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Dual Orlicz geominimal surface area

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Abstract

The L_p -geominimal surface area was introduced by Lutwak in 1996, which extended the important concept of the geominimal surface area. Recently, Wang and Qi defined the p -dual geominimal surface area, which belongs to the dual Brunn-Minkowski theory. In this paper, based on the concept of the dual Orlicz mixed volume, we extend the dual geominimal surface area to the Orlicz version and give its properties. In addition, the isoperimetric inequality, a Blaschke-Santaló type inequality, and the monotonicity inequality for the dual Orlicz geominimal surface areas are established.

MSC: 52A30; 52A40**Keywords:** convex bodies; star bodies; dual Orlicz mixed volume; dual Orlicz geominimal surface area

1 Introduction

Let \mathcal{K}^n denote the set of convex bodies (compact, convex subsets with non-empty interiors) in Euclidean space \mathbb{R}^n . For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroids lie at the origin, and the set of convex bodies having their Santaló point at the origin in \mathbb{R}^n , we write \mathcal{K}_o^n , \mathcal{K}_c^n , and \mathcal{K}_s^n , respectively. \mathcal{S}_o^n denotes the set of star bodies (about the origin) in \mathbb{R}^n . Let $V(K)$ denote the n -dimensional volume of a body K , and let B denote the standard Euclidean unit ball in \mathbb{R}^n and write $\omega_n = V(B)$ for its volume, as well as let S^{n-1} denote the unit sphere for B .

The notion of geominimal surface area was introduced by Petty [1]. For $K \in \mathcal{K}_o^n$, the geominimal surface area, $G(K)$, of K is defined by

$$\omega_n^{\frac{1}{n}} G(K) = \inf \{ n V_1(K, Q) V(Q^*)^{\frac{1}{n}} : Q \in \mathcal{K}^n \}.$$

Here Q^* denotes the polar of body Q , and $V_1(K, Q)$ denotes the mixed volume of $K, Q \in \mathcal{K}_o^n$ (see [2]). This concept has already attracted considerable interest; see, for example, [3–5].

The extension of the classical Brunn-Minkowski theory is the L_p -Brunn-Minkowski theory initialized by Lutwak [6], which has had an enormous impact, providing stronger affine isoperimetric inequalities than the classical counterparts (see, e.g., [7, 8]). In particular, Lutwak [6] introduced the p -geominimal surface area: For $K \in \mathcal{K}_o^n$, $p \geq 1$, the p -geominimal surface area, $G_p(K)$, of K is defined by

$$\omega_n^{\frac{p}{n}} G_p(K) = \inf \{ n V_p(K, Q) V(Q^*)^{\frac{p}{n}} : Q \in \mathcal{K}_o^n \}.$$

Here $V_p(K, Q)$ denotes the p -mixed volume of $K, Q \in \mathcal{K}_o^n$ (see [6]).

Combining the homogeneousness of the volume with the p -mixed volume, the p -geominimal surface area could also be defined by

$$G_p(K) = \inf\{nV_p(K, Q) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\}.$$

The recent extension of the Brunn-Minkowski theory is the Orlicz-Brunn-Minkowski theory launched by Lutwak *et al.* [9, 10]. This theory is far more general than the L_p -Brunn-Minkowski theory, and has already attracted considerable interest; see, for example, [9–14]. In particular, Yuan *et al.* [15] introduced the Orlicz geominimal surface area $G_\phi(K)$ of $K \in \mathcal{K}_o^n$. Let Φ denote the set of convex functions $\phi : [0, \infty) \rightarrow [0, \infty)$ that satisfy $\phi(0) = 0$, $\phi(1) = 1$, and $\lim_{t \rightarrow \infty} \phi(t) = +\infty$. For $K \in \mathcal{K}_o^n$ and $\phi \in \Phi$,

$$G_\phi(K) = \inf\{nV_\phi(K, Q) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\},$$

where $V_\phi(K, Q)$ denotes Orlicz mixed volume of K and Q (see [16]).

A dual theory to the L_p -Brunn-Minkowski theory was also developed by Lutwak (see [17, 18]). More recently, Wang and Qi [19] gave a definition of the p -dual geominimal surface area which is a dual concept for L_p -geominimal surface area and belongs to the dual L_p -Brunn-Minkowski theory: For $K \in \mathcal{S}_o^n$, the p -dual geominimal surface area, $\tilde{G}_{-p}(K)$, of K is defined by

$$\omega_n^{-\frac{p}{n}} \tilde{G}_{-p}(K) = \inf\{n\tilde{V}_{-p}(K, Q)V(Q^*)^{-\frac{p}{n}} : Q \in \mathcal{K}_o^n\}.$$

Here $\tilde{V}_{-p}(M, N)$ denotes the p -dual mixed volume of $M, N \in \mathcal{S}_o^n$ (see [6]).

Further, they [19] established the affine isoperimetric inequality, the Blaschke-Santaló inequality and the monotone inequality for the p -dual geominimal surface area.

