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Two inequalities for the Hadamard product of matrices

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Abstract

Using an estimate on the Perron root of the nonnegative matrix in terms of paths in the associated directed graph, two new upper bounds for the Hadamard product of matrices are proposed. These bounds improve some existing results and this is shown by numerical examples.

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1 Introduction

Let M_n denote the set of all $n \times n$ complex matrices and N denote the set $\{1, 2, \dots, n\}$. Let $A = (a_{ij})$, $B = (b_{ij}) \in M_n$. If $a_{ij} - b_{ij} \geq 0$, we say that $A \geq B$, and if $a_{ij} \geq 0$, we say that A is nonnegative. The spectral radius of A is denoted by $\rho(A)$. If A is a nonnegative matrix, the Perron-Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$, where $\sigma(A)$ denotes the spectrum of A .

If there does not exist a permutation matrix P such that

$$P^T A P = \begin{pmatrix} A_1 & A_{12} \\ 0 & A_2 \end{pmatrix},$$

where A_1, A_2 are square matrices, then A is called irreducible. Let A be an irreducible nonnegative matrix. It is well known that there exists a positive vector u such that $Au = \rho(A)u$. The Hadamard product of A, B is defined as $A \circ B = (a_{ij}b_{ij}) \in M_n$. Let $A \in M_m$, and let

$$r_i(A) = \sum_{j=1}^n |a_{ij}|, \quad 1 \leq i \leq n, \quad R_i(A) = \sum_{j \neq i}^n |a_{ij}|, \quad 1 \leq i \leq n,$$

denote the absolute row sums and the deleted absolute row sums of A , respectively.

Let $\zeta(A)$ represent the set of all simple circuits in the digraph $\Gamma(A)$ of A . Recall that a circuit of length k in $\Gamma(A)$ is an ordered sequence $\gamma = (i_1, \dots, i_k, i_{k+1})$, where $i_1, \dots, i_k \in N$ are all distinct, $i_{k+1} = i_1$. The set $\{i_1, \dots, i_k\}$ is called the support of γ and is denoted by $\bar{\gamma}$. The length of the circuit is denoted by $|\gamma|$.

In [1], there is a simple estimate for $\rho(A \circ B)$: if $A \geq 0, B \geq 0$, then $\rho(A \circ B) \leq \rho(A)\rho(B)$.

Recently, using the Gersgorin theorem that involves only elements in one row or column of the matrix, Fang [2] and Huang [3] gave new estimates for $\rho(A \circ B)$ that were better than the result of [1]. Using the Brauer theorem that involves elements in two

rows of the matrix at a time, the authors of [4,5] derived new upper bounds for $\rho(A \circ B)$ that improved the results of [2,3]. As we all know, besides Gersgorin theorem and Brauer theorem, Brualdi theorem is also an important eigenvalue inclusion theorem and it involves more elements of the matrix than the other two theorems. In view of this, Liu [4] proposed the following problem: Could we get some new estimate better than the previous results using Brualdi theorem? In this paper, we give affirmative conclusions. Two new upper bounds for $\rho(A \circ B)$ are provided. These bounds improve some existing results and numerical examples illustrate that our results are superior.

2 Main results

First, we give some lemmas which are useful for obtaining the main results.

Lemma 2.1 [6] *Let $A \in M_n$ be a nonnegative matrix. If A_k is a principal submatrix of A , then $\rho(A_k) \leq \rho(A)$. If A is irreducible and $A_k \neq A$, then $\rho(A_k) < \rho(A)$.*

Lemma 2.2 [7] *Let $A \in M_n$ be a nonnegative matrix, and let $\zeta(A) \neq \emptyset$. Then for any diagonal matrix D with positive diagonal entries, we have*

$$\min_{\gamma \in \zeta(A)} \left[\prod_{i \in \bar{\gamma}} r_i(D^{-1}AD) \right]^{1/|\gamma|} \leq \rho(A) \leq \max_{\gamma \in \zeta(A)} \left[\prod_{i \in \bar{\gamma}} r_i(D^{-1}AD) \right]^{1/|\gamma|}.$$

