamma_n\right)\|z_n - z_{n-1}\| + \left\|(I - \gamma_n F_n) z_{n-1} - (I - \gamma_{n-1} F_{n-1}) z_{n-1}\right\| \\ &= \left(1 - \frac{1}{2}\beta\varepsilon_n\gamma_n\right)\|z_n - z_{n-1}\| + \left\|(\gamma_n - \gamma_{n-1}) A z_{n-1} + (\varepsilon_n\gamma_n - \varepsilon_{n-1}\gamma_{n-1}) B z_{n-1}\right\|.\end{aligned} \tag{3.5}$$

Since $\{z_n\}$ is bounded, it turns out that, for an appropriate constant $M > 0$,

$$\|z_n - z_{n-1}\| \leq \frac{\left|\gamma_n - \gamma_{n-1}\right| + \left|\varepsilon_n\gamma_n - \varepsilon_{n-1}\gamma_{n-1}\right|}{\varepsilon_n\gamma_n} M. \tag{3.6}$$

Substituting (3.6) into (3.4) and setting $\beta_n = (1/2)\beta\varepsilon_n\gamma_n$, we get

$$\|x_{n+1} - z_n\| \leq (1 - \beta_n)\|x_n - z_{n-1}\| + \beta_n\sigma_n, \tag{3.7}$$

where

$$\sigma_n = \frac{\left|\gamma_n - \gamma_{n-1}\right| + \left|\varepsilon_n\gamma_n - \varepsilon_{n-1}\gamma_{n-1}\right|}{(\varepsilon_n\gamma_n)^2} M', \tag{3.8}$$

with $M' = 2M/\beta$. Assumptions (iii) and (iv) assure that $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sigma_n \to 0$ as $n \to \infty$, respectively. Therefore, we can apply lemma to (3.7) to conclude that $\|x_{n+1} - z_n\| \to 0$; hence, $x_n \to \xi$ in norm. □

Remark 3.3. Assume $0 < \varepsilon \leq \gamma < 1$ satisfy $2\varepsilon + \gamma < 1$, then it is not hard to see that for an appropriate constant $a > 0$,

$$\varepsilon_n := \frac{1}{(n+1)^{\varepsilon}}, \qquad \gamma_n := \frac{a}{(n+1)^{\gamma}}, \quad n \geq 0 \tag{3.9}$$

satisfy the assumptions (i)–(iv) of Theorem 3.1.

4. Application

Consider the constrained convex minimization problem:

$$\min_{x \in C} \varphi(x), \tag{4.1}$$

where C is a closed convex subset of a real Hilbert space H and $\varphi : H \to \mathbb{R}$ is a real-valued convex function. Assume that φ is continuously differentiable with a Lipschitz continuous gradient:

$$\|\nabla\varphi(x) - \nabla\varphi(y)\| \leq L\|x - y\|, \quad \forall x, y \in H, \tag{4.2}$$

where L is a constant.

It is known that the minimization (4.1) is equivalent to the variational inequality problem:

$$x^* \in C, \quad \langle \nabla\varphi(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{4.3}$$

Therefore, applying Theorems 2.2 and 3.1, we get the following result.

Theorem 4.1. *Assume the Lipschitz continuity* (4.2) *for the gradient* $\nabla\varphi$.

(a) *For* $\varepsilon \in (0,1)$, *let* $x_\varepsilon \in C$ *be the unique solution of the regularized VIP*

$$x_\varepsilon \in C, \quad \langle \nabla\varphi(x_\varepsilon) + \varepsilon x_\varepsilon, x - x_\varepsilon \rangle \geq 0, \quad \forall x \in C. \tag{4.4}$$

Equivalently, $x_\varepsilon \in C$ *is the unique solution in* C *of the regularized minimization problem:*

$$\min_{x \in C} \left\{ \varphi(x) + \frac{1}{2}\varepsilon\|x\|^2 \right\}. \tag{4.5}$$

Then, as $\varepsilon \to 0$, x_ε *remains bounded if and only if* (4.1) *has a solution, and in this case,* x_ε *converges in norm to the minimal norm solution of* (4.1).

(b) *Assume that* (4.1) *has a solution. Assume in addition that*

(i) $0 < \gamma_n < \varepsilon_n / ((L + \varepsilon_n)^2 + (\varepsilon_n^2/4))$,

(ii) $\varepsilon_n \to 0$ *as* $n \to \infty$,

(iii) $\sum_{n=1}^{\infty} \varepsilon_n \gamma_n = \infty$,

(iv) $\lim_{n\to\infty}(|\gamma_n - \gamma_{n-1}| + |\varepsilon_n\gamma_n - \varepsilon_{n-1}\gamma_{n-1}|)/(\varepsilon_n\gamma_n)^2 = 0$.

Starting $x_0 \in C$, one defines $\{x_n\}$ by the iterative algorithm

$$x_{n+1} = P_C\bigl(x_n - \gamma_n\bigl(\nabla\varphi(x_n) + \varepsilon_n x_n\bigr)\bigr). \tag{4.6}$$

Then $\{x_n\}$ converges in norm to the minimum-norm solution of the constrained minimization problem (4.1).

Proof. Apply Theorems 2.2 and 3.1 to the case where $A = \nabla\varphi$ and $B = I$ is the identity operator to get the conclusions in (a) and (b). □

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