For $K \in \mathcal{S}_o^n$ and $p \geq 1$,

$$\tilde{G}_{-p}(K) \geq n\omega_n^{-\frac{p}{n}} V(K)^{\frac{n+p}{n}}, \tag{1.1}$$

with equality if and only if K is an ellipsoid centered at the origin.

If $K \in \mathcal{K}_c^n$ and $p \geq 1$, then

$$\tilde{G}_{-p}(K)\tilde{G}_{-p}(K^*) \leq (n\omega_n)^2, \tag{1.2}$$

with equality if and only if K is an ellipsoid.

If $K \in \mathcal{S}_o^n$, $1 \leq p < q$, then

$$\left(\frac{\tilde{G}_{-p}(K)^n}{n^n V(K)^{n+p}}\right)^{\frac{1}{p}} \leq \left(\frac{\tilde{G}_{-q}(K)^n}{n^n V(K)^{n+q}}\right)^{\frac{1}{q}}. \tag{1.3}$$

Combining the homogeneousness of the volume and the p -dual mixed volume, the p -dual geominimal surface area could also be defined by

$$\tilde{G}_{-p}(K) = \inf\{n\tilde{V}_{-p}(K, Q) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\}.$$

A step toward a dual Orlicz-Brunn-Minkowski theory for star sets has already been made by Gardner *et al.* [20–22] and Zhu *et al.* [23]. In some respects, the dual Orlicz-Brunn-Minkowski theory is far more general than the Orlicz-Brunn-Minkowski theory for convex bodies.

The purpose of this article is to define a new operator $\tilde{G}_{-\phi}(K)$ of $K \in S_o^n$, called the dual Orlicz geominimal surface area, by

$$\tilde{G}_{-\phi}(K) = \inf\{n\tilde{V}_{-\phi}(K, Q) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\},$$

where $\tilde{V}_{-\phi}(K, Q)$ denotes the dual Orlicz mixed volume of K, Q (see Section 2).

Our main results are as follows.

Theorem 1.1 *If $K \in S_o^n$ and $\phi \in \Phi$, then there exists a unique body $\tilde{K} \in \mathcal{K}_o^n$ such that*

$$\tilde{G}_{-\phi}(K) = n\tilde{V}_{-\phi}(K, \tilde{K}) \quad \text{and} \quad V(\tilde{K}^*) = \omega_n. \tag{1.4}$$

Theorem 1.2 *Let $K \in S_o^n$ and $\phi \in \Phi$, then*

$$\tilde{G}_{-\phi}(K) \geq nV(K)\phi\left(\left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}}\right), \tag{1.5}$$

with equality if and only if K is an ellipsoid.

When $\phi(t) = t^p$, with $p \geq 1$, the above affine isoperimetric inequality for the dual Orlicz geominimal surface area reduces to affine isoperimetric inequality (1.1) for the p -dual geominimal surface area.

Theorem 1.3 *Let $K \in \mathcal{K}_c^n$ and $\phi \in \Phi$, then*

$$\tilde{G}_{-\phi}(K)\tilde{G}_{-\phi}(K^*) \leq (n\omega_n)^2,$$

with equality if and only if K is an ellipsoid.

When $\phi(t) = t^p$, with $p \geq 1$, the above Blaschke-Santaló type inequality for the dual Orlicz geominimal surface area reduces to the Blaschke-Santaló type inequality (1.2) for the p -dual geominimal surface area.

Theorem 1.4 *Let $\phi_1, \phi_2 \in \Phi$ and $\phi_1 \leq \phi_2$. If $K \in S_o^n$, then*

$$\phi_2^{-1}\left(\frac{(nV(K))^n}{\tilde{G}_{-\phi_2}(K)^n\phi_2(V(K)^{-1})}\right) \leq \phi_1^{-1}\left(\frac{(nV(K))^n}{\tilde{G}_{-\phi_1}(K)^n\phi_1(V(K)^{-1})}\right).$$

When $\phi_1(t) = t^p$, $\phi_2(t) = t^q$, with $1 \leq p \leq q < \infty$, the above inequality reduces to Wang’s monotonicity inequality (1.3) for the p -dual geominimal surface area ratio.

2 Background material

The support function $h_K = h(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}$ of a compact convex set $K \subset \mathbb{R}^n$ is defined, for $u \in S^{n-1}$, by

$$h_K(u) = \max\{u \cdot x : x \in K\},$$

and it uniquely determines the compact convex set. Here $u \cdot x$ denotes the standard inner product of u and x in \mathbb{R}^n .

For $K, L \in \mathcal{K}^n$, and $\lambda, \mu \geq 0$ (not both zero), the Minkowski linear combination $\lambda \cdot K + \mu \cdot L \in \mathcal{K}^n$ is defined by

$$h(\lambda \cdot K + \mu \cdot L, \cdot) = \lambda h(K, \cdot) + \mu h(L, \cdot).$$

The classical Brunn-Minkowski inequality (see [24]) states that for convex bodies $K, L \in \mathcal{K}^n$ and real $\lambda, \mu \geq 0$ (not both zero), the volume of the bodies and of their Minkowski linear combination $\lambda \cdot K + \mu \cdot L \in \mathcal{K}^n$ are related by

$$V(\lambda \cdot K + \mu \cdot L)^{\frac{1}{n}} \geq \lambda V(K)^{\frac{1}{n}} + \mu V(L)^{\frac{1}{n}}, \tag{2.1}$$

with equality if and only if K and L are homothetic.