Lemma 2.3 [4] *Let $A, B \in M_n$. If E, F are diagonal matrices of order n , then*

$$E(A \circ B)F = (EAF) \circ B = (EA) \circ (BF) = (AF) \circ (EB) = A \circ (EBF).$$

Theorem 2.1 *Let $A, B \in M_n$ and $A \geq 0, B \geq 0$. Then*

$$\rho(A \circ B) \leq \min_{\gamma \in \zeta(A \circ B)} \left[\prod_{i \in \bar{\gamma}} (2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)) \right]^{1/|\gamma|}. \quad (1)$$

Proof. If $A \circ B$ is irreducible, then A and B are irreducible. From Lemma 2.1, we have

$$\begin{aligned} \rho(A) - a_{ii} &> 0, \quad \forall i \in N, \\ \rho(B) - b_{ii} &> 0, \quad \forall i \in N. \end{aligned}$$

Since $A = (a_{ij}), B = (b_{ij})$ are nonnegative irreducible, there exist two positive vectors u, v such that $Au = \rho(A)u, Bv = \rho(B)v$. Thus, we have

$$a_{ii} + \sum_{j \neq i} \frac{a_{ij}u_j}{u_i} = \rho(A), \quad \forall i \in N, \quad (2)$$

and

$$b_{ii} + \sum_{j \neq i} \frac{b_{ij}v_j}{v_i} = \rho(B), \quad \forall i \in N. \quad (3)$$

Define $U = \text{diag}(u_1, \dots, u_n), V = \text{diag}(v_1, \dots, v_n)$. Let $\hat{A} = (\hat{A}_{ij}) = U^{-1}AU, \hat{B} = (\hat{B}_{ij}) = V^{-1}BV$. From (2) and (3), we have

$$r_i(\hat{A}) = a_{ii} + \sum_{j \neq i} \frac{a_{ij}u_j}{u_i} = \rho(A), \quad \forall i \in N.$$

and

$$r_i(\hat{B}) = b_{ii} + \sum_{j \neq i} \frac{b_{ij}v_j}{v_i} = \rho(B), \quad \forall i \in N.$$

Let $D = VU$. According to Lemma 2.2, for the positive diagonal matrix D , we have

$$\rho(A \circ B) \leq \max_{\gamma \in \zeta(A \circ B)} \left[\prod_{i \in \bar{\gamma}} r_i[D^{-1}(A \circ B)D] \right]^{1/|\gamma|}.$$

Using Lemma 2.3, we have

$$D^{-1}(A \circ B)D = U^{-1}V^{-1}(A \circ B)VU = U^{-1}(A \circ (V^{-1}BV))U = (U^{-1}AU) \circ (V^{-1}BV) = \hat{A} \circ \hat{B}.$$

Then,

$$\begin{aligned} r_i[D^{-1}(A \circ B)D] &= r_i(\hat{A} \circ \hat{B}) = a_{ii}b_{ii} + \sum_{j \neq i} \hat{A}_{ij}\hat{B}_{ij} \\ &\leq a_{ii}b_{ii} + \sum_{j \neq i} \hat{A}_{ij} \sum_{j \neq i} \hat{B}_{ij} = a_{ii}b_{ii} + (\rho(A) - a_{ii})(\rho(B) - b_{ii}). \end{aligned}$$

So, we have

$$\rho(A \circ B) \leq \max_{\gamma \in \zeta(A \circ B)} \left[\prod_{i \in \bar{\gamma}} (2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)) \right]^{1/|\gamma|}.$$

If $A \circ B$ is reducible, then one of A and B is reducible. If we denote by $P = (p_{ij})$ the $n \times n$ permutation matrix with $p_{12} = p_{23} = \dots = p_{n1} = 1$, the remaining $p_{ij} = 0$, then both $A + tP$ and $B + tP$ are nonnegative irreducible matrices for any chosen positive real numbers t . Now, we substitute $A + tP$ and $B + tP$ for A and B , respectively in the previous case, and then letting $t \rightarrow 0$, the result follows by continuity.

Two bounds for $\rho(A \circ B)$ given in [2] and [4], respectively, are

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\}, \quad (4)$$

and

$$\begin{aligned} \rho(A \circ B) &\leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + \left[\left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \right. \right. \right. \\ &\quad \left. \left. \left. + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})(\rho(A) - a_{jj})(\rho(B) - b_{jj}) \right]^{1/2} \right\}. \end{aligned} \quad (5)$$

Next, we give a simple comparison between (1) and (4). It is easy to see

$$\begin{aligned} \rho(A \circ B) &\leq \max_{\gamma \in \zeta(A \circ B)} \left[\prod_{i \in \bar{\gamma}} (2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)) \right]^{1/|\gamma|} \\ &\leq \max_{\gamma \in \zeta(A \circ B)} \left[\left(\max_{i \in N} \{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\} \right)^{|\gamma|} \right]^{1/|\gamma|} \\ &= \max_{i \in N} \{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\}. \end{aligned}$$

Then the bound (1) is better than the bound (4). From the difference between (1) and (5), we could not verify that (1) is better than (5) in theoretical analysis, but the following numerical example shows that the result derived in Theorem 2.1 is better than (4) and (5).

Example 2.1. Consider two 4×4 nonnegative matrices

$$A = \begin{pmatrix} 4 & 1 & 0 & 2 \\ 0 & 0.05 & 1 & 1 \\ 0 & 0 & 4 & 0.5 \\ 1 & 0.5 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is easy to calculate that $\rho(A \circ B) = \rho(A) = 5.4983$. By inequalities (4) and (5), we have

$$\rho(A \circ B) \leq \max_{1 \leq i \leq 4} \{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\} = 16.3949,$$

and $\rho(A \circ B) = 11.6478$, and by Theorem 2.1, we get

$$\rho(A \circ B) \leq \max_{\gamma \in \zeta(A \circ B)} \left[\prod_{i \in \bar{\gamma}} (2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)) \right]^{1/|\gamma|} = 10.0126.$$

Next, we will give the second inequality for $\rho(A \circ B)$. For $A \geq 0$, write $L = A - D$, where $D = \text{diag}(a_{11}, \dots, a_{nn})$. We denote $J_A = D_1^{-1}L$ with $D_1 = \text{diag}(d_{ii})$, where

$$d_{ii} = \begin{cases} a_{ii}, & \text{if } a_{ii} \neq 0, \\ 1, & \text{if } a_{ii} = 0. \end{cases}$$

Then, J_A is nonnegative, and $J_A = A$ if $a_{ii} = 0$ for all i . For $B \geq 0$, let $D_2 = \text{diag}(s_{ii})$, with

$$s_{ii} = \begin{cases} b_{ii}, & \text{if } b_{ii} \neq 0, \\ 1, & \text{if } b_{ii} = 0. \end{cases}$$

Then the nonnegative matrix J_B can be similarly defined.

Theorem 2.2 *Let $A, B \in M_n$ and $A \geq 0, B \geq 0$. Then*

$$\rho(A \circ B) \leq \max_{\gamma \in \zeta(A \circ B)} \left[\prod_{i \in \bar{\gamma}} (a_{ii}b_{ii} + d_{ii}\rho(J_A)s_{ii}\rho(J_B)) \right]^{1/|\gamma|}. \quad (6)$$

Proof. If $A \circ B$ is nonnegative irreducible, then A and B are irreducible. Since J_A and J_B are also nonnegative irreducible, there exist two positive vectors x, y such that $J_A x = \rho(J_A)x, J_B y = \rho(J_B)y$. So, we have

$$\sum_{j \neq i} \frac{a_{ij}x_j}{x_i} = d_{ii}\rho(J_A), \quad \sum_{j \neq i} \frac{b_{ij}y_j}{y_i} = s_{ii}\rho(J_B).$$

Let $\tilde{A} = (\tilde{a}_{ij}) = \tilde{U}^{-1}A\tilde{U}$, and $\tilde{B} = (\tilde{b}_{ij}) = \tilde{V}^{-1}B\tilde{V}$ in which \tilde{U} and \tilde{V} are nonsingular diagonal matrices $\tilde{U} = \text{diag}(x_1, \dots, x_n)$ and $\tilde{V} = \text{diag}(y_1, \dots, y_n)$.