For real $p \geq 1$, $K, L \in \mathcal{K}_o^n$, and $\lambda, \mu \geq 0$ (not both zero), the Firey linear combination $\lambda \cdot K +_p \mu \cdot L$, is defined by (see [25])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p.$$

Firey [25] also established the following L_p Brunn-Minkowski inequality. If $p > 1$, $\lambda, \mu \geq 0$ (not both zero), and $K, L \in \mathcal{K}_o^n$, then

$$V(\lambda \cdot K +_p \mu \cdot L)^{\frac{p}{n}} \geq \lambda V(K)^{\frac{p}{n}} + \mu V(L)^{\frac{p}{n}},$$

with equality if and only if K and L are dilates.

The radial function of K , $\rho_K : S^{n-1} \rightarrow [0, \infty)$, is defined by

$$\rho_K(u) = \max\{\lambda : \lambda u \in K\}.$$

A set $K \subset \mathbb{R}^n$ is said to be a star body about the origin, if the line segment from the origin to any point $x \in K$ is contained in K and K has continuous and positive radial function $\rho_K(\cdot)$.

Note that $K \in \mathcal{S}_o^n$ can be uniquely determined by its radial function $\rho_K(\cdot)$ and *vice versa*. If $\lambda > 0$, we have

$$\rho_K(\lambda x) = \lambda^{-1} \rho_K(x) \quad \text{and} \quad \rho_{\lambda K}(x) = \lambda \rho_K(x).$$

More generally, from the definition of the radial function it follows immediately that for $\Lambda \in GL(n)$ the radial function of the image $\Lambda K = \{\Lambda y : y \in K\}$ of $K \in \mathcal{S}_o^n$ is given by (see [26])

$$\rho(\Lambda K, x) = \rho(K, \Lambda^{-1}x), \quad \text{for all } x \in \mathbb{R}^n. \tag{2.2}$$

Two star bodies $K, L \in \mathcal{S}_o^n$ are said to be dilates of each other if there is a constant $\lambda > 0$ such that $L = \lambda K$, and the equation $\rho_L(u) = \lambda \rho_K(u)$ for all $u \in S^{n-1}$. Clearly, for $K, L \in \mathcal{S}_o^n$,

$$K \subseteq L \quad \text{if and only if} \quad \rho_K(u) \leq \rho_L(u), \quad \text{for all } u \in S^{n-1}.$$

The natural metric on \mathcal{S}_o^n is the radial metric $\tilde{\delta}(\cdot, \cdot) : \mathcal{S}_o^n \times \mathcal{S}_o^n \rightarrow \mathbb{R}$ defined as

$$\tilde{\delta}(K, L) = \|\rho_K - \rho_L\|_\infty = \sup_{u \in S^{n-1}} |\rho_K(u) - \rho_L(u)|, \quad \text{for } K, L \in \mathcal{S}_o^n.$$

A sequence of star bodies $\{K_j\} \subset \mathcal{S}_o^n$ is said to be convergent to $K \in \mathcal{S}_o^n$ in $\tilde{\delta}$ if $\tilde{\delta}(K_j, K) \rightarrow 0$ as $j \rightarrow \infty$, and equivalently, ρ_{K_j} is uniformly convergent to ρ_K on S^{n-1} .

If $K \in \mathcal{K}_o^n$, the polar body, K^* , of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, \text{ for all } y \in K\}.$$

Obviously, we have $(K^*)^* = K$.

The Blaschke-Santaló inequality [27] is one of the fundamental affine isoperimetric inequalities. It states that if $Q \in \mathcal{K}_c^n$ then

$$V(Q)V(Q^*) \leq \omega_n^2, \tag{2.3}$$

with equality if and only if Q is an ellipsoid.

For $\phi \in \Phi$ and $\lambda, \mu \geq 0$ (not both zero), we define the Orlicz radial sum $\lambda \circ K \tilde{+}_{-\phi} \mu \circ L$ of two star bodies $K, L \in \mathcal{S}_o^n$, by

$$\rho(\lambda \circ K \tilde{+}_{-\phi} \mu \circ L, u) = \sup \left\{ t > 0 : \lambda \phi \left(\frac{t}{\rho_K(u)} \right) + \mu \phi \left(\frac{t}{\rho_L(u)} \right) \leq 1 \right\}, \tag{2.4}$$

for all $u \in S^{n-1}$.

When $\phi(t) = t^p$, with $p \geq 1$, it is easy to show that the Orlicz radial sum reduces to Lutwak's p -harmonic radial combination (see [6]):

$$\rho(\mu \circ K \tilde{+}_{-p} \mu \circ L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}.$$

If $K, L \in \mathcal{K}_o^n$, then

$$\lambda \circ K \tilde{+}_{-p} \mu \circ L = (\lambda \cdot K^* +_p \mu \cdot L^*)^*.$$

We denote the right derivative of a real-valued function f by f'_r . For $\phi \in \Phi$, there is $\phi'_r(1) > 0$ because ϕ is convex and strictly increasing.

Let $\phi \in \Phi$. By the Orlicz radial sum (2.4), we define the dual Orlicz mixed volume $\tilde{V}_{-\phi}(K, L)$ of convex bodies $K, L \in \mathcal{S}_o^n$ by

$$\frac{n}{-\phi'_r(1)} \tilde{V}_{-\phi}(K, L) = \lim_{\varepsilon \rightarrow 0^+} \frac{V(K \tilde{+}_{-\phi} \varepsilon \circ L) - V(K)}{\varepsilon}. \tag{2.5}$$

From (2.5) we easily obtain the following integral formula of the dual Orlicz mixed volume:

$$\tilde{V}_{-\phi}(K, L) = \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_L(u)}\right) \rho_K^n(u) \, dS(u). \tag{2.6}$$

Apparently, we have

$$\tilde{V}_{-\phi}(K, K) = V(K). \tag{2.7}$$

For $\phi(t) = t^p$ with $p \geq 1$, the dual Orlicz mixed volume $\tilde{V}_{\phi}(K, L)$ reduces to Lutwak's p -dual mixed volume formula (see [6]):

$$\tilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(u) \rho_L^{-p}(u) \, dS(u),$$

for all $K, L \in \mathcal{S}_o^n$.

Using the same argument as in [23] we establish the following dual Orlicz-Minkowski inequality: Suppose $\phi \in \Phi$. If $K, L \in \mathcal{S}_o^n$, then

$$\tilde{V}_{-\phi}(K, L) \geq V(K) \phi\left(\left(\frac{V(K)}{V(L)}\right)^{\frac{1}{n}}\right), \tag{2.8}$$

with equality if and only if K and L are dilates of each other.

When $\phi(t) = t^p$, with $p \geq 1$. Then (2.8) reduces to the following L_p -dual Minkowski inequality (see [6]):

$$\tilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p},$$

with equality if and only if K and L are dilates of each other.

We also establish the dual Orlicz-Brunn-Minkowski inequality as follows: Suppose $K, L \in \mathcal{S}_o^n$, and $\lambda, \mu > 0$. If $\phi \in \Phi$, then

$$\lambda \phi\left(\left(\frac{V(\lambda \circ K \tilde{\tau}_{\phi} \mu \circ L)}{V(K)}\right)^{\frac{1}{n}}\right) + \mu \phi\left(\left(\frac{V(\lambda \circ K \tilde{\tau}_{\phi} \mu \circ L)}{V(L)}\right)^{\frac{1}{n}}\right) \leq 1,$$

with equality if and only if K and L are dilates of each other.

When $\phi(t) = t^p$, with $p \geq 1$, the above inequality reduces to Lutwak's L_p -dual Brunn-Minkowski inequality (see [6]):

$$V(\lambda \circ K \tilde{\tau}_{-p} \mu \circ L)^{-\frac{p}{n}} \geq \lambda V(K)^{-\frac{p}{n}} + \mu V(L)^{-\frac{p}{n}},$$

with equality if and only if K and L are dilates of each other.

The following results are required in the proofs of our main results.

Lemma 2.1 *If $\phi \in \Phi$, and $K, L \in \mathcal{S}_o^n$, then for $\Lambda \in \text{SL}(n)$,*

$$\tilde{V}_{-\phi}(\Lambda K, \Lambda L) = \tilde{V}_{\phi}(K, L).$$

Proof For $x \in \mathbb{R}^n \setminus \{0\}$, let $\langle x \rangle = x/|x|$. By (2.6) and (2.2), we have

$$\begin{aligned} \tilde{V}_{-\phi}(\Lambda K, L) &= \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_{\Lambda K}(u)}{\rho_L(u)}\right) \rho_{\Lambda K}^n(u) \, dS(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(\Lambda^{-1}u)}{\rho_L(\Lambda \Lambda^{-1}u)}\right) \rho_K^n(\Lambda^{-1}u) \, dS(\Lambda \Lambda^{-1}u) \\ &= \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(\langle \Lambda^{-1}u \rangle)}{\rho_{\Lambda^{-1}L}(\langle \Lambda^{-1}u \rangle)}\right) \rho_K^n(\langle \Lambda^{-1}u \rangle) |\det \Lambda| \, dS(\langle \Lambda^{-1}u \rangle) \\ &= \tilde{V}_{-\phi}(K, \Lambda^{-1}L), \end{aligned}$$

where Λ^{-1} denotes the inverse of Λ . □

It is easy to check that $\tilde{V}_{-\phi}(\lambda K, \lambda L) = \lambda^n \tilde{V}_{-\phi}(K, L)$, for $\lambda > 0$. Therefore, we have the following.