From Lemma 2.3, we have

$$(\tilde{V}\tilde{U})^{-1}(A \circ B)(\tilde{V}\tilde{U}) = (\tilde{U}^{-1}A\tilde{U}) \circ (\tilde{V}^{-1}B\tilde{V}) = \tilde{A} \circ \tilde{B},$$

and then

$$r_i(\tilde{A} \circ \tilde{B}) = a_{ii}b_{ii} + \sum_{j \neq i} \tilde{a}_{ij}\tilde{b}_{ij} \leq a_{ii}b_{ii} + \sum_{j \neq i} \frac{a_{ij}x_j}{x_i} \sum_{j \neq i} \frac{b_{ij}y_j}{y_i} = a_{ii}b_{ii} + d_{ii}\rho(J_A)s_{ii}\rho(J_B).$$

Let $W = \tilde{V}\tilde{U}$. Then for the positive diagonal matrix W , it follows from Lemma 2.2 that

$$\begin{aligned} \rho(A \circ B) &\leq \max_{\gamma \in \zeta(A \circ B)} \left[\prod_{i \in \bar{\gamma}} r_i[W^{-1}(A \circ B)W] \right]^{1/|\gamma|} \\ &= \max_{\gamma \in \zeta(A \circ B)} \left[\prod_{i \in \bar{\gamma}} r_i[\tilde{A} \circ \tilde{B}] \right]^{1/|\gamma|} \\ &\leq \max_{\gamma \in \zeta(A \circ B)} \left[\prod_{i \in \bar{\gamma}} (a_{ii}b_{ii} + d_{ii}\rho(J_A)s_{ii}\rho(J_B)) \right]^{1/|\gamma|}. \end{aligned}$$

If $A \circ B$ is reducible, then substituting $A + tP$ and $B + tP$ for A and B , respectively in the previous case, letting $t \rightarrow 0$, the result is derived.

The bounds for $\rho(A \circ B)$ obtained in [3] and [5], respectively, are

$$\rho(A \circ B) \leq \max_{1 \leq i \leq n} \{a_{ii}b_{ii} + d_{ii}\rho(J_A)s_{ii}\rho(J_B)\}, \tag{7}$$

and

$$\begin{aligned} \rho(A \circ B) &\leq \max_{i \neq j} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \right. \right. \\ &\quad \left. \left. + 4d_{ii}s_{ii}d_{jj}s_{jj}\rho^2(J_A)\rho^2(J_B) \right]^{\frac{1}{2}} \right\}. \end{aligned} \tag{8}$$

It can be easily verified that the bound (6) is better than the bound (7). Here too, we could not give the comparison between (6) and (8), but the following example shows that the result obtained in Theorem 2.2 is better than (7) and (8).

Example 2.2. Let

$$A = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 1 & 4 & 0.5 & 0.5 \\ 1 & 0 & 3 & 0.5 \\ 0.5 & 1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0.5 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 \\ 0.5 & 0 & 2 & 0.5 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Then

$$A \circ B = \begin{pmatrix} 4 & 0 & 0.5 & 0.5 \\ 1 & 4 & 0.5 & 0.5 \\ 0.5 & 0 & 6 & 0.25 \\ 0 & 1 & 1 & 4 \end{pmatrix}.$$

It is clear that $\rho(J_A) = 0.8182$, $\rho(J_B) = 1.1258$, and $\rho(A \circ B) = 6.3365$. By (7) and (8), we have

$$\rho(A \circ B) \leq \max_{1 \leq i \leq 4} \{a_{ii}b_{ii} + d_{ii}\rho(J_A)s_{ii}\rho(J_B)\} = 11.5266,$$

and $\rho(A \circ B) = 9.6221$, and by Theorem 2.2, we get

$$\rho(A \circ B) \leq \max_{\gamma \in \mathcal{S}(A \circ B)} \left[\prod_{i \in \bar{\gamma}} (a_{ii}b_{ii} + d_{ii}\rho(J_A)s_{ii}\rho(J_B)) \right]^{1/|\gamma|} = 9.4116.$$

3 Conclusions

In this paper, we propose two new upper bounds for the Hadamard product of matrices. These bounds are better than the results of [2,3] and numerical examples illustrate that our results are superior than the previous results of [2-5].

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Competing interests

The author declares that they have no competing interests.

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