Proposition 2.2 *Suppose $K, L \in S^n_o$. If $\phi \in \Phi$ and $\Lambda \in GL(n)$, then*

$$\tilde{V}_{-\phi}(\Lambda K, \Lambda L) = |\det \Lambda| \tilde{V}_{-\phi}(K, L).$$

Lemma 2.3 *Suppose f_i, f are strictly positive and continuous functions on S^{n-1} ; $\phi_j, \phi \in \Phi$; μ_k, μ are Borel probability measures on S^{n-1} ; $i, j, k \in \mathbb{N}$. If $f_i \rightarrow f$ pointwise, $\phi_j \rightarrow \phi$ uniformly, and $\mu_k \rightarrow \mu$ weakly, then*

$$\int_{S^{n-1}} \phi_j(f_i) \, d\mu_k \rightarrow \int_{S^{n-1}} \phi(f) \, d\mu, \quad \text{as } i, j, k \rightarrow \infty. \tag{2.9}$$

Proof The continuity of f_i and f , and $f_i \rightarrow f$ pointwise guarantee that $f_i \rightarrow f$ uniformly. Thus, there exists an $N_0 \in \mathbb{N}$, such that

$$\frac{1}{2} \min_{u \in S^{n-1}} f(u) \leq f_i(u) \leq 2 \max_{u \in S^{n-1}} f(u), \quad \text{for } i > N_0.$$

Let

$$c_m = \min \left\{ \frac{1}{2} \min_{u \in S^{n-1}} f(u), \min_{u \in S^{n-1}} f_i(u), \text{ with } i \leq N_0 \right\}$$

and

$$c_M = \min \left\{ 2 \max_{u \in S^{n-1}} f(u), \max_{u \in S^{n-1}} f_i(u), \text{ with } i \leq N_0 \right\}.$$

The strict positivity and the continuity of f_i and f imply that $0 < c_m \leq c_M < \infty$. Thus,

$$c_m \leq f(u) \leq c_M \quad \text{and} \quad c_m \leq f_i(u) \leq c_M, \quad \text{for } u \in S^{n-1} \text{ and } i \in \mathbb{N}. \tag{2.10}$$

Since $\phi_j \rightarrow \phi$ uniformly on $[c_m, c_M]$, by (2.10) and $f_i \rightarrow f$ uniformly, it follows that as $i, j \rightarrow \infty$, $\phi_j(f_i) \rightarrow \phi(f)$, uniformly on S^{n-1} . Combined with $\mu_k \rightarrow \mu$ weakly, one concludes

that, as $i, j, k \rightarrow \infty$,

$$\int_{S^{n-1}} \phi_j(f_i) \, d\mu_k \rightarrow \int_{S^{n-1}} \phi(f) \, d\mu,$$

as for (2.9) as desired. □

Using Lemma 2.3, we immediately obtain the following result.

Lemma 2.4 *Suppose $K, K_i, L, L_j \in \mathcal{S}_o^n$ and $\phi, \phi_k \in \Phi$. If $K_i \rightarrow K, L_j \rightarrow L$ and $\phi_k \rightarrow \phi$, then $\lim_{i,j,k \rightarrow \infty} \tilde{V}_{-\phi_k}(K_i, L_j) = \tilde{V}_{-\phi}(K, L)$.*

3 Dual Orlicz geominimal surface areas

In this section we first propose the following problem.

Problem 3.1 For $K \in \mathcal{S}_o^n$, find a convex body Q , amongst all convex bodies containing the origin in their interiors, which solves the constrained infimum problem

$$\tilde{G}_{-\phi}(K) := \inf(n\tilde{V}_{-\phi}(K, Q)) \quad \text{subject to} \quad V(Q^*) = \omega_n.$$

In order to demonstrate the existence and uniqueness of Problem 3.1, the following simple fact will be needed.

Lemma 3.2 ([6]) *Let \mathcal{C}^n denote the set of compact convex subsets of Euclidean n -space \mathbb{R}^n , and suppose $K_i \in \mathcal{K}_o^n$ such that $K_i \rightarrow L \in \mathcal{C}^n$. If the sequence $V(K_i^*)$ is bounded, then $L \in \mathcal{K}_o^n$.*

Now we answer Problem 3.1, namely, the proof of Theorem 1.1 is given.

Proof of Theorem 1.1 Problem 3.1 can be equivalently restated as: For a fixed star body $K \in \mathcal{S}_o^n$, there exists a sequence $M_i \in \mathcal{K}_o^n$ such that $V(M_i^*) = \omega_n$, with $\tilde{V}_{-\phi}(K, B) \geq \tilde{V}_{-\phi}(K, M_i)$, for all i , and $n\tilde{V}_{-\phi}(K, M_i) \rightarrow \tilde{G}_{-\phi}(K)$. To see that the $M_i \in \mathcal{K}_o^n$ are uniformly bounded, let

$$R_i = R(M_i) = \rho(M_i, u_i) = \max\{\rho(M_i, u) : u \in S^{n-1}\},$$

where u_i is any of the points in S^{n-1} at which this maximum is attained.

Let $r_K = \min_{S^{n-1}} \rho_K$. Then $r_K B \subseteq K$. From the definition (2.6) of the dual Orlicz mixed volume and the Jensen inequality, it follows that

$$\begin{aligned} \frac{\tilde{V}_{-\phi}(K, B)}{V(K)} &\geq \frac{\tilde{V}_{-\phi}(K, M_i)}{V(K)} \\ &= \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_{M_i}(u)}\right) \, d\tilde{V}_K^* \\ &\geq \phi\left(\int_{S^{n-1}} \frac{\rho_K(u)}{\rho_{M_i}(u)} \, d\tilde{V}_K^*\right) \\ &\geq \phi\left(\int_{S^{n-1}} \frac{\rho_K(u)}{R_i} \, d\tilde{V}_K^*\right) \\ &= \phi\left(\frac{1}{nV(K)R_i} \int_{S^{n-1}} \rho_K(u)^{n+1} \, dS(u)\right) \end{aligned}$$

$$\begin{aligned} &\geq \phi\left(\frac{r_K}{nV(K)R_i} \int_{S^{n-1}} \rho_K(u)^n \, dS(u)\right) \\ &= \phi\left(\frac{r_K}{R_i}\right). \end{aligned}$$

Namely,

$$\omega_n r_K^n \phi\left(\frac{r_K}{R_i}\right) \leq \tilde{V}_{-\phi}(K, M_i) \leq \tilde{V}_{-\phi}(K, B) < \infty.$$

Since the M_i are uniformly bounded, the Blaschke selection theorem guarantees the existence of a subsequence of the M_i , which will also be denoted by M_i , and a compact convex $L \in \mathcal{C}^n$, such that $M_i \rightarrow L$. Since $V(M_i^*) = \omega_n$, Lemma 3.2 gives $L \in \mathcal{K}_o^n$. Now, $M_i \rightarrow L$ implies that $M_i^* \rightarrow L^*$, and since $V(M_i^*) = \omega_n$, it follows that $V(L^*) = \omega_n$. Lemma 2.4 can now be used to conclude that L will serve as the desired body \tilde{K} .

The uniqueness of the minimizing body is easily demonstrated as follows. Suppose $L_1, L_2 \in \mathcal{K}_o^n$ and $L_1 \neq L_2$, such that $V(L_1^*) = \omega_n = V(L_2^*)$, and

$$\tilde{V}_{-\phi}(K, L_1) = \tilde{V}_{-\phi}(K, L_2).$$

Define $L \in \mathcal{K}_o^n$, by

$$L = \frac{1}{2} \circ L_1 \tilde{\tau}_{-1} \frac{1}{2} \circ L_2.$$

Since obviously

$$L^* = \frac{1}{2} \cdot L_1^* + \frac{1}{2} \cdot L_2^*,$$

and $V(L_1^*) = \omega_n = V(L_2^*)$, it follows from the Brunn-Minkowski inequality (2.1) that

$$V(L^*) \geq \omega_n,$$

with equality if and only if $L_1 = L_2$.

By the definition (2.6) of the dual Orlicz mixed volume, together with the convexity of ϕ , we have

$$\begin{aligned} \tilde{V}_{-\phi}(K, L) &= \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_{\frac{1}{2} \circ L_1 \tilde{\tau}_{-1} \frac{1}{2} \circ L_2}(u)}\right) \rho_K(u)^n \, dS(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\left(\frac{1}{2\rho_{L_1}(u)} + \frac{1}{2\rho_{L_2}(u)}\right)^{-1}}\right) \rho_K(u)^n \, dS(u) \\ &= \frac{1}{n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{2\rho_{L_1}(u)} + \frac{\rho_K(u)}{2\rho_{L_2}(u)}\right) \rho_K(u)^n \, dS(u) \\ &\leq \frac{1}{2n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_{L_1}(u)}\right) \rho_K(u)^n \, dS(u) + \frac{1}{2n} \int_{S^{n-1}} \phi\left(\frac{\rho_K(u)}{\rho_{L_2}(u)}\right) \rho_K(u)^n \, dS(u) \\ &= \frac{1}{2} \tilde{V}_{-\phi}(K, L_1) + \frac{1}{2} \tilde{V}_{-\phi}(K, L_2) \end{aligned}$$

$$\begin{aligned}
 &= \tilde{V}_{-\phi}(K, L_1) \\
 &= \tilde{V}_{-\phi}(K, L_2),
 \end{aligned}$$

with equality if and only if $L_1 = L_2$. Thus,

$$\tilde{V}_{-\phi}(K, L) < \tilde{V}_{-\phi}(K, L_1) = \tilde{V}_{-\phi}(K, L_2),$$

is the contradiction that would arise if it were the case that $L_1 \neq L_2$. This completes the proof. \square

In view of Theorem 1.1, naturally, we introduce the following definition.

Definition 3.3 For $K \in \mathcal{S}_o^n$, and $\phi \in \Phi$, we define the dual Orlicz geominimal surface area, $\tilde{G}_{-\phi}(K)$, of K by

$$\tilde{G}_{-\phi}(K) = \inf\{n\tilde{V}_{-\phi}(K, Q) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\}.$$

When $\phi(t) = t^p$, with $p \geq 1$, the dual Orlicz geominimal surface area reduces to Wang’s p -geominimal surface area.

We will show that the dual Orlicz geominimal surface area of a stat body is invariant under unimodular centro-transformations of the body.

Proposition 3.4 Suppose $K \in \mathcal{S}_o^n$. If $\phi \in \Phi$ and $\Lambda \in \text{SL}(n)$, then

$$\tilde{G}_{-\phi}(\Lambda K) = \tilde{G}_{-\phi}(K).$$

Proof From Definition 3.3 of the dual Orlicz geominimal surface area, and Lemma 2.1, we have for $\Lambda \in \text{SL}(n)$

$$\begin{aligned}
 \tilde{G}_{-\phi}(\Lambda K) &= \inf\{n\tilde{V}_{-\phi}(\Lambda K, Q) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\} \\
 &= \inf\{n\tilde{V}_{-\phi}(K, \Lambda^{-1}Q) : \Lambda^{-1}Q \in \mathcal{K}_o^n \text{ and } V((\Lambda^{-1}Q)^*) = V(\Lambda^t Q^*) = \omega_n\} \\
 &= \tilde{G}_{-\phi}(K).
 \end{aligned}$$

\square

The unique body whose existence is guaranteed by Theorem 1.1 will be denoted by $\tilde{T}_\phi K$, and it is called the dual Orlicz-Petty body of K . The polar body of $\tilde{T}_\phi K$ will be denoted by $\tilde{T}_\phi^* K$, rather than $(\tilde{T}_\phi K)^*$. Thus, for $K \in \mathcal{S}_o^n$, the body $\tilde{T}_\phi K$ is defined by

$$\tilde{G}_{-\phi}(K) = n\tilde{V}_{-\phi}(K, \tilde{T}_\phi K) \quad \text{and} \quad V(\tilde{T}_\phi^* K) = \omega_n.$$

The next proposition shows that the mapping $\tilde{T}_\phi : \mathcal{S}_o^n \rightarrow \mathcal{K}_o^n$ is a unimodular centro-affine invariant mapping.

Proposition 3.5 If $K \in \mathcal{S}_o^n$, then for $\Lambda \in \text{SL}(n)$,

$$\tilde{T}_\phi \Lambda K = \Lambda \tilde{T}_\phi K.$$

Proof From the definition of \tilde{T}_ϕ and Proposition 3.4,

$$n\tilde{V}_{-\phi}(K, \tilde{T}_\phi K) = \tilde{G}_{-\phi}(K) = \tilde{G}_{-\phi}(\Lambda K) = n\tilde{V}_{-\phi}(\Lambda K, \tilde{T}_\phi \Lambda K).$$

By Lemma 2.1,

$$\tilde{V}_{-\phi}(K, \tilde{T}_\phi K) = \tilde{V}_{-\phi}(\Lambda K, \tilde{T}_\phi \Lambda K) = \tilde{V}_{-\phi}(K, \Lambda^{-1}\tilde{T}_\phi \Lambda K).$$

The uniqueness part of Theorem 1.1 shows that $\tilde{T}_\phi K = \Lambda^{-1}\tilde{T}_\phi \Lambda K$, which is the desired result. \square

An immediate consequence of the definition of $\tilde{G}_{-\phi}$ and Lemma 2.4 is the following.

Proposition 3.6 *If $\phi \in \Phi$, then the functional $\tilde{G}_{-\phi} : \mathcal{S}_o^n \rightarrow (0, \infty)$, is continuous.*

For $Q \in \mathcal{K}^n$, let $\text{Cen}(Q) \in \text{int } Q$ denote the centroid of Q . Associated with each $Q \in \mathcal{K}^n$ is a point $s = \text{San}(Q) \in \text{int } Q$, called the Santaló point of Q , the Santaló point also may be defined as the unique $s \in Q$, such that

$$V((-s + Q)^*) = \min\{V((-x + Q)^*) : x \in \text{int } Q\},$$

or equivalently, as the unique $s \in Q$, such that

$$\int_{S^{n-1}} uh(-s + Q, u)^{-(n+1)} dS(u) = 0.$$

We recall that \mathcal{K}_s^n denotes the set of convex bodies having their Santaló point at the origin. Thus (see [6]),

$$Q \in \mathcal{K}_s^n \quad \text{if and only if} \quad Q^* \in \mathcal{K}_c^n.$$

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2 Suppose $Q \in \mathcal{K}_o^n$, and s is the Santaló point of Q . Let $Q_0 = -s + Q \in \mathcal{K}_s^n$. Since $V(Q_0^*) \leq V(Q^*)$ (see [6], p.263), then by the definition of the dual Orlicz geominimal surface area, the dual Orlicz-Minkowski inequality (2.8) and the Blaschke-Santaló inequality (2.3), it follows that

$$\begin{aligned} \tilde{G}_{-\phi}(K) &= \inf\{n\tilde{V}_{-\phi}(K, Q) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\} \\ &\geq \inf\left\{nV(K)\phi\left(\left(\frac{V(K)}{V(Q)}\right)^{\frac{1}{n}}\right) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\right\} \\ &= \inf\left\{nV(K)\phi\left(\left(\frac{V(K)}{V(s + Q_0)}\right)^{\frac{1}{n}}\right) : Q_0 \in \mathcal{K}_s^n \text{ and } V(Q_0^*) \leq \omega_n\right\} \\ &= \inf\left\{nV(K)\phi\left(\left(\frac{V(K)}{V(Q_0)}\right)^{\frac{1}{n}}\right) : Q_0 \in \mathcal{K}_s^n \text{ and } V(Q_0^*) \leq \omega_n\right\} \\ &\geq nV(K)\phi\left(\left(\frac{V(K)}{\omega_n}\right)^{\frac{1}{n}}\right). \end{aligned}$$

By the equality conditions of the dual Orlicz-Minkowski inequality (2.8) and the Blaschke-Santaló inequality (2.3), we see that equality holds in (1.5) if and only if K is an ellipsoid centered at the origin. \square

Proof of Theorem 1.3 By the dual Orlicz-Minkowski inequality (2.8), we get

$$\begin{aligned} &\omega_n \tilde{G}_{-\phi}(K) \phi\left(\left(\frac{V(Q^*)}{V(K^*)}\right)^{\frac{1}{n}}\right) \\ &= \inf\left\{n \tilde{V}_{-\phi}(K, Q) \phi\left(\left(\frac{V(Q^*)}{V(K^*)}\right)^{\frac{1}{n}}\right) V(Q^*) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\right\} \\ &\leq \inf\{n \tilde{V}_{-\phi}(K, Q) \tilde{V}_{\phi}(Q^*, K^*) : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\}. \end{aligned} \tag{3.1}$$

Since $K \in \mathcal{K}_c^n$, taking $Q = K$ in (3.1), and combining equation (2.7) with inequality (2.3), we get

$$\omega_n \tilde{G}_{-\phi}(K) \leq n V(K) V(K^*) \leq n \omega_n^2,$$

i.e.,

$$\tilde{G}_{-\phi}(K) \leq n \omega_n. \tag{3.2}$$

Similarly,

$$\tilde{G}_{-\phi}(K^*) \leq n \omega_n. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\tilde{G}_{-\phi}(K) \tilde{G}_{-\phi}(K^*) \leq (n \omega_n)^2. \tag{3.4}$$

By the equality conditions of the dual Orlicz-Minkowski inequality (2.8) and the Blaschke-Santaló inequality (2.3), we see that equality in (3.4) holds if and only if K is an ellipsoid centered at the origin. \square

Proof of Theorem 1.4 By the definition of the dual Orlicz geominimal surface area and ϕ_i ($i = 1, 2$) being strictly increasing on $(0, \infty)$, we have

$$\begin{aligned} &\phi_1\left(\frac{n^n V(K)^n}{\tilde{G}_{-\phi_2}(K)^n \phi_2(V(K)^{-1})}\right) \\ &= \phi_1\left(\frac{n^n V(K)^n}{\inf\{n^n \tilde{V}_{-\phi_2}(K, Q)^n : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\} \phi_2(V(K)^{-1})}\right) \\ &\leq \phi_1\left(\frac{n^n V(K)^n}{\inf\{n^n \tilde{V}_{-\phi_1}(K, Q)^n : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\} \phi_1(V(K)^{-1})}\right) \\ &\leq \phi_2\left(\frac{n^n V(K)^n}{\inf\{n^n \tilde{V}_{-\phi_1}(K, Q)^n : Q \in \mathcal{K}_o^n \text{ and } V(Q^*) = \omega_n\} \phi_1(V(K)^{-1})}\right) \\ &= \phi_2\left(\frac{n^n V(K)^n}{\tilde{G}_{-\phi_1}(K)^n \phi_1(V(K)^{-1})}\right). \end{aligned}$$

According to the inverse ϕ_i^{-1} ($i = 1, 2$) of ϕ_i ($i = 1, 2$) being strictly increasing and having continuity on $(0, \infty)$, it follows that

$$\phi_2^{-1}\left(\frac{n^n V(K)^n}{\tilde{G}_{-\phi_2}(K)^n \phi_2(V(K)^{-1})}\right) \leq \phi_1^{-1}\left(\frac{n^n V(K)^n}{\tilde{G}_{-\phi_1}(K)^n \phi_1(V(K)^{-1})}\right).$$

This completes the proof. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors completed the paper, and read and approved the final manuscript.

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